Control of a quantum particle in a moving box

Pierre Rouchon *

Technical report A/325
Centre Automatique et Systèmes
Ecole des Mines de Paris
October 2002

We consider the control of a quantum system represented by a probability complex amplitude $\mathbb{R} \ni q \mapsto \psi(q,t)$ solution of

$$\imath \frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial q^2} + (V(q) - uq)\psi. \tag{1}$$

This 1-D Schrödinger equation describes the non relativistic motion of a single charged particle (mass $m = 1$, $\hbar = 1$) with a potential V in a uniform electric field $t \mapsto u(t)$. With $\ddot{v} = -u$, (1) represents also the dynamics of a particle in a non Galilean frame of absolute position v (see, e.g., [9]). A change of independent variables $(t,q) \mapsto (t,z)$ and dependent variable $\psi \mapsto \phi$, transform (1) into (2) where the control appears as a shift on the space variable. These classical transformations are as follows (see, e.g., [2]). Instead of considering u as control, take v defined by $\ddot{v} = -u$ as control. Then

$$q = z - v, \quad \psi(t, z - v) = \exp\left(\imath\left(-z\dot{v} - v\dot{v} + \frac{1}{2}\int_0^t \dot{v}^2\right)\right)\phi(t,z)$$

yields

$$\imath \frac{\partial \phi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \phi}{\partial z^2} + V(z - v)\phi. \tag{2}$$

Controllability depends strongly on the shape of the potential V. We will discuss here some preliminary results with the following potential shape.

- The harmonic oscillator, (1) with $V(q) = q^2/2$, where, using [15], the controllability is completely understood even in the $3D$ case.
- The periodic potential, (1) with $V(q) = V(q+a)$, where impulsive controls achieve iso-energy translations with amplitudes multiple of the period a.

*Ecole des Mines de Paris, Centre Automatique et Systèmes, 60 Bd Saint-Michel, 75272 Paris cedex 06 FRANCE. pierre.rouchon@ensmp.fr.

- The box potential, (2) with $V(q) = 0$ for $q \in [-1, 1]$ and $V(q) = +\infty$ for q outside $[-1, 1]$. This problem admits strong similarity with the water tank problem considered in [11]: around any state of definite energy, the linear tangent approximate system is not controllable but it is "steady-state" controllable in the sense of [11]. We guess that, as for the water-tank system [3], the nonlinear dynamics is locally controllable around any state of definite energy.

The author thanks Claude Le Bris and Gabriel Turinici for interesting discussions and reference [16].

1 The harmonic oscillator

It is proved in [16] that any modal approximation of finite dimension is controllable. We have proved in [15] that the harmonic oscillator, equation (1) with $V(q) = q^2/2$, is not controllable. It is interesting to complete such surprising difference in order to understand, from a more practical point of view, what is controllable in this infinite dimensional dynamics. It can be decomposed into two parts. The controllable part corresponds to the classical dynamics on the average position

$$\frac{d}{dt}\langle q\rangle = \langle p\rangle\,, \quad \frac{d}{dt}\langle p\rangle = -\langle q\rangle + u$$

and the non-controllable part to a harmonic oscillator without control

$$\imath\frac{\partial \chi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \chi}{\partial z^2} + \frac{z^2}{2}\chi$$

where $z = q - \langle q\rangle$ and $\psi(t, q)$ is related to $\chi(t, z)$ via

$$\psi(t,q) = \exp\left(\imath\left(\langle p\rangle(q - \langle q\rangle) - \int_0^t (\langle q\rangle^2/2 - \langle p\rangle^2/2 - u\langle q\rangle)\right)\right)\chi(t, q - \langle q\rangle).$$

Denote by $\psi_0(q)$ the state of the particle at $t = 0$ with definite energy E:

$$-\frac{1}{2}\frac{\partial^2 \psi_0}{\partial q^2} + \frac{q^2}{2}\psi_0 = E\psi_0.$$

Then $\langle q\rangle(0) = \langle p\rangle(0) = 0$. Thus, $\chi(0, z) = \psi_0(0, z)$. Use flatness based motion planning method [4, 5, 8] and take any C^2 function $[0, T] \ni t \mapsto y(t) \in \mathbb{R}$ such that

$$y(0) = \dot y(0) = \ddot y(0) = 0, \quad y(T) = a, \quad \dot y(T) = \ddot y(T) = 0.$$

