FLATNESS AND OPTIMAL CONTROL OF OSCILLATORS

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Abstract:
The aim of this paper is to present some recent developments and hints for future researches in control inspired of flatness-based ideas. We explain how explicit trajectory parameterization, a property that is central for flat systems, can be useful for the control of various oscillatory systems (linear, nonlinear, finite and/or infinite dimension) of physical and engineering interests. Such parameterization provide simple algorithms to generate optimal trajectories via exact discretization. Three key examples are detailed: a linearized Schrödinger equation describing the interaction of an electro-magnetic field (the control) with an n-levels quantum system; the heavy chain described by a 1D wave equation; an Euler Bernoulli flexion beam.

Keywords: Flat systems, oscillators, motion planing, optimal control, wave equation, Euler Bernoulli equation, Gevrey functions.

1. INTRODUCTION

The goal of this paper is to present, on three physical examples, a finite dimensional Schrödinger equation, a $1D$ wave equation and a $1D$ Euler-Bernoulli equation, a natural way, initiated in (Nieuwstadt and Murray, 1998), to merge flatness based control with optimal control. The notion of flat-systems goes back to Monge (Monge, 1784): for certain under-determined differential systems appearing in differential geometry (Monge equations), he was able to describe explicitly their general solutions in term of a finite number of arbitrary functions of one variable and their derivatives. Thus, for flat systems, it is easy to generate sets of trajectories that satisfy both the initial and final state constraints. For optimal control problems where final state constraints are difficult to satisfy numerically, such trajectory sets provide an exact discretization depending on an arbitrary numbers of real parameters that can be chosen to minimize the cost functional.

Section 2 is devoted to a basic definition of flatness for systems described by nonlinear ODE's. In section 4, we treated the linearized Schrödinger equation that governs the dynamics of a ground energy level coupled via an electro-magnetic field, the control u, with a finite number of excited levels. One recovers the fact that the smallest control in L^2 norm that steers from one state to another one is the superposition of pulses with fixed amplitudes, phases and Bohr frequencies. Section 5 is devoted to the heavy chain: we propose a simple method to compute numerically with finite P_3 elements the control steering from one steady-state to another one, that minimize the L^2 norm of the trolley acceleration. A similar problem is addressed in section 6 for a flexion beam described by an Euler Bernoulli system:

the P_3 finite elements used for the heavy chain are here transformed (convolution) into C^∞ time function of Gevrey order less than 1 to guaranty the convergence of the series appearing in the flatness-based exact discretization.

2. FLAT SYSTEMS

More than 10 years ago, Michel Fliess and coworkers (Fliess *et al.*, 1992; Fliess *et al.*, 1995; Fliess *et al.*, 1999) introduced a special class of non-linear control systems described by ordinary equations: differential flat systems form a special class of nonlinear control systems for which systematic control methods are available once a flat-output is explicitly known. We just sketch a tutorial definition of flatness for state-space control system. The smooth system $\frac{d}{dt}x = f(x,u)$ with m scalar control $u = (u_1, \ldots, u_m)$ is flat, if and only if, there exist m real smooth functions $h = (h_1, ..., h_m)$ depending on x and a finite number of u derivatives, says α, such that, generically, the solution (x,u) of the square differential-algebraic system ($t \mapsto y(t)$ is given)

$$\dot{x} = f(x,u), \quad y(t) = h(x, u, \dot{u}, ..., u^{(\alpha)})$$

does not involve any differential equation and thus is of the form

$$x = \Phi(y, \dot{y}, ..., y^{(\beta)}), \quad u = \Psi(y, \dot{y}, ..., y^{(\beta+1)})$$

where Ψ and Φ are smooth functions and β is some finite number. The quantity y is of fundamental importance: it is called flat-output or linearizing-output. In control language, the flat output y is such that, the inverse of $\dot{x} = f(x,u), y = h(x, u, ..., u^{(\alpha)})$ has no dynamics (Isidori *et al.*, 1986).

Flatness is related to state feedback linearization and in fact has a long history. Such notion goes back to Hilbert (Hilbert, 1912) with his work on the general solution of Monge equations, work that has been prolonged by Cartan (Cartan, 1914) with a characterization via the derived flag of solvable (in the sense of Hilbert) Monge equations of any-order. In general, the problem of flatness characterization is fully open for multi-input systems ($\dim(u) > 1$). There is no algorithm to decide once the equations $\dot{x} = f(x,u)$ are given, if there exists such map h, called flat-output map.

