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FLATNESS OF HEAVY CHAIN SYSTEMS*

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Abstract. In this paper the *flatness* [M. Fliess, J. Lévine, P. Martin, and P. Rouchon, *Internat. J. Control*, 61 (1995), pp. 1327–1361, M. Fliess, J. Lévine, P. Martin, and P. Rouchon, *IEEE Trans. Automat. Control*, 44 (1999), pp. 922–937] of heavy chain systems, i.e., trolleys carrying a fixed length heavy chain that may carry a load, is addressed in the partial derivatives equations framework. We parameterize the system trajectories by the trajectories of its free end and solve the motion planning problem, namely, steering from one state to another state. When considered as a finite set of small pendulums, these systems were shown to be flat [R. M. Murray, in *Proceedings of the IFAC World Congress*, San Francisco, CA, 1996, pp. 395–400]. Our study is an extension to the infinite dimensional case.

Under small angle approximations, these heavy chain systems are described by a one-dimensional (1D) partial differential wave equation. Dealing with this infinite dimensional description, we show how to get the explicit parameterization of the chain trajectory using (distributed and punctual) advances and delays of its free end.

This parameterization results from symbolic computations. Replacing the time derivative by the Laplace variable s yields a second order differential equation in the spatial variable where s is a parameter. Its fundamental solution is, for each point considered along the chain, an entire function of s of exponential type. Moreover, for each, we show that, thanks to the Liouville transformation, this solution satisfies, modulo explicitly computable exponentials of s, the assumptions of the Paley–Wiener theorem. This solution is, in fact, the transfer function from the flat output (the position of the free end of the system) to the whole state of the system. Using an inverse Laplace transform, we end up with an explicit motion planning formula involving both distributed and punctual advances and delays operators.

Key words. wave equation, delay systems, flatness, motion planning

AMS subject classification. 99C20

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Introduction. The notion of *flatness* [3, 4] has proven to be relevant in many problems where motion planning problems have been solved [10, 5]. The existence of a *flat output* is the key to explicit formulas that can be implemented as open-loop controllers. Many systems of engineering interest are flat. So far the dynamics under consideration have been nonlinear ordinary differential equations, constant of varying delay equations, or even partial differential equations. In these cases the open-loop controller expression involved algebraic computations, punctual advances and delays [11, 6, 12], distributed advance and delay operators [12, 5, 14, 16], composition of functions [15], etc. In this paper we use both distributed and punctual advances and delays operators.

The heavy chain systems under consideration in this paper are defined by a trolley carrying a fixed length heavy chain to which a load may be attached. The dynamics are studied in a fixed vertical plane. When approximated as a finite set of small pendulums, such heavy chain systems were shown to be flat (see [13]). Their trajectories can be explicitly parameterized by the trajectories of their free ends. These parameterizations involve numerous derivatives (twice as many as the number of pendulums). When this number goes to infinity, the derivative order goes to infinity as

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well, yielding series expansions. This makes these relations difficult to handle and to use in practice.

In order to overcome these difficulties, we consider infinite dimensional descriptions of heavy chain systems. Around the stable vertical steady-state and under the small angle assumption, the dynamics are described by second order ordinary differential equations (dynamics of the load at position $y(t)$) coupled with one-dimensional (1D) wave equations (dynamics of the chain $X(x,t)$), where wave speed depends on x, the spatial variable along the chain length.

This combined ordinary and partial differential equation description turns out to be a significant shortcut to an explicit motion planning formula. Instead of an infinite number of derivatives, the explicit parameterization of the trajectories involves a small number of both distributed and punctual advances and delays. The controllability of such hybrid systems could be analyzed via Hilbert's uniqueness method [8, 9], as done in [7]. The work presented here is also a constructive proof of the controllability of these systems in the sense that it provides the open-loop control for steering the system from any given state to any other state. In a real application it should be used as a feedforward term complemented by a closed-loop controller using the energy method as proposed in [2].

