TRAJECTORY GENERATION FOR QUANTUM SYSTEMS BASED ON LYAPOUNOV TECHNIQUES

Mazyar Mirrahimi [*] **Pierre Rouchon** [**]

[*] *Ecole des Mines de Paris, Centre Automatique et Systèmes, 60 Bd Saint-Michel, 75272 Paris cedex 06, FRANCE. Email: mazyar.mirrahimi@ensmp.fr*
[**] *Ecole des Mines de Paris, Centre Automatique et Systèmes, 60 Bd Saint-Michel, 75272 Paris cedex 06, FRANCE. Email: pierre.rouchon@ensmp.fr*

Abstract: The convergence of a Lyapounov based control of the Schrödinger equation (finite dimensional) is analyzed via Lasalle invariance principle. When the linear tangent approximation around the goal eigen-state is controllable, such a feedback ensures global asymptotic convergence. When this linear tangent system is not controllable, the stability of the closed-loop system is not asymptotic. To overcome such lack of convergence we propose a modification based on adiabatic invariance. Simulations illustrate the simplicity and also the interest of these Lyapounov based controls for trajectory generation. Such control methods can also be adapted to tracking.

Keywords: Nonlinear systems, quantum systems, Control Lyapounov function, LaSalle's invariant set, adiabatic invariant.

1. INTRODUCTION

Controllability of a finite dimensional quantum system:

$$\imath \dot{\Psi} = (H_0 + u(t)H_1)\Psi$$

where H_0 and H_1 are $n \times n$ Hermitian matrices with coefficients in $\mathbb{C}$, can be studied via the general accessibility criteria proposed in (Sussmann and Jurdjevic, 1972) and based on Lie-Brackets. More specific results might be found in e.g. (Ramakrishna *et al.*, 1995) and (Turinici and Rabitz, 2003). In particular, the system is controllable if and only if the Lie algebra generated by the skew-symmetric matrices $H_0/\imath$ and $H_1/\imath$ is $su(n)$. Thus controllability of such systems is well characterized. However, such a characterization does not provide a simple and efficient way to generate trajectories between two different states.

Optimal control techniques (see, e.g., (Maday and Turinici, 2003) and the reference herein) provides a first set of methods to generate trajectories. Another set consisting in using inversion techniques such as in (Rabitz, 2003) where the control task is to transfer the population from the ground vibrational state to the first vibrational state in a Morse potential modeling the vibrational motion of O-H.

In the same perspective, we propose here another method for generating the trajectories based on Lyapounov based techniques and closely related to (Jurdjevic and Quinn, 1978). Exploiting linearity with respect to the state, we show here that we can fully analyze the convergence and the links between asymptotic stability of the closed-loop system and its controllability. We completely characterize the largest invariant Lasalle subset.

This yields to a very simple necessary and sufficient condition for global asymptotic convergence of the closed-loop system: the tangent linear system must be controllable around the goal eigen-state.

In section 2, we present the Lyapounov-based design and illustrate the method via simulations on two 3-states examples: for the first example, the closed-loop is asymptotically stable whereas for the second one it is stable but not asymptotically stable. In section 3, we perform the convergence analysis based on Lasalle invariance principle. Asymptotic and exponential convergence is shown to be equivalent to the controllability of the tangent system around the goal eigen-state. Section 4 deals with the degenerate case where the tangent is not controllable. We add to the control an adiabatic open-loop control. Simple arguments based on adiabatic invariance indicates that we can recover the asymptotic convergence of the closed-loop system. Simulations on the second example confirms the interest of such simple modification of the basic feedback scheme.

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2. THE TIME-VARYING FEEDBACK

2.1 Lyapounov based design

Consider the quantum system ($\hbar = 1$)

$$\imath \frac{d}{dt}\Psi = (H_0 + u(t)H_1)\Psi, \qquad (1)$$

where H_0 and H_1 are $n \times n$ Hermitian matrices with coefficients in $\mathbb{C}$. Here H_0 is a time independent Hamiltonian, corresponding to the free evolution of the system in the absence of any external fields. The external interaction here is taken as a control field amplitude $u(t) \in \mathbb{R}$ coupled to the system through the time independent Hamiltonian H_1. The wave function $\Psi = (\Psi_i)_{i=1}^n$ is a vector in $\mathbb{C}^n$, verifying the conservation of probability:

$$\sum_{i=1}^n \mid \Psi_i \mid^2 = 1. \qquad (2)$$

An important characteristic is that the choice of the global phase is arbitrary: physically, the probability amplitudes Ψ and $e^{\imath\theta(t)}\Psi$ describe the same physical state for any global phase $t \mapsto \theta(t) \in \mathbb{R}$.