The situation is somehow comparable to integrable Hamiltonian systems: there is no algorithm to decide whether a given Hamiltonian $H(q,p)$ yields an integrable system; many examples of physical interest are integrable and for these systems we have the form of their general solution in terms of the initial conditions; only necessary conditions are available (see, e.g. the Morales-Ramis theorem (Morales-Ruiz and Ramis, 2001)). For flat systems, the situation is very similar: no algorithm to decide whether a system is flat or not; many examples of engineering interest are flat and their general solution reads in term of the derivatives of a flat-output y that has a clear physical interpretation (Martin *et al.*, 2003); few necessary conditions are available (see, e.g., the ruled-manifold criterion (Rouchon, 1995)). To summarize: the role of flat-systems within the set of under-determined ordinary differential systems is very similar to the role of integrable systems within the set of determined ordinary-differential systems.

3. FLATNESS AND OPTIMAL CONTROL

Consider the standard optimal control problem

$$\min_u J(u) = \int_0^T L(x(t), u(t))dt$$

together with $\dot{x} = f(x,u)$, $x(0) = a$ and $x(T) = b$, for known a, b and T.

Assume that $\dot{x} = f(x,u)$ is flat with $y = h(x, u, \ldots, u^{(\alpha)})$ as flat output,

$$x = \Phi(y, \ldots, y^{(\beta)}), \ u = \Psi(y, \ldots, y^{(\beta+1)}).$$

A numerical resolution of $\min_u J(u)$ a priori requires discretization of the state space, i.e., a finite dimensional approximation. A better way is to discretize the flat output space. Set $y_i(t) = \sum_1^N z_{ij}\varphi_j(t)$. The initial and final conditions on x provide then initial and final constraints on y and its derivatives up to order $\beta+1$. These constraints define an affine sub-space V of the vector space spanned by the z_{ij}'s. We are thus left with the nonlinear programming problem

$$\min_{\in V} J(z) = \int_0^T L\left(\Phi(y, \ldots, y^{(\beta)}), \Psi(y, \ldots, y^{(\beta+1)})\right) dt,$$

where the $y_i^{(\nu)}$'s must be replaced by $\sum_1^N z_{ij}\varphi_j^{(\nu)}(t)$.

This methodology has been first used in (Nieuwstadt and Murray, 1998) for trajectory generation and optimal control. It should also be very useful for predictive control. The main expected benefit is a dramatic improvement in computing time and numerical stability. Indeed the exact quadrature of the dynamics –corresponding here to exact discretization via well chosen input signals through the mapping Ψ– avoids the usual numerical sensitivity troubles during integration of $\dot{x} = f(x,u)$ and the problem of satisfying $x(T) = b$.

Numerical experiments (Milam *et al.*, 2001; Murray *et al.*, 2003; Milam, 2003) indicate that substantial computing gains are obtained when flatness based parameterizations are employed. A systematic method exploiting flatness for predictive control is proposed in (Fliess and Marquez, 2000) (see also (Findeisen and Allgöwer, 2002) for nonlinear predictive control). See also (Petit

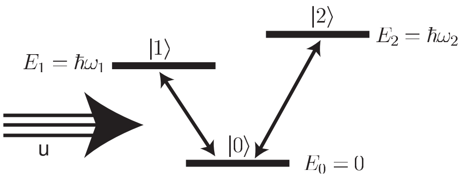

Fig. 1. A three levels system interacting with a linearly polarized electric field $u \in \mathbb{R}$, the control.

et al., 2001) for an industrial application of such methodology on a chemical reactor. See also (Oldenburg and Marquardt, 2002) where limitations of such techniques are presented. Extension to infinite dimensional systems has also been proposed in (Petit *et al.*, 2002).

4. LINEARIZED QUANTUM SYSTEMS

Take an atom with three energy levels coupled to a linearized polarized electromagnetic flied $u(t) \in \mathbb{R}$, the control. Assume that its probability amplitude wave function ψ obeys, under the dipolar approximation, the following finite dimensional Schrödinger equation ($\imath = \sqrt{-1}$):

$$\imath\hbar\frac{d}{dt}\psi = (H_0 + u(t)H_1)\psi$$

where $\psi \in \mathbb{C}^3$ is the wave function belonging to the Hilbert space $\mathbb{C}^3$ and where H_0 and H_1 are Hermitian matrices

$$H_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \hbar\omega_1 & 0 \\ 0 & 0 & \hbar\omega_2 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0 & \mu_1 & \mu_2 \\ \mu_1 & 0 & 0 \\ \mu_2 & 0 & 0 \end{pmatrix}.$$

The pulsations $\omega_1 > 0$, $\omega_2 > 0$ and the coupling parameter (μ_1, μ_2) are real quantities. With the standard notations,

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

we see that $|0\rangle$ is the quantum state of energy $E_0 = 0$, $|1\rangle$ of energy $E_1 = \hbar\omega_1$ and $|2\rangle$ and energy $E_2 = \hbar\omega_2$, $\hbar = h/(2\pi)$ being the Plank constant.

