Motion planing, equivalence, infinite dimensional systems

Pierre Rouchon
École des Mines de Paris,
Centre Automatique et Systèmes,
60, bd. Saint-Michel, 75272 Paris Cedex 06
rouchon@cas.ensmp.fr

Abstract

Motion planning, i.e., steering a system from one state to another, is a basic question in automatic control. For a certain class of systems described by ordinary differential equations and called flat systems [7, 8], motion planning admits simple and explicit solutions. This stems from an explicit description of the trajectories by an arbitrary time function y, the flat output, and a finite number of its time derivatives. Such explicit descriptions are related to old problems on Monge equations and equivalence investigated by Hilbert and Cartan. The study of several examples (the car with n-trailers and the non-holonomic snake, pendulums in series and the heavy chain, the heat equation and the Euler-Bernoulli flexible beam) indicates that the notion of flatness and its underlying explicit description can be extended to infinite-dimensional systems. As in the finite dimensional case, this property yields simple motion planning algorithms via operators of compact support. For the nonholonomic snake, such operators involve nonlinear delays. For the heavy chain, they are defined via distributed delays. For heat and Euler-Bernoulli systems, their supports are punctual and their definition domain coincides with the set of Gevrey functions of order 2.

Keywords: infinite dimensional control systems, motion planning, flatness, absolute equivalence, Pfaffian systems, delays systems, Gevrey functions.

1 Introduction

The idea underlying equivalence and flatness [8] –a one-to-one correspondence between trajectories of systems– is not restricted to control systems described by *ordinary* differential equations. It can be adapted to delay differential systems and to partial differential equations with boundary control. Of course, there are many more technicalities and the picture is far from clear. Nevertheless, this new point of view seems promising for the design of control laws. In this paper, we sketch some recent developments in this direction.

We consider three kinds of systems: nonholonomic, pendulum and diffusion systems. Each of them admits two families of models: finite dimensional ones and infinite dimensional ones. The flat output, well defined in the finite dimensional case, admits also a natural equivalent in the infinite dimensional case. In a certain sense, the finite and infinite descriptions are thus equivalent and similar motion planning algorithms can be constructed.

Thus flatness admits an infinite dimensional extension. The systems examined here in details suggest to us the following setting in terms of operators admitting compact supports with respect to the time t. Take $\Omega \ni x \mapsto X(x,t) \in \mathbb{R}^q$, defined on an open and smooth domain Ω of $\mathbb{R}^n$. Assume that X is solution of a square partial differential system $P(X) = 0$ on Ω satisfying on $\partial\Omega$ the boundary condition $L(X,u) = 0$, with $t \mapsto u(t) \in \mathbb{R}^m$ the control. Roughly speaking, an explicit parameterization consists in compact support operators $\mathcal{A}_x$, $x \in \Omega$, and $\mathcal{B}$ with a common domain of definition $\mathcal{D}$ including enough time functions $t \mapsto y(t) \in \mathbb{R}^m$ (density, partition of unity, stability by addition and multiplication, ...) such that $\Xi : \ (x,t) \mapsto \mathcal{A}_x y|_t$ and $\upsilon : \ t \mapsto \mathcal{B}y|_t$ with $t \mapsto y(t) \in \mathbb{R}^m$ belonging to $\mathcal{D}$ satisfy automatically $P(\Xi) = 0$ and $L(\Xi, \upsilon) = 0$. Finding $[0,T] \ni t \mapsto u(t)$ $(T > 0)$, steering X from $X(x,0) = X_0(x)$ to $X(x,T) = X_1(x)$ reduces then to find $t \mapsto y(t)$ when $y(t)$ is prescribed by the initial state X_0 (resp. final state X_1) for t small enough (resp. large enough). Similarly, equivalence between to systems $(P(X) = 0, L(X,u) = 0)$ and $(Q(Z) = 0, M(Z,v) = 0)$ could be defined via compact support operators exchanging solutions.

Such a setting requires precise definitions. Using the module theory and the notion of π-freeness [25] is a first possibility. Notice that parameterizations developed in [36, 37] deals with under-determined systems of PDE's whereas here $P(X) = 0$ admits the same number of unknowns and equations. We will concentrate here on three typical kind of systems where flatness admits a direct infinite dimensionnal extension.

2 Non-holonomic systems

Many mobile robots such as considered in [2, 32, 44] admit the same structure. They are flat and the flat output corresponds to the Cartesian coordinates of a special point. Starting from the classical n-trailer systems [41, 7, 40, 6] we show that, when n, the number of trailers, tends to infinity the system tends to a trivial delay system, the non-holonomic snake. Invariance with respect to rotations and translations make very natural the use of Frénet formulae and curve parameterization with respect to arc length instead of time (see [23, 43] for relations between flatness and physical symmetries) . The study of such systems gives us the opportunity to recall links with an old problem stated by Hilbert [16] and investigated by Cartan [3], on Pfaffian systems, Goursat normal forms and (absolute) equivalence.

2.1 The car

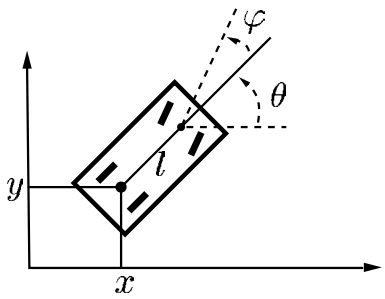

Figure 1: kinematic of a car.

The rolling without slipping conditions yield (see figure 1 for the notations)

$$\left\{ \begin{array}{lcl} \dot{x} & = & v\cos\theta \\ \dot{y} & = & v\sin\theta \\ \dot{\theta} & = & \dfrac{v}{l}\tan\varphi \end{array} \right. \tag{1}$$

where v, the velocity, and φ, the steering angle are the two controls. These equations mean geometrically that the angle θ gives the direction of the tangent to the curve followed by P, the point of coordinates (x, y), and that $\tan\varphi/l$ corresponds to the curvature of this curve:

$$v = \pm\|\dot{P}\|, \quad \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \dot{P}/v, \quad \tan\varphi = l\det(\ddot{P}\dot{P})/v\sqrt{|v|}.$$

There is thus a one to one correspondence between arbitrary smooth curves and the solutions of (1). It provides, as shown in [7], a very simple algorithm to steer the car from on a configuration to another one.

2.2 Car with n-trailers [7]

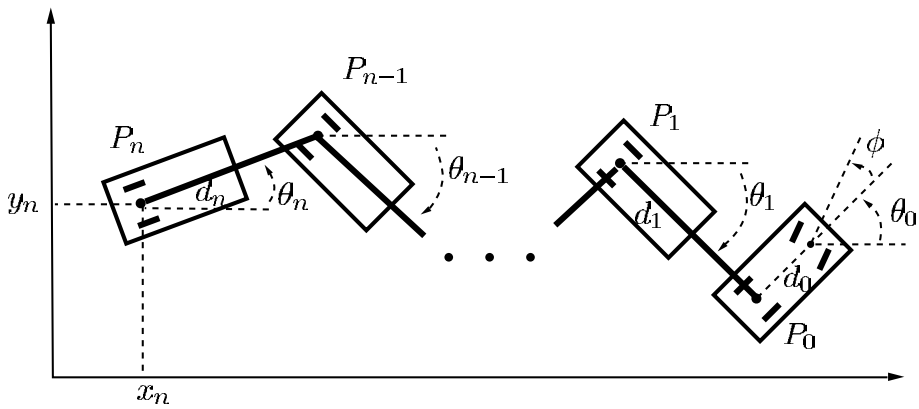

Figure 2: car with n trailers.

Take the single car here above and hitch, as display on figure 2, n trailers. The resulting system still admits two control: the velocity of the car v and the steering angle ϕ. As for the single car, modelling is based on the rolling without slipping conditions.

