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Periodic input estimation for linear periodic systems: Automotive engine applications☆

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Abstract

In this paper, we consider periodic linear systems driven by T_0-periodic signals that we desire to reconstruct. The systems under consideration are of the form $\dot{x} = A(t)x + A_0(t)w(t)$, $y = C(t)x$, $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $(m \leqslant p \leqslant n)$ where $A(t)$, $A_0(t)$, and $C(t)$ are T_0-periodic matrices. The period T_0 is known. The T_0-periodic input signal $w(t)$ is unknown but is assumed to admit a finite dimensional Fourier decomposition. Our contribution is a technique to estimate w from the measurements y. In both full state measurement and partial state measurement cases, we propose an efficient observer for the coefficients of the Fourier decomposition of $w(t)$. The proposed techniques are particularly attractive for automotive engine applications where sampling time is short. In this situation, standard estimation techniques based on Kalman filters are often discarded (because of their relative high computational burden). Relevance of our approach is supported by two practical cases of application. Detailed convergence analysis is also provided. Under standard observability conditions, we prove asymptotic convergence when the tuning parameters are chosen sufficiently small.

Keywords: Time-periodic linear systems; Observers; Automotive engine control; Lyapunov function; Monodromy matrix; Averaging

1. Introduction

Performance and environmental requirements have imposed advanced control strategies for automotive applications. Ideally, pressures, temperatures, and flows would be measured at numerous places in the engine, enabling accurate control strategies. Unfortunately, their cost and reliability often prevent these sensors from reaching commercial products. As a result, observer design has been garnering increasing attention in recent years. In this context, several common threads can be found. In particular, many observation problems may be seen as inverse filtering problems for periodic systems driven by periodic inputs. This periodicity stems from a fundamental property of the engines. At various levels of modelling, automotive engine dynamics can be considered as a linear periodic system being mechanically coordinated and synchronized by the revolution of the crankshaft. Let us now present some observation topics in more details. A first example is the sensor dynamics inversion problem (Zone 1 in Fig. 1) (see Hammerschmidt & Leteinturier, 2004; Heywood, 1988 for more details). A usual model for such sensors is a first order linear system driven by a periodic signal which can be, depending on the application, the intake pressure, the intake temperature, the exhaust pressure, or the mass air flow. A second example is the estimation of the combustion torque, using as only sensor the instantaneous crankshaft angle speed (Zone 2 in Fig. 1) (see Chauvin et al., 2004a, 2004b; Gyan, Ginoux, Champoussin, & Guezennec, 2000; Jianqiu, Minggao, Ming, & Xihao, 2002; Rizzoni, 1989; Rizzoni & Connolly, 1993 for more details). More advanced topics are the estimation of the flow from the intake manifold to the cylinders (Zone 3 in Fig. 1) (see Chevalier, Vigild, & Hendricks, 2000; Hendricks et al., 1996; Heywood, 1988; Stotsky & Kolmanovsky, 2002 for more details) or the transmission dynamics inversion (Zone 2 in Fig. 1). Online estimation of the

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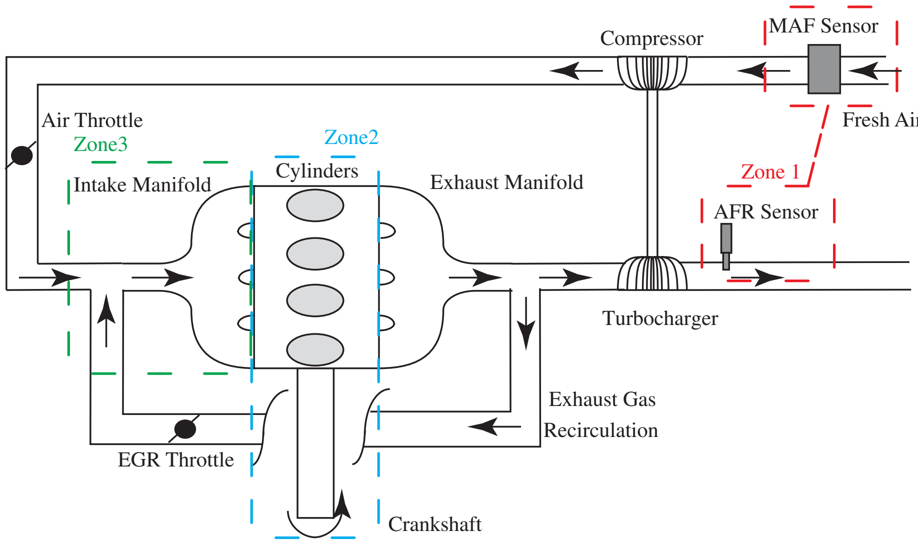

Fig. 1. Engine scheme.

frequencies of a signal which is the sum of a finite number of sinusoids with unknown magnitudes, frequencies, and phases has been addressed by numerous authors (one can refer to Hsu, Ortega, & Damn, 1999; Marino & Tomei, 2000; Xia, 2002 for instance). The problem we address is different. First, the signal we wish to estimate is not directly measured. This signal, which is also assumed to admit a finite dimensional Fourier decomposition, is filtered through a linear periodic system. The output of this system represents the only available data. Secondly (and very importantly), its frequency is precisely known. This particularity suggests a dedicated observation technique could be worth developing. In this paper, we formulate the reconstruction of periodic inputs into the general framework of time-periodic linear systems driven by T_0-periodic signals. We consider $\dot{x} = A(t)x + A_0(t)w(t)$, $y = C(t)x$, $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $(m \leqslant p \leqslant n)$ where $A(t)$, $A_0(t)$, and $C(t)$ are T_0-periodic matrices. The T_0-periodic input signal $w(t)$ is unknown but is assumed to admit a finite dimensional Fourier decomposition. Our contribution is a technique to estimate w from the measurements y. In both full state and partial state measurement cases, we propose an efficient observer for the coefficients of the Fourier decomposition of $w(t)$. This technique is particularly attractive for automotive engine applications where sampling time is very short. In this situation, standard estimation techniques based on Kalman filters are sometimes discarded because of their relative high computational burden. By contrast, the proposed technique is well suited to such real-time system requirements.

The paper is organized as follows. In Section 2, we detail the problem statement and notations. Then, we present the observers in three distinct cases. In the various situations under consideration, we propose specific sets of gains. They require only a very small number of tuning parameters to be chosen. Two practical cases of application are presented in Section 3. They stress the relevance of the proposed approach and illustrate the relative ease of tuning. Finally, the major part of the paper is dedicated to convergence analysis. Under standard observability conditions, we prove asymptotic convergence, when the tuning parameters are chosen sufficiently small. In detail, convergence in the full state measurement case is proven in Section 4 with Proposition 1. Then, we prove convergence in the case of partial state measurement. Time-invariant systems are treated in Section 5, while time-periodic systems are addressed in Section 6. Convergence proof is achieved by computing a monodromy matrix and investigating its stability. Several cascaded changes of coordinates and an averaging reduction of the dynamics are used. The main result of these two sections are Propositions 2 and 3.

2. Problem statement and observer design

2.1. Notations and problem statement

Consider the periodic system driven by an unknown periodic input signal $w(t)$,

$$\dot{x} = A(t)x + A_0(t)w(t), \quad y = C(t)x, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state and $A(t)$, $A_0(t)$, $C(t)$ are T_0-periodic matrices $(T_0 > 0)$ in $\mathscr{M}_{n,n}(\mathbb{R})$, $\mathscr{M}_{n,m}(\mathbb{R})$ and $\mathscr{M}_{p,n}(\mathbb{R})$, respectively. The goal of our study is the estimation of the T_0-periodic continuous input signal $w(t) \in \mathbb{R}^m$, with $m = dim(w) \leqslant p = dim(y) \leqslant n = dim(x)$, through its Fourier decomposition over a finite number h of harmonics

$$w(t) \triangleq \sum_{k \in \mathscr{I}_h} c_k e^{ik\omega_0 t}, \quad \omega_0 \triangleq \frac{2\pi}{T_0},$$

where $\mathscr{I}_h \triangleq \bigcup_{\sigma=1}^{h} \{\rho(\sigma), -\rho(\sigma)\}$ indexes the h modes, and $\rho : \mathbb{N}\backslash\{0\} \to \mathbb{N}$ is strictly increasing. We note $\mathscr{I}_h^+ \triangleq \{k \in \mathscr{I}_h, k \geqslant 0\}$, $d_h \triangleq \mathrm{card}(\mathscr{I}_h)$. With these notations, (1) can be rewritten as

$$\begin{cases} \dot{x} = A(t)x + A_0(t)(\sum_{k\in\mathscr{I}_h} c_k e^{ik\omega_0 t}), \quad y = C(t)x, \\ \dot{c}_k = 0, \ \forall k \in \mathscr{I}_h^+, \quad c_{-k} = c_k^\dagger, \end{cases} \tag{2}$$

where each vector c_k admits m complex entries. We denote by † the Hermitian transpose. All along the paper, we assume that the two following assumptions hold:

H 1. *For all* t, $\ker A_0(t) = \{0\}$, *and* $\ker C^\dagger(t) = \{0\}$.