Then the control

$$[0,T] \ni t \mapsto u(t) \begin{cases} 0 & \text{for } t < 0 \\ \ddot y(t) + y(t) & \text{for } t \in [0,T] \\ a & \text{for } t > T. \end{cases}$$

steers the particle to the new bounded state of definite energy and centered around $q = a$. More precisely, up to a phase shift θ,

$$\chi(T, q) = \exp(\imath\theta)\psi_0(q - a).$$

Similar computations can be obtained for the $3D$ harmonic oscillator:

$$\imath \frac{\partial \psi}{\partial t} = -\sum_{k=1}^{3} \left(\frac{1}{2} \frac{\partial^2 \psi}{\partial {q_k}^2} + \frac{q_k^2}{2} \psi - u_k q_k \psi \right).$$

The only controllable parts are the average positions $\langle q_k \rangle$ satisfying

$$\frac{d}{dt} \langle q_k \rangle = \langle p_k \rangle, \quad \frac{d}{dt} \langle p_k \rangle = - \langle q_k \rangle + u_k$$

and the transformation

$$\psi(t, q_1, q_2, q_3) = \prod_{k=1}^{3} \exp\left(\imath \left(\langle p_k \rangle (q_k - \langle q_k \rangle) - \int_0^t (\langle q_k \rangle^2 /2 - \langle p_k \rangle^2 /2 - u_k \langle q_k \rangle) \right) \right) \ldots$$
$$\ldots \chi(t, q_1 - \langle q_1 \rangle, q_2 - \langle q_2 \rangle, q_2 - \langle q_2 \rangle)$$

leads to an autonomous oscillator

$$\imath \frac{\partial \chi}{\partial t} = -\sum_{k=1}^{3} \left(\frac{1}{2} \frac{\partial^2 \chi}{\partial {z_k}^2} + \frac{z_k^2}{2} \chi \right)$$

with $z_k = q_k - \langle q_k \rangle$.

2 Periodic potential

Take (1) with a periodic potential (period $a > 0$):

$$V(q + a) = V(q), \forall q.$$

The goal is to solve approximatively the transition between two bounded states of the same energy ψ_1 and ψ_2 such that

$$\psi_2(q) = \psi_1(q - ka)$$

where $k \in \mathbb{Z}$.

Take the form (2) with v as control. Take any C^2 function $[0, 1] \ni \alpha \mapsto y(\alpha) \in \mathbb{R}$ such that

$$y(0) = \dot{y}(0) = \ddot{y}(0) = 0, \quad y(1) = ka, \quad \dot{y}(1) = \ddot{y}(1) = 0.$$

Then, for $\varepsilon > 0$ small enough the control

$$[0, T] \ni t \mapsto u(t) \begin{cases} 0 & \text{for } t < 0 \\ -y''(t/\varepsilon)/\varepsilon^2 & \text{for } t \in [0, \varepsilon] \\ -a & \text{for } t > \varepsilon. \end{cases}$$

steers, approximatively, from ψ_1 to ψ_2. This is obvious with (2): since $\ddot{v} = -u$, $v(t) = y(t/\varepsilon)$ is close to a step between 0 and ka; since $V(z - ka) = V(z)$, the influence of such variation of v on ϕ solution of (2) remains small ($O(\varepsilon)$). Thus ϕ remains closed to $\psi_1(z)$ during the impulse. Thus, up to a phase shift the real state $\psi(\varepsilon, q)$ corresponds to $\phi(\varepsilon, z) = \psi_1(z) = \psi_1(q - ka) = \psi_2(q)$.

This simple impulsive control overcomes the following difficulty: such transitions necessarily requires to reach energies in the continuous part of the spectrum. Moreover straightforward extensions to $2D$ or $3D$ periodic potentials can be done.