There is a one to one correspondence between smooth curves of arbitrary shapes and the system trajectories. It suffices to consider the curve followed by P_n, the cartesian position of the last trailer. It is not necessary to write down explicitly the system equations in the state-space form as (1). Just remember that the kinematic constraints say that the velocity of each trailer (more precisely of the middle of its wheel axle) is parallel to the direction of its hitch.

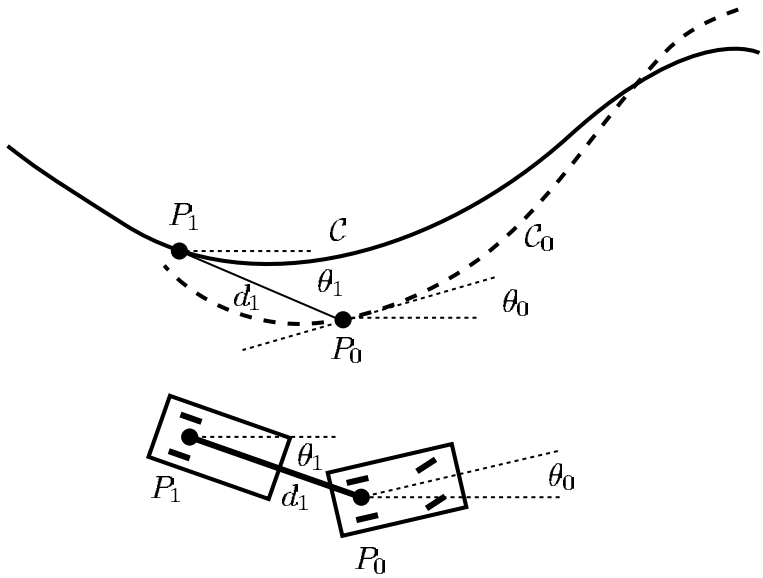

Figure 3: case $n = 1$.

Take $n = 1$ and have a look at figure 3. Assume that the curve $\mathcal{C}$ followed by P_1 is smooth. Take $s \to P(s)$ an arc length parameterization. Then $P_1 = P(s)$, θ_1 is the angle of $\vec{\tau}$, the unitary tangent vector to $\mathcal{C}$. Since $P_0 = P + d_1\vec{\tau}$ derivation with respect to s provides

$$\frac{d}{ds}P_0 = \vec{\tau} + d_1\kappa\vec{\nu}$$

with $(\vec{\tau}, \vec{\nu})$ the Frénet frame of $\mathcal{C}$ and κ its curvature. Thus $\dfrac{d}{ds}P_0 \neq 0$ is tangent to $\mathcal{C}_0$, the curve followed by P_0. This curve is necessary smooth and

$$\tan(\theta_0 - \theta_1) = d_1\kappa, \quad \vec{\tau}_0 = \frac{1}{\sqrt{1 + (d_1\kappa)^2}}\ (\vec{\tau} + d_1\kappa\vec{\nu}).$$

Derivation with respect to s_0, ($ds_0 = \sqrt{1 + (d_1\kappa)^2}\ ds$) yields the steering angle ϕ :

$$\tan\phi = d_0\kappa_0 = d_0\ \frac{1}{\sqrt{1 + (d_1\kappa)^2}}\left(\kappa + \frac{d_1}{1 + (d_1\kappa)^2}\frac{d\kappa}{ds}\right).$$

The car velocity v is then given by

$$v(t) = \sqrt{1 + d_1^2\kappa^2(s(t))}\ \dot{s}(t)$$

for any C^1 time function, $t \to s(t)$. Notice that ϕ and $\theta_0 - \theta_1$ always remain in $]-\pi/2, \pi/2[$. These computations prove the one to one correspondence between the system trajectories respecting ϕ and $\theta_0 - \theta_1$ in $]-\pi/2, \pi/2[$, and regular planar curves of arbitrary shape with an arbitrary C^1 time parameterization.

The case $n > 1$ is just a direct generalization. The correspondence between arbitrary smooth curves $s \mapsto P(s)$ (tangent $\vec{\tau}$, curvature κ) with a C^1 time parameterization $t \mapsto s(t)$ is then defined by a smooth invertible map

$$\begin{array}{ccc} \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}^{n+2} & \mapsto & \mathbb{R}^2 \times \mathbb{S}^1 \times]-\pi/2, \pi/2[^{n+1} \times \mathbb{R} \\ \left(P, \vec{\tau}, \kappa, \frac{d\kappa}{ds}, \ldots, \frac{d^n\kappa}{ds^n}, \dot{s}\right) & \mapsto & (P_n, \theta_n, \theta_n - \theta_{n-1}, \ldots, \theta_1 - \theta_0, \phi, v) \end{array}$$

where v is the car velocity. More details are given in [7].

With such a correspondence, motion planning reduces to a trivial problem: to find a smooth curve with prescribed initial and final positions, tangents, curvatures κ and curvature derivatives, $d^i\kappa/ds^i$, $i = 1, \ldots, n$.

2.3 The general one-trailer system [41]

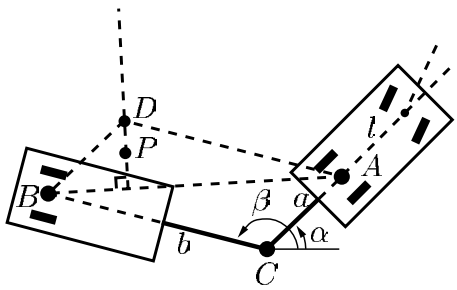

Figure 4: car with one trailer.

This nonholonomic system is displayed on figure 4: here the trailer is not directly hitched to the car at the center of the rear axle, but more realistically at a distance a of this point. The equations are

$$\begin{cases} \dot{x} &= \cos\alpha\ v \\ \dot{y} &= \sin\alpha\ v \\ \dot{\alpha} &= \dfrac{1}{l}\tan\varphi\ v \\ \dot{\beta} &= \dfrac{1}{b}\left(\dfrac{a}{l}\tan\varphi\cos(\alpha-\beta)-\sin(\alpha-\beta)\right)\ v. \end{cases} \tag{2}$$

Controls are the car velocity v and the steering angle φ.

There still exists a one to one correspondence between the trajectories of (2) and arbitrary smooth curves with a C^1 time parameterization. Such curves are followed by the point P (see figure 4) of coordinates

$$\begin{aligned} X &= x + b\cos\beta + L(\alpha-\beta)\frac{b\sin\beta - a\sin\alpha}{\sqrt{a^2+b^2-2ab\cos(\alpha-\beta)}} \\ Y &= y + b\sin\beta + L(\alpha-\beta)\frac{a\cos\alpha - b\cos\beta}{\sqrt{a^2+b^2-2ab\cos(\alpha-\beta)}} \end{aligned} \tag{3}$$

where L is defined by an elliptic integral :

$$L(\alpha-\beta) = ab\int_{\pi}^{2\pi+\alpha-\beta}\frac{\cos\sigma}{\sqrt{a^2+b^2-2ab\cos\sigma}}\ d\sigma. \tag{4}$$

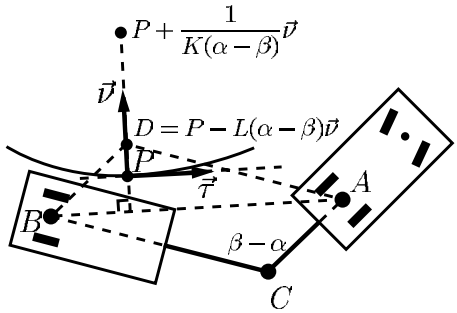

Figure 5: geometric construction with the Frénet frame.