Thus $\psi = \psi_0\,|0\rangle + \psi_1\,|1\rangle + \psi_2\,|2\rangle$ where ψ_0, ψ_1 and ψ_2 are 3 complex numbers those square modulus represent the probability for being states $|0\rangle$, $|1\rangle$ or $|2\rangle$, respectively. Schrödinger equation reads thus:

$$\begin{aligned} \imath\frac{d}{dt}\psi_0 &= u\frac{\mu_1}{\hbar}\psi_1 + u\frac{\mu_1}{\hbar}\psi_2 \\ \imath\frac{d}{dt}\psi_1 &= \omega_1\psi_1 + u\frac{\mu_1}{\hbar}\psi_0 \\ \imath\frac{d}{dt}\psi_2 &= \omega_2\psi_2 + u\frac{\mu_2}{\hbar}\psi_0. \end{aligned}$$

The ground state $\psi_0 = 1$, $\psi_1 = \psi_2 = 0$ with the control $u = 0$ is a steady state. Let us linearize the above equations around this equilibrium. Denoting by $\delta\psi_k$, $k = 0, 1, 2$, and δu the small deviations, we get:

$$\begin{aligned} \imath\frac{d}{dt}\delta\psi_0 &= 0 \\ \imath\frac{d}{dt}\delta\psi_1 &= \omega_1\delta\psi_1 + \frac{\mu_1}{\hbar}\,\delta u \\ \imath\frac{d}{dt}\delta\psi_2 &= \omega_2\delta\psi_2 + \frac{\mu_2}{\hbar}\,\delta u \end{aligned}$$

Thus $\delta\psi_0$ remains constant. This non controllable part corresponds to the conservation of probability and will by ignored in the sequel (the length of ψ is always equal to 1). We concentrate thus on following linearized dynamics in $\mathbb{C}^2$, i.e. in $\mathbb{R}^4$:

$$\imath\frac{d}{dt}\delta\psi_1 = \omega_1\delta\psi_1 + \frac{\mu_1}{\hbar}\,\delta u, \; \imath\frac{d}{dt}\delta\psi_2 = \omega_2\delta\psi_2 + \frac{\mu_2}{\hbar}\delta u$$

Replace $\psi_1, \psi_2 \in \mathbb{C}$ and by $(x_1, v_1, x_2, v_2) \in \mathbb{R}^4$ via

$$\delta\psi_1 = z_1 + \frac{\imath\; v_1}{\omega_1}, \quad \delta\psi_2 = z_2 + \frac{\imath\; v_2}{\omega_2}$$

to get the following dynamics:

$$\frac{d}{dt}z_k = v_k, \quad \frac{d}{dt}v_k = -(\omega_k)^2 z_k - \frac{\mu_k\omega_k}{\hbar}\,\delta u, \quad k = 1, 2$$

Thus for $k = 1, 2$, $\delta\psi_k = z_k + \frac{\imath \frac{d}{dt} z_k}{\omega_k}$ and

$$\frac{d^2}{dt^2}z_k = -(\omega_k)^2 z_k - \frac{\mu_k\omega_k}{\hbar}\,\delta u.$$

More generally, the linearized dynamics around a ground state $|0\rangle$, of a quantum system with n excited states $|k\rangle$, $k = 1, ..., n$, in interaction with an electro-magnetic field u, read:

$$\frac{d^2}{dt^2}z_k = -(\omega_k)^2 z_k + b_k u, \quad k = 1, \ldots, n \tag{1}$$

where for each $k = 1, \ldots, n$, $b_k = -\frac{\mu_k\omega_k}{\hbar}$ and

$$\delta\psi_k = z_k + \frac{\imath\frac{d}{dt}z_k}{\omega_k}$$

represents the complex probability amplitude to be in the excited state $|k\rangle$. This model is valid only for $\|\delta\psi_k\| \ll 1$ and for small u. We assume here that $|0\rangle$ is coupled to $n > 0$ excited state $(|k\rangle)_{1\leq k\leq n}$ via first order transition of frequencies $(\omega_k)_{1\leq k\leq n}$ and coupling real parameters $(\mu_k)_{1\leq k\leq n}$. We assume in the sequel that $\omega_k \neq \omega_l$ when $k \neq l$ and $\mu_k \neq 0$. This means that the linear tangent system (1) is controllable.