3 The moving box

Take (2), with $V(z) = 0$ for $z \in [-\frac{1}{2}, \frac{1}{2}]$ and $V(z) = +\infty$ for z outside $[-\frac{1}{2}, \frac{1}{2}]$. The dynamics reads:

$$\begin{aligned} \imath \frac{\partial \phi}{\partial t} &= -\frac{1}{2} \frac{\partial^2 \phi}{\partial z^2}, \quad z \in [v - \frac{1}{2}, v + \frac{1}{2}], \\ \phi(v - \frac{1}{2}, t) &= \phi(v + \frac{1}{2}, t) = 0 \end{aligned}$$

where v is the position of the box and z is an absolute position (Galilean frame). Otherwise stated (see (1))

$$\begin{aligned} \imath \frac{\partial \psi}{\partial t} &= -\frac{1}{2} \frac{\partial^2 \psi}{\partial q^2} + \ddot{v} q \psi, \quad q \in [-\frac{1}{2}, \frac{1}{2}], \\ \psi(-\frac{1}{2}, t) &= \psi(\frac{1}{2}, t) = 0 \end{aligned}$$

where $q = z - v$ is the relative position with respect to the box. ψ and ϕ are related via

$$\psi(t, z - v) = \exp\left(\imath \left(-z\dot{v} - \nu\dot{v} + \frac{1}{2}\int_0^t \dot{v}^2\right)\right) \phi(t, z).$$

3.1 Modal decomposition

For $v = 0$, the system admits a non-degenerate discrete spectrum (see, e.g.,[9]):

$$\omega_{2n} = 2n^2\pi^2 \qquad \psi_{2n}(q) = 2\sin(2n\pi q) \tag{3}$$

$$\omega_{2n+1} = 2\left(n + \frac{1}{2}\right)^2 \pi^2 \qquad \psi_{2n+1}(q) = 2\cos\left((2n+1)\,\pi q\right). \tag{4}$$

Set $\psi(t,q) = \sum_{n\geq 1} a_n(t)\psi_n(q)$ in $\imath\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial q^2} + \ddot{v}q\psi$, to obtain, for each integer $n \geq 1$,

$$\imath\frac{d}{dt}a_{2n} = -\omega_{2n}a_{2n} + \ddot{v}\left(\sum_{k\geq 0} a_{2k+1}\int_{-\frac{1}{2}}^{\frac{1}{2}} q\psi_{2n}(q)\psi_{2k+1}(q)\ dq\right)$$

$$\imath\frac{d}{dt}a_{2n+1} = -\omega_{2n+1}a_{2n+1} + \ddot{v}\left(\sum_{k\geq 1} a_{2k}\int_{-\frac{1}{2}}^{\frac{1}{2}} q\psi_{2n+1}(q)\psi_{2k}(q)\ dq\right).$$

For any integers $\alpha \geq 1$ and $\beta \geq 0$:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} q\psi_{2\alpha}(q)\psi_{2\beta+1}(q)\ dq = (-1)^{\alpha+\beta}\left(\frac{1}{\left[\left(\alpha+\beta+\frac{1}{2}\right)\pi\right]^2} + \frac{1}{\left[\left(\alpha-\beta-\frac{1}{2}\right)\pi\right]^2}\right).$$

Notice that odd (resp. even) modes are connected via the control v to odd (resp. even) modes.

3.2 The two-modes approximation

The controllability criteria proposed in [17, 18, 19] can be useful when only a finite number of modes is considered. Nevertheless this criteria has to be adapted because of the double integrator $\ddot{v} = u$.

A good short-cut model to understand the controllability of such system will be two modes model:

$$\begin{aligned}
\imath\frac{d}{dt}a_1 &= -\frac{\pi^2}{2}a_1 - \frac{8}{9\pi^2}ua_2 \\
\imath\frac{d}{dt}a_2 &= -2\pi^2 a_2 - \frac{8}{9\pi^2}ua_1 \\
\ddot{v} &= u
\end{aligned}$$

where a_1 and a_2 are complex, u and v are real. The sub-system

$$\begin{aligned}
\imath\frac{d}{dt}a_1 &= -\frac{\pi^2}{2}a_1 - \frac{8}{9\pi^2}ua_2 \\
\imath\frac{d}{dt}a_2 &= -2\pi^2 a_2 - \frac{8}{9\pi^2}ua_1
\end{aligned}$$

with u as control is a two-states system where the Bloch sphere appears naturally. Set $\omega_0 = \frac{3\pi^2}{2}$, $\chi = \exp(\imath\frac{5\pi^2}{4}t)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ then

$$\imath\frac{d}{dt}\chi = \left(\begin{pmatrix} -\omega_0/2 & 0 \\ 0 & \omega_0/2 \end{pmatrix} + u\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}\right)\chi \text{ with } b = -\frac{8}{9\pi^2}$$