We have also a geometrical construction (see figure 5): the tangent vector $\vec{\tau}$ is parallel to AB. Its curvature κ depends on $\delta = \alpha - \beta$:

$$\kappa = K(\delta) = \frac{\sin\delta}{\cos\delta\ \sqrt{a^2+b^2-2ab\cos\delta} - L(\delta)\ \sin\delta} \tag{5}$$

Function K is an increasing bijection between $]\gamma, 2\pi-\gamma[$ and $\mathbb{R}$. The constant $\gamma \in [0, \pi/2]$ is defined by the implicit equation

$$\cos\gamma\ \sqrt{a^2+b^2-2ab\cos\gamma} = ab\sin\gamma\ \int_{\pi}^{\gamma}\frac{\cos\sigma}{\sqrt{a^2+b^2-2ab\cos\sigma}}\ d\sigma.$$

For $a = 0$, $\gamma = \pi/2$ and P coincides with B. Then D is given by $D = P - L(\delta)\vec{\nu}$ with $\vec{\nu}$ the unitary normal vector. Thus (x, y, α, β) depends on $(P, \vec{\tau}, \kappa)$. The steering angle φ depends on κ and $d\kappa/ds$ where s is the arc length. Car velocity v is then computed from κ, $d\kappa/ds$ and $\dot{s}$, the velocity of P.

2.4 Contact structure, equivalence and Pfaffian systems

Contrarily to the standard n-trailers systems, formulas attached to the general one-trailer system are not obvious. They cannot be found by some physical intuition. In fact they rely on old questions and results relative to Pfaffian systems.

Nonholonomic systems with two controls, as the above trailer systems, are driftless systems of the form

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2$$

defined by two vector fields f_1 and f_2. Elimination of u_1 and u_2 yields a linear systems in $\dot{x}$ with coefficients depending on x. If n is the dimension of x, we have then $n-2$ equations corresponding to a Pfaffian system of codimension 2, says

$$\omega_i \equiv \sum_{j=1}^{n} a_i^j(x)\ dx_j = 0, \quad i = 1, \ldots, n-2.$$

The $n-2$ differential forms ω_i are independent and such that $(\omega_i, f_1) = (\omega_i, f_2) = 0$. Equivalence of Pfaffian systems via changes of x-coordinates is an old question firstly stated by Pfaff. Weber [46], Goursat [14] and Cartan [4] have given the conditions of equivalence (around a generic x) to the contact system

$$dx_2 - x_3 dx_1 = 0, \quad dx_3 - x_4 dx_1 = 0, \quad \ldots dx_{n-1} - x_n dx_1 = 0.$$

The interest of such systems is mainly due to the following fact: their general solution is given by an arbitrary function of one variable $z \mapsto w(z)$ and a finite number of its derivatives:

$$x_1 = z, \quad x_2 = w(z), \quad , x_3 = \frac{dw}{dz}, \quad \ldots \quad , x_n = \frac{d^{n-2}w}{dz^{n-2}}.$$

This means that we can parameterize the general solution of $\dot{x} = f_1(x)u_1 + f_2u_2$ without integrating the control $t \mapsto u(t)$. Just consider the above relations with $t \mapsto z(t)$ any C^1 time function and any C^{n-2} function of z, $z \mapsto w(z)$. The quantities $x_1 = z(t)$ and $x_2 = w(z(t))$ play here a special role. We call them the flat output [7]. For trailers systems we have sketched here above similar parameterizations: they are based on Frénet formulas and written in a special way in order to exploit the invariant with respect to planar isometries.

Coordinate free characterization of contact systems was originally written in term of the derived flag and differential form. In a dual way, it reads for the two vector fields f_1 and f_2 as follows: the generic rank of E_k has to be equal to $k+2$ for $k = 0, \ldots, n-2$ where $E_0 := \text{span}\{f_1, f_2\}$, $E_{k+1} := \text{span}\{E_k, [E_k, E_k]\}$, $k \geq 0$. (see [13, 32, 30, 33] for complementary results on chained normal forms and singularities classification of Goursat systems).

This characterization is much more general. Cartan [3, 4] (see also [24] for dynamic feedback refinements) has proved that the E_k's characterize systems that can be solvable without any integration, a notion introduced few years earlier by Hilbert [16]. Hilbert considers the second order Monge equation

$$\frac{d^2y}{d^2x} = F\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}\right),$$

an under-determined differential system (a single differential equation relating two x-functions, $y(x)$ and $z(x)$). He wonders if its general solution can be expressed without integrals as for the first order Monge equation. Hilbert also shows that this question is related to the classification of under-determined systems under a transformations group much more general than punctual transformations. Hilbert proposes a nice analogy with the group of birational transformations and the classification of algebraic manifolds.

Hilbert's original idea of "integrallos Auflösung" can be extended to some infinite dimensional control systems. Such extensions require advances and delays operators in complement to derivation and are based on several examples such as the nonholonomic snake here below.

2.5 The nonholonomic snake

When the number of trailers is large, it is natural, as displayed on figure 6, to introduce the continuous approximation of the "non-holonomic snake". The trailers are now indexed by a continuous variable $l \in [0, L]$ and their positions are given by a map $[0, L] \ni l \mapsto M(l, t) \in \mathbb{R}^2$ satisfying the following partial differential equations:

$$\left\| \frac{\partial M}{\partial l} \right\| = 1, \quad \frac{\partial M}{\partial l} \wedge \frac{\partial M}{\partial t} = 0.$$

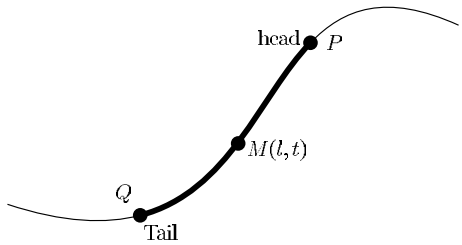

Figure 6: the non-holonomic snake, a car with an infinite number of small trailers.

The first equation says that $l \mapsto M(l,t)$ is an arc length parameterization. The second one is just the rolling without slipping conditions : velocity of trailer l is parallel to the direction of the plan of its wheels, i.e., the tangent to the curve $l \mapsto M(l,t)$. It is then obvious that the general solution of this system is

$$M(l,t) = P(s(t) - l), \quad l \in [0, L]$$

where P is the snake head and $s \mapsto P(s)$ an arc length parameterization of the curve followed by P. Similarly,

$$M(l,t) = Q(s(t) + l), \quad l \in [0, L]$$

where Q is the snake tail. It corresponds to the flat output of the finite dimensional approximation, the n-trailers system of figure 2, with n large and $d_i = L/n$. Derivatives up to order n are in the infinite case replaced by advances in the arc length scale. This results from the formal relation

$$Q(s + l) = \sum_{i \geq 0} Q^{(i)}(s) l^i / i!$$

and the series truncation up to the first n terms. Nevertheless, $M(l,t) = Q(s(t) + l)$ is much more simple to use in practice. When n is large, the series admit convergence troubles for $s \mapsto Q(s)$ smooth but not analytic.

When the number of trailers is large and the curvature radius $1/\kappa$ of $s \mapsto Q(s)$ is much larger than the length of each small trailer, such infinite dimensional approximation is valid. It reduces the dynamics to trivial delays. It is noteworthy that, in this case, an infinite dimensional description yields to a much better reduced model than a finite dimensional description that gives complex nonlinear control models and algorithms [1].

3 Pendulums and heavy chains

3.1 $2k\pi$ the juggling robot [20]

The robot $2k\pi$ is developed at École des Mines de Paris and consists of a manipulator carrying a pendulum, see figure 7. There are five degrees of freedom (dof's): 3 angles for the manipulator and 2 angles for the pendulum. The 3 dof's of the manipulator are actuated by electric drives, while the 2 dof's of the pendulum are *not* actuated.