H 2. *The only solution* $t \mapsto (x(t), \{c_k(t)\}_{k\in\mathscr{I}_h^+})$ *of Eq.* (2) *for which the output* $y(t) = C(t)x(t)$ *is identically zero over* $[0, T_0]$, *is the zero solution.*

2.2. Observer structure

Corresponding to state-space model (2), we define a time-varying Luenberger type observer:

$$\begin{cases} \dot{\hat{x}} = A(t)\hat{x} + A_0(t)(\sum_{k\in\mathscr{I}_h} \hat{c}_k e^{ik\omega_0 t}) \\ \qquad -L(t)(C(t)\hat{x} - y), \\ \dot{\hat{c}}_k = -e^{-ik\omega_0 t} L_k(t)(C(t)\hat{x} - y(t)), \quad \forall k \in \mathscr{I}_h^+. \end{cases} \tag{3}$$

To construct a real signal $\hat{w}$, we define $\hat{c}_{-k} \triangleq \hat{c}_k^\dagger$ for all $k \in \mathscr{I}_h^+$. The gain matrices $L(t)$ (with real entries) and $\{L_k\}_{k\in\mathscr{I}_h^+}$ (with complex entries) are T_0-periodic functions defined in the following section. The errors $\tilde{x} = x - \hat{x}$ and $\tilde{c}_k = c_k - \hat{c}_k$ satisfy

$$\begin{cases} \dot{\tilde{x}} = (A(t) - L(t)C(t))\tilde{x} + A_0(t)(\sum_{k\in\mathscr{I}_h} \tilde{c}_k e^{ik\omega_0 t}), \\ \dot{\tilde{c}}_k = -e^{-ik\omega_0 t} L_k(t)C(t)\tilde{x}, \quad \forall k \in \mathscr{I}_h^+. \end{cases} \tag{4}$$

2.3. Overview of main practical results: observer gains design guidelines

2.3.1. Full state measurement case

We assume here that $m \leqslant p = n$. Thus, for all t, $C(t)$ is invertible. In this case, we choose

$$L(t) \triangleq (A(t) - \bar{A})C^{-1}(t), \tag{5}$$

where $\bar{A}$ is any asymptotically stable matrix in $\mathscr{M}_{n,n}(\mathbb{R})$ and for all $k \in \mathscr{I}_h^+$, we choose the gains $L_k(t)$ as

$$L_k(t) \triangleq \beta_k A_0^\dagger(t) P C^{-1}(t), \tag{6}$$

where P is the symmetric positive definite solution of the Lyapunov equation (which is uniquely defined because $\bar{A}$ is stable)

$$P\bar{A} + \bar{A}^\dagger P = -I_n \tag{7}$$

and $\{\beta_k\}_{k\in\mathscr{I}_h^+}$ are strictly positive reals. Asymptotic convergence with such gains is proven in Section 4. An application for an automotive engine is reported in Section 3.2.

2.3.2. Time-invariant partial measurement case

We consider here that $m \leqslant p < n$ and that A, A_0, and C do not depend on t. In this case, hypothesis H2 implies that (A, C) is observable. A gain L can be chosen such that $A - LC$ is asymptotically stable. Then, for all $k \in \mathscr{I}_h^+$, one can choose the gains $L_k(t)$ under the form

$$L_k(t) \triangleq \varepsilon\beta_k[(ik\omega_0 - (A - LC))^{-1}A_0]^\dagger C^\dagger, \tag{8}$$

where $\{\beta_k\}_{k\in\mathscr{I}_h^+}$ are strictly positive reals. For small enough $\varepsilon > 0$, convergence is proven in Section 5. An illustrative example is presented in Section 3.1.

2.3.3. Time periodic partial measurement case

In this part, we consider that $m \leqslant p < n$ and assume that we have at our disposal a periodic gain $L(t)$ such that the time-periodic system $\dot{\xi} = (A(t) - L(t)C(t))\xi$ is asymptotically stable. As will appear in the proof in Section 6, this assumption is valid provided H1 and H2 hold. This means that we have already solved the real-time estimation of x from y measurements when $w = 0$. We propose here an observer design, based on perturbation theory. For $k \in \mathscr{I}_h^+$, denote by $W_k(t)$, an $n \times m$ matrix with complex entries, solution of

$$\begin{cases} \dot{W}_k(t) = (A(t) - L(t)C(t))W_k(t) + e^{ik\omega_0 t}A_0(t), \\ W_k(0) = W_k(T_0). \end{cases} \tag{9}$$

Existence of W_k is proven in Section 6. For implementation purposes, one can remark that, since $\dot{\xi}(t) = (A(t) - L(t)C(t))\xi(t)$ is asymptotically stable, the initial conditions on W_k are exponentially forgotten. Thus, a numerical approximation of $W_k(t)$ can be derived from the asymptotic solution of (9) from a zero initial condition. For any $k \in \mathscr{I}_h^+$, we set

$$L_k(t) \triangleq \varepsilon\beta_k W_k^\dagger(t)C^\dagger(t), \tag{10}$$

where $\{\beta_k\}_{k\in\mathscr{I}_h^+}$ are strictly positive reals. We prove in Proposition 3 that, for small enough $\varepsilon > 0$, the state of the observer (3) converges asymptotically toward the state of reference system (2). Notice that when $(A - LC)$ and A_0 are constant matrices, $W_k = (ik\omega_0 - (A - LC))^{-1}A_0$ and we recover the gain designed for the previously considered time-invariant case.

3. Motivating automotive engine applications

3.1. Transmission rod dynamics inversion

Modelling the transmission and inverting its dynamics is necessary for an accurate torque combustion estimation. For technical reasons, the instantaneous engine speed sensor is not located next to the cylinders but at the end of a transmission rod. When engine speed and torque increase, the excitation on the transmission rises in magnitude, yielding misleading information about the combustion.

3.1.1. Model description

Crankshaft dynamics modelling has been addressed previously in the literature (see Rizzoni, 1989; Rizzoni & Connolly, 1993 for example). In a first approach, the system can be modelled by a second order dynamics. Variables x_1 and x_2 refer to

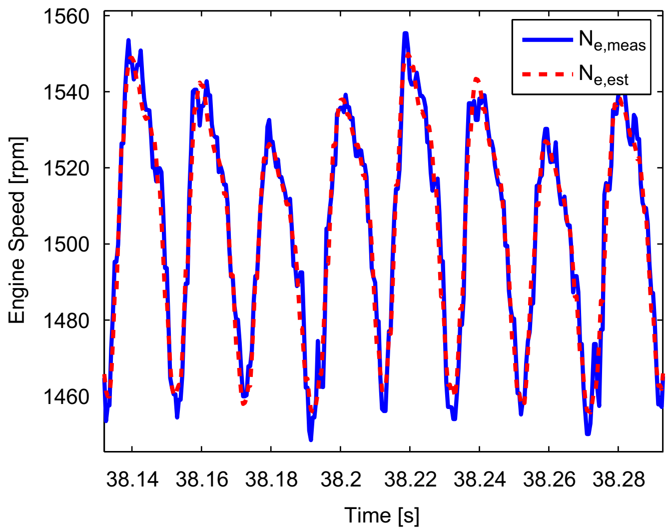

Fig. 2. Testbench results. Engine speed reconstruction over 2 cycles at 1500 rpm and 8 bar of indicated mean effective pressure (IMEP): measured engine speed (solid) $N_{e,\mathrm{meas}}$, reconstructed engine speed (dot) $N_{e,\mathrm{est}}$.

