# ON THE CONTROLLABILITY OF A WAVE EQUATION WITH STRUCTURAL DAMPING

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Abstract: We study the boundary controllability properties of a wave equation with structural damping $y_{tt} - y_{xx} - \epsilon y_{txx} = 0$, $y(0,t) = 0$, $y(1,t) = h(t)$ where $\epsilon$ is a strictly positive parameter depending on the damping strength. We prove that this equation is not spectrally controllable and that the approximate controllability depends on the functional space in which the initial value Cauchy problem is studied. 

Keywords: structural damping, wave equation, spectral controllability, approximate controllability

## 1. INTRODUCTION

In this paper we consider the following perturbed wave equation

$$y_{tt} - y_{xx} - \varepsilon y_{txx} = 0 \tag{1}$$

where $\varepsilon$ is a small positive parameter corresponding to the strength of some *structural damping* (also called internal damping). This equation has been proposed in (Pellicer and Solà-Morales, 2004) as an alternative model for the classical spring-mass-damper ODE.

We stress that the structural damping term $-\varepsilon y_{txx}$ acts in a much stronger way than the classical viscous damping term $-\varepsilon y_t$, and that the principal symbol of (1) differs from the one for the wave equation. With such structural damping, the spectrum of (1) (supplemented with Dirichlet boundary conditions on the domain $I = (0,1)$) admits two accumulation real points: $-\infty$ and $-\varepsilon$. As illustrated on Figure 1, the eigenvalues split up into two sequences of complex numbers: the first one is composed of negative real numbers which accumulate at $-\infty$ in the same way as for the heat equation, and the second one is constituted of negative real numbers (except for the first values) which accumulate at $-\varepsilon$. Therefore, the control properties of (1) are expected to deeply differ from the ones for the wave equation. (For the control of the wave equation see e.g. (Lions, 1988), (Bardos *et al.*, 1992)). The purpose of this paper

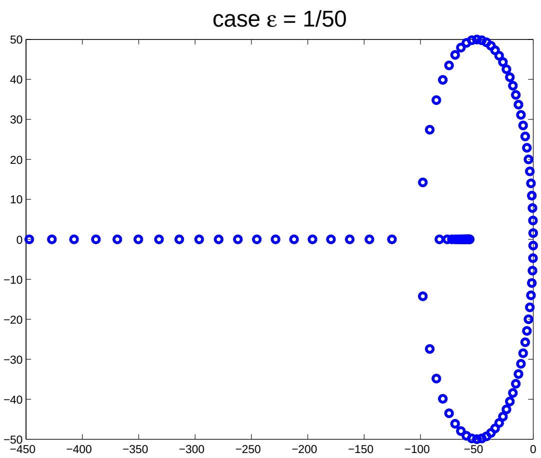


Fig. 1. Spectrum with structural damping.

is to discuss the impact on controllability of such unusual spectrum. The first issue of interest is the exact boundary controllability of the structurally damped wave equation (1) posed on a finite interval. More precisely, given $T > 0$ and some functions $(y^0, y^1)$, $(y^{0,T}, y^{1,T})$ in an appropriate space $B$, we wonder whether it is possible to find a control input $h = h(t)$ such that the solution of the initial boundary value problem (IBVP)

$$y_{tt} - y_{xx} - y_{txx} = 0, \qquad (2)$$
$$y(0,t) = 0,\ y(1,t) = h(t), \qquad (3)$$
$$y(x,0) = y^0(x),\ y_t(x,0) = y^1(x) \qquad (4)$$

satisfies $y(x,T) = y^{0,T}(x)$, $y_t(x,T) = y^{1,T}(x)$. We will show that (1) is not *spectrally controllable*, which means that no nontrivial finite linear combination of eigenvectors can be driven to zero in finite time. In particular, this equation is not exactly controllable in any reasonable functional space. On the other hand, we will see that (1) is approximately controllable in $H_\alpha \times H_{\alpha-2}$ if $\alpha < 1/2$, and not in $E_\alpha$ for $\alpha < 4$ (see below for the definition of these spaces). As far as we know, this system is the first known system with a physical meaning having such strange behavior versus the approximate controllability.

