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Flatness based control of oscillators

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The aim of this paper is to present some recent developments and hints for future researches in control inspired of flatness-based ideas. We focus on motion planning: Steering a system from one state to another. We do not consider stabilization and trajectory tracking. We explain how explicit trajectory parameterization, a property that is central for flat systems, can be useful for the feed-forward control of various oscillatory systems (linear, non-linear , finite and infinite dimensional) of physical and engineering interests. Such parameterization provide simple algorithms to generate in real-time trajectories.

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1 Introduction

More than 10 years ago, Michel Fliess and coworkers [9–11] introduced a special class of non-linear control systems described by ordinary equations: Differential flat systems form a special class of under-determined differential systems (also called implicit control systems or differential/algebraic control systems) for which systematic control methods are available once a flat-output is explicitly known. For clarity's sake, we just sketch a tutorial definition of flatness for state-space control system. The smooth system $\frac{d}{dt}x = f(x,u)$ is flat, if and only if, there exists $m = \dim(u)$ real smooth functions $h = (h_1, \ldots, h_m)$ depending on x and a finite number of u derivatives, says α, such that, generically, the solution (x,u) of the square differential-algebraic system ($t \mapsto y(t)$ is given)

$$\dot{x} = f(x,u), \qquad y(t) = h\left(x, u, \dot{u}, \ldots, u^{(\alpha)}\right)$$

does not involve any differential equation and thus is of the form

$$x = \Phi\left(y, \dot{y}, \ldots, y^{(\beta)}\right), \qquad u = \Psi\left(y, \dot{y}, \ldots, y^{(\beta+1)}\right)$$

where Ψ and Φ are smooth functions and β is some finite number. The quantity y is of fundamental importance: It is called flat-output or linearizing-output. In control language, the flat output y is such that, the inverse of $\dot{x} = f(x,u), y = h(x,u,\ldots,u^{(\alpha)})$ has no dynamics [17]. Flatness is related to state feedback linearization and in fact has a long history. Such notion goes back to Hilbert [16] with his work on the general solution of Monge equations, work that has been prolonged by Cartan [4] with a characterization via the derived flag of solvable (in the sense of Hilbert) Monge equations of any-order. In a control setting, Cartan considers in fact driftless systems with two controls $\dot{x} = f_1(x)u_1 + f_2(x)u_2$: He provides, as shown in [27], a necessary and sufficient flatness condition that is closely related to conversion of such systems into chain-form (the dual of a contact Pfaffian system, see [33, 34]). But in general, the problem of flatness characterization is fully open for multi-input systems ($\dim(u) > 1$). There is no algorithm to decide once the equations $\dot{x} = f(x,u)$ are given, if there exists such map h, called flat-output map. The situation is somehow comparable to integrable Hamiltonian systems: There is no algorithm to decide whether a given Hamiltonian $H(q,p)$ yields an integrable system; many examples of physical interest are integrable and for these systems we have the form of their general solution in terms of the initial conditions; only necessary conditions are available (see, e.g. the Morales-Ramis theorem [31]). For flat systems, the situation is very similar: No algorithm to decide whether a system is flat or not; many examples of engineering interest are flat and their general solution is in term of the derivatives of a flat-output y that has a clear physical interpretation [26]; few necessary

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conditions are available (see, e.g., the ruled-manifold criterion [38]). To summarize such analogy: The role of flat-systems within the set of under-determined ordinary differential systems is very similar to the role of integrable systems within the set of determined ordinary-differential systems.

We discuss and present here on several examples some recent developments in control inspired of flatness-based idea (see also [41] for a complementary point of view). We focus on trajectory generation. We do not consider stabilization and trajectory tracking (see, e.g., [12,15,22]).

In Sect. 2, we consider linear systems. We show for n oscillators in parallel, how to compute directly and explicitly y, h, Ψ and Φ via operational calculus with the Laplace variable $s = d/dt$. This algebraic computation directly extends to the wave equation with boundary control. Such extensions to infinite dimensional systems were first pointed out by Mounier [32] for delay systems and prolonged later to several 1D partial differential equations with boundary control (see, e.g., [13,14,18,35,36,39,40]. We explain why, as noticed in [5], for truly 2D or 3D systems with a finite number of controls, exact controllability in finite time is impossible and must be replaced by approximated controllability. Such obstruction to exact controllability relies on classical relationships between the zeros distribution of an entire function and its order at infinity [2]. We finally discuss this limitation for a 1D example of a flexible system with internal damping where exact controllability in finite time is also impossible.

In Sect. 3 we show via the example of the juggling robot $2k\pi$ [20] that flatness can be seen as an extension to under-actuated robots of the classical computed torque method widely used in robotics. We explain also how to overcome the singularities that automatically appear for trajectories between the stable and unstable equilibrium of the spherical pendulum carried by $2k\pi$. The main idea relies on deformations (here a change of time-scale) of the smooth flat output trajectory $t \mapsto y(p,t)$ via a parameter p. Then a careful choice of p provides smooth state and control trajectories even if, during the transient and for some isolated value of t, the implicit algebraic system relating (x,u) to the derivatives of y becomes singular. We suggest that idea from singularity theory could be use to generalize such open-loop design method. Finally, we conclude an example inspired from actual researches in physics: Manipulation of atom via photons. A straight-forward adaptation of the computed torque method to a classical model could be of some interest for designing the shape of Laser pulses to control the cartesian position of a charged particle or an atom. We wonder if such point of view can also be useful when quantum descriptions of the dynamics are considered.

2 Linear systems

2.1 Oscillators in parallel

Consider n oscillators in generalized coordinates z_i satisfying:

$$\frac{d^2 z_i}{dt^2} = -(\omega_i)^2 z_i + b_i u, \quad i = 1, \ldots, n \tag{1}$$

where $u \in \mathbb{R}$ is the control and where $\omega_i > 0$ and $b_i \neq 0$ are parameters with $\omega_i \neq \omega_j$ for $i \neq j$. Take $T > 0$ and $D \neq 0$. We want to describe all the open-loop controls $[0,T] \ni t \mapsto u(t)$ steering the system from the steady-state $z_i = 0$ at $t = 0$ to the steady-state $z_i = \frac{b_i D}{(\omega_i)^2}$. This means that with such open-loop control $[0,T] \ni t \mapsto u(t)$, the solution $t \mapsto z_i(t)$ of (1) with initial condition $z_i(0) = 0$, $\dot{z}_i(0) = 0$ satisfies $z_i(T) = \frac{b_i D}{(\omega_i)^2}$ and $\dot{z}_i(T) = 0$. Let us show how to obtain such control u explicitly without any integration. The difficulty is that when D/T^2 is similar to the pulsation ω_i, one has to design carefully $[0,T] \ni t \mapsto u(t)$ to avoid the residual oscillations that can occur when u stops at D for $t \geq T$. This must be done for all the z_i's simultaneously.