In the case of a single homogeneous heavy chain as depicted in Figure 1.1 (see section 1 for details), our explicit parameterization shows that the general solution of

$$\frac{\partial}{\partial x}\left(gx\frac{\partial X}{\partial x}\right) - \frac{\partial^2 X}{\partial t^2} = 0$$

is given by the integral

$$X(x,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} y(t + 2\sqrt{x/g}\sin\theta)\, d\theta, \tag{0.1}$$

where $t \mapsto y(t)$ is any smooth-enough time function: $X(0,t) = y(t)$ corresponds then to the free end position; the control $u(t) = X(L,t)$ is the trolley position.

For the general cases, we show here that relationships similar to (0.1) exist. They are expressed by (2.2) and (3.2). The structure is similar, but the moving averages involve weights (i.e., kernels) depending on the mass distribution. More precisely, given any mass distribution along the chain and any punctual mass at $x = 0$, we prove that there is a one-to-one correspondence between the trajectory of the load $t \mapsto y(t) = X(0,t)$ and the trajectory of the whole system (namely, the cable and the trolley): $t \mapsto X(x,t)$ and $t \mapsto u(t) = X(L,t)$. This correspondence yields the explicit parameterization of the trajectories: $X(x,\cdot) = \mathcal{A}_x y$, where $\{\mathcal{A}_x\}$ is a set of operators including time derivations, advances, and delays. In other words, $(x,t) \mapsto (\mathcal{A}_x y)(t)$ verifies the system equations for any smooth function $t \mapsto y(t)$. For each x, the operator $\mathcal{A}_x$ admits compact support. Thus it is possible to steer the system from any initial point to any other point in finite time.

This parameterization results from symbolic computations. Replacing the time derivative by the Laplace variable s yields a second order differential equation in x with s as a parameter. For each x, its fundamental solution A_x is an entire function of s of exponential type. Furthermore, for each x we show, thanks to the Liouville transformation, that $s \mapsto A_x(s)$ satisfies the assumptions of the Paley–Wiener theorem, modulo explicitly computable exponentials of s.

The paper is organized as follows.

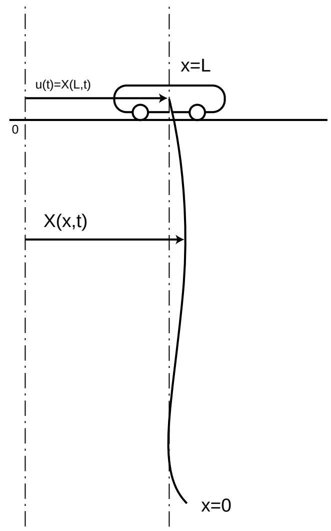

FIG. 1.1. *The homogeneous chain without any load.*

1. In section 1 we consider the case of a homogeneous chain without any load. Although it is the easiest case by far, it is explanatory, and it helps in understanding the meaning and control interest of our results.
2. In section 2 we address the case of an inhomogeneous chain without any load. The problem of the singularity at $x = 0$ of the second order differential equation receives special treatment. We prove the flatness of this system by Theorem 1.
3. In section 3 we solve the general problem of an inhomogeneous chain carrying a punctual load. By contrast with the previous case, the corresponding second order differential is not singular. Flatness of this system is proven by Theorem 2.

1. The homogeneous chain without any load. The computations are simple and explicit and summarize the goal of this paper.

Consider a heavy chain in stable position as depicted in Figure 1.1. Under the small angle approximation it is ruled by the dynamics[1]

$$\left\{\begin{aligned} &\frac{\partial}{\partial x}\left(gx\frac{\partial X}{\partial x}\right) - \frac{\partial^2 X}{\partial t^2} = 0, \\ &X(L,t) = u(t), \end{aligned}\right. \tag{1.1}$$

where $x \in [0, L]$, $t \in \mathbb{R}$, $X(x,t) - X(L,t)$ is the deviation profile, g is the gravitational acceleration, and the control u is the trolley position.