This system is typical of underactuated, nonlinear and unstable mechanical systems such as the PVTOL [22], Caltech's ducted fan [31, 23], the gantry crane [7], Champagne flyer [19]. As shown in [21, 7, 23] the robot $2k\pi$ is flat, with the center of oscillation of the pendulum as a flat output. Let us recall some elementary facts

The cartesian coordinates of the suspension point S of the pendulum can be considered here as the control variables : they are related to the 3 angles of the manipulator θ_1, θ_2, θ_3 via static relations. Let us concentrated on the pendulum dynamics. This dynamics is similar to the ones of a punctual pendulum with the same mass m located at point H, the oscillation center (Huygens theorem). Denoting by $l = \|SH\|$ the length of the isochronous punctual pendulum, Newton equation and geometric constraints yield the following differential-algebraic system ($\vec{T}$ is the tension, see figure 8):

$$m\ddot{H} = \vec{T} + m\vec{g}, \quad \vec{SH} \wedge \vec{T} = 0, \quad \|SH\| = l.$$

[1] The finite dimensional system does not require to be flat (trailer i can be hitched to trailer $i-1$ not directly at the center of its wheel axle, but more realistically at a positive distance of this point [24]).

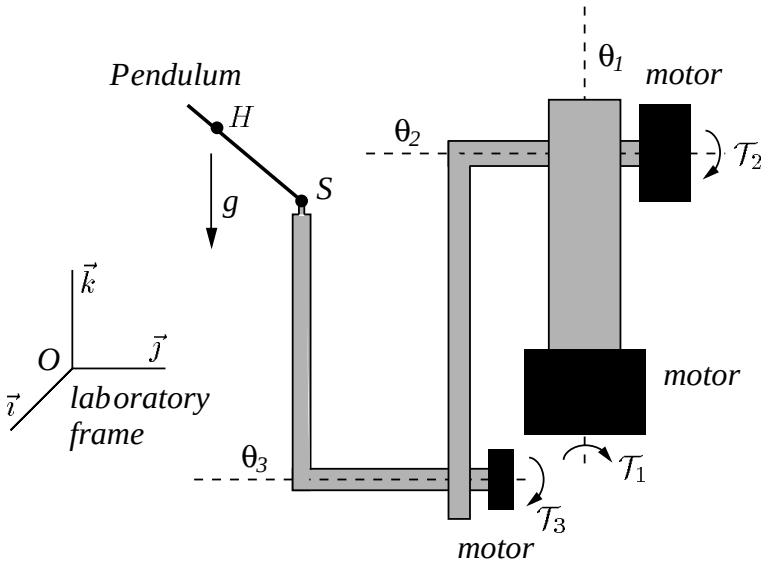

Figure 7: the robot $2k\pi$.

If, instead of setting $t \mapsto S(t)$, we set $t \mapsto H(t)$, then $\vec{T} = m(\ddot{H} - \vec{g})$. S is located at the intersection of the sphere of center H and radius l with the line passing through H of direction $\ddot{H} - \vec{g}$:

$$S = H \pm \frac{1}{\|\ddot{H} - \vec{g}\|} (\ddot{H} - \vec{g}).$$

These formulas are crucial for designing a control law steering the pendulum from the lower equilibrium to the upper equilibrium, and also for stabilizing the pendulum while the manipulator is moving around [20].

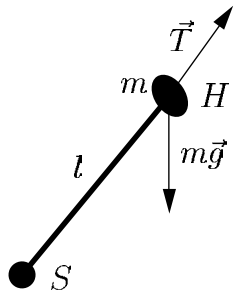

Figure 8: the isochronous pendulum.

3.2 Towed cable systems [29, 23]

This system consists of an aircraft flying in a circular pattern while towing a cable with a tow body (drogue) attached at the bottom. Under suitable conditions, the cable reaches a relative equilibrium in which the cable maintains its shape as it rotates. By choosing the parameters of the system appropriately, it is possible to make the radius at the bottom of the cable much smaller than the radius at the top of the cable. This is illustrated in Figure 9.

The motion of the towed cable system can be approximately represented using a finite element model in which segments of the cable are replaced by rigid links connected by spherical joints. The forces acting on the segment (tension, aerodynamic drag and gravity) are lumped and applied at the end of each rigid link. In addition to the forces on the cable, we must also consider the forces on the drogue and the towplane. The drogue is modeled as a sphere and essentially acts as a mass attached to the last link of the cable, so that the forces acting on it are included in the cable dynamics. The external forces on the drogue again consist of gravity and aerodynamic drag. The towplane is attached to the top of the cable and is subject to drag, gravity, and the force of the attached cable. For simplicity, we simply model the towplane as a pure force applied at the top of the cable. Our goal is to generate trajectories for this system that allow operation away from relative equilibria as well as transition between one equilibrium point and another. Due to the high dimension of the model for the system (128 states is typical), traditional approaches to solving

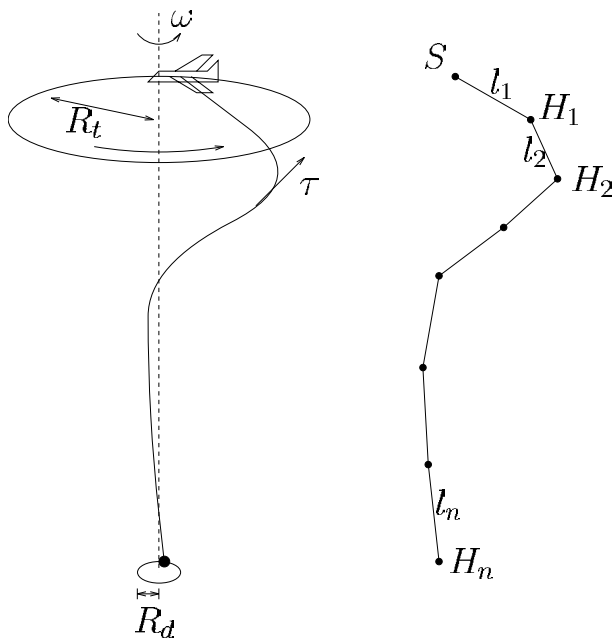

Figure 9: towed cable system and finite link approximate model.

this problem, such as optimal control theory, cannot be easily applied. However, it can be shown that this system is differentially flat using the position of the bottom of the cable H_n as the differentially flat output. See [29] for a more complete description and additional references.

Assume no friction and only gravity. Then, as for the pendulum of $2k\pi$, we have

$$H_{n-1} = H_n + \frac{1}{\|m_n \ddot{H}_n - m_n \vec{g}\|} \, (m_n \ddot{H}_n - m_n \vec{g})$$

where m_n is the mass of link n. Newton equation for link $n-1$ yields (with obvious notations)

$$H_{n-2} = H_{n-1} + \frac{1}{\|m_n \ddot{H}_n + m_{n-1} \ddot{H}_{n-1} - (m_n + m_{n-1})\vec{g}\|} \, (m_n \ddot{H}_n + m_{n-1} \ddot{H}_{n-1} - (m_n + m_{n-1})\vec{g}).$$

More generally, we have at link i

$$H_{i-1} = H_i + \frac{1}{\|\sum_i^n m_k (\ddot{H}_k - \vec{g})\|} \, \left(\sum_i^n m_k (\ddot{H}_k - \vec{g}) \right).$$

These relations imply that S is function of H_n and its time derivatives up to order $2n$. Thus H_n is the flat output. Let us consider now an infinite dimensional description. It could provide simpler computations, as for the nonholonomic snake and the car with n trailers.