Take the density matrix

$$\rho = \chi\chi^* = \begin{pmatrix} |a_1|^2 & a_1^* a_2 \\ a_1 a_2^* & |a_2|^2 \end{pmatrix} = 1 + \lambda\sigma_x + \mu\sigma_y + \nu\sigma_z$$

with the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Set $\vec{S} = (\lambda, \mu, \nu) \in \mathbb{S}^2$ (the Bloch sphere, where the meaningless absolute phase is removed):

$$\frac{d}{dt}\vec{S} = \vec{S} \wedge (\omega_0 \vec{B}_0 + \frac{u}{\hbar}\vec{B}_1), \quad \vec{B}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{B}_1 = \begin{pmatrix} -2b \\ 0 \\ 0 \end{pmatrix}.$$

Set $\tau = \omega_0 t$, $' = d/d\tau$ and $\bar{u} = \frac{2b}{\omega_0}u$ the new control. The dynamics becomes

$$\vec{S}' = \vec{S} \wedge (\vec{B}_0 + \bar{u}\vec{J})$$

where $\vec{J}$ is the unitary vector $\frac{1}{\|\vec{B}_1\|}\vec{B}_1$. $\vec{B}_0$ is orthogonal to $\vec{J}$ and consider the ortho-normal frame $(\vec{I}, \vec{J}, \vec{K})$ with $\vec{K} = \vec{B}_0 \wedge \vec{J}$ and $\vec{I} = \vec{J} \wedge \vec{K}$. Set $\vec{S} = x\vec{I} + y\vec{J} + z\vec{K}$ ($(x, y, z) \in \mathbb{R}^3$ with $x^2 + y^2 + z^2 = 1$). Up to scaling on the box position,v, the two modes model reads now

$$x' = -z\bar{u}, \quad y' = z, \quad z' = x\bar{u} - y, \quad v'' = \bar{u} \tag{5}$$

where all quantities are real now. From this formulation one can prove that the system is not flat: its defect is two [4]. Using the defect-2 output y, we have for x around 1,

$$z = y', \quad x = \sqrt{1 - y^2 - (y')^2}, \quad \bar{u} = \frac{y + y''}{\sqrt{1 - y^2 - (y')^2}}$$

and we recover $\bar{v}$ via the double integral

$$\bar{v} = \int\int \frac{y + y''}{\sqrt{1 - y^2 - (y')^2}}.$$

Such formulae can be used to prove that for (x, y, z) close to $(1, 0, 0)$ on the sphere and $(\bar{v}, \bar{v}')$ close to 0, the nonlinear two-modes model (5) is locally controllable.

3.3 Tangent linearization

Denote by $\bar{\psi}$ any state of definite energy $\bar{\omega}$ in (3) or (4). Set

$$\psi(t, q) = \exp(-\imath\bar{\omega}t)(\bar{\psi}(q) + \Psi(q, t))$$

in (2). Then Ψ satisfies

$$\imath\frac{\partial \Psi}{\partial t} + \bar{\omega}\Psi = -\frac{1}{2}\frac{\partial^2 \Psi}{\partial q^2} + \ddot{v}q(\bar{\psi} + \Psi)$$
$$0 = \Psi(-\frac{1}{2}, t) = \Psi(\frac{1}{2}, t).$$

The tangent linear system is obtained, assuming Ψ and $\ddot{v}$ small and neglecting the second order term $\ddot{v}q\Psi$:

$$\imath \frac{\partial \Psi}{\partial t} + \bar{\omega}\Psi = -\frac{1}{2}\frac{\partial^2 \Psi}{\partial q^2} + \ddot{v}q\bar{\psi}, \quad \Psi(-\frac{1}{2}, t) = \Psi(\frac{1}{2}, t) = 0. \tag{6}$$

We prove here below via operational computations that (6) is not controllable but steady-state controllable. We give explicit formulae for the control $[0,T] \ni t \mapsto \ddot{v}(t)$, steering in finite time from $\Psi = 0$, $v = \dot{v} = 0$ at $t = 0$ to $\Psi = 0$, $v = a$, $\dot{v} = 0$ at $t = T$ for any $T > 0$. Computations are similar to those we have proposed for heat or Euler-Bernouilli dynamics where ultra-distributions and Gevrey functions of order ≤ 1 appear [7, 6, 14].