Thanks to the classical mapping $y = 2\sqrt{\frac{x}{g}}$, we get

$$y\frac{\partial^2 X}{\partial y^2}(y,t) + \frac{\partial X}{\partial y}(y,t) - y\frac{\partial^2 X}{\partial t^2}(y,t) = 0.$$

[1]This model was used in the historical work of D. Bernoulli on a heavy chain system where the zero-order Bessel functions appear for the first time; see [18, pp. 3–4].

Use Laplace transform of X with respect to the variable t (denoted by $\hat{X}$ and with zero initial conditions, i.e., $X(.,0) = 0$ and $\frac{\partial X}{\partial t}(.,0) = 0$) to get

$$y \frac{\partial^2 \hat{X}}{\partial y^2}(y,s) + \frac{\partial \hat{X}}{\partial y}(y,s) - ys^2 \hat{X}(y,s) = 0.$$

Less classically, the mapping $z = \imath s y$ gives

$$z \frac{\partial^2 \hat{X}}{\partial z^2}(z,s) + \frac{\partial \hat{X}}{\partial z}(z,s) + z\hat{X}(z,s) = 0. \tag{1.2}$$

This is a Bessel equation. Its solution writes in terms of J_0 and Y_0 the zero-order Bessel functions. Using the inverse mapping $z = 2\imath s\sqrt{\frac{x}{g}}$, we get

$$\hat{X}(x,s) = A\ J_0(2\imath s\sqrt{x/g}) + B\ Y_0(2\imath s\sqrt{x/g}).$$

Since we are looking for a bounded solution at $x = 0$, we have $B = 0$. Then

$$\hat{X}(x,s) = J_0(2\imath s\sqrt{x/g})\hat{X}(0,s), \tag{1.3}$$

where we can recognize the Clifford function $\mathcal{C}_\prime$ (see [1, p. 358]). Using Poisson's integral representation of J_0 [1, formula 9.1.18],

$$J_0(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(\imath z \sin\theta)\ d\theta,$$

we have

$$J_0(2\imath s\sqrt{x/g}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(2s\sqrt{x/g}\sin\theta)\ d\theta.$$

In terms of Laplace transforms, this last expression is a combination of delay operators. Turning (1.3) back into the time-domain, we get

$$X(x,t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(t + 2\sqrt{x/g}\sin\theta)\ d\theta \tag{1.4}$$

with $y(t) = X(0,t)$.

Relation (1.4) means that there is a one-to-one correspondence between the (smooth) solutions of (1.1) and the (smooth) functions $t \mapsto y(t)$. For each solution of (1.1), set $y(t) = X(0,t)$. For each function $t \mapsto y(t)$, set X by (1.4) and u as

$$u(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(t + 2\sqrt{L/g}\sin\theta)\ d\theta \tag{1.5}$$

to obtain a solution of (1.1).

Finding $t \mapsto u(t)$, steering the system from the steady-state $X \equiv 0$ at $t = 0$ to the other one $X \equiv D$ at $t = T$ becomes obvious. Our analysis shows that T must be larger than 2Δ, where $\Delta = 2\sqrt{L/g}$ is the travelling time of a wave between $x = L$ and $x = 0$. It consists only in finding $t \mapsto y(t)$ that is equal to 0 for $t \leq \Delta$ and to D for $t > T - \Delta$ and in computing u via (1.5).