With $s = d/dt$, (1) reads formally:

$$\left(s^2 + (\omega_i)^2\right) z_i = b_i u, \quad i = 1, \ldots, n$$

that can be seen as a linear under-determined system with $n+1$ unknown variables (the z_i and the control u) and n equations. Following [22,28], this system admits an explicit formulation in the following sense:

$$z_i = Q_i(s) y, \qquad u = Q(s) y \quad \text{with} \quad y = \sum_{k=1}^{n} c_k z_k$$

where ($\jmath = \sqrt{-1}$):

$$Q_i(s) = \frac{b_i}{(\omega_i)^2} \prod_{\substack{k=1\\ k\neq i}}^{n} \left(1 + \left(\frac{s}{\omega_k}\right)^2\right), \qquad Q(s) = \prod_{k=1}^{n} \left(1 + \left(\frac{s}{\omega_k}\right)^2\right), \qquad c_k = \frac{1}{Q_k(\jmath\omega_k)} \in \mathbb{R}\,. \tag{2}$$

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The justification of such formulae is mainly as follows: The n even polynomials $(Q_1, \dots, Q_n)$ are independent and of degree less or equal to $2n-2$; they form a basis of the even polynomials of degree less than $2n-2$; the constant polynomial 1 can be expressed as a linear combination of $(Q_1, \dots, Q_n)$ that defines c_i: $1 = \sum c_k Q_k(s)$. Let us define the real coefficients q_k^i and q_k via $Q_i(s) = \sum_{k=0}^{n-1} q_k^i s^{2k}$ and $Q(s) = \sum_{k=0}^{n} q_k s^{2k}$.

All the formal manipulations above imply that, for any arbitrary smooth function $t \mapsto y(t)$, the smooth functions χ_i and $\upsilon(t)$ derived from $y(t)$ via

$$\chi_i(t) = \sum_{k=0}^{n-1} q_k^i y^{(2k)}(t), \qquad \upsilon(t) = \sum_{k=0}^{n} q_k y^{(2k)}(t)$$

automatically satisfy (1) with $z_i = \chi_i$ and $u = \upsilon$. Moreover

$$y(t) = \sum_k c_k \chi_k(t) .$$

There is a one to one linear correspondence between the trajectories of (1) and the arbitrary smooth time function y. More precisely, any piecewise continuous open-loop control $[0,T] \ni t \mapsto u(t)$ steering from the steady-state $z_i(0) = 0$ to the steady-state $z_i(T) = \frac{b_i}{(\omega_i)^2} D$ can be described via

$$u(t) = \sum_{k=0}^{n} q_k y^{(2k)}(t)$$

for all KC^{2n}-functions[1] $[0,T] \ni t \mapsto y(t)$ such that

$$y(0) = 0, \qquad y(T) = D, \qquad \forall i \in \{1, \dots, 2n-1\}, \quad y^{(i)}(0) = y^{(i)}(T) = 0 .$$

Such formulae provide efficient real-time motion planning algorithms.

Let us give similar formulae when damping effect are included for each oscillator. We replace (1) by

$$\frac{d^2 z_i}{dt^2} = -(\omega_i)^2 z_i - 2\xi_i \omega_i \dot{z}_i + b_i u, \quad i = 1, \dots, n \tag{3}$$

where the additional parameter is $\xi_i \geq 0$, the damping factor. We have

$$z_i = Q_i(s) y, \qquad , u = Q(s) y$$

with

$$Q_i(s) = \frac{b_i}{(\omega_i)^2} \prod_{k=1,\ k \neq i}^{n} \left(1 + 2\xi_k \left(\frac{s}{\omega_k}\right) + \left(\frac{s}{\omega_k}\right)^2\right), \qquad Q(s) = \prod_{k=1}^{n} \left(1 + 2\xi_k \left(\frac{s}{\omega_k}\right) + \left(\frac{s}{\omega_k}\right)^2\right) .$$

The polynomials Q_i and Q are no longer made with even power of s. Assume that for any $i \neq k$ the second order polynomials $1 + 2\xi_k \left(\frac{s}{\omega_k}\right) + \left(\frac{s}{\omega_k}\right)^2$ and $1 + 2\xi_i \left(\frac{s}{\omega_i}\right) + \left(\frac{s}{\omega_i}\right)^2$ have no common roots (the system (3) is controllable). This implies that the $2n$ polynomials $Q_1(s), sQ_1(s), \dots, Q_n(s), sQ_n(s)$ are independent. Since they are of degree $2n-1$ at least, they form a basis of the polynomials of degree less or equal to $2n-1$. Thus $1 = \sum_{k=1}^{n} c_k Q_k(s) + d_k s Q_k(s)$ and we have

$$y = \sum_{k=1}^{n} c_k z_k + d_k s z_k .$$

Let us explain how to compute c_k and d_k explicitly. Denote by α_k and β_k the roots of $\omega_k^2 + 2\xi_k \omega_k s + s^2$. When $\alpha_k \neq \beta_k$, c_k and d_k are given by the following invertible linear system

$$c_k + \alpha_k d_k = \frac{1}{Q_k(\alpha_k)}, \qquad c_k + \beta_k d_k = \frac{1}{Q_k(\beta_k)} .$$

[1] KC^{2n} means that $t \mapsto y(t)$ is $2n-1$ continuously differentiable and that $t \mapsto y^{(2n)}(t)$ is piece-wise continuous.

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When $\alpha_k = \beta_k$, c_k and d_k are given by the following invertible linear system

$$c_k + \alpha_k Q_k(\alpha_k) d_k = 1, \qquad c_k Q'_k(\alpha_k) + \big(\alpha_k Q'_k(\alpha_k) + Q_k(\alpha_k)\big)\, d_k = 0\,.$$

In both cases, c_k and d_k are symmetric rational functions with real coefficients of α_k, β_k. Thus c_k and d_k are necessarily real since they are rational functions of $\alpha_k + \beta_k = -2\xi_k\omega_k$ and $\alpha_k\beta_k = \omega_k^2$.