3.3 Nonlinear heavy chain systems

The nonlinear conservative model of an homogenous heavy chain with an end mass is the following.

$$\begin{cases} \rho \dfrac{\partial^2 M}{\partial t^2} = \dfrac{\partial}{\partial s} \left(T \dfrac{\partial M}{\partial s} \right) + \rho \vec{g} \\ \left\| \dfrac{\partial M}{\partial s} \right\| = 1 \\ M(L,t) = u(t) \\ T(0,t) \dfrac{\partial M}{\partial s}(0,t) = m \dfrac{\partial^2 M}{\partial t^2}(0,t) - m\vec{g}. \end{cases} \tag{6}$$

where $[0, L] \ni s \mapsto M(s,t) \in \mathbb{R}^3$ is an arc length parameterization of the chain and $T(s,t) > 0$ is the tension. The control u is the position of the suspension point. If we use

$$N(s,t) = \int_0^s M(\sigma, t) \, d\sigma$$

instead of $M(s,t)$ (Bäcklund transformation) we have

$$\begin{aligned}
\rho\frac{\partial^2 N}{\partial t^2} &= T(s,t)\frac{\partial^2 N}{\partial s^2}(s,t) - T(0,t)\frac{\partial^2 N}{\partial s^2}(0,t) + \rho s\vec{g}\\
\left\|\frac{\partial^2 N}{\partial s^2}\right\| &= 1\\
\frac{\partial N}{\partial s}(L,t) &= u(t)\\
T(0,t)\frac{\partial^2 N}{\partial s^2}(0,t) &= m\frac{\partial^3 N}{\partial t^2 \partial s}(0,t) - m\vec{g}\\
N(0,t) &= 0.
\end{aligned}$$

Assume that the load trajectory is given

$$t \mapsto y(t) = \frac{\partial N}{\partial s}(0,t).$$

Then (we take the positive branch)

$$T(s,t) = \left\|\rho\frac{\partial^2 N}{\partial t^2}(s,t) - (\rho s + m)\vec{g} + m\ddot{y}(t)\right\|$$

and we have the Cauchy-Kovalevsky problem

$$\begin{aligned}
\frac{\partial^2 N}{\partial s^2}(s,t) &= \frac{1}{T(s,t)}\left(\rho\frac{\partial^2 N}{\partial t^2}(s,t) - (\rho s + m)\vec{g} + m\ddot{y}(t)\right)\\
N(0,t) &= 0\\
\frac{\partial N}{\partial s}(0,t) &= y(t).
\end{aligned}$$

Formally, its series solution expresses in terms of y and its derivatives of infinite order. This could be problematic since y must be analytic and the series converge for $s \geq 0$ small enough.

We will see here below that the solution of the tangent linearization of this Cauchy-Kovalevsky system around the stable vertical steady-state can be expressed via advances and delays of y. Such a formulation avoids series with y derivatives of arbitrary orders. For the nonlinear system here above, we conjecture a solution involving nonlinear delays and advances of y.

3.4 Linear heavy chain systems [35]

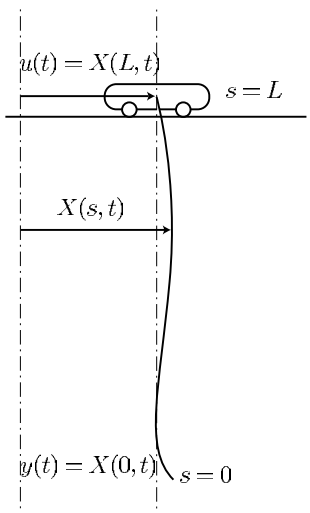

Figure 10: the homogeneous chain without any load.

Small angle approximation of (6) with $m = 0$ yields the following dynamics around the stable vertical steady-state:

$$\begin{cases} \frac{\partial}{\partial s}(gs\frac{\partial X}{\partial s}) - \frac{\partial^2 X}{\partial t^2} = 0 \\ X(L,t) = u(t). \end{cases} \tag{7}$$

where $X(s,t)$ is the horizontal coordinate of M. In this case, the vertical and horizontal dynamics are decoupled. The two horizontal dynamics are also decoupled. The tension T equals $g\rho s$. The control u is the trolley horizontal position.

We prove in [35] that the general solution of (7) is given by the following formulas where y is the free end position $X(0,t)$:

$$X(s,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} y(t + 2\sqrt{s/g}\sin\theta)\ d\theta. \tag{8}$$

Simple computations show that (8) corresponds to the series solution of the (singular) Cauchy-Kovalesky problem:

$$\begin{cases} \dfrac{\partial}{\partial s}(gs\dfrac{\partial X}{\partial s}) = \dfrac{\partial^2 X}{\partial t^2} \\ X(0,t) = y(t). \end{cases}$$

Relation (8) means that there is a one to one correspondence between the (smooth) solutions of (7) and the (smooth) functions $t \mapsto y(t)$. For each solution of (7), set $y(t) = X(0,t)$. For each function $t \mapsto y(t)$, set X via (8) and u via

$$u(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} y(t + 2\sqrt{L/g}\sin\theta)\ d\theta \tag{9}$$

to obtain a solution of (7).

Finding $t \mapsto u(t)$ steering the system from a steady-state $X \equiv 0$ to another one $X \equiv D$ becomes obvious. It just consists in finding $t \mapsto y(t)$ that is equal to 0 for $t \leq 0$ and to D for t large enough (at least for $t > 4\sqrt{L/g}$) and in computing u via (9).

For example take

$$y(t) = \begin{cases} 0 \text{ if } t < 0 \\ \frac{3L}{2}\left(\frac{t}{T}\right)^2\left(3 - 2\left(\frac{t}{T}\right)\right) \text{ if } 0 \leq t \leq T \\ \frac{3L}{2} \text{ if } t > T \end{cases}$$

where the chosen transfer time T equals 2Δ with $\Delta = 2\sqrt{L/g}$, the travelling time of a wave between $x = L$ and $x = 0$. For $t \leq 0$ the chain is vertical at position 0. For $t \geq T$ the chain is vertical at position $D = 3L/2$.

When $m > 0$, small angle approximation of (6) gives

$$\begin{cases} \dfrac{\partial}{\partial s}\left(g(s+a)\dfrac{\partial X}{\partial s}\right) - \dfrac{\partial^2 X}{\partial t^2} = 0 \\ \dfrac{\partial^2 X}{\partial t^2}(0,t) = g\dfrac{\partial X}{\partial s}(0,t) \\ X(L,t) = u(t) \end{cases}$$

where $a = m/\rho$ is homogenous to a length. We prove also in [35], that its general solution depends on advances and delays of y and its first derivative.

4 Diffusion and Gevrey functions

4.1 Finite dimensional models

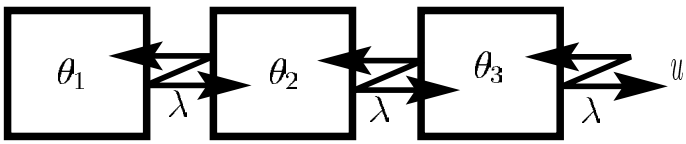

Figure 11: a finite volume model of a heating system.

Consider the 3-compartment model described on figure 11. Its dynamics is based on the following energy balance equations ((m, ρ, C_p, λ) are physical constants)

$$\begin{cases} m\rho C_p\ \dot{\theta}_1 = \lambda(\theta_2 - \theta_1) \\ m\rho C_p\ \dot{\theta}_2 = \lambda(\theta_1 - \theta_2) + \lambda(\theta_3 - \theta_2) \\ m\rho C_p\ \dot{\theta}_3 = \lambda(\theta_2 - \theta_3) + \lambda(u - \theta_3). \end{cases} \tag{10}$$

Its is obvious that this linear system is controllable with $y = \theta_1$ as Brunovsky output: it can be transformed via linear change of coordinates and linear static feedback into $y^{(3)} = v$.

