Technical Notes and Correspondence

Flatness-Based Control of a Single Qubit Gate

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***Abstract*—This work considers the open-loop control problem of steering a two-level quantum system from any initial to any final condition. The model of this system evolves on the state space $\mathcal{X} = SU(2)$, having two inputs that correspond to the complex amplitude of a resonant laser field. A symmetry preserving flat output is constructed using a fully geometric construction and quaternion computations. Simulation results of this flatness-based open-loop control are provided.**

***Index Terms*—Flatness, geometric control, nonlinear systems, quantum control, qubit gate.**

I. Introduction

Take a single qubit, i.e., a two-level quantum system. Denote by ω_0 its transition frequency. Assume that it is controlled via a resonant laser field $v \in \mathbb{R}$:

$$v = u \exp(-\imath\omega_0 t) + u^* \exp(\imath\omega_0 t) \tag{1}$$

where $u = u_1 + \imath u_2 \in \mathbb{C}$, $(u_1, u_2) \in \mathbb{R}^2$ is its complex amplitude. In general, the frequency ω_0 is large and the time variation of u is slow: $|\dot{u}| \ll \omega_0 |u|$. In the interaction frame, after the rotating wave approximation and up to some scaling (see, e.g., [1]), the Hamiltonian reads $u_1\sigma_1 + u_2\sigma_2$, where σ_1 and σ_2 are the first two Pauli matrices (see appendix). The gate generation problem then reads: take a transition time $T > 0$ such that $\omega_0 T \ll 1$ and a goal matrix $\bar{U} \in SU(2)$; find a smooth laser impulsion $[0, T] \ni t \mapsto u(t) \in \mathbb{C}$ with $u(0) = u(T) = 0$ such that the solution $[0, T] \ni t \mapsto U(t) \in SU(2)$ of the initial value problem

$$\imath \frac{d}{dt} U(t) = (u_1(t)\sigma_1 + u_2(t)\sigma_2) U(t) \qquad U(0) = I_2 \tag{2}$$

reaches $\bar{U}$ at time T, i.e., $U(T) = \bar{U}$. This motion planning problem admits a well-known elementary solution.[1] It relies on the fact that $\bar{U} = \exp(-\imath\gamma\sigma_1)\exp(-\imath\beta\sigma_2)\exp(-\imath\alpha\sigma_1)$ for all $\bar{U} \in SU(2)$, for convenient $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ (see, e.g., [2]). An obvious steering control $u(t)$ is decomposed into three elementary and successive pulses: for the first (resp. third) pulse, $u_2 = 0$ and u_1 is such that its integral over the pulse interval equals α (resp. γ); for the second pulse, $u_1 = 0$ and the integral of u_2 is β.

Another possibility is to use optimal control techniques to solve this motion planning problem. As in [3] and [4], one can minimize the control energy or transition time when the control is bounded. Despite the fact that these small-dimensional optimal problems are well understood from a mathematical point of view (see, e.g., [5] and the references therein) and the fact that optimal control techniques provide, for specific initial and final states, very elegant and natural solutions (such as for the 1/2 spin flipping problem), this approach has not provided, up until now, solutions as explicit as the one we propose in theorem 2 of Section III and valid for any initial and final states in $SU(2)$. Moreover, the motion planning proposed here provides controls that can be chosen to be C^ω or C^∞ functions of t, and the geometric path followed the state on $SU(2)$ is always smooth. As far as we know, such explicit and smooth solution is new and could be of some interest for reducing the transition time T, while still respecting the rotating wave approximation. Our approach combines flatness-based motion planning techniques [6] and symmetries to obtain a globally defined flat-output map: this controllable driftless system with three states and two controls is automatically flat (it is locally equivalent to a contact system); the system lives on a compact Lie Group $SU(2)$ and is invariant versus right translations. The flat output constructed in this note has a clear geometrical interpretation: the flat output map is globally defined, goes from $SU(2)$ to $\mathbb{S}^2$, and is compatible (equivariance) with respect to right multiplication on $SU(2)$ in the sense of [7].

In Section II, theorem 1 proposes, using a quaternion description of (2), a coordinate-free definition of the flat output that lives in the homogenous space $SU(2)/\exp(\imath\mathbb{R}\sigma_1) \equiv \mathbb{S}^2$. This geometric construction preserves invariance with respect to right translations. The proposed construction can be seen as the analogue of the geometric construction based on the Frenet formula for the car system, where the steering angle is directly related to the curvature of the path followed by the flat-output curve [6], [8]. The geometric and globally defined flat output of theorem 1 is a first important but not sufficient property to solve analytically, and explicitly the motion planning problem corresponding to an arbitrary quantum gate. It shows that the state is an algebraic function of the flat-output y and its time derivatives. This means that, generically, one can locally express the state as a smooth function of y and $\dot{y}$ (there are four branches of solutions). Theorem 1 does not imply that, for any initial and final states, one can find a smooth trajectory $t \mapsto y(t)$, such that, for any time t, the corresponding state trajectory remains on the same solution branch. This is precisely the object of theorem 2 in Section III that defines explicitly, for any initial and final states, a specific smooth trajectory $t \mapsto y(t)$ that provides a state trajectory that remains on the same algebraic branch (with no singulary crossing). Simulations illustrate the explicit and analytic solution elaborated in theorem 2 and the smoothness of the open-loop steering control, and the smoothness the geometric path followed by the state. In Section IV, we conclude and recall how to eliminate a possible drift; thus, our approach can be used for any two-state quantum systems with two independent controls. Some materials have been deferred to the appendix. In Appendix A, one finds the basic properties of Pauli matrices and their associated quaternions as well as the correspondence between $SU(2)$ and quaternions of length one. In Appendix B, one finds a proof of the fact that the motion planning algorithm has no singularities.