Such explicit descriptions via linear combinations of derivatives of an arbitrary function (the so called flat-output since its can be expressed via the state) is always possible for any controllable linear system of finite dimension. They can be systematized as shown in [22] and also extended to other type of operators than d/dt as in [6] where effective algorithms implemented in Maple are developed.

2.2 1D wave equation

Let us consider the following prototype of an infinite dimensional system with boundary control, namely the wave equation on the segment $[0, 1]$:

$$\frac{\partial^2 Z}{\partial t^2}(x,t) = \frac{\partial^2 Z}{\partial x^2}(x,t) \quad \text{for} \quad 0 < x < 1, \quad Z(0,t) = 0, \quad Z(1,t) = u(t)\,. \tag{4}$$

Replacing the partial derivation with respect to t by multiplication via the Laplace variable s yields

$$s^2 Z = \frac{d^2 Z}{dx^2}, \qquad Z(0) = 0, \qquad Z(1) = u$$

where s is a parameter. The general solution of this second order equation is $Z(x) = a(s)\cosh(sx) + b(s)\frac{\sinh(sx)}{s}$. The boundary conditions imply that $a(s) = 0$ and $b(s)\frac{\sinh(s)}{s} = u(s)$. Pulling back such relations in the time domain we get

$$Z(x,t) = \tfrac{1}{2}\int_{t-x}^{t+x} b(\tau)d\tau, \qquad u(t) = \tfrac{1}{2}\int_{t-1}^{t+1} b(\tau)d\tau\,.$$

For b, an arbitrary KC^2 function in time, such relationships yield to the general solution of (4). Notice that $b(t) = \frac{\partial Z}{\partial x}(0,t)$.

This wave equation can be seen as an infinite number of oscillators in parallel when a modal decomposition is used:

$$Z(x,t) = u(t)x + \sum_{i=1}^{+\infty} z_i(t) Z_i(x)$$

where $Z_i(x) = \sqrt{2}\sin(i\pi x)$ is the spatial mode associated to the eigen-value $\jmath\omega_i = \jmath(i\pi)$ of (4) ($\jmath = \sqrt{-1}$). In the "coordinates" z_i, the dynamics reads

$$\frac{d^2 z_i}{dt^2} = -(\omega_i)^2 z_i + b_i \ddot{u}.$$

with $b_i = -\frac{\sqrt{2}}{i\pi}$. Following the computations for (1), we consider the generalization to infinite products of (2):

$$Q_i(s) = \frac{b_i}{(\omega_i)^2} \prod_{\substack{k=1\\ k\neq i}}^{\infty} \left(1 + \left(\frac{s}{\omega_k}\right)^2\right), \qquad Q(s) = \prod_{k=1}^{\infty}\left(1 + \left(\frac{s}{\omega_k}\right)^2\right), \qquad c_k = \frac{1}{Q_k(\jmath\omega_k)}\,.$$

Such infinite product are convergent since the series $\sum \frac{1}{\omega_k^2}$ is absolutely convergent [2]. They define the entire functions of $s \in \mathbb{C}$, $Q_i(s)$ and $Q(s)$. This yields to the following formal description

$$z_i = Q_i(s)y, \qquad s^2 u = Q(s)y \quad \text{with} \quad y = \sum_{k=1}^{\infty} c_k z_k\,. \tag{5}$$

Since $Q_i = \frac{b_i}{(\omega_i)^2}\frac{Q(s)}{1+\left(\frac{s}{\omega_k}\right)^2}$, we just consider the infinite product defining $Q(s)$. But, it is well known that

$$\sinh(s) = s\prod_{k=1}^{+\infty}\left(1 + \left(\frac{s}{k\pi}\right)^2\right)$$

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and we recognize that $Q(s) = \sinh(s)/s$, that $y = b/s^2$ and that, in (5), $z_i(s) = Q_i(s)y(s)$ is the modal counterpart of $Z(x,s) = \frac{\sinh(sx)}{s} b(s)$. For more elaborate examples with wave dynamics see, e.g., [32,35,36,42].

Similar computations can be made with first order dynamics in parallel: It suffices to replace in the above computations s by $\sqrt{s}$: (1) becomes $\frac{d}{dt} z_i = -(\omega_i)^2 z_i + b_i u$ and the wave equation becomes the heat equation $\frac{\partial Z}{\partial t} = \frac{\partial^2 Z}{\partial x^2}$. The operators Q_i and Q remain polynomials in s in the finite dimensional case. In the infinite dimension case the distributed delay operator $\sinh(s)/s$ becomes $\sinh(\sqrt{s})/\sqrt{s}$, an entire function of s of order 1/2 at infinity. This operator can be interpreted in the time domain as a convergent sum of derivatives of δ_t, the Dirac distribution, $\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \delta_t^{(2k)}$. Its domain of definition is then the set of C^∞ functions such that such series are absolutely convergent. Such smooth functions are called Gevrey functions of order less than 1 (Gevrey function of order 0 are analytic function).

For the classical $1D$-linear partial differential equations with boundary control (wave, Euler-Bernoulli, heat equation), such formal computations with the Laplace variable s lead to explicit trajectory generation algorithms. A formalization based on module theory (y is the generator of the "free-module" associated to the dynamics) and operational calculus can be found in [40], an up-to-date presentation of the theory with many examples and simulations.

2.3 Higher dimensional wave equation

For infinite dimensional linear system such descriptions remain formal and must be completed by a careful convergence analysis of the resulting operator obtained via such infinite product. We have seen the two key examples where such an analysis is possible and provides explicit descriptions via distributed delay operators (1D-wave) or ultra-distribution (1D-diffusion). Such explicit descriptions can be exploited to generate trajectory, i.e., to steer the system from one state to another state in *finite* time.