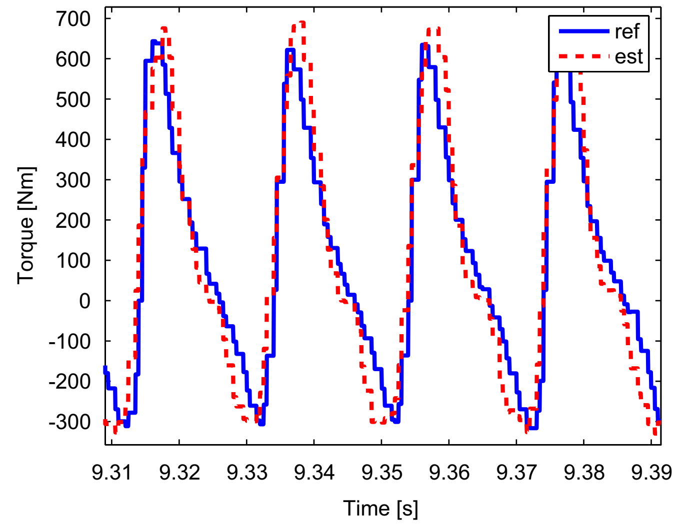

Fig. 3. Testbench results. Combustion torque reconstruction over 1 cycle at 1500 rpm and 8 bar of Indicated Mean Effective Pressure (IMEP): reference via in-cylinder pressure sensor (solid), reconstructed signal (dot).

the angular speed at the end of the transmission and next to the cylinders, respectively. The coupled dynamics are

$$\frac{\mathrm{d}^2x_1}{\mathrm{d}\alpha^2} - \frac{\mathrm{d}^2x_2}{\mathrm{d}\alpha^2} + 2\bar{\xi}\bar{\omega}\left(\frac{\mathrm{d}x_1}{\mathrm{d}\alpha} - \frac{\mathrm{d}x_2}{\mathrm{d}\alpha}\right) + \bar{\omega}^2(x_1 - x_2) = 0$$

with $y=x_1$, and where $\bar{\xi}$ is a damping coefficient and $\bar{\omega}$ the natural frequency of the transmission. Let $w_0(t)\triangleq\mathrm{d}^2x_2(t)/\mathrm{d}\alpha^2 + 2\bar{\xi}\bar{\omega}\mathrm{d}x_2(t)/\mathrm{d}\alpha+\bar{\omega}^2x_2(t)$. Since x_2 is a periodic signal, so does w_0. Some rewriting yields $\mathrm{d}^2x_1/\mathrm{d}\alpha^2 = -2\bar{\xi}\bar{\omega}\mathrm{d}x_1/\mathrm{d}\alpha - \bar{\omega}^2x_1 + w_0$, $y = x_1$.

3.1.2. Observer definition

The state is $x = [x_1\ \frac{\mathrm{d}x_1}{\mathrm{d}\alpha}]^{\mathrm{T}}$. The state-space model is of the form (2) with $A = [\begin{smallmatrix}0 & 1\\ -\bar{\omega}^2 & -2\bar{\xi}\bar{\omega}\end{smallmatrix}]$, $A_0 = [0\ 1]^{\mathrm{T}}$, and $C = [1\ 0]$. The observer is designed according to (3) with the gains defined by Eq. (8). We take $L = [2\bar{\xi}\bar{\omega}\ 2\bar{\omega}^2]^{\mathrm{T}}$, $\mathscr{I}_h = \{-2, \dots, 2\}$ and $L_k(t)=\varepsilon\beta_k((\mathrm{i}k\omega_0 - A + LC)^{-1}A_0)^{\dagger}C^{\mathrm{T}}$. We choose $\varepsilon=0.1$ and $\beta_k=1/(k^2+1)$. The coefficients of the decomposition of x_2 are obtained from the coefficients of w_0 by matrix multiplication with

$$\begin{bmatrix}\bar{\omega}^2 - (k\omega_0)^2 & k\omega_0 2\bar{\xi}\bar{\omega}\\ -k\omega_0 2\bar{\xi}\bar{\omega} & \bar{\omega}^2 - (k\omega_0)^2\end{bmatrix}^{-1}.$$

3.1.3. Testbench results

The observer reconstructs the angular speed next to the cylinders. Experimental results are given in Fig. 2. Implementation was done in discrete time with a sample angle of 6° crank angle. These provide insights on the work produced by each cylinder. This information can be used to infer a diagnosis, and eventually to balance the cylinders by applying a closed loop control on the cylinder individual injection masses.

3.1.4. Convergence

To check convergence of the observer designed according to (3) with the gains defined by Eq. (8), we simply have to check that H1 and H2 hold. Clearly, H1 is true. Let $(x, \{c_k\}_{k\in\mathscr{I}_h^+})$ be a nontrivial solution of (2): $y(t) \equiv 0$ implies $x_1=0$ and $\dot{x}_1=0$, and then $x_2=0$. Moreover $\dot{c}_k=0$, $\forall k \in \mathscr{I}_h$ implies $c_k(t)=\bar{c}_k$, $\forall k \in \mathscr{I}_h^+$. By substitution in (2) we get $\sum_{k\in\mathscr{I}_h}\bar{c}_k\mathrm{e}^{\mathrm{i}k\omega_0 t} = 0$. Then, $(x, \{c_k\}_{k\in\mathscr{I}_h}) = 0$ and H2 holds.

3.2. Combustion torque estimation

Here, we consider the design of a real-time observer for the combustion torque using the reliable and available instantaneous engine speed as only measurement. Following Kiencke and Nielsen, 2000, an energy balance yields (α is the crankshaft angle)

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{1}{2}J(\alpha)\dot{\alpha}^2\right) = T_{\mathrm{comb}}(\alpha) - T_{\mathrm{load}}. \tag{11}$$

Derivation of the (4π) periodic function $\alpha \mapsto J(\alpha)$ is described in Kiencke and Nielsen (2000). This function is usually perfectly known for a particular engine geometry. In Eq. (11), $\alpha \mapsto T_{\mathrm{comb}}(\alpha) - T_{\mathrm{load}}$ is 4π-periodic with zero mean. We can approximatively decompose it on a Fourier basis. Introducing $\phi(\alpha)\triangleq\frac{1}{2}J(\alpha)\dot{\alpha}^2$, the dynamics read $\mathrm{d}\phi/\mathrm{d}\alpha = \sum_{k\in\mathscr{I}_h}c_k\mathrm{e}^{\mathrm{i}k\alpha/2}$. The measurement y is the instantaneous engine speed $\dot{\alpha}^2$. The reference model is $\mathrm{d}\phi/\mathrm{d}\alpha = \sum_{k\in\mathscr{I}_h}c_k\mathrm{e}^{\mathrm{i}k\alpha/2}$, $y = (2/J(\alpha))\phi$. This system is similar to (2) with $A=0$, $A_0=1$, $C(\alpha)=2/J(\alpha)$, and $\mathscr{I}_h\triangleq\{-4, \dots, 4\}\backslash\{0\}$. The observer is designed according to (3) with the gains defined by Eqs. (5) and (6). We use $\bar{A}=-15$ and $\beta_k = 10/(k^2 + 1)$. The implementation is achieved in discrete time with a sample angle of 6° crankshaft angle. Testbench results are given in Fig. 3, see Chauvin et al. (2004a) for further details. In the context of combustion real-time control, this observer is a handy tool. It does not suffer from any phase shift, and can thus be used in a closed loop control strategy on the fuel injectors.