II. Symmetry Preserving Flat Output

The dynamics (2) read in quaternion notation (see Appendix A)

$$\frac{d}{dt} q = (u_1 e_1 + u_2 e_2) q \tag{3}$$

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[1]The so-called ZYZ quantum logic gate.

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where $q \in \mathbb{H}_1$ is a quaternion of length one and where $(u_1, u_2) \in \mathbb{R}^2$ is the control relative to the modulation of a coherent laser field ($u_1 + \imath u_2$ is the complex field amplitude). This system is a driftless system on the Lie Group $\mathbb{H}_1$. It is controllable (see, e.g., [9]). Moreover, this control system is invariant with respect to right translations in the sense of [7] and [10]:

1) the group $G = \mathbb{H}_1$ acts on the state space $\mathcal{X} = \mathbb{H}_1$ via right multiplication $\phi_g : q \mapsto qg$, where $q \in \mathbb{H}_1$;
2) the dynamics is G-invariant: if $t \mapsto (q(t), u_1(t), u_2(t))$ is a solution of (3), then $t \mapsto (q(t)g, u_1(t), u_2(t))$ is also a solution of (3) for any $g \in G$.

The controllability structure of this system is, in fact, of a very special kind. Around any point $\bar{q} \in \mathbb{H}_1$, (3) can be seen in local coordinates as a driftless controllable system with three states[2] and two controls. Thus, as known since [11] (see also [12]), such a system is differentially flat, and the flat output function can be chosen to depend only on the state. More precisely, the flat output for the controllable system $(d/dt)x = u_1 f_1(x) + u_2 f_2(x)$ with $\dim(x) = 3$ is obtained by the rectifying coordinates of any vector field $f(x) = \alpha_1(x) f_1(x) + \alpha_2(x) f_2(x)$, which is a linear combination of the two control vector fields f_1 and f_2 (α_1 and α_2 are any scalar functions of x).

We propose here a coordinate-free and symmetry-preserving construction of the flat output via the previous procedure. Thus, we are looking for a flat output map $h : \mathbb{H}_1 \mapsto \mathcal{Y}$, where $\mathcal{Y}$ is the output space, a compact manifold of dimension 2, and G-compatible in the sense of [7]. This means that the output map h must satisfy the following constraint: there exists an action of $G = \mathbb{H}_1$ on the flat output space $\mathcal{Y}$ described by the transformation group $\varrho_g : y \mapsto \varrho_g(y)$ such that $\varrho_g(h(q)) = h(qg)$ for any $q \in \mathbb{H}_1$. The following construction will be based on the control vector field associated to u_1, and hence, to e_1.

Denoted by $K = \{\exp(\phi e_1)\}_{\phi \in [0, 2\pi]}$ the 1-D subgroup of $\mathbb{H}_1$ generated by e_1. We can consider the action of K on $\mathbb{H}_1$ via left multiplication: to any $k \in K$, we have the diffeomorphism $\mathbb{H}_1 \ni q \mapsto kq \in \mathbb{H}_1$. Two elements of $\mathbb{H}_1$, q and p, belong to the same orbit if and only if there exists $k \in K$ such that $kq = p$. Denote by $\mathcal{Y}$ the set of the orbits. This set is a compact manifold of dimension 2, and the output function h is the map that associates to any q, the orbit $h(q)$ to which q belongs. This map is a smooth submersion, and $\mathcal{Y}$ is called an homogenous space (see, e.g., [13]). If q and p belong to the same orbit, qg and pg also belong to the same orbit for any $g \in \mathbb{H}_1$. Therefore, this output map is G-compatible in the sense of [7]. Notice that $\mathcal{Y}$ can be identified with the unit sphere of $\mathbb{R}^3$: $\mathcal{Y} \equiv \mathbb{S}^2$.

Assume that $y(t)$ is a curve on $\mathcal{Y} = \mathbb{H}_1/K$, at least of class C^2. Since the map $h : \mathbb{H}_1 \to \mathcal{Y}$ is a submersion, in adequate local coordinates, one has $h(x_1, x_2, x_3) = (x_1, x_2)$. Assume, without the loss of generality, that the open neighborhood of definition of h is *rectangular* and contains $(0, 0, 0)$. Define locally the smooth map $g : U \subset \mathcal{Y} \to V \subset \mathbb{H}_1$, where U and V are open sets and $g(x_1, x_2) = (x_1, x_2, 0)$. Note that g is smooth, and $Y(t) = g(y(t))$ is such that $h(Y(t)) = y(t)$. Then, locally, there exist smooth maps $g^{(1)}$ and $g^{(2)}$ such that $\dot{Y}(t) = g^{(1)}(y(t), \dot{y}(t))$ and $\ddot{Y}(t) = g^{(2)}(y(t), \dot{y}(t), \ddot{y}(t))$.