Let us now emphasize the basic limitation that can be encountered in such convergence analysis for 2D and 3D wave dynamics. The major ingredient relies on $Q(s)$, an entire function of the complex variables s that vanishes on the open-loop spectrum. When the system is finite dimensional, $Q(s)$ is the characteristic polynomial. When the system is of infinite dimension, $Q(s)$ can be defined via an infinite Weierstrass product [2]. Let us consider a wave equation on a compact domain $\Omega \subset \mathbb{R}^d$ of dimension d with a scalar control on $\Lambda \subset \partial\Omega$:

$$\frac{\partial^2 Z}{\partial t^2} = \Delta Z \quad \text{in} \quad \Omega, \qquad Z = u(t) \quad \text{on} \quad \Lambda, \qquad Z = 0 \quad \text{on} \quad \partial\Omega/\Lambda\,. \tag{6}$$

Notice that such control is not distributed on the boundary Λ: We have a single input system. Denote by $\pm \jmath w_i$ the open-loop eigenvalues, i.e., the eigen-frequencies of the wave equation in the cavity Ω with Dirichlet boundary condition. Since the famous result of H. Weyl, it is well-known that the counting function $\mathbb{N} \ni n \mapsto C(n) \in \mathbb{N}$, the number of eigen-frequencies countered with their multiplicities of modulus less or equal to n, admits the following asymptotics for n large: $C(n) \sim K_\Omega n^d$, for some constant K_Ω depending only on the bounded domain Ω (see [1] for a very suggestive presentation when $d = 3$ and also complementary results). This implies that exists $R_\Omega > 0$ such that $\omega_i \sim R_\Omega i^{1/d}$ for i large.

Thus the infinite product $\prod_{i=1}^{\infty}(1 + s^2/\omega_i^2)$ is convergent only for $d = 1$. For $d = 2$ and $d = 3$, one has to introduce (see [21], Chap. 1) primary Weierstrass factor $(1+s^2/\omega_i^2)\exp(-s^2/\omega_i^2)$ to have a convergent product defining the "simplest" entire function whose zeros are exactly $\pm\jmath\omega_i$:

$$Q(s) = \prod_{i=1}^{\infty} \left(1 + s^2/\omega_i^2\right) \exp\left(-s^2/\omega_i^2\right) \,.$$

In this case, $Q(s)$ is a entire function of order at infinity d. For $d = 2$, $Q(s)$ is not of finite type. For $d = 3$, $Q(s)$ is of finite type. For $d = 2$, this means the following properties; for all $\rho > 2$, exists $A, B > 0$ such that for all $s \in \mathbb{C}$, $|Q(s)| \le A\exp(B|s|^\rho)$ (order 2); but there does not exist $A, B > 0$ such that $\forall s \in \mathbb{C}$, $|Q(s)| \le A\exp(B|s|^2)$ (infinite type). For $d = 3$, this means that exist $A, B > 0$ such that $\forall s \in \mathbb{C}$, $|Q(s)| \le A\exp(B|s|^3)$ (order 3 and finite type).

All this property are classical and are related to the asymptotics $\omega_i \sim R_\Omega i^{1/d}$, the fact that for $d = 2$, the series $\sum_i (1/(\omega_i)^d + 1/(-\omega_i)^d)$ and $\sum 1/\omega_i^d$ are divergent whereas, for $d = 3$, the first converges whereas the second diverges (see, e.g. [2], Chap. 2).

If for example $\omega_i = \sqrt{i}$ then

$$Q(s) = \prod_{i \ge 1}(1 + s^2/i)\exp(-s^2/i) = \frac{\exp(-\gamma s^2)}{s^2\Gamma(s^2)}$$

where γ is the Euler constant and Γ the Gamma function. On this example we understand why the order of Q is 2 and the type is infinite (via the Stirling formulae, the growth of $1/|\Gamma(s)|$ is similar to $\exp(|s|\log|s|)$ for $|s|$ large).

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Let us see the implication of such properties on the modal representation of the dynamics $(1+s^2/\omega_i^2)z_i(s) = \frac{b_i}{\omega_i^2}u(s)$:

$$z_i(s) = Q_i(s)y(s), \qquad u(s) = Q(s)y(s)\,.$$

As noticed in [5] if $Q(s)$ is an entire function that is not of exponential type (order 1 and finite type), then there does not exist $T>0$ and a smooth control $[0,T] \in t \mapsto u(t)$ steering all the z_i from the steady-state 0 associated to $u=0$ to the steady b_i/ω_i^2 associated to $u=1$ in any finite time $T>0$. The argument is roughly speaking as follows. If this is the case, then $\dot{u}$ admits a compact support and thus its Laplace transform is an entire function of exponential type. Thus its distributions of zeros is of order 1 at most (i.e., exists $A>0$ such that the magnitude of zero number n of the entire function $su(s)$ is less than An). Similarly $sz_i(s)$, the Laplace transform of $\dot{z}_i$ is also of exponential type and we have $(1+s^2/\omega_i^2)sz_i(s) = b_i su(s)$. Thus each $\pm \jmath\omega_i$ is a zero of $su(s)$. We obtain a contradiction since, in this case, $su(s)$ has "too many zeros" to be an entire function of exponential type.

Similar arguments indicates that any truly $2D$ or $3D$ heat or wave equations with a finite number of controls (the control cannot be distributed on the boundary) are not steady-state controllable in the sense of [36]. This means that exact controllability in finite time for such system is impossible. It has to be relaxed to approximated controllability in finite time.

Let us consider another but less classical $1D$ wave equation with internal damping[2]. The dynamics reads

$$\frac{\partial^2 Z}{\partial t^2}(x,t) = \frac{\partial^2 Z}{\partial x^2}(x,t) + \varepsilon\frac{\partial^3 Z}{\partial x^2 \partial t}, \qquad Z(0,t)=0, \qquad Z(1,t)=u(t)$$

where $\varepsilon > 0$ is the internal damping coefficient. With the Laplace variable s we have the following formal system

$$(1+\varepsilon s)Z'' = s^2 Z, \qquad Z(0)=0, \qquad Z(1)=u(s)$$

with general solution

$$Z(x,s) = \frac{\sinh\left(\frac{sx}{\sqrt{1+\varepsilon s}}\right)}{\frac{s}{\sqrt{1+\varepsilon s}}}y(s), \qquad u(s) = \frac{\sinh\left(\frac{s}{\sqrt{1+\varepsilon s}}\right)}{\frac{s}{\sqrt{1+\varepsilon s}}}y(s)$$

where $y(s)$ is arbitrary. Denote by $Q(s) = \frac{\sinh\left(\frac{s}{\sqrt{1+\varepsilon s}}\right)}{\frac{s}{\sqrt{1+\varepsilon s}}}$ the analytic function defined on $\mathbb{C}/\{-1/\varepsilon\}$. Arguments similar to the previous ones indicate that there does not exist a smooth trajectory steering the system form $Z(x,0)=0$ to $Z(x,T)=x$ in any finite time $T>0$ because this system has an accumulation of poles around the essential singularity $-1/\varepsilon$. They are given by the zeros of the $Q(s)$:

$$\lambda_k = \frac{-\varepsilon k^2\pi^2 - \sqrt{k^2\pi^2\left(\varepsilon^2 k^2\pi^2 - 4\right)}}{2}, \qquad \mu_k = \frac{-\varepsilon k^2\pi^2 + \sqrt{k^2\pi^2\left(\varepsilon^2 k^2\pi^2 - 4\right)}}{2}, \quad \text{for} \quad k=1,2,\ldots\,.$$

It is clear that for $\varepsilon^2 k^2\pi^2 < 4$, λ_k and μ_k are complex conjugated with negative real parts. Otherwise, they are real and negative. When k tend to infinity λ_k tend to $-\infty$ as $-\varepsilon k^2\pi^2$ and μ_k tend to $-1/\varepsilon$. Such accumulation at finite distance of poles is a clear obstruction to exact controllability in finite time. Nevertheless, we think that such formal computations can be very useful to prove approximated controllability in any time $T>0$ and to construct approximate steering control. We wonder if summability techniques [29] can be useful here.