Throughout the paper we will take $\varepsilon = 1$ for the sake of simplicity. All the results can be extended without difficulty to $\varepsilon > 0$ arbitrary. The authors thank Philippe Martin for interesting discussions.

## 2. BASIC PROPERTIES OF (1)

### 2.1 Well-posedness

Let us first investigate the well-posedness of the homogeneous IBVP

$$y_{tt} - y_{xx} - y_{txx} = 0, \qquad (5)$$
$$y(0,t) = 0,\ y(1,t) = 0, \qquad (6)$$
$$y(x,0) = y^0(x),\ y_t(x,0) = y^1(x). \qquad (7)$$

A straightforward computation shows that any smooth solution of (5)-(7) fulfills

$$\int_0^1 (|y_t(x,t)|^2 + |y_x(x,t)|^2)\,dx = -\int_0^t \int_0^1 |y_{txx}(x,s)|^2\,dxds. \qquad (8)$$

This suggests to investigate the well-posedness of (1) in the energy space $H = H_0^1(0,1) \times L^2(0,1)$. Let $A : D(A) \subset H \to H$ denote the operator with domain

$$D(A) = \{(y,z) \in H_0^1(0,1)^2; y + z \in H^2(0,1)\}$$

and which is defined by

$$A\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y_{xx} + z_{xx} \end{pmatrix}.$$

Notice that (1) is equivalent to $\begin{pmatrix} y \\ y_t \end{pmatrix}_t = A\begin{pmatrix} y \\ y_t \end{pmatrix}$. Then the following result holds true.

*Proposition 1.* $A$ generates a strongly semigroup of contractions in the energy space $H$.

*Proof:* The easy proof is left to the reader. ■

### 2.2 Spectral properties

The $n^{th}$ Fourier coefficient (with respect to the orthonormal basis $(\sqrt{2}\sin(n\pi x))_{n\geq 1}$ of $L^2(0,1)$) of any integrable function $y : (0,1) \to \mathbb{R}$ is defined as

$$\hat{y}_n = \int_0^1 y(x)\sqrt{2}\sin(n\pi x)\,dx.$$

For any $\alpha \in \mathbb{R}$, let

$$H_\alpha := \{y : (0,1) \to \mathbb{R};\ \sum_{n\geq 1} n^{2\alpha}|\hat{y}_n|^2 < \infty\}.$$

Endowed with the scalar product

$$(y,z)_\alpha = \sum_{n\geq 1} n^{2\alpha}\hat{y}_n\hat{z}_n$$

$H_\alpha$ is a Hilbert space. Moreover, $H_1 = H_0^1(0,1)$, $H_2 = H^2(0,1) \cap H_0^1(0,1)$, and $H_{-\alpha} = {H_\alpha}'$ (the dual space of $H_\alpha$ with respect to the pivot space $H_0 = L^2(0,1)$) for any $\alpha \geq 0$. Finally, for any $f = \sum_{n\geq 1} c_n\sqrt{2}\sin(n\pi x) \in H_{-\alpha}$ and any $g = \sum_{n\geq 1} d_n\sqrt{2}\sin(n\pi x) \in H_\alpha$ , we have that

$$\langle f, g\rangle_{H_{-\alpha},H_\alpha} = \sum_{n\geq 1} c_n d_n, \qquad (9)$$

where $\langle .,.\rangle_{H_{-\alpha},H_\alpha}$ stands for the duality pairing between $H_{-\alpha}$ and $H_\alpha$. To obtain a representation formula for the solutions of (5)-(7) we have to solve the eigenvalue/eigenvector equation

$$A\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y_{xx} + z_{xx} \end{pmatrix} = \lambda\begin{pmatrix} y \\ z \end{pmatrix}.$$

We obtain that $z = \lambda y$ and $y_{xx} = \mu y$ with $\mu = \frac{\lambda^2}{1+\lambda}$, hence $\mu = -(n\pi)^2$, $\begin{pmatrix} y \\ z \end{pmatrix} \in \mathrm{Span}(f_n^\pm)$, where $f_n^\pm = \begin{pmatrix} \sqrt{2}\sin(n\pi x) \\ \lambda_n^\pm\sqrt{2}\sin(n\pi x) \end{pmatrix}$ and $\lambda_n^\pm = (-n^2\pi^2 \pm n\pi\sqrt{n^2\pi^2 - 4})/2$. Notice that

$$\lambda_n^+ \sim -1 \quad \text{and} \quad \lambda_n^- \sim -n^2\pi^2 \qquad (10)$$

as $n \to \infty$. Any solution $y = y(x,t)$ of (1) may be written as

$$\begin{pmatrix} y(x,t) \\ y_t(x,t) \end{pmatrix} = \sum_{n\geq 1}(a_n^+ e^{\lambda_n^+ t} f_n^+ + a_n^- e^{\lambda_n^- t} f_n^-). \qquad (11)$$