The conservation of probability and global phase invariance have important consequences on the geometry of the physical state space: Ψ lives on the unit sphere of $\mathbb{C}^n$; two probability amplitudes Ψ_1 and Ψ_2 are identified when exists $\theta \in \mathbb{R}$ such

that $\Psi_1 = \exp(\imath\theta)\Psi_2$. Thus the geometry of the state space does not coincides with the unit sphere of $\mathbb{C}^n$, i.e., $\mathbb{S}^{2n-1}$.

The usual way to take into account such geometry is to reduce the dynamics on the minimal state space: when $n = 2$, the dynamics reduces on $\mathbb{S}^2$, the unit sphere of $\mathbb{R}^3$ called the Bloch sphere. This corresponds to the geometric representation in terms of a fictitious spin $1/2$ (see, e.g.,(Cohen-Tannoudji *et al.*, 1977)). For $n > 2$ such reduction is less simple and the state-space geometry corresponds to the complex projective space of $\mathbb{C}^n$, $\mathbb{PC}^n$.

In this paper, we propose another way to take into account such non trivial geometry of the physical state-space. Instead of reducing the state dimension, we increase the number of controls by one. To u we will add a second control ω corresponding to the time derivative of the global phase. Thus we consider instead of (1) the following control system

$$\imath\dot{\Psi} = (H_0 + uH_1 + \omega)\Psi \qquad (3)$$

where $\omega \in \mathbb{R}$ is a new control playing the role of a gauge degree of freedom. We can choose it arbitrarily without changing the physical quantities attached to Ψ. With such additional fictitious control ω, we will assume in the sequel that the state space is $\mathbb{S}^{2n-1}$ and the dynamics given by (3) admits two independent controls u and ω.

Remark 1. Adding controls to take into-account symmetry is not new. It has been already proposed for induction motors by Blaschke (Blaschke, 1972). This point has been re-explained in (Martin and Rouchon, 1998) and is widely used for induction motors (see e.g. (Espinosa and Ortega, 1995)).

Our goal is to steer the initial state Ψ^0 to a pure state associated to an eigen-vector $\phi \in \mathbb{C}^n$ of H_0 associated to the eigen-value (energy) $\lambda \in \mathbb{R}$. Thus we have $H_0\phi = \lambda\phi$ and $|\phi| = 1$. Take the following real-value function $V(\Psi)$:

$$V(\Psi) = \langle \Psi - \phi | \Psi - \phi \rangle \qquad (4)$$

where $\langle .|. \rangle$ denotes the hermitian product. V is positive for all $\Psi \in \mathbb{C}^n$ and vanishes when $\Psi = \phi$. Simple computations show that V is a control Lyapounov function:

$$\frac{d}{dt}V = 2u\Im(\langle H_1\Psi(t) \mid \phi\rangle) + 2(\omega + \lambda)\Im(\langle \Psi(t) \mid \phi\rangle) \qquad (5)$$

where $\Im$ denotes the imaginary part. By choosing u and $\omega + \lambda$ with the opposite signs of $\Im(\langle H_1\Psi \mid \phi\rangle)$ and $\Im(\langle \Psi(t) \mid \phi\rangle)$ respectively, V will decrease along the trajectories. Any feedback of the form

$$u = -a\Im(\langle H_1\Psi \mid \phi\rangle)$$
$$\omega = -\lambda - b\Im(\langle \Psi \mid \phi\rangle). \qquad (6)$$

where a and b are positive constants ensures $dV/dt \leq 0$: with such feedback, the distance between the actual state Ψ and the goal state ϕ decreases.