Taking an arbitrary number n of compartments yields

$$\begin{cases} m\rho C_p\ \dot{\theta}_1 = \lambda(\theta_2 - \theta_1) \\ m\rho C_p\ \dot{\theta}_2 = \lambda(\theta_1 - \theta_2) + \lambda(\theta_3 - \theta_2) \\ \qquad\vdots \\ m\rho C_p\ \dot{\theta}_i = \lambda(\theta_{i-1} - \theta_i) + \lambda(\theta_{i+1} - \theta_i) \\ \qquad\vdots \\ m\rho C_p\ \dot{\theta}_{n-1} = \lambda(\theta_{n-2} - \theta_{n-1}) + \lambda(\theta_n - \theta_{n-1}) \\ m\rho C_p\ \dot{\theta}_n = \lambda(\theta_{n-1} - \theta_n) + \lambda(u - \theta_n). \end{cases} \tag{11}$$

$y = \theta_1$ remains the Brunovsky output: via linear change of coordinates and linear static feedback we have $y^{(n)} = v$.

When n tends to infinity, m and λ tend to zeros as $1/n$ and (11) tends to the classical heat equation (12) consider here below. We will see that the temperature on the opposite side to u, i.e., $y = \theta(0,t)$, still plays a special role.

4.2 Heat equation [18]

Consider the linear heat equation

$$\begin{cases} \partial_t\theta(x,t) = \partial_x^2\theta(x,t), \quad x \in [0,1] \\ \partial_x\theta(0,t) = 0 \\ \quad \theta(1,t) = u(t), \end{cases} \tag{12}$$

where $\theta(x,t)$ is the temperature and $u(t)$ is the control input. We claim that

$$y(t) := \theta(0,t)$$

is a "flat" output. Exchange the role of time t and space x and consider the following Cauchy-Kovalesky system

$$\begin{cases} \dfrac{\partial^2\theta}{\partial x^2} = \dfrac{\partial\theta}{\partial t} \\ \theta(0,t) = 0 \\ \dfrac{\partial\theta}{\partial x}(0,t) = 0 \end{cases} \tag{13}$$

Its series solution reads:

$$\begin{cases} \theta(x,t) = \displaystyle\sum_{i=1}^{\infty} \frac{y^{(i)}(t)}{(2i)!}\ x^{2i} \\ u(t) = \displaystyle\sum_{i=1}^{\infty} \frac{y^{(i)}(t)}{(2i)!}. \end{cases} \tag{14}$$

Whenever $t \mapsto y(t)$ is an arbitrary function (i.e., a trajectory of the trivial system $y = v$), $t \mapsto \big(\theta(x,t), u(t)\big)$ defined by (14) is a (formal) trajectory of (12), and vice versa. This is exactly the idea underlying our definition of flatness in [8]. Notice these calculations have been known for a long time, see [45, pp. 588 and 594].

To make the statement precise, we now turn to convergence issues. On the one hand, $t \mapsto y(t)$ must be a smooth function such that

$$\exists\ K, M > 0, \quad \forall i \geq 0, \forall t \in [t_0, t_1], \quad |y^{(i)}(t)| \leq M(Ki)^{2i}$$

to ensure convergence.

On the other hand $t \mapsto y(t)$ cannot in general be analytic. Indeed, if the system is to be steered from an initial temperature profile $\theta(x,t_0) = \alpha_0(x)$ at time t_0 to a final profile $\theta(x,t_1) = \alpha_1(x)$ at time t_1, equation (12) implies

$$\forall t \in [0,1], \forall i \geq 0, \quad y^{(i)}(t) = \partial_t^i\theta(0,t) = \partial_x^{2i}\theta(0,t),$$

and in particular

$$\forall i \geq 0, \qquad y^{(i)}(t_0) = \partial_x^{2i}\alpha_0(0) \quad \text{and} \quad y^{(i)}(t_1) = \partial_x^{2i}\alpha_1(1).$$

If for instance $\alpha_0(x) = c$ for all $x \in [0,1]$ (i.e., uniform temperature profile), then $y(t_0) = c$ and $y^{(i)}(t_0) = 0$ for all $i \geq 1$, which implies $y(t) = c$ for all t when the function is analytic. It is thus impossible to reach any final profile but $\alpha_1(x) = c$ for all $x \in [0,1]$.

Smooth functions $t \in [t_0, t_1] \mapsto y(t)$ that satisfy

$$\exists\ K, M > 0, \quad \forall i \geq 0, \qquad |y^{(i)}(t)| \leq M(Ki)^{\sigma i}$$

are known as Gevrey functions of order σ [38] (they are also closely related to class S functions [15]). The Taylor expansion of such functions is convergent for $\sigma \leq 1$ and divergent for $\sigma > 1$ (the larger σ is, the "more divergent" the Taylor expansion is). Analytic functions are thus Gevrey of order ≤ 1.

In other words we need a Gevrey function on $[t_0, t_1]$ of order > 1 but ≤ 2, with initial and final Taylor expansions imposed by the initial and final temperature profiles. With such a function, we can then compute open-loop control steering the system from one profile to the other by the formula (14).

For instance, we steered the system from uniform temperature 0 at $t = 0$ to uniform temperature 1 at $t = 1$ by using the function

$$\mathbb{R} \ni t \mapsto y(t) := \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 1 \\ \dfrac{\int_0^t \exp\big(-1/(\tau(1-\tau))^\gamma\big)d\tau}{\int_0^1 \exp\big(-1/(\tau(1-\tau))^\gamma\big)d\tau} & \text{if } t \in [0,1], \end{cases}$$

with $\gamma = 1$ (Gevrey order $1 + 1/\gamma$).

More details can be found in [18] where numerical tests indicate the practical interest of using Gevrey functions of order > 2 and divergente series.

4.3 Flexible beam (Euler-Bernoulli) [1, 10]

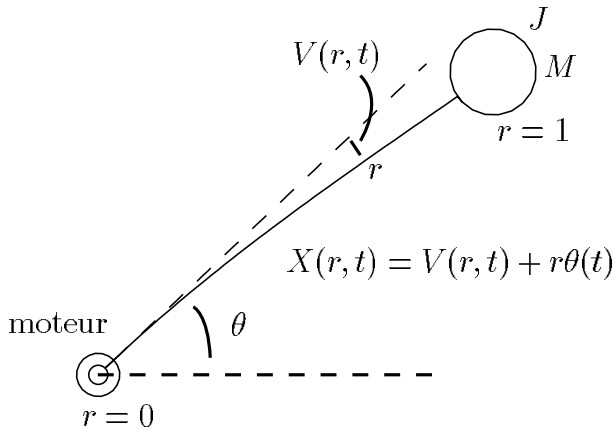

Figure 12: a flexible beam rotating around a control axle

Symbolic computations " à la Heaviside" with s instead of $\frac{\partial}{\partial t}$ are here important. We will not develop the formal aspect with Mikusiǹski operational calculus as in [10]. We just concentrate on the computations. We have the following $1D$ modelling:

$$\partial_{tt}X = -\partial_{xxxx}X$$

$$X(0,t) = 0, \quad \partial_x X(0,t) = \theta(t)$$
$$\ddot{\theta}(t) = u(t) + k\partial_{xx}X(0,t)$$

$$\partial_{xx}X(1,t) = -\lambda\partial_{ttx}X(1,t)$$
$$\partial_{xxx}X(1,t) = \mu\partial_{tt}X(1,t)$$

where the control is the motor torque u, $X(r,t)$ is the deformation profile, k, λ and μ are physical parameters (t and r are in reduced scales).