Let us show now that the map h defines a flat output. This means that the inverse of system (3) with output $y = h(q)$ has no dynamics.[3] Thus, we have to consider the following implicit system

$$\frac{d}{dt} q = (u_1 e_1 + u_2 e_2) q \qquad y = h(q)$$

where $t \mapsto y(t)$ is a known function of time and where the quaternion $q(t) \in \mathbb{H}_1$ and the control $(u_1(t), u_2(t))$ are the unknown quantities.

The problem is how to manipulate h, since only a geometric construction for h is available. Knowing the function $t \mapsto y(t)$ means that we have at our disposal a smooth function $t \mapsto Y(t) \in \mathbb{H}_1$ such that $y(t) = h(Y(t))$. Hence, to have $y(t) = h(q(t))$ means that q and Y belong to the same orbit for each time t. Therefore, there exists $k(t) = \exp(\phi(t) e_1)$ in K such that $q = kY$. Since $k(t) = q(t) Y^*(t)$, then $k(t)$ is smooth. Thus, we have

$$\frac{d}{dt} q = \left(\frac{d}{dt} k\right) Y + k \frac{d}{dt} Y.$$

But, $(d/dt)k = \omega e_1 k$ where $\omega = (d/dt)\phi$. Using (3), we get the following equation $k(d/dt)Y = ((u_1 - \omega)e_1 + u_2 e_2) kY$, that is

$$k \left(\frac{d}{dt} Y\right) Y^* k^* = (u_1 - \omega) e_1 + u_2 e_2.$$

This quaternion equation gives, in fact, k as a function of $\left(\frac{d}{dt} Y\right) Y^*$. Left and right multiplication by e_3 yields

$$e_3 k \left(\frac{d}{dt} Y\right) Y^* k^* e_3 = (u_1 - \omega) e_1 + u_2 e_2$$

since $e_3 e_i e_3 = e_i$ for $i = 1, 2$. Hence, we have the following relation (without the controls and ω)

$$e_3 k \left(\frac{d}{dt} Y\right) Y^* k^* e_3 = k \left(\frac{d}{dt} Y\right) Y^* k^*. \tag{4}$$

Assume that

$$\left(\frac{d}{dt} Y\right) Y^* = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 \tag{5}$$

where the ω_i's are known smooth real functions of time. Thus, we get

$$k \left(\frac{d}{dt} Y\right) Y^* k^* = \omega_1 e_1 + k^2 (\omega_2 e_2 + \omega_3 e_3)$$

since $e_1 k^* = k^* e_1$, $kk^* = 1$, and $e_i k^* = k e_i$ for $i = 2, 3$. Therefore, (4) reads

$$k^4 (\omega_2 e_2 + \omega_3 e_3) = (\omega_2 e_2 - \omega_3 e_3)$$

since $e_3 k^2 = (k^*)^2 e_3$ and $k^{-1} = k^*$.

Right multiplication by e_2 yields the following algebraic equation defining k

$$k^4 (\omega_2 + \omega_3 e_1) = (\omega_2 - \omega_3 e_1).$$

Since $k = \cos\phi + \sin\phi e_1$, we have the following equation for the angle ϕ

$$(\cos 4\phi + \sin 4\phi e_1)(\omega_2 + \omega_3 e_1) = (\omega_2 - \omega_3 e_1)$$

which is equivalent to $\exp(4\phi\imath) = z^2/|z|^2$, where $z = \omega_2 - \omega_3 \imath$ is a known complex number. Thus, there are four distinct possibilities for k:

$$k = \pm \exp\left(\frac{\theta}{2} e_1\right) \qquad k = \pm e_1 \exp\left(\frac{\theta}{2} e_1\right) \tag{6}$$

where θ is the argument of $\omega_2 - \omega_3 \imath$. The controls u_1 and u_2 associated to one of these four trajectories are obtained by

$$e_3 k \frac{d}{dt} Y Y^* k^* e_3 = (u_1 - \omega) e_1 + u_2 e_2$$

[2]Take, e.g., the exponential map: $(x_1, x_2, x_3) \in \mathbb{R}^3 \mapsto \exp(x_1 e_1 + x_2 e_2 + x_3 e_3)\bar{q}$ that maps a neighborhood of $0 \in \mathbb{R}^3$ to a neighborhood of $\bar{q}$ in $\mathbb{H}_1$.