2.2 Tutorial examples and simulations
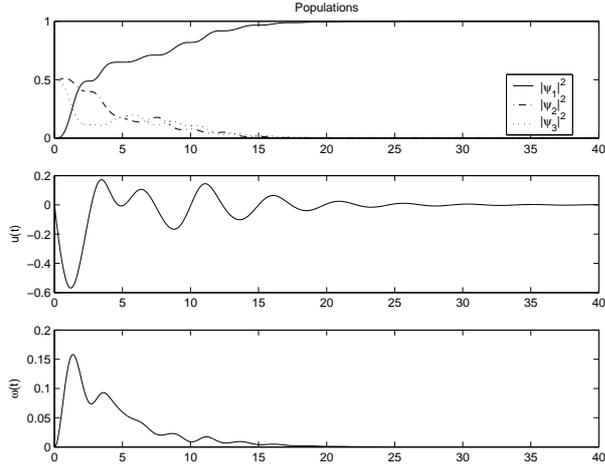
Fig. 1. populations $((|\Psi_1|^2, |\Psi_2|^2, |\Psi_3|^2)$ and controls u and ω; initial condition $(0, 1/\sqrt{2}, 1/\sqrt{2})$; system defined by (7) with feedback (8).

Take $n = 3$, $\Psi = (\Psi_1, \Psi_2, \Psi_3)^T$ and

$$H_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad (7)$$

Let us use the previous Lyapunov control in order to trap our system in the first eigen-state $\phi = (1, 0, 0)$ of energy $\lambda = 0$. We have (* means complex conjugate)

$$\Im(\langle H_1\Psi|\phi\rangle) = \Im\left(\Psi_2^\star + \Psi_3^\star\right)$$

and we take (6) with $a = b = 1$:

$$u = -\frac{1}{2}\Im\left(\Psi_2^\star + \Psi_3^\star\right) \qquad (8)$$
$$\omega = -\frac{1}{2}\Im\left(\Psi_1^\star\right)$$

Simulations of Figure 1 describes the trajectory with $\Psi^0 = (0, 1/\sqrt{2}, 1/\sqrt{2})$ as initial state. Other simulations indicate that the trajectories always converges to ϕ. It appears that such Lyapunov based technics is quite efficient for system (7). In Theorem 2, it is shown that almost global convergence is equivalent to the controllability of the linear tangent system around ϕ.

Let us consider another example that clearly illustrates the basic limitation of such Lyapunov
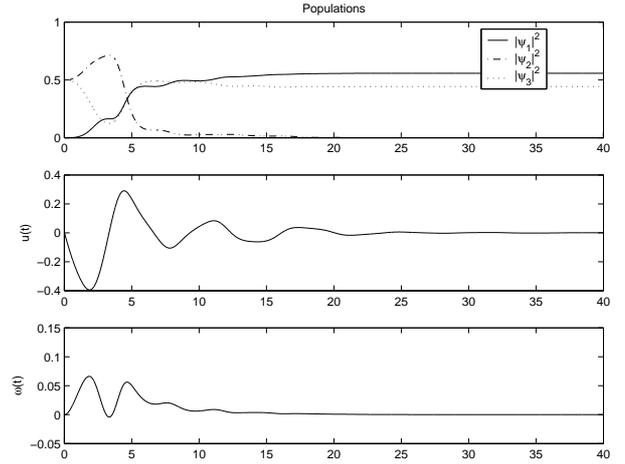
Fig. 2. populations $((|\Psi_1|^2, |\Psi_2|^2, |\Psi_3|^2)$ and controls u and ω; initial condition $(0, 1/\sqrt{2}, 1/\sqrt{2})$; H_0 defined by (7), H_1 by (9) and feedback by (10).

based technique: H_0 and the goal state ϕ remain unchanged but H_1 becomes:

$$H_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (9)$$

The feedback becomes

$$u = -\frac{1}{2}\Im(\Psi_2^\star), \quad \omega = -\frac{1}{2}\Im(\Psi_1^\star). \qquad (10)$$

Simulations of Figure 2 start with $(0, 1/\sqrt{2}, 1/\sqrt{2})$ as initial condition for Ψ. We clearly realize that such a feedback reduces the distance with the first state but does not ensure its convergence to 0. This is not due to a lack of controllability. This system is controllable since the Lie algebra spanned by $H_0/\imath$ and $H_1/\imath$ coincides with $su(3)$ (Ramakrishna *et al.*, 1995). As explained in Theorem 2, such convergence deficiency comes form the fact that the linear tangent system around ϕ is not controllable.

3. CONVERGENCE ANALYSIS

The goal of this section is to prove the following theorem that underlies simulations of figures 1 and 2.