As explained in (Lévine and Nguyen, 2003; Lévine, 2004; Rouchon, 2005), we have for any n, the following explicit flatness based parameterization: with $s = d/dt$, (1) reads formally:

$$\left(s^2 + (\omega_k)^2\right) z_k = b_k u, \quad k = 1, ..., n$$

that can be seen as a linear under-determined system with $n + 1$ unknown variables (the z_k and the control u) and n equations. This system admits an explicit formulation in the following sense:

$$z_k = Q_k(s)y, \quad u = Q(s)y \quad \text{with} \quad y = \sum_{l=1}^{n} c_l z_l$$

where

$$Q_k(s) = \frac{b_k}{(\omega_k)^2} \prod_{\substack{l=1 \\ l \neq k}}^{n} \left(1 + \left(\frac{s}{\omega_l}\right)^2\right)$$

$$Q(s) = \prod_{l=1}^{n} \left(1 + \left(\frac{s}{\omega_l}\right)^2\right)$$

$$c_k = \frac{1}{Q_k(\imath\omega_k)} \in \mathbb{R}.$$

Take $T > 0$, the initial probability amplitudes $\delta\psi_k^0 = z_k^0 + \imath\frac{\dot{z}_k^0}{\omega_k}$ and the final probability amplitudes $\delta\psi_k^T = z_k^T + \imath\frac{\dot{z}_k^T}{\omega_k}$. Let us compute with the above polynomials Q et Q_k, the control of minimum energy (L^2 norm) steering from the initial to final conditions fixed here above. Thus we minimize $\int_0^T u^2(t)dt$ subject to these initial and final constraints. Since $u = Q(\frac{d}{dt})y$, This problem is equivalent to the following one

$$\min_{\substack{[0,T] \ni t \mapsto y(t) \\ \text{s.t. for } l = 0,...,2n-1 \\ y^{(l)}(0) = y_0^l \\ y^{(l)}(T) = y_T^l}} \int_0^T \left[Q\left(\frac{d}{dt}\right)y(t)\right]^2 dt.$$

The initial et final derivatives of y up to order $2n-1$ are related to the initial and final probability amplitudes via $z_k^\nu = \sum_{l=0}^{n-1} q_l^k y_\nu^{(2l)}$ and $\dot{z}_k^\nu = \sum_{l=0}^{n-1} q_l^k y_\nu^{(2l+1)}$, $\nu = 0, T$ ($Q_k(s) = \sum q_k^l s^{2l}$). This problem can be solved by the Euler-Lagrange differential equation of order $4n$ satisfied by y: $Q^2(\frac{d}{dt})y = 0$. Its general solution reads (c.c. means 'complex conjugate')

$$y(t) = \sum_{l=1}^{n} (a_l + tb_l)e^{\imath\omega_l t} + \text{c.c.}$$

where a_l and b_l are the complex integration constants implicitly fixed by the initial and final constraints. We have for such $y(t)$,

$$Q_k(\frac{d}{dt})y(t) = \sum_{l=1}^{n-1} \left(Q_k(\imath\omega_l)(a_l + tb_l) + Q_k'(\imath\omega_l)b_l\right) e^{\imath\omega_k t} + \text{c.c.}.$$

But $Q_k(\imath\omega_l) = 0$ for $l \neq k$, $Q_k(\imath\omega_k) = p_k \in \mathbb{R}$, $Q_k'(\imath\omega_l) = \imath r_k^l$, with $r_l^l \in \mathbb{R}$. With such notations we have

$$z_k(t) = p_k(a_k + tb_k)e^{\imath\omega_k t} + \imath\sum_{k=1}^{n} r_k^l b_l e^{\imath\omega_l t} + \text{c.c.}.$$

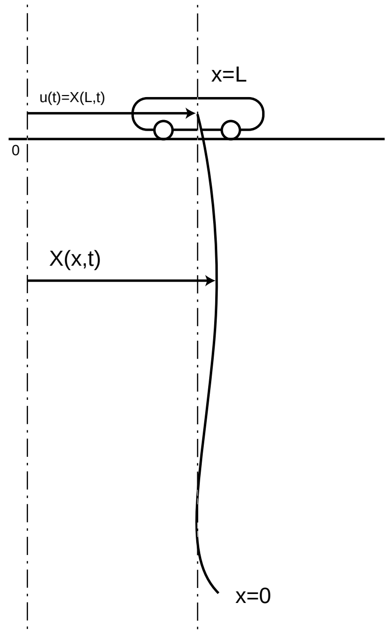

Fig. 2. The homogeneous chain without any load.