Set $s = d/dt$. Standard computations show that the general solution of

$$(\imath s + \bar{\omega})\Psi = -\frac{1}{2}\Psi'' + s^2 v q \bar{\psi}$$

is

$$\Psi = A(s,q)a(s) + B(s,q)b(s) + C(s,q)v(s)$$

where

$$\begin{aligned} A(s,q) &= \cos\left(q\sqrt{2\imath s + 2\bar{\omega}}\right) \\ B(s,q) &= \frac{\sin\left(q\sqrt{2\imath s + 2\bar{\omega}}\right)}{\sqrt{2\imath s + 2\bar{\omega}}} \\ C(s,q) &= (-\imath s q \bar{\psi}(q) + \bar{\psi}'(q)). \end{aligned}$$

Case $q \mapsto \bar{\phi}(q)$ even. The boundary conditions imply

$$A(s,1/2)a(s) = 0, \quad B(s,1/2)b(s) = -\psi'(1/2)v(s).$$

The element $a(s)$ is a torsion element [10], thus the system is not controllable. Nevertheless, for steady-state controllability, we have $a \equiv 0$ (as for the water tank [11]) and we have the following parameterization[1]:

$$\begin{aligned} b(s) &= -\bar{\psi}'(1/2)\frac{\sin\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right)}{\sqrt{-2\imath s + 2\bar{\omega}}}\; y(s) \\ v(s) &= \frac{\sin\left(\frac{1}{2}\sqrt{2\imath s + 2\bar{\omega}}\right)}{\sqrt{2\imath s + 2\bar{\omega}}}\frac{\sin\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right)}{\sqrt{-2\imath s + 2\bar{\omega}}} y(s) \\ \Psi(s,q) &= B(s,q)b(s) + C(s,q)v(s) \end{aligned} \tag{7}$$

The entire functions of s appearing in this formulae are of order less than 1/2, i.e., their module for s large is bounded by $\exp(M\sqrt{|s|})$ for some $M > 0$, independent of $s \in \mathbb{C}$ and $q \in [-1, 1]$. The above formulae (7) admit then a clear

[1]Remember that v is associated to a real quantity and the operator $\frac{\sin\left(\frac{1}{2}\sqrt{2\imath s+2\bar{\omega}}\right)}{\sqrt{2\imath s+2\bar{\omega}}}\frac{\sin\left(\frac{1}{2}\sqrt{-2\imath s+2\bar{\omega}}\right)}{\sqrt{-2\imath s+2\bar{\omega}}}$ is a real operator.

interpretation in the time domain, as for the heat equation with the Holmgren series solution [20], when y is a C^∞ time function of Gevrey order less than 1: i.e. $\exists M > 0$ and $\exists \sigma \in [0,1]$ such that, $\forall t$, $\forall n$, $|y^{(n)}(t)| \leq M^n \Gamma(1+(\sigma+1)n)$ where Γ is the classical Gamma function[2]. This results from the following fact: to an entire function of s, $F(s)$, of order $\leq 1/2$ is associated a series $\sum_{n\geq 0} a_n s^n$ with coefficient a_n satisfying $|a_n| \leq K/\Gamma(1+2n)$ for all n, with some constant $K > 0$ independent of n. To $F(s)y(s)$ corresponds in the time domain, the series,

$$\sum_{n\geq 0} a_n y^{(n)}(t)$$

that is absolutely convergent when y is a Gevrey function of order $\sigma < 1$. Take $T > 0$ and $D \in \mathbb{R}$. Steering (6) from $\Psi = 0$, $v = 0$ at time $t = 0$, to $\Psi = 0$, $v = D$ at $t = T$ is possible with the following Gevrey function of order σ:

$$[0,T] \ni t \mapsto y(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ \bar{D} \frac{\exp\left(-\left(\frac{T}{t}\right)^{\frac{1}{\sigma}}\right)}{\exp\left(-\left(\frac{T}{t}\right)^{\frac{1}{\sigma}}\right) + \exp\left(-\left(\frac{T}{T-t}\right)^{\frac{1}{\sigma}}\right)} & \text{for } 0 < t < T \\ \bar{D} & \text{for } t \geq T \end{cases}$$

with $\bar{D} = \frac{2\bar{\omega} D}{\sin^2(\sqrt{\bar{\omega}/2})}$. The fact that this function is of Gevrey order σ results from its exponential decay of order σ around 0 and 1 (see, e.g., [13, 12]).