Theorem 2. Consider (3) with $\Psi \in \mathbb{S}^{2n-1}$ and an eigen-state $\phi \in \mathbb{S}^{2n-1}$ of H_0 associated to the eigenvalue λ. Take the static feedback (6) with $a, b > 0$. Then the two following propositions are true:

(1) If the spectrum of H_0 is not degenerate (all eigen-values are distinct), the Ω-limit set of the closed loop system is the intersection of $\mathbb{S}^{2n-1}$ with the real vector-space E spanned by the eigen-vectors Φ of H_0 such that $\langle H_1\Phi|\phi\rangle = 0$ and $\Im(\langle\Phi|\phi\rangle) = 0$.

(2) The Ω-limit set reduces to $\{\phi, -\phi\}$ if and only if H_0 is not degenerate and $E = \mathbb{R}\phi$. In this case: the equilibrium ϕ is exponentially stable (on $\mathbb{S}^{2n-1}$); the equilibrium $-\phi$ is unstable; the attractor set of ϕ is exactly $\mathbb{S}^{2n-1}/\{-\phi\}$. This case corresponds to the controllability of the linear-tangent system at ϕ, a time-invariant linear system that lives on the $2n-1$ plane tangent to $\mathbb{S}^{2n-1}$ at ϕ.

For example of Figure 1, it becomes clear that $E = \mathbb{R}\phi$ since H_0 is not degenerate and $\phi = (1,0,0)$ is almost globally asymptotically stable. Notice the condition $E = \mathbb{R}\phi$ says that, physically, the goal-state ϕ is connected to all other excited states via mono-photonic transition (see, e.g., (Messiah, 1962)).

For example of Figure 2, element of E are of the form $(x, 0, z)$ where $x \in \mathbb{R}$ and $z \in \mathbb{C}$; we observe effectively that the Ω-limit set contains elements of the form $(x, 0, r \exp(\imath\theta))$ with x, r and θ in $\mathbb{R}$ such that $x^2 + r^2 = 1$. Physically, the transition between ϕ and state of energy $3/2$ necessitates at least two photons: the feedback (10) cannot find such multi-photonic processes.

The proof of Theorem 2 mainly relies on the characterization of the Ω-limit set via Lasalle invariance principle. It provides here a complete description of the invariant subset via the linear system $M\Xi = 0$ where M is defined by (12) here below. Such description becomes very simple when H_0 is not degenerate.

3.1 Proof of proposition (1) of Theorem 2

Up to a shift on ω and H_0, we can assume that $\lambda = 0$. Lasalle's principle (see, e.g., theorem 3.4, page 115 of (Khalil, 1992)) says that the trajectories of the closed-loop system converge to the largest invariant set contained in $dV/dt = 0$ where V is defined by (4). The equation $dV/dt = 0$ means that
$$\Im(\langle H_1\Psi|\phi\rangle) = \Im(\langle\Psi|\phi\rangle) = 0,$$
Thus $u = 0$ and $\omega = 0$. Invariance means that with $\imath\frac{d}{dt}\Psi = H_0\Psi$, $\frac{d}{dt}\Im(\langle H_1\Psi|\phi\rangle) = 0$ and $\frac{d}{dt}\Im(\langle\Psi|\phi\rangle) = 0$. Clearly $\frac{d}{dt}\Im(\langle\Psi|\phi\rangle) = 0$ does not give any additional information since $H_0\phi = 0$. Only $\frac{d}{dt}\Im(\langle H_1\Psi|\phi\rangle) = 0$ provides a new independent equation: $\Re(\langle H_1 H_0\Psi|\phi\rangle) = 0$ that reads
$$\Re(\langle[H_0, H_1]\Psi|\phi\rangle) = 0.$$
Similarly $\frac{d}{dt}\Re(\langle[H_0, H_1]\Psi|\phi\rangle) = 0$ implies
$$\Im(\langle[H_0, [H_0, H_1]]\Psi|\phi\rangle) = 0.$$
And so on. Finally, the largest invariant set is characterized by $\Im(\langle\Psi|\phi\rangle) = 0$ with the following conditions,

$$\Im(\langle H_1\Psi|\phi\rangle) = 0$$
$$\Re(\langle[H_0, H_1]\Psi|\phi\rangle) = 0$$
$$\Im(\langle[H_0, [H_0, H_1]]\Psi|\phi\rangle) = 0$$
...

that corresponds to the "ad-conditions" obtained in (Jurdjevic and Quinn, 1978). At each step, we have the Lie bracket of the Hamiltonian H_0 with the Hamiltonian of the last step.