Similarly, we have

$$\dot{z}_k(t) = \imath\omega_k p_k(a_k + tb_k)e^{\imath\omega_k t} - \sum_{k=1}^{n} \omega_l r_k^l b_l e^{\imath\omega_l t} + \text{c.c.}.$$

It is thus simple to recover the a_l and the b_l from z_k^0, $\dot{z}_k^0$, z_k^T and $\dot{z}_k^T$ by inverting a linear system. The optimal control is then of the form

$$u(t) = \sum_{k=1}^{n} Q'(\imath\omega_k)b_k e^{\imath\omega_k t} + \text{c.c.}$$

and thus is of the form

$$u(t) = \sum_{k=1}^{n} u_k \cos(\omega_k t + \alpha_k)$$

where u_k and α_k are amplitudes and phases parameters. We recover the usual fact that physicists manipulate atoms with laser lights resonant with the transition frequencies ω_k. Such manipulations are usually justified with the rotating wave approximation and averaging techniques. In such context, the relative phase between two different excited state $|k\rangle$ and $|l\rangle$ is not controlled, only populations, the module of each ψ_k, is controlled via the amplitude and time window of the pulse with frequency ω_k. Here, we can also controlled the relative phase between ψ_k and ψ_l.

5. THE HEAVY CHAIN

Consider a heavy chain in stable position as depicted in figure 2. Under the small angle approximation it is ruled by the following dynamics

$$\begin{cases} \frac{\partial}{\partial x}(gx\frac{\partial X}{\partial x}) - \frac{\partial^2 X}{\partial t^2} = 0 \\ X(L,t) = u(t). \end{cases} \quad (2)$$

where $x \in [0, L]$, $t \in \mathbb{R}$, $X(x,t) - X(L,t)$ is the deviation profile, g is gravitational acceleration and the control u is the trolley position. Let us recall the explicit parameterization of this system given in (Petit and Rouchon, 2001) and based on symbolic computations in the Laplace domain:

$$X(x,t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(t + 2\sqrt{x/g} \sin\theta) \; d\theta, \qquad (3)$$

with $y(t) = X(0,t)$. Relation (3) means that there is a one to one correspondence between the (smooth) solutions of (2) and the (smooth) functions $t \mapsto y(t)$. For each solution of (2), set $y(t) = X(0,t)$. For each function $t \mapsto y(t)$, set X by (3) and u as

$$u(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(t + 2\sqrt{L/g} \sin\theta) \; d\theta, \qquad (4)$$

to obtain a solution of (2). Denote by $\Delta = 2\sqrt{L/g}$, the travelling time between $x = 0$ to $x = L$.

For $T > 2\Delta$ and $D \in \mathbb{R}$, let us consider the following problem

$$\begin{array}{c} \min \\ [0,T] \ni t \mapsto u(t) \\ X(x,0) = 0, \; x \in [0,L] \\ X(x,T) = D, \; x \in [0,L] \end{array} \int_0^T (\ddot{u})^2(t) \; dt.$$

The initial and final state impose $y(t) = 0$ for $t \in [-\Delta, +\Delta]$ and $y(t) = D$ for $t \in [T - \Delta, T + \Delta]$. This explains why we take $T > 2\Delta$. For $t \in]\Delta, T - \Delta[$, $y(t)$ is free and thus, the above problem reduced

$$\begin{array}{c} \min \\ y \\ y(t \leq \Delta) = 0 \\ y(t \geq T - \Delta) = D \end{array} \int_0^T \left[\int_{-\pi}^{\pi} \ddot{y}(t + \Delta \sin\theta) \; d\theta, \right]^2 dt.$$

To get a numerical solution to this problem, we can approximate y via finite elements P_3 over a time grid of step $h = (T - 2\Delta)/N$, N a large integer. Denote by $\phi : \mathbb{R} \mapsto \mathbb{R}$ the P_3 generating function:

$$\phi(\rho) = \begin{cases} 0, & \text{for } \rho \leq -1; \\ 3(1+\rho)^2 - 2(1+\rho)^3, & \text{for } -1 \leq \rho \leq 0; \\ 3(1-\rho)^2 - 2(1-\rho)^3, & \text{for } 0 \leq \rho \leq 1; \\ 0, & \text{for } 1 \leq \rho; \end{cases}$$