[3]This is equivalent to say that the state q and the input $u = (u_1, u_2)$ can be written, respectively, as $q = \mathcal{A}(y, \dot{y}, \ddot{y}, \ldots, y^{(\alpha)})$ and $u = \mathcal{B}(y, \dot{y}, \ddot{y}, \ldots, y^{(\beta)})$ for convenient smooth maps $\mathcal{A}$ and $\mathcal{B}$.

where $2\omega = (d/dt)\theta$ is given via simple algebraic formulae based on $\omega_2, \omega_3, (d/dt)\omega_2$, and $(d/dt)\omega_3$

$$\omega = \frac{\omega_3(d/dt)\omega_2 - \omega_2(d/dt)\omega_3}{2(\omega_2^2+\omega_3^2)}.$$

For the two branches $k = \pm\exp[(\theta/2)e_1]$, we get

$$\begin{cases} u_1 = \omega_1 + \dfrac{\omega_3(d/dt)\omega_2 - \omega_2(d/dt)\omega_3}{2(\omega_2^2+\omega_3^2)} \\ u_2 = \sqrt{\omega_2^2+\omega_3^2} \end{cases}$$

and for the two other ones $k = \pm e_1\exp[(\theta/2)e_1]$, we get

$$\begin{cases} u_1 = \omega_1 + \dfrac{\omega_3(d/dt)\omega_2 - \omega_2(d/dt)\omega_3}{2(\omega_2^2+\omega_3^2)} \\ u_2 = -\sqrt{\omega_2^2+\omega_3^2} \end{cases}$$

where just the sign of u_2 is changed. All the previous computations are valid when $\omega_2 - \omega_3\imath \neq 0$, i.e., when $(d/dt)y \neq 0$: $(\omega_2^2+\omega_3^2)$ does not depend on $Y(t)$ such that $h(Y(t)) = y(t)$; it depends only on $y(t)$ and vanishes if and only if $(d/dt)y(t) = 0$. To summarize, we have proved the following result.

Theorem 1: Take $T > 0$ and an arbitrary C^2 curve $[0,T] \ni t \mapsto y(t)$ on $\mathcal{Y}$ such that $(d/dt)y(t) \neq 0$ for any $t \in [0,T]$. For any smooth curve $t \mapsto Y(t) \in \mathbb{H}_1$ such that $h(Y(t)) = y(t)$, set $z = \omega_2(t) - \omega_3(t)\imath \neq 0$ for all $t \in [0,T]$ where $((d/dt)Y)Y^* = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$. Then, there exists a smooth function $[0,T] \ni t \mapsto \theta(t) \in \mathbb{R}$ such that $\exp(\theta\imath) = z/|z|$ and any smooth solution $t \mapsto (q(t), u_1(t), u_2(t))$ of (3) satisfying $h(q(t)) = y(t)$ for all $t \in [0,T]$ is one of the four following trajectories indexed by $n \in \{0,1,2,3\}$

$$\begin{cases} q(t) = (e_1)^n \exp\left(\dfrac{\theta(t)}{2}e_1\right) Y(t) \\ u_1 = \omega_1 + \dfrac{\omega_3(d/dt)\omega_2 - \omega_2(d/dt)\omega_3}{2(\omega_2^2+\omega_3^2)} \\ u_2 = (-1)^n\sqrt{\omega_2^2+\omega_3^2} \end{cases}. \tag{7}$$

This theorem proves that $y = h(q)$ is a flat output since locally the state q and the control u can be expressed as a function of $(y, \dot{y}, \ddot{y})$. Notice that the relationships (7) describing, in fact, four branches of solutions can be seen as an inverse kinematics problem, a problem usually encountered in robotics such as the conversion from Cartesian to angular coordinates where different families of angular configurations provide the same Cartesian position. Here, the problem is a bit different since, additionally, we use for describing the given data, i.e., the flat output y, more degrees of freedom (in fact 3) than necessary (in fact 2): the curve $t \mapsto y(t) \in \mathcal{Y} \equiv \mathbb{S}^2$ is described by a curve $t \mapsto Y(t) \in \mathbb{H}_1$ where for each t, $h(Y(t)) = y(t)$, but where $t \mapsto Y(t)$ is not necessarily a trajectory of the system ($\omega_3 \neq 0$ in general). This redundancy can be seen as a convenient and simple way to maintain the computations global since two curves $t \mapsto Y_1(t)$ and $t \mapsto Y_2(t)$ on $\mathbb{H}_1$ such that $h(Y_1) \equiv h(Y_2)$ (i.e., for each t, there exists $\alpha \in \mathbb{R}$ such that $Y_1(t) = \exp(\alpha e_1)Y_2(t)$) leads via (7) to the same values for q and u. A minimal parametrization of the flat output would lead to nonintrinsic singularities in the system inversion procedure required by the flatness approach.

The flat output $y = h(q)$ is obtained with e_1 playing a specific role. In fact, one can see that any map $h_\eta : \mathbb{H}_1 \mapsto \mathbb{H}_1/K_\eta$ ($\eta \in \mathbb{S}^1$) corresponding to the subgroup $K_\eta = \exp(\mathbb{R}(\cos\eta e_1 + \sin\eta e_2))$ also defines a flat output. It just corresponds to a rotation by the angle η of (q_1, q_2) and (u_1, u_2). If we set

$$e_1 = \cos\eta\tilde{e}_1 + \sin\eta\tilde{e}_2 \qquad e_2 = -\sin\eta\tilde{e}_1 + \cos\eta\tilde{e}_2$$