More precisely, if the initial condition $(y^0, y^1) \in H$ is given by

$$y^0 = \sum_{n\geq 1} c_n \sqrt{2}\sin(n\pi x), \tag{12}$$

$$y^1 = \sum_{n\geq 1} d_n \sqrt{2}\sin(n\pi x), \tag{13}$$

we have that

$$c_n = a_n^+ + a_n^-, \tag{14}$$

$$d_n = \lambda_n^+ a_n^+ + \lambda_n^- a_n^-, \tag{15}$$

hence

$$a_n^+ = \frac{c_n\lambda_n^- - d_n}{\lambda_n^- - \lambda_n^+}, \tag{16}$$

$$a_n^- = \frac{d_n - \lambda_n^+ c_n}{\lambda_n^- - \lambda_n^+}. \tag{17}$$

As an easy consequence of the above representation formula, we may prove the following result.

*Proposition 2.* Let $\alpha \in \mathbb{R}$. If $(y^0, y^1) \in H_\alpha \times H_{\alpha-2}$, then $(y, y_t) \in C(\mathbb{R}^+; H_\alpha \times H_{\alpha-2})$. If in addition $\alpha > 3/2$, then $\sum_{n\geq 1} n(|a_n^+| + |a_n^-|) < \infty$ and $y_x(1,.) \in C(\mathbb{R}^+)$.

*Proof:* Using (10) and (16)-(17), it is easy to see that

$$n^{2\alpha}(|a_n^+|^2 + |a_n^-|^2) \leq Cn^{2\alpha}(|c_n|^2 + |d_n|^2/n^4)$$
$$n^{2(\alpha-2)}(|\lambda_n^+ a_n^+|^2 + |\lambda_n^- a_n^-|^2) \leq Cn^{2\alpha}(|c_n|^2 + |d_n|^2/n^4),$$

hence $(y, y_t) \in C(\mathbb{R}^+; H_\alpha \times H_{\alpha-2})$. On the other hand, taking the derivative w.r.t. $x$ in (11) we obtain that

$$y_x(1,t) = \sum_{n\geq 1}(a_n^+ e^{\lambda_n^+ t} + a_n^- e^{\lambda_n^- t})\sqrt{2}(n\pi)(-1)^n.$$

As Re $\lambda_n^\pm < 0$ for each $n$ and

$$\sum_{n\geq 1} n|a_n^\pm| \leq \left(\sum_{n\geq 1} n^{-2(\alpha-1)}\right)^{1/2} \cdot \left(\sum_{n\geq 1} n^{2\alpha}|a_n^\pm|^2\right)^{1/2}$$

we see that $\sum_{n\geq 1} n(|a_n^+| + |a_n^-|) < \infty$ and that $y_x(1,.) \in C(\mathbb{R}^+)$ provided that $\alpha > 3/2$. ■

## *2.3 Boundary initial value problem*

Let us now turn to the IBVP (2)-(4). Performing the change of unknown functions $y(x,t) = z(x,t) + xh(t)$, we readily obtain that $z$ is a solution of the system

$$z_{tt} - z_{xx} - z_{txx} = xh''(t), \tag{18}$$

$$z(0,t) = 0,\ z(1,t) = 0, \tag{19}$$

$$y(x,0) = y^0(x) - xh(0), \tag{20}$$

$$y_t(x,0) = y^1(x) - xh'(0). \tag{21}$$

An application of the classical semigroup theory gives that (18)-(21) has a unique strong solution $(z, z_t) \in C([0,T], D(A)) \cap C^1([0,T], H)$ whenever $h'' \in W^{1,1}(0,T)$ and $h(0) = h'(0) = 0$.