We will show that the general solution expresses in term of an arbitrary C^∞ function y (Gevrey order ≤ 2 for convergence):

$$X(x,t) = \sum_{n\geq 0} \frac{(-1)^n\ y^{(2n)}(t)}{(4n)!} P_n(x) + \sum_{n\geq 0} \frac{(-1)^n\ y^{(2n+2)}(t)}{(4n+4)!} Q_n(x) \tag{15}$$

with $\imath = \sqrt{-1}$,

$$P_n(x) = \frac{x^{4n+1}}{2(4n+1)} + \frac{(\Im - \Re)(1-x+\imath)^{4n+1}}{2(4n+1)} + \mu\Im(1-x+\imath)^{4n}$$

and

$$\begin{aligned} Q_n(x) = \frac{\lambda\mu}{2}(4n+4)(4n+3)(4n+2)\left((\Im-\Re)(1-x+\imath)^{4n+1} - x^{4n+1}\right) \\ - \lambda(4n+4)(4n+3)\Re(1-x+\imath)^{4n+2} \end{aligned}$$

($\Re$ and $\Im$ stand for real part and imaginary part). Notice that θ and u result from (15): it suffices to derive term by term.

We just show here how to get these formulas with $\lambda = \mu = 0$ (no inertia at the free end $r = 1$, $M = J = 0$). The method remains unchanged in the general case. The question is: how to get

$$X(x,t) = \sum_{n\geq 0} \frac{y^{(2n)}(t)(-1)^n}{(4n)!} \pi_n(x) \tag{16}$$

with

$$\pi_n(x) = \frac{x^{4n+1}}{2(4n+1)} + \frac{(\Im-\Re)(1-x+\imath)^{4n+1}}{2(4n+1)}.$$

With the Laplace variable s, we have the ordinary differential system

$$X^{(4)} = -s^2 X$$

where

$$X(0) = 0, \quad X^{(2)}(1) = 0, \quad X^{(3)}(1) = 0.$$

Derivatives are with respect to the space variable r and s is here a parameter. The general solution depends on an arbitrary constant, i.e., an arbitrary function of s, since we have 3 boundary conditions. With the 4 elementary solutions of $X^{(4)} = -s^2 X$

$$\begin{aligned} C_+(x) &= (\cosh((1-x)\sqrt{s}\xi) + \cosh((1-x)\sqrt{s}/\xi))/2 \\ C_-(x) &= (\cosh((1-x)\sqrt{s}\xi) - \cosh((1-x)\sqrt{s}/\xi))/(2\imath) \\ S_+(x) &= (\imath\sinh((1-x)\sqrt{s}\xi) + \sinh((1-x)\sqrt{s}/\xi))/(2\xi\sqrt{s}) \\ S_-(x) &= \xi(\imath\sinh((1-x)\sqrt{s}\xi) - \sinh((1-x)\sqrt{s}/\xi))/(2\sqrt{s}) \end{aligned}$$

where $\xi = \exp(\imath\pi/4)$, X reads

$$X(x) = aC_+(x) + bC_-(x) + cS_+(x) + dS_-(x).$$

The 3 boundary conditions provide 3 equations relating the constant a, b, c and d :

$$\begin{aligned} aC_+(0) + bC_-(0) + cS_+(0) + dS_-(0) &= 0 \\ sb &= 0 \\ sc &= 0. \end{aligned}$$

Thus $b = c = 0$ and we have just one relation between a and d

$$aC_+(0) + dS_-(0) = 0.$$

Since

$$C_+(0) = \Re(\cosh(\xi\sqrt{s}), \quad S_-(0) = \Im(\xi\sinh(\xi\sqrt{s}/\sqrt{s})$$

are entire functions of s very similar to $\cosh(\sqrt{s})$ and $\sinh\sqrt{s}/\sqrt{s}$ appearing for the heat equation (12), we can associate to them two operators, algebraically independent and commuting,

$$\delta_+ = C_+(0), \quad \delta_- = S_-(0).$$

They are in fact ultra-distributions belonging to the dual of Gevrey function of order less than ≤ 2 and with a punctual support [15]. We have thus a module generated by two elements (a, d) satisfying $\delta_+ a + \delta_- d = 0$. This is a $\mathbb{R}[\delta_+, \delta_-]$-module. This module is not free but δ_+-free [25]:

$$a = \delta_- y, \quad d = -\delta_+ y$$

with $y = -\delta_+^{-1} d$.

The basis y plays the role of flat output since

$$X(x) = (S_-(0)C_+(x) - S_-(x)C_+(0))y.$$

Simple but tedious computations using hyperbolic trigonometry formulas yield then to

$$X(x) = -\frac{1}{2}[S_-(x) + \Im(S_-(1 - x + \imath))]y.$$

The series of the entire function S_- provides (16). We conjecture that the quantity y admits a physical sense as an explicit expression with integrals of X over $r \in [0, 1]$ (center of flexion).

5 Conclusion

The above infinite dimensional examples can be completed by several other ones. For advance/delay parameterizations we have :

- water-tanks systems [5];
- telegraph equation [28, 9];
- flexible beams [27, 12];
- Burger equation and nonlinear delays [34] (see also [26] for flatness based control of nonlinear delay systems).

For series parameterization with Gevrey functions, we have

- tubular chemical reactors (multi-inputs case) [11, 42];
- heat equation with variable coefficients [39, 17].

All the above examples are 1D systems. Such explicit parameterization also exists for higher space dimensions. Take, e.g., the 2D wave equation corresponding, in the linear approximation, to the surface wave generated by the horizontal motions of a cylindric tank containing a fluid (linearized Saint-Venant equations (shallow water approximation)):

$$\begin{cases} \dfrac{\partial^2 \xi}{\partial t^2} = g\bar{h}\ \Delta\xi & \text{on } \Omega \\ g\nabla\xi \cdot n = -\ddot{D} \cdot n & \text{on } \partial\Omega \end{cases} \tag{17}$$

where Ω is the interior of a circle of radius R and of center $D(t) \in \mathbb{R}^2$, the control (n is the normal to the boundary $\partial\Omega$), $\bar{h} + \xi$ is the height of liquid, g is the gravity. A family of solutions of (17) is given by the following formulas ((r, θ) are the polar coordinates)

$$\xi(r, \theta, t) = \frac{1}{\pi}\sqrt{\frac{\bar{h}}{g}}\ \left(\int_0^{2\pi} \cos\alpha \left[\dot{a}\left(t - \frac{r\cos\alpha}{c}\right)\cos\theta + \dot{b}\left(t - \frac{r\cos\alpha}{c}\right)\sin\theta\right]\ d\alpha\right)$$

and ((u, v) are the Cartesian coordinates of D)

$$u = \frac{1}{\pi}\int_0^{2\pi} a\left(t - \frac{R\cos\alpha}{c}\right)\cos^2\alpha\ d\alpha, \quad v = \frac{1}{\pi}\int_0^{2\pi} b\left(t - \frac{R\cos\alpha}{c}\right)\cos^2\alpha\ d\alpha$$

where $t \mapsto (a(t), b(t)) \in \mathbb{R}^2$ is an arbitrary smooth function.

References

[1] Y. Aoustin, M. Fliess, H. Mounier, P. Rouchon, and J. Rudolph. Theory and practice in the motion planning and control of a flexible robot arm using mikusiński operators. In *Proc. of the Fifth IFAC Symposium on Robot Control*, pages 287–293, Nantes, France, 1997.

[2] G. Campion, B. d'Andrea Novel, and G. Bastin. Structural properties and classification of kinematic and dynamic models of wheeled mobile robots. *IEEE Trans. Robotics Automation*, 12(1):47–62, 1996.

[3] E. Cartan. Sur l'équivalence absolue de certains systèmes d'équations différentielles et sur certaines familles de courves. *Bull. Soc. Math. France*, 42:12–48, 1914. Also in Œuvres Complètes, part II, vol. 2, pp.1133–1168, CNRS, Paris, 1984.

[4] E. Cartan. Sur l'intégration de certains systèmes indéterminés d'équations différentielles. *J. für reine und angew. Math.*, 145:86–91, 1915. Also in Œuvres Complètes, part II, vol. 2, pp.1164–1174, CNRS, Paris, 1984.

[5] F. Dubois, N. Petit, and P. Rouchon. Motion planing and nonlinear simulations for a tank containing a fluid. In *European Control Conference, Karlsruhe*, 1999.