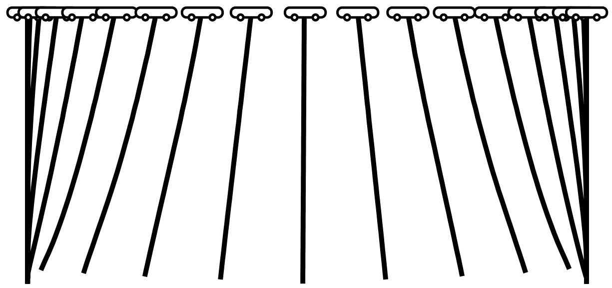

FIG. 1.2. *Steering from 0 to $3L/2$ in finite time $T = 4\Delta$. Regularly time-spaced positions of the heavy chain system are represented. The Matlab simulation code can be obtained from the second author via email.*

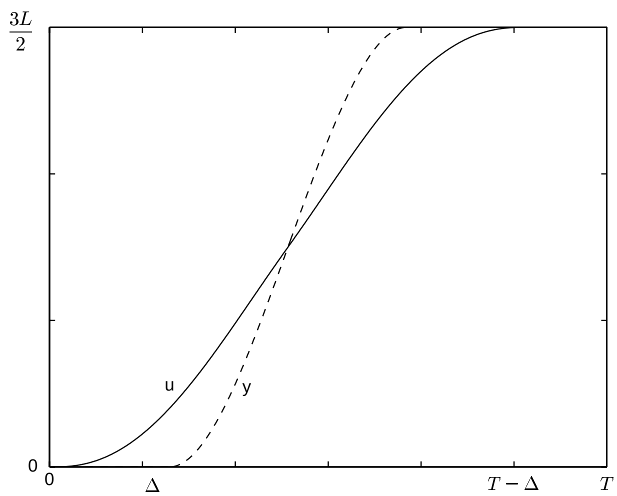

FIG. 1.3. *The steering control, trolley position u, and the "flat output," the free end y.*

Figure 1.2 illustrates computations based on (1.4) with

$$y(t) = \begin{cases} 0 \text{ if } t < \Delta, \\ \frac{3L}{2}\left(\frac{t-\Delta}{T-2\Delta}\right)^2 \left(3 - 2\left(\frac{t-\Delta}{T-2\Delta}\right)\right) \text{ if } \Delta \leq t \leq T - \Delta, \\ \frac{3L}{2} \text{ if } t > T - \Delta, \end{cases}$$

where the chosen transfer time T equals 4Δ. For $t \leq 0$ the chain is vertical at position 0. For $t \geq T$ the chain is vertical at position $D = 3L/2$.

Plots of Figure 1.3 show the control $[0, T] \ni t \mapsto u(t)$ required for such motion. Notice that the support of $\dot{u}$ is $[0, T]$, while the support of $\dot{y}$ is $[\Delta, T - \Delta]$. To be consistent with the small angle approximation, the horizontal acceleration of the end point $\ddot{y}$ must be much smaller than g. In our computations the maximum of $|\ddot{y}|$ is chosen rather large, $9g/16$. This is just for tutorial reasons. In practice, a reasonable transition time is $T = 5\Delta$ yielding $|\ddot{y}| \leq g/4$.

2. The inhomogeneous (i.e., variable section) chain without any load. Formula (1.4) can be extended to a heavy chain with variable section and carrying no load (see Figure 2.1). Such an extension deserves special consideration because of the singularity of the partial differential system at $x = 0$.

Such a system is governed by the equations

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\tau(x)\dfrac{\partial X}{\partial x}\right) - \dfrac{\tau'(x)}{g}\dfrac{\partial^2 X}{\partial t^2} = 0, \\ X(L,t) = u(t), \end{cases} \tag{2.1}$$

where $x \in [0, L]$, $t \in \mathbb{R}$, and u is the control. The tension of the chain is $\tau(x)$ with $\tau(0) = 0$ and $\tau(x) = gx + \mathcal{O}(x^2)$, while $\tau'(x)/g > 0$ is the mass distribution along the chain. Furthermore, we assume that there exists $a > 0$ such that $\tau(x) \geq ax \geq 0$.