*Proposition 3.* Assume that $(y^0, y^1) \in D(A)$, $h \in W^{3,1}(0,T)$ and that $h(0) = h'(0) = 0$. Then (2)-(4) admits a unique solution $y \in C([0,T], H^2(0,1)) \cap C^1([0,T]; H^1(0,1) \cap C^2([0,T]; L^2(0,1))$.

*Corollary 1.* If $(y^0, y^1) \in H_0^1(0,1) \times L^2(0,1)$ and $h \in W^{2,1}(0,1)$ satisfies $h(0) = 0$, then (2)-(4) admits a mild solution $y \in C([0,T], H^1(0,1)) \cap C^1([0,T]; L^2(0,1))$.

## *2.4 Moment problem*

The adjoint system to (2)-(4) is found to be

$$z_{tt} - z_{xx} + z_{txx} = 0, \tag{22}$$

$$z(0,t) = 0,\ z(1,t) = 0, \tag{23}$$

$$z(x,T) = z^{0,T}(x),\ z_t(x,T) = z^{1,T}(x). \tag{24}$$

Notice that (22)-(24) is a *backwards* Cauchy problem. If $z^{0,T}$ and $z^{1,T}$ are decomposed as

$$z^{0,T} = \sum_{n\geq 1} \tilde{c}_n \sqrt{2}\sin(n\pi x) \tag{25}$$

$$z^{1,T} = \sum_{n\geq 1} \tilde{d}_n \sqrt{2}\sin(n\pi x) \tag{26}$$

and if $\tilde{a}_n^+$ and $\tilde{a}_n^-$ are given by (16)-(17) (with $c_n$, $d_n$ replaced by $\tilde{c}_n$, $\tilde{d}_n$), then the solution to (22)-(24) is given by

$$z(x,t) = \sum_{n\geq 1}(\tilde{a}_n^+ e^{\lambda_n^+(T-t)} + \tilde{a}_n^- e^{\lambda_n^-(T-t)})\sqrt{2}\sin(n\pi x),$$

which yields

$$z_x(1,t) = \sum_{n\geq 1}(\tilde{a}_n^+ e^{\lambda_n^+(T-t)} + \tilde{a}_n^- e^{\lambda_n^-(T-t)})\sqrt{2}(n\pi)(-1)^n. \tag{27}$$

Let $y = y(x,t)$ and $z = z(x,t)$ be smooth solutions of (2)-(4) and (22)-(24), respectively. Scaling in (2)-(4) by $z$, we obtain after some integrations by parts that

$$\int_0^1 [-y_t z + y z_t - y z_{xx}]_0^T \, dx = \int_0^T (y + y_t)(1,t) z_x(1,t) \, dt. \quad (28)$$

Notice that (28) may be rewritten as

$$\left[\langle -y_t, z\rangle_{H_{\alpha-2},H_{2-\alpha}} + \langle z_t - z_{xx}, y\rangle_{H_{-\alpha},H_\alpha}\right]_0^T = \int_0^T (h(t) + h'(t)) z_x(1,t) \, dt \quad (29)$$

where $\alpha$ is chosen so that the terms in the l.h.s. of (29) are meaningful.

*Definition 1.* Let $\alpha \in \mathbb{R}$ and $B := H_\alpha \times H_{\alpha-2}$. The system (2)-(4) is said to be

- *exactly controllable in B* if for any $(y^0, y^1) \in B$ and any $(y^{0,T}, y^{1,T}) \in B$, there exists a control input $h = h(t)$ such that the solution of (2)-(4) satisfies $y(.,T) = y^{0,T}$, $y_t(.,T) = y^{1,T}$;
- *null controllable in B* if for any $(y^0, y^1) \in B$, there exists a control input $h = h(t)$ driving the solution of (2)-(4) to $(y^{0,T}, y^{1,T}) = (0,0)$;
- *approximately controllable in B* if for any $(y^{0,T}, y^{1,T}) \in B$ and any $\varepsilon > 0$, there exists a control input $h = h(t)$ such that the solution of (2)-(4) issued from $(y^0, y^1) = (0,0)$ satisfies $||y(T,.) - y^{0,T}||_{H_\alpha} + ||y_t(T,.) - y^{1,T}||_{H_{\alpha-2}} < \varepsilon$;
- *spectrally controllable* if any finite linear combination of eigenvectors (i.e., $y^0 = \sum_{n=1}^N c_n \sqrt{2}\sin(n\pi x)$, $y^1 = \sum_{n=1}^N d_n \sqrt{2}\sin(n\pi x)$, $N \ge 1$ arbitrary) may be driven to zero by a control input $h = h(t)$.