3 Nonlinear systems

3.1 Under-actuated robots

The dynamics of a mechanical system with as many independent control inputs $u=(u_1,\ldots,u_n)$ as configuration variables $q=(q_1,\ldots,q_n)$ with Lagrangian $L(q,\dot{q})$ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = M(q)u + D\left(q,\dot{q}\right),$$

with $M(q)$ is an $n\times n$ invertible matrix and $D(q,\dot{q})$ corresponds to non conservative forces. This system admits q as a flat output – even when $\frac{\partial^2 L}{\partial \dot{q}^2}$ is singular –: Indeed, u can be expressed in function of $q,\dot{q}$ and $\ddot{q}$ by the *computed torque* formula

$$u = M(q)^{-1}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} - D\left(q,\dot{q}\right)\right).$$

[2] This example steems from discussion of the author with Philippe Martin who points out that the spectrum admits two accumulation points.

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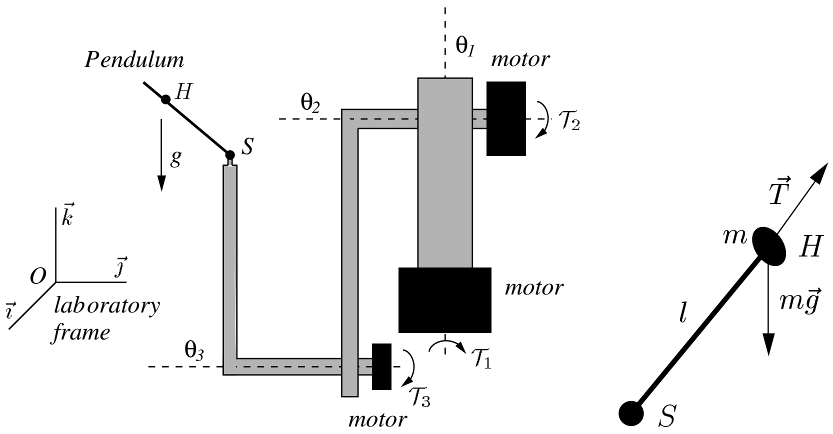

Fig. 1 The robot $2k\pi$ and the isochronous pendulum.

In general, for under-actuated mechanical systems ($\dim u < \dim x$), such computation is no-more possible because there is a constraint independent of u between q, $\dot{q}$, and $\ddot{q}$. Thus one cannot choose arbitrary $t \mapsto q(t)$ to derive the control via the Lagrange equations. However, for flat under-actuated mechanical systems, this is still possible in a certain sense.

The robot $2k\pi$ is developed at École des Mines de Paris and consists of a manipulator carrying a pendulum, see Fig. 1. There are five degrees of freedom (dof's): 3 angles for the manipulator and 2 angles for the pendulum. The 3 dof's of the manipulator are actuated by electric drives, while the 2 dof's of the pendulum are *not* actuated.

This system is typical of underactuated, nonlinear, and unstable mechanical systems such as the PVTOL [24], Caltech's ducted fan [25,30], the gantry crane [10], Champagne flyer [19]. As shown in [10,23,25] the robot $2k\pi$ is flat, with the center of oscillation of the pendulum as a flat output. Let us recall some elementary facts.

The cartesian coordinates of the suspension point S of the pendulum can be considered here as the control variables: They are related to the 3 angles of the manipulator θ_1, θ_2, θ_3 via static relations. Let us concentrate on the pendulum dynamics. This dynamics is similar to the one of a punctual pendulum with the same mass m located at point H, the oscillation center (Huygens theorem). Denoting by $l = \|SH\|$ the length of the isochronous punctual pendulum, Newton equation, and geometric constraints yield the following differential-algebraic system ($\vec{T}$ is the tension, see Fig. 1):

$$m\ddot{H} = \vec{T} + m\vec{g}, \qquad \vec{SH} \times \vec{T} = 0, \qquad \|SH\| = l\,. \tag{7}$$

If, instead of setting $t \mapsto S(t)$, we set $t \mapsto H(t)$, then $\vec{T} = m(\ddot{H} - \vec{g})$. S is located at the intersection of the sphere of center H and radius l with the line passing through H of direction $\ddot{H} - \vec{g}$:

$$S = H \pm \frac{1}{\|\ddot{H} - \vec{g}\|}\left(\ddot{H} - \vec{g}\right).$$

These formulas are crucial for designing a control law steering the pendulum from the lower equilibrium to the upper equilibrium, and also for stabilizing the pendulum while the manipulator is moving around [20].

Let us detail the method to find a trajectory from the stable to unstable equilibrium. The goal is to find explicitly a solution $[0,T] \ni t \mapsto (H(t), \vec{T}(t), S(t))$ of (7) from the stable equilibrium $H = 0 - l\vec{k}$, $\vec{T} = -m\vec{g}$ and $S = 0$, to the unstable equilibrium $H = 0 + l\vec{k}$, $\vec{T} = -m\vec{g}$ and $S = 0$ where 0 is the origine of coordinate and $\vec{k}$ is the unitary vertical vector pointing upwards. We will consider the motion in a vertical plane $(0, \vec{\jmath}, \vec{k})$ with coordinates (y, z).

Assume that the geometric path followed by H is the half-circle of radius l of center O:

$$H(t) = 0 + l\begin{bmatrix}\sin\theta(s)\\ -\cos\theta(s)\end{bmatrix} \quad \text{with} \quad \theta(s) = \mu(s)\pi, \quad s = t/T \in [0,1]$$

where T is the transition time and $\mu(s) = \frac{s^5}{s^5 + (1-s)^5}$ is the smooth bijection on $[0,1]$ with $\mu^{(i)}(0) = \mu^{(i)}(1) = 0$ for $i = 1, 2, 3, 4$.