the imaginary quaternions (e_1, e_2, e_3) and $(\tilde{e}_1, \tilde{e}_2, e_3)$ satisfy exactly the same commutation relations. Thus, if $t \mapsto (q_0(t), q_1(t), q_2(t), q_3(t))$ is a solution of (3) with the control $(u_1(t), u_2(t))$, then $t \mapsto (\tilde{q}_0(t), \tilde{q}_1(t), \tilde{q}_2(t), \tilde{q}_3(t))$ is also a solution of (3) with the control $(\tilde{u}_1(t), \tilde{u}_2(t))$, where $\tilde{q}_0(t) = q_0(t)$, $\tilde{q}_1(t) = (q_1(t)\cos\eta - q_2(t)\sin\eta)$, $\tilde{q}_2(t) = (q_1(t)\sin\eta + q_2(t)\cos\eta)$, $\tilde{q}_3(t) = q_3$, $\tilde{u}_1 = u_1(t)\cos\eta - u_2(t)\sin\eta$, $\tilde{u}_2 = u_1(t)\sin\eta + u_2(t)\cos\eta$. This symmetry and the fact that, as stated in theorem 1, $h = h_0$ is a flat output, implies directly that h_η is also a flat output. The family $(h_\eta)_{\eta\in\mathbb{S}^1}$ is made of flat outputs all compatible with right translations.

III. Motion Planning

In this section, we will use (7) with $n = 0$ to propose an explicit solution for the motion planning problem stated in the introduction: for any $T > 0$ and any final state $\bar{q} \in \mathbb{H}_1$, find a smooth control $[0,T] \ni t \mapsto u(t) = (u_1(t), u_2(t)) \in \mathbb{R}^2$ with $u(0) = u(T) = 0$, such that the solution $[0,T] \ni t \mapsto q(t) \in \mathbb{H}_1$ of (3) starting from $q(0)$ reaches $\bar{q}$ at time T: i.e., $q(T) = \bar{q}$.

As the system is driftless, every time reparameterization of a solution is also a solution. In fact, consider the equation

$$\frac{d}{ds}\tilde{q}(s) = (\tilde{u}_1(s)e_1 + \tilde{u}_2(s)e_2)\tilde{q}(s).$$

Let $\varsigma : [0,T] \to [0,1]$ be an increasing diffeomorphism. Then, $\tilde{q}(s)$ is a solution of the previous equation defined on $[0,1]$ with input $(\tilde{u}_1(s), \tilde{u}_2(s))$ if and only if $q(t) = \tilde{q}(\varsigma(t))$ is a solution of (3) defined on $[0,T]$ with input $(u_1(t), u_2(t)) = (d\varsigma/dt)(\tilde{u}_1(\varsigma(t)), \tilde{u}_2(\varsigma(t)))$. One concludes that, without the loss of generality, one may always state the motion planning problem with the (virtual) time s belonging to the interval $[0,1]$, and, after that, one may "control the clock" by choosing a convenient bijection $s = \varsigma(t)$. Thus, it is enough to solve the motion planning problem in the s-scale, where we can disregard the fact that the control has to vanish at the beginning and at the end: it is enough to take for example $\varsigma(t) = 3(t/T)^2 - 2(t/T)^3$ to get u equal to zero at $t = 0$ and at $t = T$, since $(d/dt)\varsigma(0) = (d/dt)\varsigma(T) = 0$.

In the sequel, we propose a solution in the s-scale. For clarity's sake, we will remove the $\tilde{}$ when u and q are considered as function of s. The derivation in s will be denoted by $'$: $du/ds = u'$, $dq/ds = q'$, ...

Thus, we have to find a smooth control $[0,1] \ni s \mapsto u(s)$ such that the solution of

$$q'(s) = (u_1(s)e_1 + u_2(s)e_2)q(s) \qquad q(0) = 1$$

satisfies $q(1) = \bar{q}$, where $\bar{q}$ is any goal state in $\mathbb{H}_1$.

We can always assume that

$$\bar{q} = \bar{q}_0 + \sqrt{\bar{q}_1^2 + \bar{q}_2^2}(\sin\bar{\eta}e_1 + \cos\bar{\eta}e_2) + \bar{q}_3$$

for some angle $\bar{\eta} \in [0, 2\pi]$. Thus, as explained at the end of last section, up to a rotation of angle $\bar{\eta}$ of the control, we can assume that $\bar{q}_1 = 0$. More precisely, if $\bar{q}_1 \neq 0$, set $\bar{\eta}$ to be the argument of the complex $\bar{q}_2 + \bar{q}_1\imath$. If $s \mapsto (u_1(s), u_2(s))$ steers q from $q(0) = 1$ to $q(1) = \bar{q}_0 + \sqrt{\bar{q}_1^2 + \bar{q}_2^2}e_2 + \bar{q}_3 e_3$, then the control

$$s \mapsto (u_1(s)\cos\bar{\eta} + u_2(s)\sin\bar{\eta}, -u_1(s)\sin\bar{\eta} + u_2(s)\cos\bar{\eta})$$

steers q from $q(0) = 1$ to $q(1) = \bar{q}$.