4. Convergence results in the case of full state measurement

We now present convergence proofs. In the first case under consideration $p=n$, and C is a square invertible matrix. To prove convergence of the error dynamics (4) for the gain design in Section 2.3.1, we exhibit a Lyapunov function and use LaSalle's invariance principle.[1]

4.1. Lyapunov function candidate

Let A be an asymptotic stable matrix in $\mathscr{M}_{n,n}(\mathbb{R})$, we set $L(t)=(A(t)-\bar{A})C^{-1}(t)$ and P the symmetric definite solution of the Lyapunov equation $P\bar{A}+\bar{A}^{\dagger}P=-I_n$. Consider strictly positive real numbers $\{\beta_k\}_{k\in\mathscr{I}_h^+}$. For $k<0$, we define $\beta_k=\beta_{-k}$. A Lyapunov function candidate for (4) is

$$V(\tilde{x},\{\tilde{c}_k\}_{k\in\mathscr{I}_h^+})=\tilde{x}^{\mathrm{T}}P\tilde{x}+\sum_{k\in\mathscr{I}_h}\frac{1}{\beta_k}\tilde{c}_k^{\dagger}\tilde{c}_k. \tag{12}$$

By differentiation w.r.t. t, we get $\dot{V}=-\tilde{x}^{\mathrm{T}}\tilde{x}+\sum_{k\in\mathscr{I}_h}(\tilde{x}^{\mathrm{T}}M_k(t)\tilde{c}_k+\tilde{c}_k^{\dagger}M_k^{\dagger}(t)\tilde{x})$, where

$$M_k(t)\triangleq\left(PA_0(t)-\frac{1}{\beta_k}C^{\dagger}(t)L_k(t)^{\dagger}\right)\mathrm{e}^{\mathrm{i}k\omega_0 t}. \tag{13}$$

According to (6), for all $k\in\mathscr{I}_h^+$, we use $L_k(t)\triangleq\beta_k A_0^{\dagger}(t)PC(t)^{-1}$. Thus, for all $k\in\mathscr{I}_h$, $M_k=0$ and

$$\dot{V}=-\tilde{x}^{\dagger}\tilde{x}. \tag{14}$$

In summary, V is continuously differentiable and satisfies $V(0)=0$, $V(\tilde{x},\{\tilde{c}_k\}_{k\in\mathscr{I}_h^+})>0$ for $(\tilde{x},\{\tilde{c}_k\}_{k\in\mathscr{I}_h^+})\neq 0$, and $\dot{V}(\tilde{x},\{\tilde{c}_k\}_{k\in\mathscr{I}_h^+})\leqslant 0$. Thus, V is a Lyapunov function for system (4).

4.2. Application of LaSalle's theorem

To conclude, we now use LaSalle's invariance principle. It is usually exposed for time-invariant systems (see for instance Khalil, 1992, Theorem 4.4). Nevertheless, the result can be extended to periodic systems where the notion of invariance set is easily transposed (see, e.g. Vidyasagar, 1992). Let I_c be the largest invariant set in $\{(\tilde{x},\{\tilde{c}_k\}_{k\in\mathscr{I}_h^+})|\dot{V}(\tilde{x},\{\tilde{c}_k\}_{k\in\mathscr{I}_h^+})=0\}$. From Vidyasagar (1992, Section 5.2), if I_c does not contain any trajectory but the trivial trajectory, then the equilibrium 0 is uniformly asymptotically stable. We now characterize I_c. From (14), the set $\{(\tilde{x},\{\tilde{c}_k\}_{k\in\mathscr{I}_h^+})|\dot{V}(\tilde{x},\{\tilde{c}_k\}_{k\in\mathscr{I}_h^+})=0\}$ is equal to $\{(\tilde{x},\{\tilde{c}_k\}_{k\in\mathscr{I}_h^+})|\tilde{x}=0\}$. We apply the error dynamics (4) to an element of this last set. To remain in I_c, the variation of the first coordinate of the dynamics must equal zero. This implies

$$\forall t\in\mathbb{R},\quad A_0(t)\left(\sum_{k\in\mathscr{I}_h}\mathrm{e}^{\mathrm{i}k\omega_0 t}\tilde{c}_k\right)=0.$$

Yet, from H1, $A_0(t)$ is injective for all t. Thus, for all t, $\sum_{k\in\mathscr{I}_h}\mathrm{e}^{\mathrm{i}k\omega_0 t}\tilde{c}_k=0$. The functions $\{\mathrm{e}^{\mathrm{i}k\omega_0 t}\}_{k\in\mathscr{I}_h}$ are linearly independent. Then, the previous equation implies $\tilde{c}_k=0$, for all $k\in\mathscr{I}_h$. Therefore, the set I_c is reduced to $\{0\}$. The observation error is asymptotically stable. We have proven the following result.

Proposition 1. *Consider* (2). *Assume that $m\leqslant p=n$, and that* H1 *and* H2 *hold. Consider the observer* (3) *with gains L and L_k as defined in Eqs.* (5) *and* (6), *then the error dynamics* (4) *asymptotically converge to* 0.

5. Convergence results in the case of time-invariant partial state measurement

In this part, we consider that $m\leqslant p<n$. By contrast with Section 4, it is not always possible to find L_k and P_k yielding $M_k=0$ in (13). Further investigations are required. The key idea is to assume that the tuning parameters $\{L_k\}_{k\in\mathscr{I}_h}$ are small, so that the dynamics of $\tilde{x}$ in (4) converge much faster than the $(\tilde{c}_k)_{k\in\mathscr{I}_h^+}$ dynamics. We note $\forall k\in\mathscr{I}_h^+$, $L_k\triangleq\varepsilon l_k$, $0<\varepsilon\ll 1$. In this section, we perform a perturbation analysis and conclude toward convergence of the proposed observer (3) with the gains defined in Section 2.3.2 when $\varepsilon>0$ is sufficiently small (Proposition 2). The proof is based on several changes of coordinates represented in Fig. 4. We now sketch what are the benefits of these cascaded transformations. In the original coordinates, the error dynamics involve a matrix of the form $\left[\begin{smallmatrix}(A-LC)+\mathscr{O}(\varepsilon) & f(A-LC,A_0)\\ \mathscr{O}(\varepsilon) & 0\end{smallmatrix}\right]$. It is impossible to infer the stability property when $\varepsilon\to 0$. But, with carefully chosen coordinates changes, a new matrix of the form $\left[\begin{smallmatrix}(A-LC)+\mathscr{O}(\varepsilon) & \mathscr{O}(\varepsilon)\\ \mathscr{O}(\varepsilon^2) & \varepsilon E+\mathscr{O}(\varepsilon^2)\end{smallmatrix}\right]$ appears. For ε small enough, this matrix is asymptotically stable if the time-invariant matrices $A-LC$ and E are.

5.1. A first change of coordinates: $(\bar{z},z_k)\mapsto(\bar{\bar{z}},z_k)$

Since (A,C) is observable, we can find L such that $A-LC$ is asymptotically stable. Let $\bar{z}=z=\tilde{x}$, $z_k=\tilde{c}_k$, and $P=A-LC$. The error dynamics (4) rewrite

$$\begin{cases}\dot{\bar{z}}=P\bar{z}+\sum_{k\in\mathscr{I}_h}A_0\mathrm{e}^{\mathrm{i}k\omega_0 t}z_k,\\ \dot{z}_k=-\varepsilon\mathrm{e}^{-\mathrm{i}k\omega_0 t}l_kC\bar{z},\quad\forall k\in\mathscr{I}_h^+.\end{cases} \tag{15}$$

At zero order ($\varepsilon=0$), the matrix of system (15) is $\left[\begin{smallmatrix}P & f(P,A_0)\\ 0 & 0\end{smallmatrix}\right]$ which does not allow us to conclude toward convergence. This first change of variables aims at cancelling the term $f(P,A_0)$ and replace it by an $\mathscr{O}(\varepsilon)$-term. As P is asymptotically stable, $\forall k\in\mathscr{I}_h$, $\det(P-\mathrm{i}k\omega_0 I_n)\neq 0$. We set $\bar{A}_k\triangleq(\mathrm{i}k\omega_0-P)^{-1}A_0$

[1] We would like to mention that this case can also be addressed using the notion of persistent excitation in the context proposed in Panteley, Loría, & Teel, 2001 (persistency is guaranteed by H1).