Denote by $'$ derivation with respect to s. Then $\ddot{H} = H''/T^2$. We know that S is given by the intersection of the vertical circle of center H and radius l with the line passing through H of direction $\ddot{H} - \vec{g}$.

As displayed on Fig. 2, let us consider in the plane moving with H (H is the origin of this vertical plane) the curve C_T followed by the point $P = H + \ddot{H} - \vec{g}$ when t goes from 0 to T. This curve is closed: $\ddot{H}(0) = \ddot{H}(T) = 0$ and

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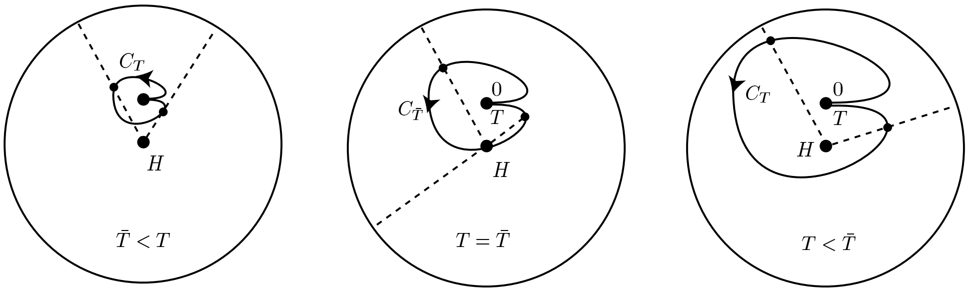

Fig. 2 Construction of trajectories steering from the stable to the unstable equilibrium; a change of time-scale induces a dilatation of the curve C_T followed by the point $H + \ddot{H} - \vec{g}$.

$P(0) = P(T) = H - \vec{g}$. This results from the fact that the derivatives of μ vanish at $s = 0$ and $s = 1$. Changing T to αT yields to a dilation of factor $1/\alpha^2$ of C_T, the dilation center being $H - \vec{g}$. In the picture on the left of Fig. 2, time T is large and C_T is a small closed path around $H - \vec{g}$ and H lies outside of the interior of C_T. Then T decreases, C_T becomes larger and larger and for $T = \bar{T}$, H belongs to $C_{\bar{T}}$. For $T < \bar{T}$ then H remains in the interior of C_T.

For $T \neq \bar{T}$, $\ddot{H} - \vec{g}$ never vanishes. Then we have only one smooth solutions of (7) with such trajectory for H and starting from the stable position:

$$\begin{cases} \vec{T} = m\left(\ddot{H} - \vec{g}\right) , \\ S = H + \frac{l}{\|\ddot{H}-\vec{g}\|}\left(\ddot{H} - \vec{g}\right) . \end{cases}$$

This solution does not connect the stable to the unstable equilibrium. Take now $T = \bar{T}$. As displayed on the central picture of Fig. 2, there exists a $\bar{t} \in]0, \bar{T}[$ such that for $t \in [0, \bar{t}[\cup]\bar{t}, \bar{T}]$, $\ddot{H}(t) - \vec{g} \neq 0$ and for $t = \bar{t}$, $\ddot{H} = \vec{g}$. The direction of $\ddot{H}(t) - \vec{g}$ for t close to $\bar{t}$ and different of $\bar{t}$, evolves smoothly around the tangent direction at point H to $C_{\bar{T}}$. This means that the following map

$$[0, \bar{t}[\cup]\bar{t}, \bar{T}] \ni t \mapsto S(t) = \begin{cases} H + \frac{l}{\|\ddot{H}-\vec{g}\|}\left(\ddot{H} - \vec{g}\right) & \text{for} \quad 0 \leq t < \bar{t} \\ H - \frac{l}{\|\ddot{H}-\vec{g}\|}\left(\ddot{H} - \vec{g}\right) & \text{for} \quad \bar{t} < t \leq \bar{T} \end{cases}$$

is smooth and can be extended by continuity at $t = \bar{t}$ with all its derivatives. This provides a smooth trajectory from the stable to the unstable equilibrium. The above method is in fact generic and work for almost any path followed by H.

Such deformation of the flat-output trajectory (here we use time-scaling) can be useful in other situations when one has to deal with singularity in the map between the flat-output and the remaining variables. One can deform, via tuning parameters whose number will depend of the singularity co-dimension, the flat-output trajectories in order to obtain trajectories for the other variables depending smoothly on the time t and satisfying at each time the implicit algebraic relationships with the flat output and its times derivatives.

3.2 Particle and atom in an electro-magnetic field

This type of system is directly inspired from actual researches in physics related to manipulation of atoms with photons (see, e.g., [7, 8]). We just consider here the basic classical and non-relativistic dynamics of a charged particle or a neutral atom in a linearly polarized electromagnetic field.

As displayed on Fig. 3, assume that the particle P of coordinates (X, Y, Z) is submitted to the fields $\vec{E}(P, t) = E(Z, t)\vec{\imath}$ and $\vec{B}(P, t) = B(Z, t)\vec{\jmath}$ (we neglect the influence of the particle on the fields). The Maxwell equation for $\vec{E}$ and $\vec{B}$ is just a wave equation (c is the velocity of light):

$$\frac{\partial E}{\partial z} = -\frac{\partial B}{\partial t}, \qquad \frac{\partial B}{\partial z} = -\frac{1}{c^2}\frac{\partial E}{\partial t}$$

those general solution is the superposition of two arbitrary waves $u(t - z/c)$ and $v(t + z/c)$:

$$E(z, t) = u(t - z/c) + v(t + z/c), \qquad B(z, t) = \frac{1}{c}\left(u(t - z/c) - v(t + z/c)\right) .$$

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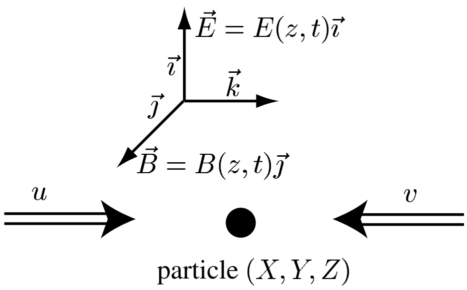

Fig. 3 Particle in a linearly polarized electro-magnetic field.