Thus, up to a rotation of angle $\bar{\eta}$ of the control, we can assume that $\bar{q}_1 = 0$ and $\bar{q}_2 \geq 0$. Thus, we can define two angles $\bar{\alpha} \in]0, \pi]$ and $\bar{\beta} \in [-\pi/2, \pi/2]$, such that

$$\bar{q} = \cos\bar{\alpha} + \sin\bar{\alpha}(\cos\bar{\beta}e_2 + \sin\bar{\beta}e_3).$$

If the control $s \mapsto u(s)$ steers the system from $q(0) = 1$ to $q(1) = \cos\bar{\alpha} + \sin\bar{\alpha}(\cos\bar{\beta}e_2 + \sin\bar{\beta}e_3)$, the same control steers the system from

$$q(0) = \cos\bar{\lambda} + \sin\bar{\lambda}(\cos\bar{\beta}e_2 + \sin\bar{\beta}e_3)$$

to

$$q(1) = \cos(\bar{\lambda} + \bar{\alpha}) + \sin(\bar{\lambda} + \bar{\alpha})(\cos\bar{\beta}e_2 + \sin\bar{\beta}e_3).$$

This is a direct consequence of right translation invariance of (3) and right multiplication by $\cos\bar{\lambda} + \sin\bar{\lambda}(\cos\bar{\beta}e_2 + \sin\bar{\beta}e_3)$.

Take now the formulae (7) in the s-scale with

$$Y(s) = \cos(\alpha(s)) + \sin(\alpha(s))(\cos(\beta(s))e_2 + \sin(\beta(s))e_3 \quad (8)$$

where $\alpha(s)$ and $\beta(s)$ are smooth functions such that

$$\alpha(0) = \bar{\lambda} \qquad \alpha(1) = \bar{\lambda} + \bar{\alpha} \qquad \beta(0) = \beta(1) = \bar{\beta}. \quad (9)$$

Set, as in theorem 1

$$Y'Y^* = \omega_1(s)e_1 + \omega_2(s)e_2 + \omega_3(s)e_3.$$

Simple computations show that

$$z = \omega_2 - \omega_3\imath = \exp(-\imath\beta)(\alpha' - \imath\beta'\cos\alpha\sin\alpha).$$

Now, we shall construct (8) such that $q(s) = \exp(\phi(s)e_1)Y(s), s \in [0, 1]$ is a trajectory of the system. We will assume that $q(0) = Y(0)$ and $q(1) = Y(1)$. So, we must have $\phi(0) = \phi(1) = 0$. Furthermore, if we can ensure that $s \mapsto z(s)$ never vanishes and $\theta(0) = \theta(1) = 0$, then the trajectory of (7) with $n = 0$ will provide a steering control u.

Let us now show in detail how to design the functions $\alpha(s)$ and $\beta(s)$ satisfying these constraints. First of all, we have the initial and final constraints (9). By taking

$$\bar{\lambda} = \begin{cases} -\dfrac{\bar{\alpha}}{2}, & \text{for } \bar{\alpha} \in [\dfrac{\pi}{4}, \dfrac{3\pi}{4}] \\ \dfrac{\pi}{4} - \dfrac{\bar{\alpha}}{2}, & \text{otherwise} \end{cases}$$

we always have $\cos\alpha\sin\alpha$ far from 0 when $s = 0$ and $s = 1$. Thus, we can impose the following initial and final constraints for β'

$$\beta'(0) = -\frac{\bar{\alpha}\sin\bar{\beta}}{\sin\bar{\lambda}\cos\bar{\lambda}} \qquad \beta'(1) = -\frac{\bar{\alpha}\sin\bar{\beta}}{\sin(\bar{\lambda} + \bar{\alpha})\cos(\bar{\lambda} + \bar{\alpha})}$$

and for α'

$$\alpha'(0) = \alpha'(1) = \bar{\alpha}\cos\bar{\beta}.$$

Then, $\alpha(s)$ and $\beta(s)$ are the polynomials of degree ≤ 3 satisfying these initial and final constraints. Since $\bar{\alpha} > 0$ and $|\bar{\beta}| \leq \pi/2$, $s \mapsto \alpha(s)$ can be a strictly increasing function on $[0, 1]$ and $\alpha' > 0$ for $s \in]0, 1[$ (see Appendix B). Thus, the complex number

$$z = \exp(-\imath\beta)(\alpha' - \imath\beta'\cos\alpha\sin\alpha)$$

never vanishes for $s \in]0, 1[$. For $s = 0$ and $s = 1$, we have

$$\alpha' - \imath\beta'\cos\alpha\sin\alpha = \exp(\imath\bar{\beta})\bar{\alpha}.$$

Thus, $z(0) = z(1) = \bar{\alpha} > 0$. To summarize, the closed path $[0, 1] \ni s \mapsto z(s) \in \mathbb{C}$ never passes through 0 nor turns around 0. We satisfy the assumption of theorem 1 in the s-scale. Moreover, we can set $z(s) = r(s)\exp(\imath\theta(s))$ with $r(s) > 0$ and $\theta(s)$ smooth functions on $[0, 1]$ with $\theta(0) = \theta(1) = 0$. We avoid with such design of $\alpha(s)$ and $\beta(s)$ the monodromy problem associated to the resolution of $(\exp(\imath\phi))^4 = z^2/|z|^2$. Finally, we have proved the following result.