Noticing that the application $(z^{0,T}, z^{1,T}) \in H_{2-\alpha} \times H_{-\alpha} \mapsto (z^{0,T}, z^{1,T} - z^{0,T}_{xx}) \in H_{2-\alpha} \times H_{-\alpha}$ is invertible, and that for the exact or the approximate controllability $(y^0, y^1)$ may be given the value $(0,0)$ without loss of generality, we obtain the following criterion for the various controllability notions.

*Proposition 4.* The system (2)-(4) is

- *exactly controllable in B* if and only if for each target function $(y^{0,T}, y^{1,T}) \in B$ there exists some control input $h = h(t)$ such that the solution $z(x,t)$ of (22)-(24) satisfies
$$\langle -y^{1,T}, z^{0,T}\rangle_{H_{\alpha-2},H_{2-\alpha}} + \langle z^{1,T} - z^{0,T}_{xx}, y^{0,T}\rangle_{H_{-\alpha},H_\alpha} = \int_0^T (h(t) + h'(t)) z_x(1,t) \, dt \quad (30)$$
for each pair $(z^{0,T}, z^{1,T}) \in H_{2-\alpha} \times H_{-\alpha}$;
- *null controllable in B* if and only if for each initial state $(y^0, y^1) \in B$, there exists some control input $h = h(t)$ such that the solution $z(x,t)$ of (22)-(24) satisfies
$$\langle -y^0, z(.,0)\rangle_{H_{\alpha-2},H_{2-\alpha}} + \langle z_t(.,0) - z_{xx}(.,0), y^1\rangle_{H_{-\alpha},H_\alpha} = -\int_0^T (h(t) + h'(t)) z_x(1,t) \, dt \quad (31)$$
for each pair $(z^{0,T}, z^{1,T}) \in H_{2-\alpha} \times H_{-\alpha}$;
- *approximatively controllable in B* if and only if $(0,0)$ is the only pair $(z^{0,T}, z^{1.T}) \in H_{2-\alpha} \times H_{-\alpha}$ for which the solution $z(x,t)$ of (22)-(24) fulfills $\int_0^T (h(t) + h'(t)) z_x(1,t) \, dt = 0$ for any function $h = h(t)$.

Using (9) and (27), we may express the above conditions as moment problems. For instance, the null controllability is equivalent to the existence of a function $h = h(t)$ such that

$$\int_0^T (h(t) + h'(t)) e^{-\lambda_n^+ t} \sqrt{2}(n\pi)(-1)^n \, dt = d_n + c_n(\lambda_n^+ - (n\pi)^2) \quad (32)$$

and

$$\int_0^T (h(t) + h'(t)) e^{-\lambda_n^- t} \sqrt{2}(n\pi)(-1)^n \, dt = d_n + c_n(\lambda_n^- - (n\pi)^2) \quad (33)$$

for each $n \ge 1$. We are now in a position to state the first main result in the paper. Its proof is inspired by the one of a similar result for the BBM equation (Micu, 2001, Thm 3.2).

*Theorem 1.* The system (2)-(4) is not spectrally controllable. Consequently, it is not exactly controllable nor null controllable in any space $B = H_\alpha \times H_{\alpha-2}$.

*Proof.* As each eigenvector of $A$ belongs to $B$ for each $\alpha$, we only have to prove the first assertion. Actually, we prove that *no nontrivial* finite linear combination of eigenvectors may be driven to zero in finite time. To this end, consider any pair of sequences $(c_n)_{n\ge1}$, $(d_n)_{n\ge1}$ with $c_n = d_n = 0$ for $n > N$, for which there exists a function $h \in H_0^1(0,T)$ such that (32)-(33) holds true. Let $F(z) := \int_0^T (h(t) + h'(t)) e^{izt} \, dt$. Then $F$ is an entire function (according to Paley-Wiener theorem), which satisfies $F(i\lambda_n^\pm) = 0$ for $n > N$. As $i\lambda_n^+ \to -i$ as $n \to \infty$, we infer that $F$ is zero on a set with a finite accumulation point, hence $F \equiv 0$. It follows that $c_n = d_n = 0$ for each $n \ge 1$. ■

## 3. APPROXIMATE CONTROLLABILITY

*Theorem 2.* For any $\alpha < 1/2$, the system (2)-(4) is approximatively controllable in $B = H_\alpha \times H_{\alpha-2}$.