Case $q \mapsto \bar{\phi}(q)$ odd. The boundary conditions imply

$$B(s, 1/2)b(s) = 0, \quad A(s, 1/2)a(s) = -\psi'(1/2)v(s).$$

b is a torsion element and thus the system is not controllable. Nevertheless, as for the even case, we have the following parameterization:

$$\begin{aligned} a(s) &= -\bar{\psi}'(1/2)\cos\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right) y(s) \qquad (8) \\ v(s) &= \cos\left(\frac{1}{2}\sqrt{2\imath s + 2\bar{\omega}}\right)\cos\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right) y(s) \\ \Psi(s,q) &= A(s,q)a(s) + C(s,q)v(s). \end{aligned}$$

As for the even case, with

$$[0,T] \ni t \mapsto y(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ \bar{D} \frac{\exp\left(-\left(\frac{T}{t}\right)^{\frac{1}{\sigma}}\right)}{\exp\left(-\left(\frac{T}{t}\right)^{\frac{1}{\sigma}}\right) + \exp\left(-\left(\frac{T}{T-t}\right)^{\frac{1}{\sigma}}\right)} & \text{for } 0 < t < T \\ \bar{D} & \text{for } t \geq T \end{cases}$$

where $\bar{D} = \frac{D}{\cos^2(\sqrt{\bar{\omega}/2})}$, we can steer (6) from $\Psi = 0$, $v = 0$ at time $t = 0$, to $\Psi = 0$, $v = D$ at $t = T$.

[2] Analytic functions are Gevrey functions of order $\sigma = 0$.

Practical computations The above method for computing the steering control requires to develop in series of s and to calculate high order time derivatives of y. All these calculations can be bypassed with Cauchy formula. Take a bounded measurable function $t \mapsto Y(t)$ corresponding to the position set-point for v. From this function, we deduce a complex entire function $\zeta \mapsto y(\zeta)$ via convolution with a Gaussian kernel with standard deviation ε

$$y(\zeta) = \frac{1}{\varepsilon\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(\zeta - t)^2}{2\varepsilon^2}\right) Y(t)\, dt$$

Consider, e.g, the relation giving the control v in the even case: $v(s) = F(s)y(s)$ where

$$F(s) = \frac{\sin\left(\frac{1}{2}\sqrt{2\imath s + 2\bar{\omega}}\right)}{\sqrt{2\imath s + 2\bar{\omega}}} \frac{\sin\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right)}{\sqrt{-2\imath s + 2\bar{\omega}}}$$

is an entire function of order less than 1 (order 1/2 in fact but 1 is enough here). Thus $F(s) = \sum_{n\geq 0} a_n s^n$ where $|a_n| \leq K^n/\Gamma(1+n)$ with $K > 0$ independent of n. In the time domain $F(s)y(s)$ corresponds to $\sum_{n\geq 0} a_n y^{(n)}(t)$. But

$$y^{(n)}(t) = \frac{\Gamma(n+1)}{2\imath\pi} \oint_\gamma \frac{y(t+\xi)}{\xi^{n+1}}\, d\xi$$

where γ is a closed path around zero. Thus $\sum_{n\geq 0} a_n y^{(n)}(t)$ becomes

$$\sum_{n\geq 0} a_n \frac{\Gamma(n+1)}{2\imath\pi} \oint_\gamma \frac{y(t+\xi)}{\xi^{n+1}}\, d\xi = \frac{1}{2\imath\pi} \oint_\gamma \left(\sum_{n\geq 0} a_n \frac{\Gamma(n+1)}{\xi^{n+1}} \right) y(t+\xi)\, d\xi$$

where[3]

$$\sum_{n\geq 0} a_n \frac{\Gamma(n+1)}{\xi^{n+1}} = \int_{D_\delta} F(s) \exp(-s\xi) ds = B_1(F)(\xi).$$

is the Borel transform (see, e.g., [1]) of the F that is defined for $\xi \in \mathbb{C}$ large enough, $|\xi| > K$. In the time domain $F(s)y(s)$ corresponds to

$$\frac{1}{2\imath\pi} \oint_\gamma B_1(F)(\xi) y(t+\xi)\, d\xi$$

where γ is a closed path around zero. Since $y(\zeta) = \frac{1}{\varepsilon\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-(\zeta - t)^2/2\varepsilon^2) Y(t)\, dt$ we have the following filter for the control

$$v(t) = \int_{-\infty}^{+\infty} \left[\frac{1}{\imath\varepsilon(2\pi)^{\frac{3}{2}}} \oint_\gamma B_1(F)(\xi) \exp(-(\xi - \tau)^2/2\varepsilon^2)\, d\xi \right] Y(t-\tau)\, d\tau.$$