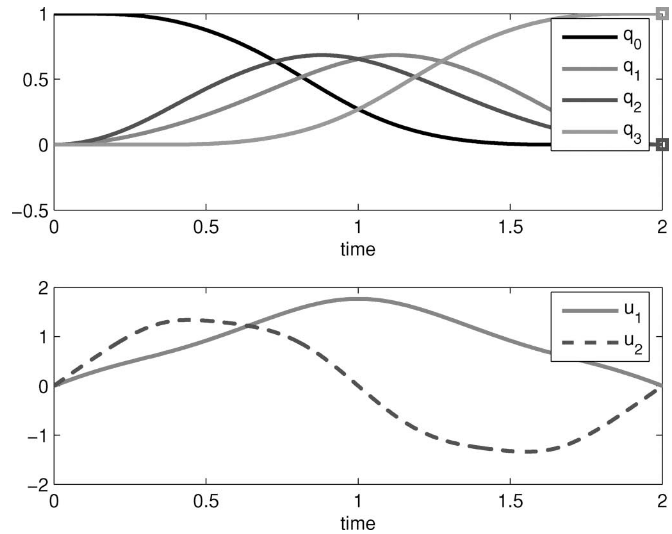

Fig. 1. Steering control and trajectory derived from theorem 2 with $T = 2$, $\bar{q} = e_3$, and $\varsigma(t) = 3(t/T)2 - 2(t/T)3$. The control magnitude is very close to an ZYZ control design with two separated $\pi/2$ pulses. The simulation code (Matlab m-file and scilab sci-file) can be downloaded from http://arxiv.org/.

Theorem 2: Take $\bar{q} = \bar{q}_0 + \bar{q}_1e_1 + \bar{q}_2e_2 + \bar{q}_3e_3 \in \mathbb{H}_1$ with $\bar{q} \neq 1$. Chose $\bar{\eta} \in [0, 2\pi[$ such that $q_1e_1 + q_2e_2 = \sqrt{\bar{q}_1^2 + \bar{q}_2^2}(\sin\bar{\eta}e_1 + \cos\bar{\eta}e_2)$. Define $\bar{\alpha} \in]0, \pi]$ and $\bar{\beta} \in [-\pi/2, \pi/2]$ such that

$$\bar{q}_0 + \sqrt{\bar{q}_1^2 + \bar{q}_2^2}e_2 + \bar{q}_3e_3 = \cos\bar{\alpha} + \sin\bar{\alpha}(\cos\bar{\beta}e_2 + \sin\bar{\beta}e_3).$$

Set $\bar{\lambda} = -\bar{\alpha}/2$ if $\bar{\alpha} \in [\pi/4, 3\pi/4]$ and $\bar{\lambda} = \pi/4 - \bar{\alpha}/2$ otherwise. Define $\alpha(s)$ and $\beta(s))$ as being the unique polynomial functions of degree ≤ 3 such that ($'$ stands for d/ds)

$$\alpha(0) = \bar{\lambda} \qquad \alpha(1) = \bar{\lambda} + \bar{\alpha} \qquad \alpha'(0) = \alpha'(1) = \bar{\alpha}\cos\bar{\beta}$$
$$\beta(0) = \beta(1) = \bar{\beta}$$
$$\beta'(0) = -\frac{\bar{\alpha}\sin\bar{\beta}}{\sin\bar{\lambda}\cos\bar{\lambda}} \qquad \beta'(1) = -\frac{\bar{\alpha}\sin\bar{\beta}}{\sin(\bar{\lambda} + \bar{\alpha})\cos(\bar{\lambda} + \bar{\alpha})}.$$

Define $\omega_1(s)$, $\omega_2(s)$, and $\omega_3(s)$ by

$$\omega_1 = (1 - 2\cos^2(\alpha))\beta'$$
$$\omega_2 - \imath\omega_3 = \exp(-\imath\beta)(\alpha' - \imath\beta'\sin\alpha\cos\alpha).$$

Then, ω_2 and ω_3 never vanish simultaneously, and the control $(u_1(t), u_2(t))$ given by

$$\frac{d\varsigma(t)}{dt}\begin{pmatrix} \cos\eta & \sin\eta \\ -\sin\eta & \cos\eta \end{pmatrix}\begin{pmatrix} \omega_1 + \dfrac{\omega_3\omega_2' - \omega_2\omega_3'}{2(\omega_2^2 + \omega_3^2)} \\ \sqrt{\omega_2^2 + \omega_3^2} \end{pmatrix}\Bigg|_{s=\varsigma(t)}$$

steers system (3) from $q(0) = 1$ to $q(T) = \bar{q}$ with $t \mapsto \varsigma(t) \in [0, 1]$ being a C^k increasing bijection between $[0, T]$ and $[0, 1]$ $k \geq 1$. When, in addition, $d^n\varsigma/dt^n|_s = 0$ for $s = 0$ and $s = 1$, and $n = 1, \ldots, k$, the control $t \mapsto u(t)$ is C^{k-1} with $d^{n-1}u/dt^{n-1} = 0$ for $s = 0$ and $s = 1$.