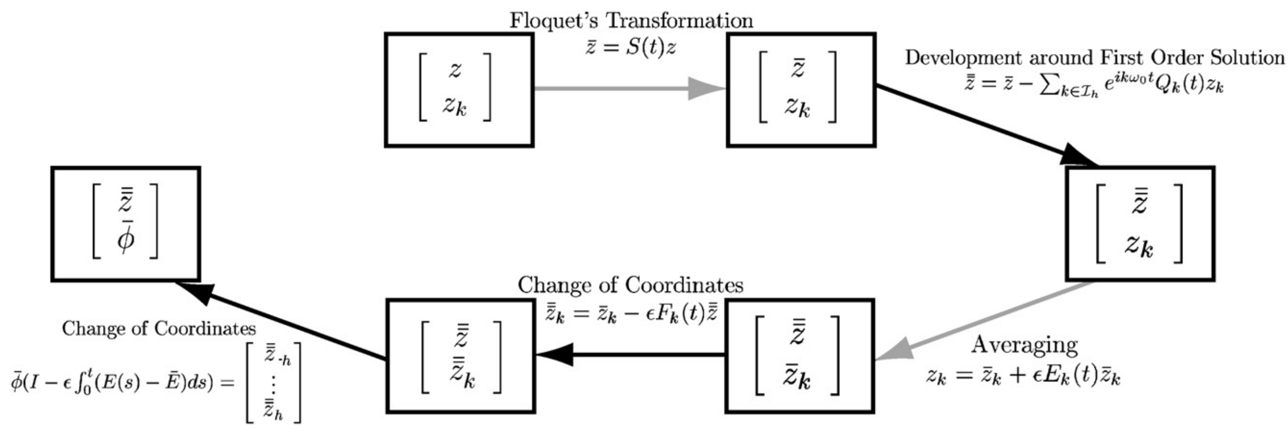

Fig. 4. Proof of Propositions 2 and 3 organization. Proposition 3 is an extension of Proposition 2 to the periodic matrices. Grey arrows represent changes of coordinates that are used only in the time-periodic case. They reduce to identity in the time-invariant case.

and, then, $\mathrm{i}k\omega_0\bar A_k = A_0 + P\bar A_k$. We define

$$\bar{\bar z} \triangleq \bar z - \sum_{k\in\mathscr{I}_h} \mathrm{e}^{\mathrm{i}k\omega_0 t}\bar A_k z_k. \tag{16}$$

Note $Q\triangleq\sum_{k\in\mathscr{I}_h}\bar A_k l_k C$. In the $(\bar{\bar z}, \{z_k\}_{k\in\mathscr{I}_h^+})$ coordinates, system (15) reads (after some computations)

$$\begin{cases}\dot{\bar{\bar z}} = (P+\varepsilon Q)\bar{\bar z} + \varepsilon\sum_{k\in\mathscr{I}_h}\mathrm{e}^{\mathrm{i}k\omega_0 t}Q\bar A_k z_k,\\ \dot z_k = -\varepsilon\mathrm{e}^{-\mathrm{i}k\omega_0 t}l_k C\bar{\bar z} - \varepsilon\sum_{l\in\mathscr{I}_h}\mathrm{e}^{\mathrm{i}(l-k)\omega_0 t}R_{k,l}z_l, \quad \forall k\in\mathscr{I}_h^+,\end{cases} \tag{17}$$

where $R_{k,l}\triangleq l_k C\bar A_l$. This change of coordinates stresses the first part of the dynamics as an asymptotically linear stable system with an $\mathscr{O}(\varepsilon)$ perturbation.

5.2. Second change of coordinates: $(\bar{\bar z}, z_k)\mapsto(\bar{\bar z}, \bar{\bar z}_k)$

The purpose of this second change of variables, bearing on the $\{z_k\}_{k\in\mathscr{I}_h^+}$ variables only, is to make a $\mathscr{O}(\varepsilon^2)$ coupling term appear through which $\bar{\bar z}$ impacts on z_k. Let $\forall k\in\mathscr{I}_h$, $F_k\triangleq l_k C(\mathrm{i}k\omega_0 - P)^{-1}$. This gives $F_k\mathrm{i}k\omega_0 = l_k C + F_k P$. Then, with $\bar{\bar z}_k\triangleq z_k - \varepsilon\mathrm{e}^{-\mathrm{i}k\omega_0 t}F_k\bar{\bar z}$, we have, after some calculus, that for all $k\in\mathscr{I}_h$,

$$\dot{\bar{\bar z}}_k = -\varepsilon\sum_{l\in\mathscr{I}_h}\mathrm{e}^{\mathrm{i}(l-k)\omega_0 t}R_{k,l}\bar{\bar z}_l + \varepsilon^2(f_{k,\bar{\bar z}}(t)\bar{\bar z} + \textstyle\sum_{l\in\mathscr{I}_h}f_{k,l}(t)\bar{\bar z}_l) + \varepsilon^3 f_{\varepsilon,k,\bar{\bar z}}(t)\bar{\bar z}, \tag{18}$$

where the functions $\{f_{k,\bar{\bar z}}, f_{k,l}, f_{\varepsilon,k,\bar{\bar z}}\}_{(k,l)\in\mathscr{I}_h^2}$ are regular, and T_0-periodic in t.

5.3. Final change of coordinates: $(\bar{\bar z}, \bar{\bar z}_k)\mapsto(\bar{\bar z}, \bar\phi)$

Gathering $\{\bar{\bar z}_k\}_{k\in\mathscr{I}_h^+}$ in $\phi = [\bar{\bar z}_{-h}\ \ldots\ \bar{\bar z}_h]^{\mathrm{T}}$, we can finally regroup system (17) and (18) under the form

$$\begin{cases}\dot{\bar{\bar z}} = (P+\varepsilon Q)\bar{\bar z} + \varepsilon\sum_{k\in\mathscr{I}_h}\mathrm{e}^{\mathrm{i}k\omega_0 t}Q\bar A_k\bar{\bar z}_k - \varepsilon^2 f_{\bar{\bar z}}(t)\bar{\bar z},\\ \dot\phi = -\varepsilon E(t)\phi + \varepsilon^2((f_{\phi,\bar{\bar z}}(t) + \varepsilon f_{\varepsilon,\phi,\bar{\bar z}}(t))\bar{\bar z} + f_\phi(t)\phi),\end{cases} \tag{19}$$

where the functions $f_{\phi,\bar{\bar z}}$, $f_{\varepsilon,\phi,\bar{\bar z}}$ and f_ϕ are computed from (18). These are regular, and T_0-periodic in t. Finally, $E(t) = (E_{\mu,\nu})$ is a $d_h\times d_h$ matrix. Its coefficients are of the form

$$E_{\mu,\nu}(t)\triangleq\mathrm{e}^{\mathrm{i}(\rho(\nu)-\rho(\mu))\omega_0 t}l_{\rho(\mu)}C(\mathrm{i}\rho(\nu)\omega_0 - P)^{-1}A_0. \tag{20}$$

Theorem 1 (*Averaging theorem (Guckenheimer & Holmes, 1983)*). *There exists a $\mathscr{C}^r$ change of coordinates $\phi = \bar\phi + \varepsilon w(\bar\phi, t, \varepsilon)$ such that $\dot\phi = \varepsilon f(\phi, t, \varepsilon)$, where f is a $\mathscr{C}^r$, $r>1$ bounded function of period $T_0>0$ w.r.t. t, becomes $\dot{\bar\phi} = \varepsilon\bar f(\bar\phi) + \varepsilon^2 f_1(\bar\phi, t, \varepsilon)$ where f_1 is of period T_0 w.r.t. t, $\bar f(\bar\phi)\triangleq 1/T_0\int_0^{T_0} f(\bar\phi, t, 0)\,\mathrm{d}t$, and $w(\bar\phi, t, \varepsilon) = \int_0^t f(\bar\phi, s, \varepsilon) - \bar f(\bar\phi)\,\mathrm{d}s$.*

In our case, $f(\phi, t, 0) = -E(t)\phi$, then $\bar f(\bar\phi) = -\bar E\bar\phi$ where

$$\bar E\triangleq\frac{1}{T_0}\int_0^{T_0}E(t)\,\mathrm{d}t = \mathrm{diag}(E_{-\rho(h),-\rho(h)},\ldots,E_{\rho(h),\rho(h)}).$$

We set $\bar\phi(I-\varepsilon\int_0^t(E(s)-\bar E)\,\mathrm{d}s)\triangleq\phi$, then we have $\dot{\bar\phi} = -\varepsilon\bar E\bar\phi + \varepsilon^2(g_{\bar\phi,\bar{\bar z}}(\bar\phi,\varepsilon,t)\bar{\bar z} + g_{\bar\phi}(\bar\phi,t,\varepsilon))$ where the functions $g_{\bar\phi,\bar{\bar z}}$ and $g_{\bar\phi}$ are regular, bounded, and T_0-periodic w.r.t. t.