Set

$$y(t) = \sum_{k=1}^{k=N-1} y_k h^2 \phi\left(\frac{t + \Delta - kh}{h}\right) + DH\left(\frac{t + \Delta - T}{T - 2\Delta}\right)$$

where the y_k's are parameters and where

$$H(\rho) = \begin{cases} 0, & \text{for } \rho \leq -1; \\ 3(1+\rho)^2 - 2(1+\rho)^3, & \text{for } -1 \leq \rho \leq 0; \\ 1 & \text{for } 0 \leq \rho; \end{cases}$$

It is easy to seen that y is KC^2, $y(t \leq \Delta) = 0$ and $y(t > T - \Delta) = D$. Thus we have to minimize the following quadratic function in $(y_k)_{k=1,N-1}$:

$$\int_0^T \left[\int_0^{2\pi} \left(\frac{D}{(T-2\Delta)^2} \ddot{H}\left(\frac{t + \Delta(\sin\theta + 1) - T}{T - 2\Delta}\right) + \sum_{k=0}^{N-1} y_k \ddot{\phi}\left(\frac{t + \Delta(\sin\theta + 1) - kh}{h}\right) \right) d\theta \right]^2 dt.$$

The coefficient of this quadratic function can be computed explicitly via simple integrals.

6. FLEXION BEAM

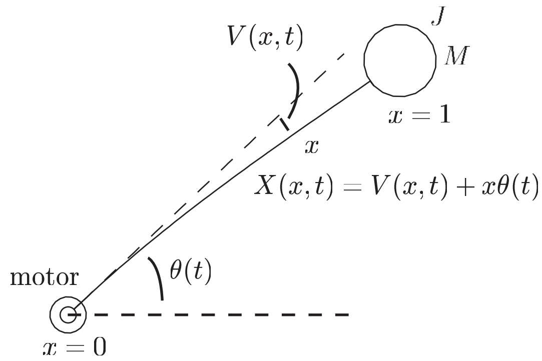

Fig. 3. a flexible beam rotating around a control axle

Consider the flexible beam of figure 3 that rotates around a motorized axis of angle θ and equipped with a punctual end load of mass M and inertia moment J. Up to some scaling, it obeys the following Euler Bernoulli dynamics:

$$\partial_{tt} X = -\partial_{xxxx} X$$

$$X(0,t) = 0, \quad \partial_x X(0,t) = \theta(t)$$
$$\ddot{\theta}(t) = u(t) + k \partial_{xx} X(0,t)$$

$$\partial_{xx} X(1,t) = -\lambda \partial_{ttx} X(1,t)$$
$$\partial_{xxx} X(1,t) = \mu \partial_{tt} X(1,t)$$

where the control is the motor torque u, $X(x,t)$ is the deformation profile, k, λ and μ are physical parameters (t and x are in reduced scales). Symbolic computations provide, as shown in (Fliess *et al.*, 1996), the following parameterization:

$$X(x,t) = \sum_{n \geq 0} \frac{(-1)^n \; y^{(2n)}(t)}{(4n)!} P_n(x) + \frac{(-1)^n \; y^{(2n+2)}(t)}{(4n+4)!} Q_n(x) \qquad (5)$$

with $\imath = \sqrt{-1}$,

$$P_n(x) = \frac{(\Im - \Re)(1 - x + \imath)^{4n+1}}{2(4n+1)} + \mu \Im (1 - x + \imath)^{4n} + \frac{x^{4n+1}}{2(4n+1)}$$

and

$$\frac{Q_n(x)}{(4n+4)(4n+3)} = -\lambda \Re (1 - x + \imath)^{4n+2} + \frac{\lambda\mu(4n+2)}{2} \left((\Im - \Re)(1 - x + \imath)^{4n+1} - x^{4n+1} \right)$$

($\Re$ and $\Im$ stand for real part and imaginary part). The values of $\theta = \partial_x X(0,t)$ and $u = \partial_{xtt} X(0,t) -$

$k\partial_{xx}X(0,t)$ result from (5): it suffices to derive term by term the above series. The series obtained for u is

$$u(t) = \sum_{n=1}^{+\infty} c_n \frac{y^{(2n)}(t)}{(4n)!} \tag{6}$$

where c_n are real coefficients such that exists $R > 0$ with $|c_n| \leq R^n$ for all n.