We can always assume that H_0 is diagonal. Then we can easily compute the commutator $[H_0, B]$ where $B = (B_{ij})$ is a $n \times n$ matrix. With $H_0 = \mathrm{diag}(\lambda_1, ..., \lambda_n)$, we have
$$[H_0, A]_{i,j} = (\lambda_i - \lambda_j)B_{ij}.$$
Let take $B = H_1$ in order to simplify the notations. So:
$$[H_0, B] = ((\lambda_i - \lambda_j)B_{ij})$$
$$[H_0, [H_0, B]] = ((\lambda_i - \lambda_j)^2 B_{ij})$$
$$\vdots$$
$$\underbrace{[H_0, [H_0, ..., [H_0}_{k \quad times}, B]]...] = ((\lambda_i - \lambda_j)^k B_{ij}).$$
Thus the previous system reads:
$$\Im(\Sigma_j B_{1j}\Psi_j) = 0,$$
$$\Re(\Sigma_j(\lambda_1 - \lambda_j)B_{1j}\Psi_j) = 0,$$
$$\vdots$$
$$\Im(\Sigma_j(\lambda_1 - \lambda_j)^{2k}B_{1j}\Psi_j) = 0,$$
$$\Re(\Sigma_j(\lambda_1 - \lambda_j)^{2k+1}B_{1j}\Psi_j) = 0,$$
$$\cdots \quad\quad (11)$$
Set
$$M = \begin{pmatrix} 1 & \cdots & 1 \\ (\lambda_1 - \lambda_2)^2 & \cdots & (\lambda_1 - \lambda_n)^2 \\ (\lambda_1 - \lambda_2)^4 & \cdots & (\lambda_1 - \lambda_n)^4 \\ \vdots & \vdots & \vdots \\ (\lambda_1 - \lambda_2)^{2(n-2)} & \cdots & (\lambda_1 - \lambda_n)^{2(n-2)} \end{pmatrix} \quad (12)$$
and
$$\Xi = ((1 + \imath(\lambda_1 - \lambda_2))B_{12}\Psi_2, \ldots, (1 + \imath(\lambda_1 - \lambda_n))B_{1n}\Psi_n)^T.$$
Then (11) implies that $M\Xi = 0$. Indeed even if (11) corresponds to an infinite number of linear relations, using the Vandermonde structure of these equations one can easily see that it suffices to consider its first $n - 1$ equations: higher order equations are necessarily linear combinations of the first $n - 1$ equations. Thus the linear system $M\Xi = 0$ with $\Im(\langle\Psi|\phi\rangle) = 0$ provides a complete characterization of the Ω-limit set.

In the particular case where the free Hamiltonian H_0 has a non-degenerate spectrum, M is invertible and thus $\Xi = 0$. Then $\Psi \in \mathbb{S}^{2n-1}$ is in the Ω-limit set if and only if
$$B_{1j}\Psi_j = 0, \forall j \in \{2, ..., n\}.$$
and $\Im(\Psi_1) = 0$.

3.2 Proof of proposition (2) of Theorem 2

Notice first that in any case the Ω-limit set contains ϕ and $-\phi$.

If H_0 has a non-degenerate spectrum and $E = \mathbb{R}\phi$ then proposition (1) implies that the Ω-limit set is just $\{\pm\phi\}$. Now let us suppose that at least one of these two conditions is not fulfilled.

Assume that $E \neq \mathbb{R}\phi$. Thus exists an eigen-vector Φ of H_0 not co-linear to ϕ such that $\langle H_1\Phi|\phi\rangle = 0$. With $\Psi(0) = \Phi$ as initial state, we have $u(t) = 0$ and $\omega(t) = -\lambda$ and $\Psi(t) = \Phi$ for all $t > 0$. The Ω-limit set contains Φ.

Assume $E = \mathbb{R}\phi$ but that H_0 has a degenerate spectrum. We will consider two cases:

(1) There exists an eigen-vector ϕ_k with length 1 of H_0 orthogonal to ϕ but with the same eigen-value λ. Since $E = \mathbb{R}\phi$, $B_{1k} = \langle H_1\phi_k|\phi\rangle) \neq 0$. With $\Psi(0) = \frac{B_{1k}}{|B_{1k}|}\phi_k$ as initial state, we have $u(t) = 0$, $\omega = -\lambda$ and $\Psi(t) = \frac{B_{1k}}{|B_{1k}|}\phi_k$ belongs to the Ω-limit set.