5.4. Conclusion toward convergence

In the obtained $(\bar{\bar z}, \bar\phi)$-coordinates, system (19) reads

$$\begin{cases}\dot{\bar{\bar z}} = (P+\varepsilon Q)\bar{\bar z} + \varepsilon g_{\bar{\bar z},\bar\phi}(t)\bar\phi - \varepsilon^2(f_{\bar{\bar z}}(t)\bar{\bar z} + f_{\bar{\bar z},\bar\phi}(t)\bar\phi),\\ \dot{\bar\phi} = -\varepsilon\bar E\bar\phi + \varepsilon^2(g_{\bar\phi,\bar{\bar z}}(\bar\phi,\varepsilon,t)\bar{\bar z} + g_{\bar\phi}(\bar\phi,t,\varepsilon)).\end{cases} \tag{21}$$

To check convergence of the observer, we investigate the hyperbolic stability of the monodromy matrix of (21). Since this system is triangular up to second order terms, there exists $(M_{1,1}, M_{1,2})\in\mathscr{M}_n(\mathbb{R})\times\mathscr{M}_{n,md_h}(\mathbb{C})$ such that the considered monodromy matrix writes

$$\Phi_{\bar{\bar z},\bar\phi,\varepsilon} = \begin{bmatrix}P-\varepsilon M_{1,1} & -\varepsilon M_{1,2}\\ 0 & -\varepsilon\bar E\end{bmatrix} + \mathscr{O}(\varepsilon^2).$$

Further, this expression is (up to second order terms in ε) time-invariant. Because P is asymptotically stable, (21) is asymptotically stable for $0<\varepsilon\ll 1$ if and only if the system

$$\dot{\bar{\phi}} = -\varepsilon\bar{E}\bar{\phi} \tag{22}$$

is hyperbolically stable. All the changes of coordinates are linear, time-periodic and smooth, and thus uniformly continuous. Therefore, convergence toward 0 of $(\bar{\bar{z}}, \bar{\phi})$ leads to the convergence toward 0 of $(z, \{z_k\}_{k\in\mathscr{I}_h^+})$. To impose hyperbolic stability of system (22), we set, for $k\in\mathscr{I}_h^+$,

$$l_k(t) \triangleq \beta_k((\mathrm{i}k\omega_0 - P)^{-1}A_0)^\dagger C^\dagger. \tag{23}$$

In this case, we have $E_{\mu,\mu}=\beta_{\rho(\mu)}P^\dagger_{\rho(\mu)}P_{\rho(\mu)}$ where $P_{\rho(\mu)}\triangleq C(\mathrm{i}\rho(\mu)\omega_0 - P)^{-1}A_0$, and $\mu\in\mathscr{I}_h$. But, for all $k\in\mathscr{I}_h$, $P_k \triangleq C(\mathrm{i}k\omega_0 - P)^{-1}A_0$ has full rank, i.e., $\ker P_k=\{0\}$ $(m\leqslant p)$. This holds because otherwise H2 would be violated. For any $\Gamma_k\in\ker P_k$, $x=\mathrm{e}^{\mathrm{i}k\omega_0 t}(\mathrm{i}k\omega_0-P)^{-1}A_0\Gamma_k$ and $c_l=\delta_{l,k}\Gamma_k$, is solution of (2) with $y=Cx\equiv 0$. Thus, system (22) is asymptotically stable. We have proven the following proposition.

Proposition 2. *Consider* (2) *with $m\leqslant p<n$. Assume that the matrices A, A_0 and C are constant and that* H1 *and* H2 *hold. Consider the observer* (3) *with gain L and L_k as defined in Section* 2.3.2. *Then, for small enough $\varepsilon>0$, the error dynamics* (4) *asymptotically converge to* 0.

6. Convergence results in the case of time-periodic partial state measurement

In this last part, we assume that $m\leqslant p<n$ and we consider time-periodic matrices $A(t)$, $A_0(t)$, $C(t)$ with the gain design of Section 2.3.3. The proof has similarities with the time-invariant case treated in the previous section. A sequence of changes of variables is used, and additionally, Floquet's theorem and averaging provide rescaling under a time-invariant form. Again, Fig. 4 summarizes the organization of the proof. By H2, the observability Gramian on $[0, T_0]$ is definite positive. Thus, we can find a T_0-periodic matrix $L(t)$ such that $A-LC$ is asymptotically stable (see Anderson & Moore, 1971, Section 14.2; Ikeda, Maeda, & Kodama, 1975 for example). A constructive choice is given, for example, by the Kalman filter. In the case under consideration here, the error dynamics (4) can be formulated under the familiar expression

$$\begin{cases} \dot{z} = P(t)z + \sum_{k\in\mathscr{I}_h} A_0(t)\mathrm{e}^{\mathrm{i}k\omega_0 t}z_k, \\ \dot{z}_k = -\varepsilon\mathrm{e}^{-\mathrm{i}k\omega_0 t}l_k(t)C(t)z, \quad \forall k\in\mathscr{I}_h^+ \end{cases} \tag{24}$$

with $P(t)=A(t)-L(t)C(t)$. The matrices P, A_0, and l_kC are T_0-periodic. Further, P is asymptotically stable. We note Φ the transition matrix of P, i.e., the nonsingular matrix solution of $\dot{\Phi}(t)=P(t)\Phi(t)$, $\Phi(0)=I_n$. Analysis of linear time-periodic systems can be performed by the following result (see Brauer & Nohel, 1989; Coddington & Levinson, 1955 for example).

Theorem 2 (*Floquet's theorem*). *Consider the state-space model $\dot{z}=P(t)z$ with T_0-periodic matrix P. There exists a matrix $J\in\mathscr{M}_{n,n}(\mathbb{R})$ such that $S(t)\triangleq \mathrm{e}^{Jt}\Phi^{-1}(t)$ is a periodic nonsingular T_0-periodic matrix with $S(0)=I_n$. Generally, J is noted $J\triangleq(1/T_0)\log(\Phi(T_0))$. The following results hold*: (i) *the state transformation $\bar{z}=S(t)z$ yields a linear time-invariant system $\dot{\bar{z}}=J\bar{z}$*; (ii) *a necessary and sufficient condition for asymptotic stability is that all the eigenvalues of the monodromy matrix ($\Phi(T_0)=\mathrm{e}^{JT_0}$) lie in the open unitary disk, i.e., J is asymptotically stable.*

Our convergence analysis heavily relies on this last result. We use periodic changes of variables to find conditions on asymptotic stability of system (24).

6.1. First change of coordinates: $(z, z_k)\mapsto(\bar{z}, z_k)$

Following Theorem 2 with $P(t)=A(t)-L(t)C(t)$, we set $\bar{z}\triangleq S(t)z$. By construction, $S(t)$ is invertible and T_0 periodic. Since P is asymptotically stable, J is asymptotically stable. In the $(\bar{z}, \{z_k\}_{k\in\mathscr{I}_h^+})$ coordinates, the dynamics rewrite

$$\begin{cases} \dot{\bar{z}} = J\bar{z} + \sum_{k\in\mathscr{I}_h}\mathrm{e}^{\mathrm{i}k\omega_0 t}\bar{A}(t)z_k, \\ \dot{z}_k = -\varepsilon\mathrm{e}^{-\mathrm{i}k\omega_0 t}R_k(t)\bar{z}, \quad \forall k\in\mathscr{I}_h^+ \end{cases} \tag{25}$$

with $\bar{A}(t)=S(t)A_0(t)\in\mathscr{M}_{n,m}(\mathbb{R})$, and $R_k(t)=l_k(t)C(t)S^{-1}(t)\in\mathscr{M}_{p,n}(\mathbb{R})$. Both $\bar{A}$ and R_k are T_0-periodic, and they can be rewritten as Fourier series. We note $\bar{A}(t)=\sum_{l\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}l\omega_0 t}\mathscr{F}_l(\bar{A})$, and $R_k(t)=\sum_{l\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}l\omega_0 t}\mathscr{F}_l(R_k)$, where, for all $l\in\mathbb{Z}$, $\mathscr{F}_l(\bullet)=(1/T_0)\int_0^{T_0}\bullet(s)\mathrm{e}^{-\mathrm{i}l\omega_0 s}\,\mathrm{d}s$. Comparing (24) and (25), we notice that some progress has been made by stressing time-invariant terms. We now investigate the effects of coupling through the ε-terms.