To ensure converge of such series, the "flat output" y, a C^∞ function, has to satisfy some conditions. Roughly speaking, the growth of its derivative of order n must be comparable to $(2n)!$. More precisely, if the C^∞ function y is of Gevrey order [1] less than 1, the above series are absolutely convergent and for each t, $x \mapsto X(x,t)$ is an entire function. A C^∞ function y is of Gevrey order $\alpha \geq 0$, iff exists $K, A > 0$ such that for all t and $n > 0$

$$|y^{(n)}| \leq KA^n\Gamma(1 + n(\alpha + 1)).$$

The sum and multiplication of two Gevrey function of order α is still of order α. Gevrey functions of order 0 are analytic functions. A typical Gevrey function ϖ_α of order $\alpha > 0$ with the compact support $[-1, 1]$ is the following

$$\varpi_\alpha(\rho) = \begin{cases} 0, & \text{for } \rho \leq -1 \\ \exp\left(\dfrac{-1}{(1-\rho^2)^{\frac{1}{\alpha}}}\right), & \text{for } -1 \leq \rho \leq 1 \\ 0, & \text{for } \rho \geq 1 \end{cases}$$

See (Guelfand and Chilov, 1964; Ramis, 1978) for more details on Gevrey functions. For $T > 0$ and Θ, let us consider the following problem

$$\begin{array}{c} \min\limits_{[0,T] \ni t \mapsto u(t)} \displaystyle\int_0^T u^2(t)\ dt \\ X(x,0) = 0,\ x \in [0,1] \\ X(x,T) = x\Theta,\ x \in [0,1] \end{array}$$

where the initial and final state are steady-state at $\theta = 0$ and Θ. Such constraints impose on y, the following conditions:

$$y(0) = 0,\ y(T) = \Theta,\ y^{(k)}(0,T) = 0,\ \text{for } k > 0.$$

Take $\Delta \in]0, \frac{T}{2}[$ (typically $\Delta = \frac{T}{10}$), and consider, with the grid notations used for the heavy chain, the $KC2$ function $z : \mathbb{R} \mapsto \mathbb{R}$, depending of $N-1$ real parameters $(z_k)_{1\leq k<N}$, defined for $t \in [\Delta, T-\Delta]$ via

$$z(t) = \sum_{k=1}^{N-1} z_k \phi\left(\frac{t+\Delta-kh}{h}\right) + \Theta H\left(\frac{t+\Delta-T}{T-2\Delta}\right)$$

and $z(t < \Delta) = 0$, $z(t > T - \Delta) = \Theta$. By convolution with the positive Gevrey function χ,

$$\chi(t) = \frac{\varpi_{\frac{1}{2}}\left(\frac{t}{\Delta}\right)}{\Delta\int_{-\infty}^{+\infty}\varpi_{\frac{1}{2}}(\rho)\ \mathrm{d}\rho},$$

one always obtains a C^∞ function $y = \chi * z$ that satisfies the constraints on y for $t = 0$ and $t = T$. Moreover since the convolution kernel χ is of order $\frac{1}{2}$, the function $\chi * z$ is also of Gevrey order $\frac{1}{2}$ and its derivatives of order n is just $\chi^{(n)} * z$. Since ϖ satisfies the first order differential equation

$$(1-\rho^2)^3\frac{d\varpi_{\frac{1}{2}}}{d\rho}(\rho) = 4\rho\varpi(\rho)$$

it's easy to compute $\chi^{(n)}/(2n)!$ via a well conditioned recurrence obtained from derivations of this first order differential equation. We can thus compute numerically without any difficulties the $N-1$ parameters $(z_k)_{1\leq k<N}$, that make the following integral

$$\int_0^T \left(\sum_{n=1}^{+\infty} c_n \frac{\chi^{(2n)} * z\ (t)}{(4n)!}\right)^2 dt$$

minimum since it is non-degenerated quadratic function.

It is known that divergent series could be also very efficient numerically (see e.g. (Ramis, 1978). Thus further extensions of these computations consist in applying the above formula with a function χ constructed with ϖ_α for $\alpha > 1$ and resummation techniques (see (Laroche *et al.*, 2000) and (Meurer and Zeitz, 2004)).

[1] We use here the new convention suggested by B. Malgrange for the Gevrey order: it is just the old order minus one.

REFERENCES

Cartan, E. (1914). Sur l'équivalence absolue de certains systèmes d'équations différentielles et sur certaines familles de courves. *Bull. Soc. Math. France* **42**, 12–48. Also in Ouvres Complètes, part II, vol. 2, pp.1133–1168, CNRS, Paris, 1984.

Findeisen, R. and F. Allgöwer (2002). An introduction to nonlinear model predictive control. 21st Benelux Meeting on Systems and Control, Veldhoven.

Fliess, M. and R. Marquez (2000). Continuous time linear predictive control and flatness: a module theoretic setting with examples. *Int. Journal of Control* **73**, 606–623.