THEOREM 1. *Consider* (2.1) *with* $[O, L] \ni x \mapsto \tau(x)$ *a smooth increasing function with* $\tau(0) = 0$ *and* $\tau' > 0$. *There is a one-to-one correspondence between the solutions* $[0, L] \times \mathbb{R} \ni (x, t) \mapsto (X(x, t), u(t))$ *that are* C^3 *in* t *and the* C^3 *functions* $\mathbb{R} \ni t \mapsto y(t)$ *via the formulas*

(2.2)
$$\begin{aligned} X(x,t) =& \frac{L^{1/4}\sqrt{g}}{2\pi^{3/2}(\tau(x)\tau'(x))^{1/4}}\sqrt{G(2\sqrt{\tau(x)/g})}\ \int_{-\pi}^{\pi} y\left(t + KG(2\sqrt{\tau(x)/g})\sin\theta\right) d\theta \\ &+ \frac{1}{(\tau(x)\tau'(x)/g)^{1/4}}\int_{-2\sqrt{\frac{\tau(x)}{ag}}}^{2\sqrt{\frac{\tau(x)}{ag}}} \mathcal{K}(G(2\sqrt{\tau(x)/g}), \xi)\ \dot{y}\,(t+\xi)\ d\xi, \\ u(t) =& X(L,t) \end{aligned}$$

with

$$y(t) = X(0, t),$$

where the constant K *and the functions* G *and* $\mathcal{K}$ *are defined by the function* τ *via formulas* (2.15) *and* (2.29).

The proof of this result is organized as follows.

1. A simple time-scaling simplifies the system. We shift from X to Y.
2. Symbolic computations where time derivatives are replaced by the Laplace variable s are performed.
3. The solution $Y(x, s)$ is factorized as $Y(x, s) = Y(0, s)A(x, s)$. A partial differential system is derived for $A(x, s)$.
4. A Liouville transformation is performed.
5. In these new coordinates the preceding transformed equation is compared to an equation that we have already solved in section 1, namely, the equation of a single homogeneous chain. We denote by $D(x, s)$ the difference between these two solutions.
6. $D(x, s)$ is proven to be an entire function of s and of exponential type.
7. A careful study of the Volterra equation satisfied by $D(x, s)$ shows that, for each x, the restriction to $D(x, s)/s$ to the imaginary axis is in L^2.
8. Thanks to the Paley–Wiener theorem, we prove that, for each x, $D(x, s)/s$ can be represented as a compact sum (discrete and continuous) of exponentials in s.
9. Gathering all the terms of $A(x, s)$, we get an expression involving the Bessel function J_0 (the solution for a homogeneous chain) and exponentials in s multiplied by s. This gives (2.2).

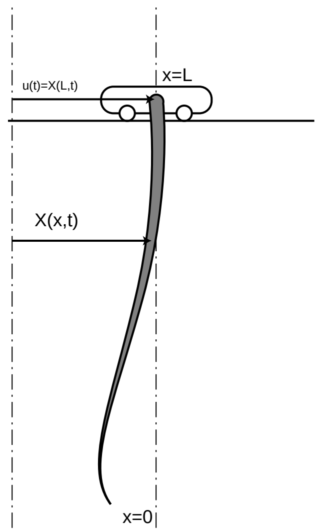

FIG. 2.1. *The inhomogeneous chain without any load.*

Proof. Simple change of coordinates Let[2] $Y(x,t) = X\left(\tau(x)/g, t\right)$.

[2]One may easily show the following result: if Y satisfies

$$\frac{\partial}{\partial x}\left(x\tau' \circ \tau^{-1}(gx)\frac{\partial Y}{\partial x}\right) - \frac{\partial^2 Y}{\partial t^2} = 0, \tag{2.3}$$

then $X(x,t) = Y(\tau(x)/g, t)$ satisfies

$$\frac{\partial}{\partial x}\left(\tau(x)\frac{\partial X}{\partial x}\right) - \frac{\tau'(x)}{g}\frac{\partial^2 X}{\partial t^2} = 0. \tag{2.4}$$

To show this, denote $\circ$ the composition operator with respect to the first variable. Thus $X = Y \circ (\tau/g)$. Then

$$\frac{\partial}{\partial x}\left(\tau\frac{\partial X}{\partial x}\right) = \frac{\partial}{\partial x}\left(\tau\tau'/g\frac{\partial Y}{\partial x} \circ (\tau/g)\right). \tag{2.5}$$