(2) There exist two orthogonal eigen-vectors ϕ_k and ϕ_l of H_0, with length one and admitting the same eigen-value $\mu \neq \lambda$. Since $E = \mathbb{R}\phi$, $B_{1k} = \langle H_1\phi_k|\phi\rangle) \neq 0$ and $B_{1l} = \langle H_1\phi_l|\phi\rangle) \neq 0$. With $\Psi(0) = (B_{1k}\phi_l - B_{1l}\phi_k)/\sqrt{|B_{1k}|^2 + |B_{1l}|^2}$, we have $u(t) = 0$, $\omega = -\lambda$ and
$$\Psi(t) = e^{-\imath(\mu-\lambda)t}\Psi(0).$$
Thus the Ω-limit set contains $(e^{\imath\alpha}\Psi(0))_{\alpha\in[0,2\pi]}$.

The proof of the first part of proposition (2) is thus done.

Let us prove now that ϕ is locally exponentially stable when H_0 is not degenerate and $E = \mathbb{R}\phi$. We will prove that the linear tangent closed-loop system is asymptotically stable. This will automatically implies that the equilibrium ϕ is hyperbolically stable. Set
$$\Psi(t) = \phi + \Delta\Psi(t)$$
with $\Delta\Psi$ small. Then up to second order terms we have
$$\imath\frac{d}{dt}\Delta\Psi = (H_0 - \lambda I)\Delta\Psi - a\Im(\langle H_1\Delta\Psi|\phi\rangle)H_1\phi$$
$$-b\Im(\langle\Delta\Psi|\phi\rangle)\phi$$
and $\Re(\langle\Delta\Psi|\phi\rangle) = 0$ (definition of the tangent space at ϕ to the unit sphere $\mathbb{S}^{2n-1}$). Set $W(\Delta\Psi) = \frac{1}{2}\langle\Delta\Psi|\Delta\Psi\rangle$. Simple computations show that $dW/dt \leq 0$ and $E = \mathbb{R}\phi$ implies that the LaSalle's invariant set of this linearized system reduces to $\Delta\Psi = 0$ on the tangent space at ϕ to $\mathbb{S}^{2n-1}$.

The fact that $-\phi$ is unstable results from the fact that the Lyapunov function V reaches 4 its maximum on $\mathbb{S}^{2n-1}$ only for $\Psi = -\phi$. Thus

if $\Psi(0) \neq -\phi$, then necessary $\Psi(t)$ must converge to the other point of the Ω-limit set. Thus $\lim_{t\mapsto+\infty}\Psi(t) = \phi$; the equilibrium $-\phi$ is unstable, the attraction region of ϕ is $\mathbb{S}^{2n-1}/\{-\phi\}$.

Let us finally proved that H_0 non degenerate and $E = \mathbb{R}\phi$ is equivalent to the controllability of the linear-tangent system at ϕ.

Set $\Psi(t) = \phi + \Delta\Psi(t)$ with $\Re(\langle\Delta\Psi|\phi\rangle) = 0$, $u = \Delta u$ and $\omega = -\lambda + \Delta\omega$ with $\Delta\Psi$, Δu and $\Delta\omega$ small. Then up to second order terms, (3) reads
$$\imath\frac{d}{dt}\Delta\Psi = (H_0 - \lambda I)\Delta\Psi + \Delta u H_1\phi + \Delta\omega\phi.$$
Take $(\phi_1, ..., \phi_n)$ an ortho-normal eigen-basis of H_0 associated to $(\lambda_1, ..., \lambda_n)$ with $\phi_1 = \phi$ and $\lambda_1 = \lambda$. Set $(z_1, ..., z_n) \in \mathbb{C}^n$ the coordinates of $\Delta\Psi$ in this basis. Then: $\Re(z_1) = 0$ and
$$\frac{d}{dt}(\Im(z_1)) = -\Delta\omega - B_{11}\Delta u$$
$$\imath\frac{d}{dt}z_2 = (\lambda_2 - \lambda_1)z_2 + B_{12}\Delta u$$
$$\vdots$$
$$\imath\frac{d}{dt}z_n = (\lambda_n - \lambda_1)z_n + B_{1n}\Delta u$$
where $B_{ij} = \langle\phi_i|H_1\phi_j\rangle$. Controllability is then equivalent to the fact that $B_{1i} \neq 0$ and $\lambda_i \neq \lambda_j$ for $i \neq j$ (use, e.g., Kalman controllability matrix). This is clearly equivalent to H_0 non degenerate and $E = \mathbb{R}\phi$.