Fliess, M., H. Mounier, P. Rouchon and J. Rudolph (1996). Systèmes linéaires sur les opérateurs de Mikusiński et commande d'une poutre flexible. In: *ESAIM Proc. "Élasticité, viscolélasticité et contrôle optimal", 8ème entretiens du centre Jacques Cartier, Lyon.* pp. 157–168.

Fliess, M., J. Lévine, Ph. Martin and P. Rouchon (1992). Sur les systèmes non linéaires différentiellement plats. *C.R. Acad. Sci. Paris* **I–315**, 619–624.

Fliess, M., J. Lévine, Ph. Martin and P. Rouchon (1995). Flatness and defect of nonlinear systems: introductory theory and examples. *Int. J. Control* **61**(6), 1327–1361.

Fliess, M., J. Lévine, Ph. Martin and P. Rouchon (1999). A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems. *IEEE AC* **44**, 922–937.

Guelfand, I.M. and G.E. Chilov (1964). *Les Distributions, tome 3*. Dunod, Paris.

Hilbert, D. (1912). Über den Begriff der Klasse von Differentialgleichungen. *Math. Ann.* **73**, 95–108. also in Gesammelte Abhandlungen, vol. III, pp. 81–93, Chelsea, New York, 1965.

Isidori, A., C.H. Moog and A. De Luca (1986). A sufficient condition for full linearization via dynamic state feedback. In: *Proceedings 25th. IEEE Conf. Decision Control*. pp. 203–208.

Laroche, B., Ph. Martin and P. Rouchon (2000). Motion planing for the heat equation. *Int. Journal of Robust and Nonlinear Control* **10**, 629–643.

Lévine, J. (2004). Systèmes plats : mode d'emploi. In: *CIFA04, Douz, Tunisie*. plenary lecture.

Lévine, J. and D.V. Nguyen (2003). Flat output characterization for linear systems using polynomial matrices. *Systems & Control Letters* **48**, 69–75.

Martin, Ph., R. Murray and P. Rouchon (2003). Flat systems, equivalence and trajectory generation. Technical Report `http://www.cds.caltech.edu/reports/`.

Meurer, T. and M. Zeitz (2004). A modal approach to flatness-based control of flexible structures.. *Proc. Appl. Math. Mech* **4**(1), 133–134.

Milam, M. B. (2003). Real-time optimal trajectory generation for constrained systems. PhD thesis. California Institute of Technology.

Milam, M. B., N. Petit and R. M. Murray (2001). Constrained trajectory generation for microsatellite formation flying. In: *AIAA Guidance, Navigation and Control Conference*. pp. 328–333.

Monge, G. (1784). *Mémoire de l'Académie Royale des Sciences*. Chap. Supplément où l'on fait voir que les équations aux différences ordinaires pour lesquelles les conditions d'intégrabilité ne sont pas satisfaites sont succeptibles d'une véritable intégration..., pp. 502–576. Paris.

Morales-Ruiz, J.J. and J.P. Ramis (2001). Galoisian obstructions to integrability of hamiltonian systems. *Methods and Applications of Analysis*.

Murray, R. M., J. Hauser, A. Jadbabaie, M. B. Milam, N. Petit, W. B. Dunbar and R. Franz (2003). Online control customization via optimization-based control. In: *Software-Enabled Control, Information technology for dynamical systems* (T. Samad and G. Balas, Eds.). Wiley-Interscience. pp. 149–174.

Nieuwstadt, M. van and R.M. Murray (1998). Real time trajectory generation for differentially flat systems. *Int. Journal of Robust and Nonlinear Control* **8**(11), 995–1020.

Oldenburg, J. and W. Marquardt (2002). Flatness and higher order differential model representations in dynamic optimization. *Computers and Chemical Engineering* **26**, 385–400.

Petit, N. and P. Rouchon (2001). Motion planning for heavy chain systems. *SIAM J. Control and Optim.* **41**(2), 475–495.

Petit, N., M. B. Milam and R. M. Murray (2002). A new computational method for optimal control of a class of constrained systems governed by partial differential equations.

Petit, N, Y. Creff and P. Rouchon (2001). Minimum time constrained control of acid strength on a sulfuric acid alkylation unit. *Chemical Engineering Science*.

Ramis, J.P. (1978). Dévissage Gevrey. *Astérisque* **59-60**, 173–204.

Rouchon, P. (1995). Necessary condition and genericity of dynamic feedback linearization. *J. Math. Systems Estim. Control* **5**(3), 345–358.

Rouchon, P. (2005). Flatness based control of oscillators. *ZAMM Z. Angew. Math. Mech* **85**(6), 411–421.