On the other hand, a factorization of (2.3) gives

$$\begin{aligned}\frac{\partial^2 Y}{\partial t^2} &= \frac{\partial}{\partial x}\left(\left(\tau/g\tau'\frac{\partial Y}{\partial x} \circ (\tau/g)\right) \circ \tau^{-1}(gx)\right)\\ &= \frac{\partial}{\partial x}\left(\tau^{-1}(gx)\right)\frac{\partial}{\partial x}\left(\tau\tau'/g\frac{\partial Y}{\partial x} \circ (\tau/g)\right) \circ \tau^{-1}(gx).\end{aligned}$$

So by using (2.5)

$$\frac{\partial}{\partial x}\left(\tau^{-1}(gx)\right)\frac{\partial}{\partial x}\left(\tau\frac{\partial X}{\partial x}\right) \circ \tau^{-1}(gx) = \frac{\partial^2 Y}{\partial t^2}.$$

Yet

$$\frac{\partial}{\partial x}\left(\tau^{-1}(gx)\right) = \frac{g}{\tau' \circ \tau^{-1}(gx)},$$

so

$$\frac{\partial}{\partial x}\left(\tau\frac{\partial X}{\partial x}\right) \circ \tau^{-1}(gx) = \frac{1}{g}\tau' \circ \tau^{-1}(gx)\frac{\partial^2 Y}{\partial t^2},$$

or, equivalently,

$$\frac{\partial}{\partial x}\left(\tau\frac{\partial X}{\partial x}\right) = \frac{\tau'}{g}\frac{\partial^2 Y}{\partial t^2} \circ (\tau/g) = \frac{\tau'}{g}\frac{\partial^2 X}{\partial t^2},$$

which gives the conclusion.

Now (2.1) gives

$$\frac{\partial}{\partial x}\left(\tau_1(x)\frac{\partial Y}{\partial x}\right) - \frac{\partial^2 Y}{\partial t^2} = 0, \tag{2.6}$$

where $\tau_1(x) = x\tau'(\tau^{-1}(gx))$.

Symbolic computations. Replacing the time derivation by s gives

$$\frac{\partial}{\partial x}\left(\tau_1(x)\frac{\partial Y}{\partial x}\right) - s^2 Y = 0. \tag{2.7}$$

Factorization. It is very easy to check that $Y(x,s) = Y(0,s)A(x,s)$ is the solution of (2.7), provided that $A(x,s)$ is solution of the following partial differential system:

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\tau_1(x)\dfrac{\partial A}{\partial x}\right) - s^2 A = 0, \\ \quad A(0,s) = 1. \end{cases} \tag{2.8}$$

Existence of a solution. System (2.8) admits a smooth solution that is an entire function of exponential type in s. This solution reads

$$A(x,s) = \sum_{i\geq 0} \frac{s^{2i}}{i!} f_i(x), \tag{2.9}$$

where

$$\begin{cases} \quad f_0 = 1, \\ f_i(x) = \displaystyle\int_0^x \frac{1}{\tau_1(l)} \int_0^l i f_{i-1}(s) ds \; dl. \end{cases} \tag{2.10}$$

It is very easy to check that, formally, $\sum_{i\geq 0} \frac{s^{2i}}{i!} f_i(x)$ is solution of (2.8): since

$$\frac{\partial}{\partial x}\left(\tau_1(x)\frac{\partial}{\partial x} f_i(x)\right) = i f_{i-1}(x),$$

we can write

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\tau_1(x)\dfrac{\partial}{\partial x}\displaystyle\sum_{i\geq 0}\frac{s^{2i}}{i!} f_i(x)\right) = s^2 \displaystyle\sum_{i\geq 0}\frac{s^{2i}}{i!} f_i(x), \\ \qquad \displaystyle\sum_{i\geq 0}\frac{s^{2i}}{i!} f_i(0) = f_0(0) = 1. \end{cases} \tag{2.11}$$