Up to some constant advance or delay, $t \mapsto (u(t), v(t))$ corresponds to the two counter-propagating laser waves considered here as control inputs u and v. Denoting by m and q the mass and charge, we assume that the particle is subject to the classical Lorenz force and thus obeys the following Newton equation

$$m\frac{d^2P}{dt^2} = q\vec{E}(P,t) + q\left(\frac{d}{dt}P\right) \times \vec{B}(P,t)$$

that reads in coordinates

$$m\ddot{X} = q\left(E(Z,t) - \dot{Z}B(Z,t)\right), \qquad m\ddot{Y} = 0, \qquad m\ddot{Z} = q\dot{X}B(Z,t)\,.$$

Thus we have

$$E(Z,t) = \frac{m}{q}\left(\ddot{X} + \frac{\dot{Z}}{\dot{X}}\ddot{Z}\right), \qquad B(Z,t) = \frac{m}{q}\frac{\ddot{Z}}{\dot{X}}\,.$$

But

$$E(Z,t) = u\left(t - Z(t)/c\right) + v\left(t + Z(t)/c\right), \qquad B(Z,t) = \frac{1}{c}\left(u(t - Z(t)/c) - v(t + Z(t)/c)\right)\,.$$

Hence we have u and v explicitly without any integration once the motion $t \mapsto (X(t), Z(t))$ has been fixed via the following formulae:

$$u(t - Z(t)/c) = \frac{m}{2q}\left(\ddot{X}(t) + \frac{\dot{Z}(t) + c}{\dot{X}(t)}\ddot{Z}(t)\right), \qquad v(t + Z(t)/c) = \frac{m}{2q}\left(\ddot{X}(t) + \frac{\dot{Z}(t) - c}{\dot{X}(t)}\ddot{Z}(t)\right)\,.$$

The structure of the trajectories is very similar to the one of a fully actuated mechanical system. The above computation is analogue to the computed torque method used in robotics. The robot position is replaced by the particle position (X, Z) and the torques corresponds to the electromagnetic field. The only difference is mainly due to propagation delay. Since $|\dot{Z}| \ll c$ the inverse of the function $t \mapsto t - Z(t)/c$ can be approximated by $t \mapsto t + Z(t)/c$ and we have

$$u(t) \approx \frac{m}{2q}\left(\ddot{X} + \frac{\dot{Z} + c}{\dot{X}}\ddot{Z}\right)_{\left(t + \frac{Z(t)}{c}\right)}, \qquad v(t) \approx \frac{m}{2q}\left(\ddot{X} + \frac{\dot{Z} - c}{\dot{X}}\ddot{Z}\right)_{\left(t - \frac{Z(t)}{c}\right)}\,.$$

Similar computations can be made for relativistic motion that can be obtained with large u and v (ultra-short and intense laser pulses) and where a self-force corresponding to radiation processes and proportional to the third spatial derivatives of P can be added in the dynamics.

Let us show also that similar computations can be made when the charged particle is replaced by a neutral atom. In this case, the Newton equation becomes (non relativistic regime, linear dipole dynamics for D with frequency $\omega_0 > 0$ and damping coefficient $\Gamma > 0$, dipole static constant $\mu > 0$):

$$m\frac{d^2P}{dt^2} = -\vec{D} \cdot \nabla_P E(P,t), \qquad \frac{d^2\vec{D}}{dt^2} = -\omega_0^2\left(\vec{D} - \mu\vec{E}\right) - \Gamma\frac{d}{dt}\vec{D}\,.$$

With a linearly polarized field $\vec{E}$ as above, only the dynamics along Z is active and we get with $\vec{D} = D(t)\vec{\imath}$,

$$m\ddot{Z}(t) = -D(t)\frac{\partial E}{\partial z}(Z(t),t), \qquad \ddot{D}(t) = -\omega_0^2(D(t) - \mu E(Z(t),t)) - \Gamma\dot{D}(t)\,.$$

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Now assume that the internal motion $t \mapsto D(t)$ and the external motion $t \mapsto Z(t)$ of the atom are given. We obtain the external field explicitly via

$$E(Z(t),t) = \frac{1}{\mu}\left(D(t) + \frac{\ddot{D}(t)}{\omega_0^2} + \frac{\Gamma\dot{D}(t)}{\omega_0^2}\right), \qquad \frac{\partial E}{\partial z}(Z(t),t) = -m\frac{\ddot{Z}(t)}{D(t)} .$$

But

$$E(Z,t) = u(t - Z(t)/c) + v(t + Z(t)/c), \qquad \frac{\partial E}{\partial z}(Z(t),t) = \frac{1}{c}\left(-u'(t - Z(t)/c) + v'(t + Z(t)/c)\right) .$$

Assuming $|\dot{Z}| \ll c$ we have u' and v' explicitly:

$$u'(t) \approx \frac{1}{2}\left(\frac{\dot{D}}{\mu} + \frac{\Gamma\ddot{D}}{\mu\omega_0^2} + + \frac{D^{(3)}}{\mu\omega_0^2} + mc\frac{\ddot{Z}}{D}\right)_{\left(t-\frac{Z(t)}{c}\right)}, \quad v'(t) \approx \left(\frac{\dot{D}}{\mu} + \frac{\Gamma\ddot{D}}{\mu\omega_0^2} + + \frac{D^{(3)}}{\mu\omega_0^2} - mc\frac{\ddot{Z}}{D}\right)_{\left(t+\frac{Z(t)}{c}\right)} .$$

To be more realistic, the above computations must take into account that the input u and v represent laser light. Thus u and v admits the following form $a(t)\cos(\omega t + \phi(t))$ where ω is the laser frequency close to $\omega_0 \gg \Gamma$ with a and ϕ slowly varying functions. In the time-scale $1/\omega$, the variation of a et ϕ is very small: $|\dot{\phi}| \ll \omega$ and $|\dot{a}/a| \ll \omega$. To obtain physically interesting trajectory planning formulae, one has in fact to derive from the above model, the average dynamics of Z and then to compute the amplitude and phase modulation for u and v from the average motion of the atom.

4 Conclusion

Many control systems of engineering interest can be described by dynamical flat dynamics (see, e.g., [26, 40, 41] and the examples herein). We have seen in this paper how to exploit this property to generate open-loop trajectory. In [11, 15, 37], it is shown also how to use use flatness for robust stabilization and trajectory tracking of finite dimensional system. We wonder if it is possible to extend such flatness-based feedback design to infinite dimensional systems. Notice that closely related backstepping techniques have recently been extended to parabolic 1-D dynamics (see [3]).