The kernel

$$f(\tau) = \frac{1}{\imath\varepsilon(2\pi)^{\frac{3}{2}}} \oint_\gamma B_1(F)(\xi) \exp(-(\xi - \tau)^2/2\varepsilon^2)\, d\xi$$

[3] D_δ is the half line starting from 0 in the complex plane with direction δ chosen to ensure the convergence of the integral.

can be computed numerically once for all. One can check that $f(\tau)$ is real and vanishes rapidly for $|\tau| \gg \varepsilon$. Since F is here of order $1/2$ implies that $B_1(F)$ is defined on $\mathbb{C}/\{0\}$: it admits an essential singularity in 0. These formulas are used in a small Matlab animation that can be obtained upon request from the author.

References

[1] R.P. Boas. *Entire functions*. Academic Press, New York, 1954.

[2] A.G. Butkovskiy and Y.I. Samoilenko. *Control of Quantum-Mechnical Processes and Systems*, volume 56 of *Mathematics and its Applications (Soviet Series*. Kluver Academic Publishers, Dortrecht/Boston/London, 1984. English edition 1990.

[3] J.M. Coron. Local controllability of a 1-d tank containing a fluid modeled by the shallow water equations. *ESAIM: COCV*, 8:513–554, 2002.

[4] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. Flatness and defect of nonlinear systems: introductory theory and examples. *Int. J. Control*, 61(6):1327–1361, 1995.

[5] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems. *IEEE AC*, 44:922–937, 1999.

[6] M. Fliess, H. Mounier, P. Rouchon, and J. Rudolph. Systèmes linéaires sur les opérateurs de Mikusiński et commande d'une poutre flexible. In *ESAIM Proc. "Élasticité, viscolélasticité et contrôle optimal", 8ème entretiens du centre Jacques Cartier, Lyon*, pages 157–168, 1996.

[7] B. Laroche, Ph. Martin, and P. Rouchon. Motion planing for the heat equation. *Int. Journal of Robust and Nonlinear Control*, 10:629–643, 2000.

[8] Ph. Martin and P. Rouchon. Systèmes plats, planification et suivi de trajectoires, 1999. Notes, Journées X-UPS, `http://math.polytechnique.fr/xups/vol99.html`.

[9] A. Messiah. *Quantum Mechanics*, volume I&II. North Holland Publ. Co., Amsterdam, 1962.

[10] H. Mounier. Algebraic interpretations of the spectral controllability of a linear delay system. *Forum Math.*, 10:39–58, 1998.

[11] N. Petit and P. Rouchon. Dynamics and solutions to some control problems for water-tank systems. *IEEE AC*, 47(4):594–609, 2002.

[12] J.P. Ramis. Dévissage Gevrey. *Astérisque*, 59-60:173–204, 1978.

[13] J.P. Ramis. *Séries Divergentes et Théories Asymptotiques.* Panoramas et synthèses. Société Mathématique de France, 1993.

[14] P. Rouchon. Motion planning, equivalence, infinite dimensional systems. *Int. J. Applied Mathematics and Computer Science*, 11(1):165–188, 2001.

[15] P. Rouchon. On the control of quantum oscillators. Technical Report A/320, École des Mines de Paris, Centre Automatique et Systèmes, June 2002.

[16] S. G. Schirmer, H. Fu, and A. I. Solomon. Complete controllability of quantum systems. *Physical Review A*, 63(063410), 2001.

[17] G. Turinici. *Analyse de méthodes numériques de simulation et contrôle en chimie quantique.* PhD thesis, Université Paris Sud, `http://www-rocq.inria.fr/Gabriel.Turinici` /, 2000.

[18] G. Turinici. Controlabilité exacte de la population des états propres dans les systèmes quantiques bilinéaires. *C. R. Acad. Sci. Paris, Sèrie 1*, 330:327–332, 2000.

[19] G. Turinici. Controllable quantities for bilinear quantum systems. In *Proceedings of the 39th IEEE Conference on Decision and Control*, pages 1364–1369, 2000.

[20] G. Valiron. *Equations Fonctionnelles.* Masson et Cie, Editeurs, Paris, 2nd edition, 1950.