Now let us address the convergence by proving that for all i

$$|f_i(x)| \leq \frac{1}{i!}\left(\frac{x}{a}\right)^i. \tag{2.12}$$

Suppose that (2.12) is true for a given i. (It is obviously the case for $i = 0$.) Let us inductively prove that it is also true for $i+1$. From (2.10) we get

$$|f_{i+1}(x)| \leq \int_0^x \frac{l^{i+1}}{\tau_1(l) a^i i!} dl.$$

Yet $\tau' \geq a$, so $\tau_1(x) \geq ax \geq 0$, and then

$$\begin{aligned}|f_{i+1}(x)| &\leq \int_0^x \frac{l^i}{a^{i+1} i!} dl \\ &\leq \frac{1}{(i+1)!}\left(\frac{x}{a}\right)^{i+1},\end{aligned}$$

which is (2.12) at rank $i+1$.

So, gathering (2.9) and (2.12) and using $\frac{1}{(i!)^2} \leq \frac{2^{2i}}{(2i)!}$, we get

$$A(x,s) \leq \sum_{i\geq 0} \frac{s^{2i} x^i}{(i!)^2 a^i} \leq \sum_{i\geq 0} \frac{s^{2i} 2^{2i} x^i}{(2i)! a^i} \leq \exp\left(2s\sqrt{\frac{x}{a}}\right). \tag{2.13}$$

This proves that, for each x, $s \mapsto A(x,s)$ is an entire function of s of exponential type.

Liouville transformation. The Liouville transformation

$$(x, A) \mapsto (z, u)$$

(see, e.g., [19, p. 110]) turns equations of the form

$$\frac{d}{dx}\left(p(x)\frac{dA}{dx}\right) + (\lambda r(x) - q(x))\, A = 0$$

with $p(x) > 0$ into

$$\frac{d^2 u}{dz^2} + (\rho^2 - h(z))u = 0,$$

where ρ is depending only on λ and can be considered as a parameter.

Here

$$p(x) = \tau_1(x), \quad \lambda = -s^2, \quad r(x) = 1, \quad q(x) = 0, \quad x \in [0, L],$$

and the transformation is defined for each $x > 0$. Nevertheless, it can be extended to $x = 0$ because around 0, $\tau_1(x) \approx gx$ with $g > 0$. It turns (2.8) into

$$\frac{d^2 u}{dz^2} - K^2 s^2 u = \bar{h}(z) u \tag{2.14}$$

with

$$z = \frac{1}{K}\int_0^x \sqrt{\frac{1}{\tau_1}} \equiv G(2\sqrt{x}), \quad K = \frac{1}{\pi}\int_0^L \sqrt{\frac{1}{\tau_1}}, \tag{2.15}$$

$$u(z,s) = (\tau_1(x))^{1/4} A(x,s), \tag{2.16}$$

$$\bar{h}(z) = \frac{F''(z)}{F(z)} \quad \text{with } F(z) \equiv (\tau_1(x))^{1/4}. \tag{2.17}$$

Notice that since $\tau_1(x) \geq ax$ with $a > 0$, $\int_0^x 1/\tau_1$ is a smooth function of $\sqrt{x}$, and thus G is well defined and invertible. Similar arguments imply that $\bar{h}$ is, in fact, a function of z^2. Thus $\bar{h}(z) = h(z^2)$, and we have the following Laurent series around 0:

$$\bar{h}(z) = h(z^2) = \frac{-1}{4z^2} + \mathcal{O}(1).$$

Comparison to a simpler solution. We know from [1, formula 9.1.49, p. 362] that

$$u_0(z,s) = (Lg)^{1/4}\sqrt{\frac{z}{\pi}}\, J_0(iKsz) \tag{2.18}$$

satisfies

$$\frac{d^2u_0}{dz^2} - K^2s^2u_0 = \left(\frac{-1}{4z^2}\right)u_0. \tag{2.19}$$