4. AN IDEA TO IMPROVE CONVERGENCE

Theorem 2 and simulation of Figure 2 show that such Lyapounov method fails to steer the state Ψ to the pure state associated to ϕ when the dimension of vector space E exceeds 1. However, the three states system of Figure 2 is controllable: the Lie algebra spanned by $H_0/\imath$ and $H_1/\imath$ is $su(3)$.

To overcome such lack of convergence observed with simulations on figure 2, we will try to track an adiabatic reference trajectory
$$\imath\frac{d}{dt}\Psi_r = (H_0 + u_r(t)H_1)\Psi_r, \quad \Psi_r(0) = (1, 0, 0),$$
(13)
where $u_r = \frac{1}{2}\sin(2\pi t/T)$ with a period $T = 300$, large compared with the natural periods of H_0 to ensure that u_r is a slowly varying time function. Take the following tracking feedback
$$u = u_r - \frac{1}{2}\Im(\langle H_1\Psi|\Psi_r(t)\rangle), \quad \omega = -\frac{1}{2}\Im(\langle\Psi|\Psi_r(t)\rangle).$$
(14)
Since u_r varies slowly, adiabatic theory ensures that Ψ_r will follow closely the first eigen-state of $H_0 + u_r H_1$ (Messiah, 1962). So when u_r returns to 0, Ψ_r will almost return to the first eigen-space spanned by $(1, 0, 0)$: we have $\Psi_r(T) \approx$

$(\exp(\imath\theta), 0, 0)$ for some phase shift θ. If during this slow motion, the reference trajectory Ψ_r is in the neighborhood of eigen-states of $H_0 + u_r H_1$ where the linear tangent system is controllable, this will strongly improve convergence. This is effectively the case as shown in figure 3 that illustrates the efficiency of combining Lyapunov design and adiabatic invariance. Rigorous proofs of such convergence can be performed and will be given in future publications. See also (Beauchard et al., 2004) for a different method based on an implicitly defined control-Lyapunov function that ensures convergence then the linear-tangent system around ϕ is not controllable.
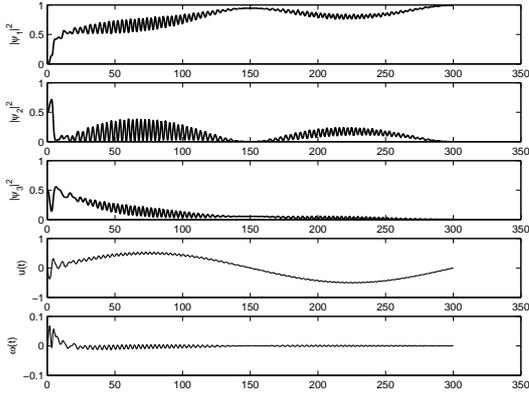
Fig. 3. system and initial conditions identical to figure 2; adiabatic trajectory (13) tracking via the feedback (14).

5. CONCLUSION

The convergence analysis presented here can be extended to trajectory tracking around generic reference trajectories. Consider a reference trajectory $t \mapsto (\Psi_r(t), u_r(t), \omega_r(t))$ of (1). Set $\Delta\Psi = \Psi - \Psi_r$, $\Delta\omega = \omega - \omega_r$ and $\Delta u = u - u_r$. Then

$$\imath \frac{d}{dt}\Delta\Psi = (H_0 + u_r(t)H_1 + \omega_r)\,\Delta\Psi + \Delta u\, H_1\Psi + \Delta\omega\,\Psi.$$

Set $V = \langle\Delta\Psi|\Delta\Psi\rangle$. Simple computations yield $\frac{d}{dt}V = 2\Delta u\Im(\langle H_1\Psi(t) \mid \Psi_r\rangle) + 2\Delta\omega\Im(\langle\Psi(t) \mid \Psi_r\rangle)$. Thus any feedback where Δu and $\Delta\omega$ are of the opposite sign with respect to $\Im(\langle H_1\Psi(t) \mid \Psi_r\rangle)$ and $\Im(\langle\Psi(t) \mid \Psi_r\rangle)$ respectively, ensures $\frac{d}{dt}V \leq 0$. A careful convergence analysis based on Lasalle invariant principle shows that, when the first variation around Ψ_r is controllable, such simple feedback provides asymptotic tracking. This point is developed in (Mirrahimi and Rouchon, 2004).

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