Fig. 1 illustrates the steering control described by theorem 2 with $T = 2$, $\bar{q}_0 = e_3$, and $\varsigma(t) = 3(t/T)^2 - 2(t/T)^3$. We see that the control is a smooth function with maxima around $\pi/2$, a value close to the ZYZ design based on two successive pulses: $(u_1, u_2) = (0, \pi/2)$ for $t \in [0, 1]$ and $(u_1, u_2) = (\pi/2, 0)$ for $t \in [1, 2]$. Thus, our flatness-based design yields, with the same transition time and control magnitude, smooth control actions.

IV. Concluding Remarks

The results of this note hold if the laser matches exactly the resonant frequency. If we have a frequency offset of Δ from resonance, then this offset leads to the following drift (see, e.g., [1])

$$\frac{d}{dt} q = \left(u_1 e_1 + u_2 e_2 + \frac{\Delta}{2} e_3 \right) q.$$

It is still interesting to notice that $h(q)$ is also a flat output. In this case, the key relation (4) becomes

$$e_3 k \left(\frac{d}{dt} Y \right) Y^* k^* e_3 = k \left(\frac{d}{dt} Y \right) Y^* k^* + \Delta e_3$$

and $k = \exp(\phi e_1)$ is a root of the following polynomial

$$k^4 (\omega_2 e_2 + \omega_3 e_3) + k^2 \Delta e_3 - (\omega_2 e_2 - \omega_3 e_3) = 0.$$

Following [5], it is also interesting to notice the drift term associated with Δ can be removed via the following time-varying change of coordinates and controls

$$q = \exp\left(\frac{\Delta t e_3}{2} \right) \tilde{q} \qquad \tilde{u}_1 + \imath \tilde{u}_2 = e^{-\imath \Delta t} (u_1 + \imath u_2).$$

With these new variables, the dynamics reads $(d/dt)\tilde{q} = (\tilde{u}_1 e_1 + \tilde{u}_2 e_2)\tilde{q}$ and theorem 2 applies directly.

Appendix A

Pauli Matrices and Quaternions

The Hermitian matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the three Pauli matrices. They satisfy $\sigma_k^2 = 1$, $\sigma_k \sigma_j = -\sigma_j \sigma_k$ for $k \neq j$, and

$$\sigma_1 \sigma_2 = \imath \sigma_3 \qquad \sigma_2 \sigma_3 = \imath \sigma_1 \qquad \sigma_3 \sigma_1 = \imath \sigma_2.$$

Any matrix U in $SU(2)$ reads

$$U = q_0 - q_1 \imath \sigma_1 - q_2 \imath \sigma_2 - q_3 \imath \sigma_3$$

with $(q_0, q_1, q_2, q_3) \in \mathbb{R}^4$ such that $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$. By setting

$$e_1 = -\imath \sigma_1 \qquad e_2 = -\imath \sigma_2 \qquad e_3 = -\imath \sigma_3$$

one can identify $SU(2)$ with the set of quaternions

$$q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$$

of length one. This set is denoted by $\mathbb{H}_1$ and corresponds to quaternions $q \in \mathbb{H}$ such that $qq^* = 1$ where $q^* = q_0 - q_1 e_1 - q_2 e_2 - q_3 e_3$ is the conjugate quaternion of q. Thus, the dynamics (2) becomes (3) with q corresponding to U. Notice that $\mathbb{H}_1$ is a compact Lie group of dimension 3.

We recall here some useful relations for $k = 1, 2, 3$, $j \neq k$ and $\phi \in \mathbb{R}$:

$$e_k^2 = -1 \qquad e_k e_j = -e_j e_k \qquad \exp(\phi e_k) = \cos\phi + e_k \sin\phi$$

$$\exp(\phi e_k) e_j = e_j \exp(-\phi e_k)$$

$$e_1 e_2 = e_3 \qquad e_2 e_3 = e_1 \qquad e_3 e_1 = e_2.$$

Appendix B

Proof that $z = \omega_2 - \imath\omega_3$ never vanishes for $s \in]0, 1[$ Since $\omega_2 - \imath\omega_3 = \exp(\imath\beta)(\alpha' - \imath\beta' \sin\alpha \cos\alpha)$, it suffices to show that $\alpha' > 0$ for $s \in]0, 1[$. For this, let $\delta = \bar{\alpha} - \alpha'(0) = \bar{\alpha}(1 - \cos\bar{\beta}) \geq 0$. A simple exercise shows that the polynomial $\alpha(s) = as^3 + bs^2 + cs + d$ meeting the restrictions $\alpha'(0) = \alpha'(1)$ and $\alpha(1) - \alpha(0) = \bar{\alpha}$ is such that $a = -2\delta$, $b = 3\delta$, $c = \alpha'(0)$, and $d = \alpha(0)$. In particular, $\alpha'(s) = -6\delta s(s-1) + \alpha'(0)$. If $\cos\bar{\beta} \neq 1$, then $-6\delta s(s-1) > 0$, for $s \in]0, 1[$. As $\alpha'(0) \geq 0$, then $\alpha' > 0$ for $s \in]0, 1[$. If $\cos\bar{\beta} = 1$, then $\delta = 0$ and $\alpha'(0) = \bar{\alpha}\cos\bar{\beta} > 0$. So, $\alpha' > 0$ for $s \in [0, 1]$.

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