*Proof.* Pick any $\alpha < 1/2$ and any pair $(z^{0,T}, z^{1,T}) \in H_{2-\alpha}\times H_{-\alpha}$, decomposed as in (25)-(26). Assume that

$$\int_0^T (h(t)+h'(t))z_x(1,t)\,dt = 0$$

for any $h \in H_0^1(0,T)$. We aim to prove that $\tilde{c}_n = \tilde{d}_n = 0$ for each $n$, or equivalently that $\tilde{a}_n^+ = \tilde{a}_n^+ = 0$ for each $n$. Set $f = h + h'$. When $h$ ranges over $H_0^1(0,T)$, $f$ ranges over the subspace of $L^2(0,T)$ constituted by the functions satisfying the condition $\int_0^T e^t f(t)\,dt = 0$. It follows that for any $f \in \mathrm{Span}(e^t)^\perp$, $(f, z_x(1,.))_{L^2(0,T)} = 0$, so $z_x(1,.) \in \mathrm{Span}(e^t)^{\perp\perp} = \mathrm{Span}(e^t)$. Therefore, there exists a constant $C\in\mathbb{R}$ such that

$$\sum_{n\ge1}(a_n^+ e^{\lambda_n^+(T-t)} + a_n^- e^{\lambda_n^-(T-t)})\sqrt{2}(n\pi)(-1)^n = Ce^t$$

for a.e. $t\in(0,T)$. In other words,

$$\sum_{n\ge1}\left(c_n^+ e^{\lambda_n^+ t} + c_n^- e^{\lambda_n^- t}\right) + c_0 e^{-t} = 0$$

for a.e. $t\in(0,T)$, where $c_n^\pm := a_n^\pm\sqrt{2}(n\pi)(-1)^n$ for each $n\ge1$ and $c_0 := -Ce^T$. The conclusion is then a direct consequence of Proposition 2 and of the next result.

*Lemma 1.* Let $(c_n)_{n\ge1}$ and $(\lambda_n)_{n\ge1}$ be two sequences of complex numbers such that $\sum_{n\ge1}|c_n| < \infty$ and $\mathrm{Re}\,\lambda_n < \Lambda$ for each $n\ge1$ and some number $\Lambda\in\mathbb{R}$. Assume that the $\lambda_n$'s are pairwise distinct, and that $\sum_{n\ge1} c_n e^{\lambda_n t} = 0$ for a.e. $t\in(0,T)$. Then $c_n = 0$ for all $n\ge1$.

*Proof.* Let $F(z) = \sum_{n\ge1} c_n e^{\lambda_n z}$. Then $F$ is an analytic function on the half plane $\mathbb{C}^+ = \{z\in\mathbb{C};\ \mathrm{Re}\ z > 0\}$. By the analytic continuation property, we obtain that $F(1+it) = 0$ for every $t\in\mathbb{R}$. Let us set $c_n' := c_n e^{\lambda_n}$ for each $n$. (Notice that $\sum_{n\ge1}|c_n'| < \infty$.) Pick any $N\ge1$. We have that

$$c_N' = -\sum_{n\ne N} c_n' e^{(\lambda_n-\lambda_N)it} \qquad \forall t\in\mathbb{R}.$$

Integrating w.r.t. time in both sides of the above equation, we obtain that

$$c_N' = -\frac{1}{2T}\sum_{n\ne N} c_n' \int_{-T}^T e^{(\lambda_n-\lambda_N)it}dt =: I(T).$$