References

[1] R. Balian and C. Bloch, Ann. Physics **60**, 401–447 (1970).
[2] R. P. Boas, Entire functions (Academic Press, New York, 1954).
[3] D. Boskovic, A. Balogh, and M. Krstic, Math. Control Signals Syst. (UK) **16**, 44–75 (2003).
[4] E. Cartan, Bull. Soc. Math. Fr. **42**, 12–48 (1914); also in Ouvres Complètes, Part II, Vol. 2 (CNRS, Paris, 1984), pp. 1133–1168.
[5] Y. Chitour, J.-M. Coron, and M. Caravello, Obstruction for steady-state controllability of some linear partial differential equations, 2004, preprint.
[6] F. Chyzak, A. Quadrat, and D. Robertz, Effective Algorithms for Parametrizing Linear Control Systems over Ore Algebra, Tech. Rep. 5181, INRIA (2004).
[7] C. Cohen-Tannoudji, Rev. Mod. Phys. **70**(3), 707–719 (1998).
[8] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, Atom-Photon interaction: Basic Processes and Applications (Wiley-Interscience, 1992).
[9] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon, C.R. Acad. Sci. I, Math. (France), **315**, 619–624 (1992).
[10] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon, Int. J. Control **61**(6), 1327–1361 (1995).
[11] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon, IEEE AC **44**, 922–937 (1999).
[12] M. Fliess and R. Marquez, Int. J. Control **73**, 606–623 (2000).
[13] M. Fliess, H. Mounier, P. Rouchon, and J. Rudolph, Systèmes linéaires sur les opérateurs de Mikusiński et commande d'une poutre flexible, in: ESAIM Proc. "Élasticité, viscolélasticité et contrôle optimal", 8ème entretiens du centre Jacques Cartier, Lyon, 1996 (EDP Sciences, France), pp. 157–168.
[14] M. Fliess, H. Mounier, P. Rouchon, and J. Rudolph, A distributed parameter approach to the control of a tubular reactor: A multi-variable case, in: Proceedings of the Control and Decision Conference, Tampa, 1998 (IEEE), pp. 439–442.
[15] V. Hagenmeyer and E. Delaleau, Automatica **76**, 1941–1946 (2003).
[16] D. Hilbert, Math. Ann. **73**, 95–108 (1912); ebenso in Gesammelte Abhandlungen, Vol. III (Chelsea, New York, 1965), pp. 81–93.
[17] A. Isidori, C. H. Moog, and A. De Luca, A sufficient condition for full linearization via dynamic state feedback, in: Proceedings of the 25th. IEEE Conf. Decision Control, 1986 (IEEE), pp. 203–208.
[18] B. Laroche, Ph. Martin, and P. Rouchon, Int. J. Robust Nonlinear Control **10**, 629–643 (2000).
[19] C. Lemon and J. E. Hause, Design and initial flight test of the Champagne Flyer, in: Proceedings of the 33nd IEEE Conf. Decision and Control, Lake Buena Vista, 1994 (IEEE), pp. 3852–3853.

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[20] Y. Lenoir, Ph. Martin, and P. Rouchon, $2k\pi$, the juggling robot, in: Proceedings of the Control and Decision Conference, Tampa, 1998 (IEEE), pp. 1995–2000.
[21] B. Ja Levin, Distribution of Zeros of Entire Functions (American Mathematical Society, Providence, Rhode Island, 1972).
[22] J. Lévine and D. V. Nguyen, Syst. Control Lett. **48**, 69–75 (2003).
[23] Ph. Martin, Contribution à l'étude des systèmes diffèrentiellement plats, Ph.D. thesis, École des Mines de Paris (1992).
[24] Ph. Martin, S. Devasia, and B. Paden, Automatica **32**, 101–108 (1995).
[25] Ph. Martin, R. Murray, and P. Rouchon, Flat systems, in: Proceedings of the of the 4th European Control Conf., Brussels, 1997, pp. 211–264.
[26] Ph. Martin, R. Murray, and P. Rouchon, Flat Systems, Equivalence and Trajectory Generation, Tech. Rep. (2003) `http://www.cds.caltech.edu/reports/`.
[27] Ph. Martin and P. Rouchon, Math. Control Signals Syst. (UK) **7**, 235–254 (1994).
[28] T. Meurer and M. Zeitz, Proc. Appl. Math. Mech **4**(1), 133–134 (2004).
[29] T. Meurer and M. Zeitz, Ind. Eng. Chem. Res. (USA), April 2005, special issue in honor of the 65th birthday of Professor W. Harmon Ray.
[30] M. B. Milam and R. M. Murray, A testbed for nonlinear flight control techniques: The Caltech ducted fan, in: Proceedings of the IEEE International Conference on Control and Applications, 1999.
[31] J. J. Morales-Ruiz and J. P. Ramis, Methods Appl. Anal. **8**, no. 1 (2001).
[32] H. Mounier, Propriétés structurelles des systèmes linéaires à retards: Aspects théoriques et pratiques, Ph.D. thesis, Université Paris Sud, Orsay (1995).
[33] R. M. Murray, Math. Control Signals Syst. (UK) **7**, 58–75 (1994).
[34] W. Pasillas-Lépine, Systèmes de Contact et Structures de Goursat: Théorie et Application au Contrôle des Systèmes Mécaniques non Holonomes, Ph.D. thesis, Université de Rouen, France (2000).
[35] N. Petit and P. Rouchon, SIAM J. Control Optim. **41**(2), 475–495 (2001).
[36] N. Petit and P. Rouchon, IEEE AC **47**(4), 594–609 (2002).
[37] R. Rothfuß, J. Rudolph, and M. Zeitz, Automatica **32**, 1433–1439 (1996).
[38] P. Rouchon, J. Math. Syst. Estim. Control **5**(3), 345–358 (1995).
[39] P. Rouchon, Int. J. Appl. Math. Comput. Sci. (Poland) **11**(1), 165–188 (2001).
[40] J. Rudolph, Flatness Based Control of Distributed Parameter Systems (Shaker, Germany, 2003).
[41] H. Sira-Ramirez and G. Silva-Navarro (eds.), Algebraic Methods in Flatness, Signal Processing and State Estimation (Innovacion Editorial Lagares, Mexico, 2003).
[42] F. Woittennek and J. Rudolph, ESAIM, Control, Optimisation Calc. Var. (France) **9**, 419–435 (2003).

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