[6] M. Fliess, J. levine, P. Martin, F. Ollivier, and P. Rouchon. Controlling nonlinear systems by flatness. In C.I. Byrnes, B.N. Datta, D.S. Gilliam, and C.F. Martin, editors, *Systems and control in the Twenty-First Century*, Progress in Systems and Control Theory. Birkhauser, 1997.

[7] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. Flatness and defect of nonlinear systems: introductory theory and examples. *Int. J. Control*, 61(6):1327–1361, 1995.

[8] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems. *IEEE AC*, 44:922–937, 1999.

[9] M. Fliess, Ph. Martin, N. Petit, and P. Rouchon. Active signal restoration for the telegraph equation. In *CDC 99*, Phenix, december 1999.

[10] M. Fliess, H. Mounier, P. Rouchon, and J. Rudolph. Systèmes linéaires sur les opérateurs de Mikusiński et commande d'une poutre flexible. In *ESAIM Proc. "Élasticité, viscolélasticité et contrôle optimal", 8ème entretiens du centre Jacques Cartier, Lyon*, pages 157–168, 1996.

[11] M. Fliess, H. Mounier, P. Rouchon, and J. Rudolph. A distributed parameter approach to the control of a tubular reactor: A multi- variable case. In *Control and Decision Conference, Tampa*, pages 439–442, 1998.

[12] M. Fliess, H. Mounier, P. Rouchon, and J. Rudolph. Tracking control of a vibrating string with an interior mass viewed as delay system. *ESAIM: COCV(www.eamth.fr/cocv*, 3:315–321, 1998.

[13] A. Giaro, A. Kumpera, and C. Ruiz. Sur la lecture d'un résultat d'Élie Cartan. *C.R. Acad. Sc. Paris, Série A*, 287:241–244, 1978.

[14] E. Goursat. *Leçons sur le Problème de Pfaff.* Hermann, Paris, 1923.

[15] I.M. Guelfand and G.E. Chilov. *Les Distributions, tome 3.* Dunod, Paris, 1964.

[16] D. Hilbert. Über den Begriff der Klasse von Differentialgleichungen. *Math. Ann.*, 73:95–108, 1912. also in Gesammelte Abhandlungen, vol. III, pp. 81–93, Chelsea, New York, 1965.

[17] B. Laroche and Ph. Martin. Motion planning for a 1-d diffusion equation using a brunovsky-like decomposition. In *MTNS 2000*, page , 2000.

[18] B. Laroche, Ph. Martin, and P. Rouchon. Motion planning for a class of partial differential equations with boundary control. In *IEEE-CDC, Tampa*, pages 3494– 3497, 1998.

[19] C. Lemon and J.E. Hause. Design and initial flight test of the Champagne Flyer. In *Proc. 33nd IEEE Conf. Decision and Control*, pages 3852–3853, Lake Buena Vista, 1994.

[20] Y. Lenoir, , Ph. Martin, and P. Rouchon. $2k\pi$, the juggling robot. In *Control and Decision Conference, Tampa*, pages 1995–2000, 1998.

[21] Ph. Martin. *Contribution à l'étude des systèmes diffèrentiellement plats.* PhD thesis, École des Mines de Paris, 1992.

[22] Ph. Martin, S. Devasia, and B. Paden. A different look at output tracking: control of a VTOL aircraft. *Automatica*, 32:101–108, 1995.

[23] Ph Martin, R. Murray, and P. Rouchon. Flat systems. In *Proc. of the 4th European Control Conf.*, pages 211–264, Brussels, 1997. Plenary lectures and Mini-courses.

[24] Ph. Martin and P. Rouchon. Feedback linearization and driftless systems. *Math. Control Signal Syst.*, 7:235–254, 1994.

[25] H. Mounier. *Propriétés structurelles des systèmes linéaires à retards: aspects théoriques et pratiques.* PhD thesis, Université Paris Sud, Orsay, 1995.

[26] H. Mounier and J. Rudolph. Flatness based control of nonlinear delay systems: A chemical reactor example. *Int. Journal of Control*, 71:871–890, 1998.

[27] H. Mounier, J. Rudolph, M. Petitot, and M. Fliess. A flexible rod as a linear delay system. In *Proc. of the 3rd European Control Conf.*, pages 3676–3681, Rome, 1995.

[28] H. P. Rouchon Mounier and J. Rudolph. π-freeness of a long electric line. In *Proc. CESA '96 IMACS Multiconference, Lille, France*, pages 28–29, 1996.

[29] R. M. Murray. Trajectory generation for a towed cable flight control system. In *Proc. IFAC World Congress*, pages 395–400, San Francisco, 1996.

[30] R.M. Murray. Nilpotent bases for a class on nonintegrable distributions with applications to trajectory generation for nonholonomic systems. *Math. Control Signal Syst.*, 7:58–75, 1994.

[31] R.M. Murray. Caltech ducted fan reference manual. Technical report, California Institute of Technology, Division of Engineering and Applied Science, 1995.

[32] R.M. Murray and S.S. Sastry. Nonholonomic motion planning: Steering using sinusoids. *IEEE Trans. Automat. Control*, 38:700–716, 1993.

[33] W. Pasillas-Lépine. *Systèmes de Contact et Structures de Goursat: Théorie et Application au Contrôle des Systèmes Mécaniques non Holonomes.* PhD thesis, Université de Rouen, France, 2000.

[34] N. Petit, Y. Creff, and P. Rouchon. Motion planning for two classes of nonlinear systems with delays depending on the control. In *Control and Decision Conference, Tampa*, pages 1007– 1011, 1998.

[35] N. Petit and P. Rouchon. Motion planning for heavy chain systems. In *Wave 2000.* to appear.

[36] J.F. Pommaret. *Systems of Partial Differential Equations and Lie Pseudogroups.* Gordon & Breach, N.Y., 1978.

[37] J.F. Pommaret. Dualité différentielle et applications. *C.R. Acad. Sci. Paris, Série I*, 320:1225–1230, 1995.

[38] J.P. Ramis. Dévissage Gevrey. *Astérisque*, 59-60:173–204, 1979.

[39] R. Rothfuß, U. Becker, and R. Rudolph. Controlling a solenoid valve: a distributed parameter approach. In *MTNS 2000*, page , 2000.

[40] P. Rouchon, M. Fliess, J. Lévine, and Ph. Martin. Flatness and motion planning: the car with n-trailers. In *Proc. ECC'93, Groningen*, pages 1518–1522, 1993.

[41] P. Rouchon, M. Fliess, J. Lévine, and Ph. Martin. Flatness, motion planning and trailer systems. In *Proc. of the 32nd IEEE Conf. on Decision and Control*, pages 2700–2705, San Antonio, 1993.

[42] P. Rouchon and J. Rudolph. *Réacteurs chimiques différentiellement plats: planification et suivi de trajectoires.* Hermès. Partie II, Chapitre III.f, du livre Automatique et procédés chimiques, coordonnateur J.P. Corriou, à paraître.

[43] P. Rouchon and J. Rudolph. *Invariant tracking and stabilization: problem formulation and examples*, pages 261–273. Lecture Notes in Control and Information Sciences 246. Springer, 1999.

[44] D.M. Tilbury. *Exterior differential systems and nonholonomic motion planning.* PhD thesis, University of California, Berkeley, 1994. Memorandum No. UCB/ERL M94/90.

[45] G. Valiron. *Equations Fonctionnelles.* Masson et Cie, Editeurs, Paris, 2nd edition, 1950.

[46] E. von Weber. Zur Invarianten Theorie der Systeme Pfaff'scher Gleigungen. *Berichte Verhandlungen der Königlich Sächsichen Gesellshaft der Wissenschaften Mathematische-Physicalische Klasse, Leipzig*, 50:207–229, 1898.