6.2. Second change of coordinates: $(\bar{z}, z_k)\mapsto(\bar{\bar{z}}, z_k)$

Following Javid (1980, 1982), we now perform a series expansion w.r.t. ε on (25). Since J is asymptotically stable, then $\forall k\in\mathscr{I}_h$, $(\mathrm{i}k\omega_0-J)^{-1}$ is nonsingular. Consider

$$Q_k(t)\triangleq\sum_{l\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}l\omega_0 t}(\mathrm{i}(l+k)\omega_0-J)^{-1}\mathscr{F}_l(\bar{A}). \tag{26}$$

This series is well defined because $\{(\mathrm{i}(l+k)\omega_0-J)^{-1}\mathscr{F}_l(\bar{A})\}_{l\in\mathbb{Z}}$ belongs to $l^1_m\triangleq\{\{u_l\}_{l\in\mathbb{Z}}\in(\mathbb{R}^m)^{\mathbb{Z}}/\sum_{l\in\mathbb{Z}}\|u_l\|^1<+\infty\}$. This can be proven by noticing that its general term is the product of the general terms of two l^2_m series. Moreover, it is periodic, and differentiable: $\forall k\in\mathscr{I}_h$, $\mathrm{d}(\mathrm{e}^{\mathrm{i}k\omega_0 t}Q_k(t))/\mathrm{d}t=J\mathrm{e}^{\mathrm{i}k\omega_0 t}Q_k(t)+\mathrm{e}^{\mathrm{i}k\omega_0 t}\bar{A}(t)$. Notice that, by construction, $W_k(t)\triangleq S^{-1}(t)Q_k(t)$ is a periodic solution of $\dot{W}_k(t)=(A(t)-L(t)C(t))W_k(t)+\mathrm{e}^{\mathrm{i}k\omega_0 t}A_0(t)$ and satisfies $W_k(0)=W_k(T_0)$. Then, we define $\bar{\bar{z}}\triangleq\bar{z}-\sum_{k\in\mathscr{I}_h}\mathrm{e}^{\mathrm{i}k\omega_0 t}Q_k(t)z_k$. In the $(\bar{\bar{z}}, \{z_k\}_{k\in\mathscr{I}_h^+})$ coordinates, system (25) rewrites

$$\begin{cases} \dot{\bar{\bar{z}}} = (J+\varepsilon\mathscr{Q}(t))\bar{\bar{z}} + \varepsilon\sum_{k\in\mathscr{I}_h}\mathscr{Q}(t)Q_k(t)z_k, \\ \dot{z}_k = -\varepsilon\mathrm{e}^{-\mathrm{i}k\omega_0 t}R_k(t)\bar{\bar{z}} - \varepsilon\sum_{l\in\mathscr{I}_h}R_{k,l}(t)\mathrm{e}^{\mathrm{i}(l-k)\omega_0 t}z_l, \\ \quad \forall k\in\mathscr{I}_h^+, \end{cases} \tag{27}$$

where $\mathscr{Q}(t) \triangleq \sum_{k\in\mathscr{I}_h} Q_k(t)R_k(t)$ and

$$R_{k,l}(t) \triangleq R_k(t)Q_l(t). \tag{28}$$

This change of coordinates stresses the first part of the dynamics as an asymptotically stable system with a perturbation in ε. We obtain the right structure for analysis of the first equation of (27), but the second equation is still time-varying. In the next section, we explicit a change of coordinates such that the whole dynamics become linear time-invariant.

6.3. Third change of coordinates: $(\bar{\bar{z}}, z_k) \mapsto (\bar{\bar{z}}, \bar{z}_k)$

The second part of (27) is factorized as ε times a periodic function. Averaging Theorem 1 will give insights here. We note, for all $k \in \mathscr{I}_h$, $f_k(z_k, t, \varepsilon) = -R_{k,k}(t)z_k$ and $\bar{f}_k(z_k) = -(1/T_0)(\int_0^{T_0} R_{k,k}(t)\,\mathrm{d}t)z_k \triangleq -\bar{R}_k z_k$. From (26) and (28), we have, for all $k \in \mathscr{I}_h$,

$$\bar{R}_k = \sum_{l\in\mathbb{Z}} \mathscr{F}_{-l}(R_k)(\mathrm{i}(l+k)\omega_0 - J)^{-1}\mathscr{F}_l(\bar{A}). \tag{29}$$

We note $E_k(t) \triangleq -\int_0^t (R_{k,k}(\tau) - \bar{R}_k)\,\mathrm{d}\tau$. By Proposition 1, for all $k \in \mathscr{I}_h^+$ the new coordinate $\bar{z}_k$ satisfies $z_k = \bar{z}_k + \varepsilon E_k(t)\bar{z}_k$. This yields, for all $k \in \mathscr{I}_h^+$, $\dot{\bar{z}}_k = -\varepsilon \mathrm{e}^{-\mathrm{i}k\omega_0 t}R_k(t)\bar{\bar{z}} + \varepsilon^2 \sum_{l\in\mathscr{I}_h} f_{k,l}(\bar{z}_k, \bar{z}_l, t, \varepsilon) - \varepsilon \sum_{l\in\mathscr{I}_h, l\neq k} R_{k,l}(t)\mathrm{e}^{\mathrm{i}(l-k)\omega_0 t}\bar{z}_l - \varepsilon \bar{R}_k \bar{z}_k$ where $\{f_{k,l}\}_{(k,l)\in\mathscr{I}_h^2}$ are bounded regular functions, T_0-periodic w.r.t. t. In the $(\bar{\bar{z}}, \{\bar{z}_k\}_{k\in\mathscr{I}_h^+})$ coordinates, system (27) rewrites

$$\begin{cases} \dot{\bar{\bar{z}}} = (J + \varepsilon\mathscr{Q}(t))\bar{\bar{z}} \\ \qquad + \varepsilon \sum_{k\in\mathscr{I}_h} \mathscr{Q}(t)Q_k(t)(1 + \varepsilon E_k(t))\bar{z}_k, \\ \dot{\bar{z}}_k = -\varepsilon \mathrm{e}^{-\mathrm{i}k\omega_0 t}R_k(t)\bar{\bar{z}} - \varepsilon \bar{R}_k \bar{z}_k \\ \qquad - \varepsilon \sum_{l\in\mathscr{I}_h, l\neq k} R_{k,l}(t)\mathrm{e}^{\mathrm{i}(l-k)\omega_0 t}\bar{z}_l \\ \qquad + \varepsilon^2 \sum_{l\in\mathscr{I}_h} f_{k,l}(\bar{z}_k, \bar{z}_l, t, \varepsilon), \quad \forall k \in \mathscr{I}_h^+. \end{cases} \tag{30}$$

Now, the $\{\bar{z}_k\}_{k\in\mathscr{I}_h^+}$ dynamics have a time-invariant self-excitation. Unfortunately, the monodromy matrix cannot be easily derived and stability remains to be proven. Yet, the constant stable matrix J in the dynamics of $\bar{\bar{z}}$ has a positive impact on convergence toward 0 and compensates the ε-perturbation. The next section exhibits a change of coordinates which triangulates the system, i.e., the excitation in the new variable on the $\bar{\bar{z}}$-dynamics will be a ε^2-term.