We claim that $I(T)\to0$ as $T\to\infty$. Indeed, for each given $\varepsilon>0$, we may pick an integer $N' > N$ such that $\sum_{n\ge N'}|c_n'| < \varepsilon/2$. Then

$$|I(T)| \le \frac{\varepsilon}{2} + \left|\sum_{n<N', n\ne N} c_n' \frac{[e^{(\lambda_n-\lambda_N)it}]_{-T}^T}{2Ti(\lambda_n-\lambda_N)}\right| < \varepsilon$$

for $T$ large enough. We conclude that $c_N = c_N' = 0$. This completes the proof of Lemma 1 and of Theorem 2. ∎

Finally, we show that the approximate controllability does no hold in another family of spaces. For any $\alpha\in\mathbb{R}$, let $E_\alpha := \{(y^0,y^1);\ ||(y^0,y^1)||_{E_\alpha} = \sup_{n\ge1}(n^\alpha|c_n| + n^{\alpha-2}|d_n|) < \infty\}$. Then it is easily seen that $E_\alpha$ is a Banach space in which the IBVP (5)-(7) is well-posed. Furthermore, we have the continuous embeddings

$$H_\alpha\times H_{\alpha-2}\subset E_\alpha\subset H_{\alpha-1}\times H_{\alpha-3}.$$

Notice that the first embedding is *not* dense.

*Theorem 3.* For any $\alpha < 4$, the system (2)-(4) is not approximatively controllable in $E_\alpha$.

*Proof.* In what follows, $C$ will denote a numerical constant (i.e., independent of any variable) which may vary from line to line. Pick any number $\alpha < 4$, and assume that (2)-(4) is approximatively controllable in $E_\alpha$. Consider the sequence $(x_n)_{n\ge1}$ defined by

$$x_n = (-1)^n n^{3-\alpha} \qquad \forall n\ge1.$$

Next, set

$$a_n^+ = \pi(-1)^{n+1}(n\pi)^{-2}(n^2\pi^2-4)^{-1/2}x_n$$

and $a_n^- = 0$ for each $n\ge1$, and define $y^0$ and $y^1$ by (12)-(13), where $c_n$ and $d_n$ are defined by (14)-(15). Clearly, $(y^0,y^1)\in E_\alpha$. According to the approximate controllability in $E_\alpha$, for any given $\varepsilon>0$ we may find a function $h\in H_0^2(0,T)$ such that the solution of (2)-(4) fulfills

$$||(y(T), y_t(T))||_{E_\alpha} < \varepsilon. \tag{34}$$

Setting $y(x,t) = z(x,t) + xh(t)$ as in Section 2.3, we note that $(z(.,0), z_t(.,0)) = (y^0,y^1)$, $(z(.,T), z_t(.,T)) = (y(.,T), y_t(.,T))$ and that $z$ fulfills (18)-(19). Therefore, we infer from Duhamel formula that

$$\begin{aligned}\begin{pmatrix} z\\ z_t\end{pmatrix} &= \sum_{n\ge1}(a_n^+(t)f_n^+ + a_n^- f_n^-)\\ &= \sum_{n\ge1}(a_n^+e^{\lambda_n^+t}f_n^+ + a_n^-e^{\lambda_n^-t}f_n^-)\\ &\quad+\int_0^t(\sum_{n\ge1}b_n^+(s)e^{\lambda_n^+(t-s)}f_n^+\\ &\quad+b_n^-(s)e^{\lambda_n^-(t-s)}f_n^-)ds\end{aligned}$$

where the coefficients $b_n^\pm$ are given by

$$\begin{pmatrix}0\\ xh''(s)\end{pmatrix} = \sum_{n\ge1}(b_n^+(s)f_n^+ + b_n^-(s)f_n^-).$$

A straightforward computation leads to

$$\begin{aligned}b_n^+(s) &= \pi(-1)^{n+1}(n\pi)^{-2}((n\pi)^2-4)^{-1/2}h''(s)\\ &= -b_n^-(s).\end{aligned}$$