6.4. Fourth change of coordinates: $(\bar{\bar{z}}, \bar{z}_k) \mapsto (\bar{\bar{z}}, \bar{\bar{z}}_k)$

A final change of coordinates leads to consider a new variable $\{\bar{\bar{z}}_k\}_{k\in\mathscr{I}_h}$ whose dynamics dependance on $\bar{\bar{z}}$ is a term in ε^2. We set for all $k \in \mathscr{I}_h$ $F_k(t) \triangleq \sum_{l\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}l\omega_0 t}\mathscr{F}_{-l}(R_k)(\mathrm{i}(l+k)\omega_0 - J)^{-1}$. Then, for all $k \in \mathscr{I}_h$, $\mathrm{d}(\mathrm{e}^{-\mathrm{i}k\omega_0 t}F_k(t))/\mathrm{d}t = -\mathrm{e}^{-\mathrm{i}k\omega_0 t}F_k(t)J - \mathrm{e}^{-\mathrm{i}k\omega_0 t}R_k(t)$. Consider, for all $k \in \mathscr{I}_h$, $\bar{\bar{z}}_k \triangleq \bar{z}_k - \varepsilon \mathrm{e}^{-\mathrm{i}k\omega_0 t}F_k(t)\bar{\bar{z}}$. Following the computation presented in Section 5.2, we easily conclude that, in the $(\bar{\bar{z}}, \{\bar{\bar{z}}_k\}_{k\in\mathscr{I}_h^+})$ coordinates, system (30) rewrites

$$\begin{cases} \dot{\bar{\bar{z}}} = (J + \varepsilon\mathscr{Q}(t))\bar{\bar{z}} + \varepsilon \sum_{k\in\mathscr{I}_h} \mathscr{Q}(t)Q_k(t)\bar{\bar{z}}_k \\ \qquad + \varepsilon^2 h_{\bar{\bar{z}},k}(t,\varepsilon)\bar{\bar{z}} + \varepsilon^2 \mathscr{Q}(t)\sum_{k\in\mathscr{I}_h} Q_k(t)E_k(t)\bar{\bar{z}}_k, \\ \dot{\bar{\bar{z}}}_k = -\varepsilon \bar{R}_k \bar{\bar{z}}_k - \varepsilon \sum_{l\in\mathscr{I}_h, l\neq k} R_{k,l}(t)\mathrm{e}^{\mathrm{i}(l-k)\omega_0 t}\bar{\bar{z}}_l \\ \qquad + \varepsilon^2 h_k(\bar{\bar{z}}, \{\bar{\bar{z}}_l\}_{l\in\mathscr{I}_h^+}, t, \varepsilon), \end{cases} \tag{31}$$

where the functions $\{h_k, h_{\varepsilon,k,\bar{\bar{z}}}\}_{k\in\mathscr{I}_h}$ are regular, bounded, and T_0-periodic w.r.t. t.

6.5. Final change of coordinates: $(\bar{\bar{z}}, \bar{\bar{z}}_k) \mapsto (\bar{\bar{z}}, \bar{\phi})$

Gathering $\phi = [\bar{\bar{z}}_{-h} \ \ldots \ \bar{\bar{z}}_h]^{\mathrm{T}}$, the $\{\bar{\bar{z}}_k\}_{k\in\mathscr{I}_h}$ dynamics of system (31) write

$$\dot{\phi} = -\varepsilon E(t)\phi + \varepsilon^2 \mathscr{K}(\phi, \bar{\bar{z}}, t, \varepsilon), \tag{32}$$

where $\mathscr{K}$ is regular, and T_0-periodic w.r.t. t,

$$E_{k,l}(t) \triangleq \mathrm{e}^{\mathrm{i}(l-k)\omega_0 t}R_{k,l}. \tag{33}$$

We use the averaging Theorem 1. In our case, $f(\phi) = -E(t)\phi$, then $\bar{f}(\bar{\phi}) = -\bar{E}\bar{\phi}$ where $\bar{E} = (1/T_0)\int_0^{T_0} E(t)\,\mathrm{d}t$. We set $\bar{\phi}(I - \varepsilon \int_0^t (E(s) - \bar{E})\,\mathrm{d}s) \triangleq \phi$, then we have $\dot{\bar{\phi}} = -\varepsilon \bar{E}\bar{\phi} + \varepsilon^2 \bar{\mathscr{K}}(\phi, \bar{\bar{z}}, t, \varepsilon)$, where $\bar{\mathscr{K}}$ is regular, and T_0-periodic w.r.t. t.

6.6. Conclusion toward the convergence of the observer

To check convergence of the observer, we check hyperbolic stability of the monodromy matrix of system (31). Gathering terms yields that there exists $(\tilde{M}_{1,1}, \tilde{M}_{1,2}) \in \mathscr{M}_{n,n}(\mathbb{R}) \times \mathscr{M}_{n,md_h}(\mathbb{C})$ such that the sought-after monodromy matrix is

$$\Phi_{\bar{\bar{z}},\{\bar{\bar{z}}_k\}_{k\in\mathscr{I}_h},\varepsilon} = \begin{bmatrix} J - \varepsilon\tilde{M}_{1,1} & -\varepsilon T_0 \tilde{M}_{1,2} \\ 0 & -\varepsilon\bar{E} \end{bmatrix} + \mathscr{O}(\varepsilon^2) \tag{34}$$

which is (up to second order terms in ε) time-invariant. Because J is asymptotically stable, (34) is asymptotically stable for $0 < \varepsilon \ll 1$ if and only if the system

$$\dot{\bar{\phi}} = -\varepsilon \bar{E}\bar{\phi} \tag{35}$$

is hyperbolically stable. All the changes of coordinates are linear, time-periodic and smooth, and thus uniformly continuous. Therefore, convergence toward 0 of $(\bar{\bar{z}}, \bar{\phi})$ leads to the convergence toward 0 of $(z, \{z_k\}_{k\in\mathscr{I}_h})$. To ensure hyperbolically stability of system (35), we set $\beta_k \neq 0$ and

$$l_k(t) \triangleq \beta_k (C(t)W_k(t))^{\dagger}, \tag{36}$$

where J and $S(t)$ are defined from Theorem 2, and $W_k = S^{-1}Q_k$, where Q_k is defined in Eq. (26). With this choice, we have for all $(k,l) \in \mathscr{I}_h^2$, $R_{l,k} = (C(t)W_l(t))^{\dagger}C(t)W_k(t)$. Then, $E(t) \geqslant 0$ and so is $\bar{E}$. Therefore, hyperbolic stability of $-\bar{E}$ is equivalent to $\ker \bar{E} = \{0\}$. Yet, necessarily, $\ker \bar{E} = \{0\}$ otherwise H2 would be violated: for any $\Gamma = [\Gamma_{-h}, \ldots, \Gamma_h]^{\mathrm{T}} \in \ker \bar{E}$, $x = \sum_k \mathrm{e}^{\mathrm{i}k\omega_0 t}W_k(t)\Gamma_k$ and $c_l = \delta_{l,k}\Gamma_k$, is solution of (2) with

$y = Cx \equiv 0$. Thus system (35) is asymptotically stable. We have proven the following proposition.

Proposition 3. *Consider* (2) *with* $m \leqslant p < n$. *Assume that the matrices* A, A_0 *and* C *are* T_0*-periodic and that hypothesis* H1 *and* H2 *hold. Consider the observer* (3) *with gains* L *and* L_k *as defined in Section* 2.3.3. *Then, for small enough* $\varepsilon > 0$, *the error dynamics* (4) *asymptotically converge to* 0.

7. Conclusion

A constructive input estimation method for a class of time-periodic linear systems is proposed. Convergence is analysed using averaging techniques along with Lyapunov arguments. Examples from the automotive engines area are presented. These examples stress that this method is an efficient alternative to Kalman filtering when real-time computational power is limited. This is especially true when h, the number of modes to reconstruct, is large. Further publications will concentrate on infinite dimensional cases (i.e., when $h = \infty$). We wish to derive a theoretical justification for the $1/(k^2+1)$-dependence of the gain β_k associated to the kth mode. This tuning rule is used in the discussed examples. In fact, it is related to an asymptotic formula that appears in the infinite dimensional setting.

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