Using the fact that the vectors $f_n^\pm$ and $f_m^\pm$ are orthogonal in $H_{\alpha-1}\times H_{\alpha-3}$ if $n\ne m$, and that the vectors $f_n^+$ and $f_n^-$ are linearly independent, we readily conclude that

$$a_n^\pm(t) = a_n^\pm e^{\lambda_n^\pm t} + \int_0^t b_n^\pm(s) e^{\lambda_n^\pm(t-s)} ds,$$

hence

$$\frac{d}{dt} a_n^\pm = \lambda_n^\pm a_n^\pm + b_n^\pm.$$

Setting

$$x_n(t) := \pi^{-1}(-1)^{n+1}(n\pi)^2(n^2\pi^2 - 4)^{1/2} a_n^+(t)$$

(hence $x_n(0) = x_n$), we arrive to

$$\frac{d}{dt} x_n = -(1+\eta_n)x_n + u(s)$$

where $u(s) := h''(s)$ and $\eta_n := -1 - \lambda_n^+ = (n\pi)^{-2} + o(n^{-2})$. Since

$$x_n(t) = x_n(0)e^{-(1+\eta_n)t} + \int_0^t e^{(s-t)(1+\eta_n)} u(s) ds,$$

we have that

$$|x_n(t)| \le |x_n(0)| e^{-(1+\eta_n)t} + \int_0^t |u(s)| ds.$$

For any integer $N$, consider the function

$$\xi_N(t) = \sum_{n=1}^{2N} (-1)^n x_n(t).$$

Notice that

$$\begin{aligned} \xi_N(0) &= \sum_{n=1}^{2N} n^{3-\alpha} \ge \int_0^{2N} x^{3-\alpha}\, dx \\ &= (2N)^{4-\alpha}/(4-\alpha) =: K_\alpha N^{4-\alpha} \end{aligned}$$

and that

$$\frac{d}{dt}\xi_N(t) = -\xi_N(t) - \sum_{n=1}^{2N} (-1)^n \eta_n x_n(t).$$

Setting

$$h_N(t) := -\sum_{n=1}^{2N} (-1)^n \eta_n x_n(t),$$

we have that

$$\begin{aligned} |h_N(t)| &\le C\left(\sum_{n=1}^{2N} n^{-2}(|x_n| + ||u||_{L^1(0,T)})\right) \\ &\le C\left(\sum_{n=1}^{\infty} n^{1-\alpha} + ||u||_{L^1(0,T)}\right) \\ &\le C\left(1 + \frac{N^{2-\alpha}}{2-\alpha} + ||u||_{L^1(0,T)}\right) =: A. \end{aligned}$$

(In $A$ the term $N^{2-\alpha}/(2-\alpha)$ has to be replaced by $\ln N$ when $\alpha = 2$.) With

$$\xi_N(t) = \xi_N(0)e^{-t} + \int_0^t e^{s-t} h_N(s) ds$$

we have

$$\left|\xi_N(t) - \xi_N(0)e^{-t}\right| \le A.$$

Thus

$$|\xi_N(T)| \ge \xi_N(0)e^{-T} - A \ge K_\alpha N^{4-\alpha} e^{-T} - A.$$

On the other hand, using (34) we obtain that

$$\begin{aligned} |\xi_N(T)| &\le C \sum_{n=1}^{2N} n^3 |c_n(T)| \\ &\le C\left(\sum_{n=1}^{2N} n^{3-\alpha}\right) \sup_{n\ge 1}(n^\alpha |c_n|) \\ &\le K'_\alpha N^{4-\alpha}\varepsilon \end{aligned}$$

for some positive constant $K'_\alpha$. Thus we have for any $N > 0$

$$K_\alpha N^{4-\alpha} e^{-T} - C\left(1 + \frac{N^{2-\alpha}}{2-\alpha} + ||u||_{L^1(0,T)}\right) \le K'_\alpha N^{4-\alpha}\varepsilon.$$

Dividing in both sides by $N^{4-\alpha}$ and letting $N \to \infty$, we arrive to $\varepsilon \ge (K_\alpha/K'_\alpha)e^{-T}$, contradicting the fact that $\varepsilon$ may be chosen arbitrarily small. Thus the system (2)-(4) is not approximatively controllable in $E_\alpha$. ■

## 4. CONCLUSION

The paper was devoted to the analysis of the control properties of the structurally damped 1D wave equation on the interval $I = (0,1)$, the control acting at the point $x = 1$. It has been proved that the spectral controllability does not hold, and that the approximate controllability holds in $H_\alpha \times H_{\alpha-2}$ for $\alpha < 1/2$. The question whether the approximate controllability holds for $\alpha \ge 1/2$ remains open.

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