According to the Laurent series of $\bar{h}$, we compare the solutions of (2.14), namely, $u(z,s)$, and (2.19), namely, $u_0(z,s)$. Let $D(z,s) = u(z,s) - u_0(z,s)$. We deduce from (2.14) and (2.19) that

$$\frac{d^2D}{dz^2} - K^2s^2D = \left(h(z^2) + \frac{1}{4z^2}\right)u_0 + h(z^2)D. \tag{2.20}$$

Since $z = G(2\sqrt{x})$ with G smooth and invertible, we have from (2.9) and (2.16)

$$u(z,s) = (Lg)^{1/4}\sqrt{\frac{z}{\pi}} + \mathcal{O}(z^{5/2}).$$

Then it is easy to check that for each s, D is a C^1 function of z around 0 with $D(0,s) = 0$ and $D'(0,s) = 0$. Equation (2.20) can be turned into the following integral equation (see [19, p. 111]):

$$\begin{aligned} D(z,s) =& \frac{1}{Ks}\int_0^z \sinh(Ks(z-t))\left(h(z^2) + \frac{1}{4t^2}\right)u_0(t,s)dt \\ &+ \frac{1}{Ks}\int_0^z \sinh(Ks(z-t))h(t^2)D(t,s)dt. \end{aligned} \tag{2.21}$$

Proving that $\mathbb{C} \ni s \mapsto D(z,s)$ *is an entire function of exponential type.* We already know that $A(x,s)$ and thus $u(z,s)$ (by (2.16)) are entire functions of exponential type in s. On the other hand, for each z, $s \mapsto u_0(z,s)$ is also an entire function of s of exponential type as J_0 is. This gives the conclusion.

Proving that $i\mathbb{R} \ni s \mapsto D(z,s)/s$ *belongs to* L^2. For each z, we need only an estimation of $D(z,iw)$ as w tends to ∞. For the sake of simplicity, we consider here $w \mapsto D(z,iw)$ for $w > 0$ large enough. The case $w < 0$ is similar. Classically (see, for instance, [19, p. 112]), let $M(z,w) = \sup_{0\le\zeta\le z}|D(\zeta,iw)|$. Using (2.21), we will get an estimation of $M(z,w)$. This gives

$$KwM(z,w) \le I_1(z,w) + I_2(z,w) \tag{2.22}$$

with

$$\begin{aligned} I_1(z,w) &= \int_0^z \left|h(t^2) + \frac{1}{4t^2}\right| |u_0(t,iw)|\, dt, \\ I_2(z,w) &= \int_0^z \left|h(t^2)\right| |D(t,iw)|\, dt. \end{aligned}$$

We know that

$$0 \le z \le \pi, \quad |u_0(t,iw)| \le (Lg)^{1/4}$$

since J_0 is bounded by 1 on the real axis. We know also that $h(t^2)+1/4t^2$ is bounded on $[0,\pi]$. Thus the integral I_1 is bounded by a constant $K_1 > 0$, independent of $z \in [0,\pi]$ and w,

$$I_1(z,w) \leq K_1. \tag{2.23}$$

Next, to majorate I_2 we split it into

$$I_2(z,w) = \underbrace{\int_0^{\gamma/w} \left|h(t^2)\right| |D(t,iw)|\, dt}_{I_2'(z,w)} + \underbrace{\int_{\gamma/w}^{z} \left|h(t^2)\right| |D(t,iw)|\, dt}_{I_2''(z,w)},$$

where $\gamma > 0$ is a parameter we will choose afterwards. A simple but quite tedious computation gives (using $J_0(z) = 1 - \frac{1}{4}z^2 + \circ(z^2)$)

$$D(z,s) = \sqrt{z}\; cs^2z^2(1+\mu(s^2z^2)),$$

where c is a constant and μ is a smooth function such that $\mu(0) = 0$. Using this last expression in I_2', we get

$$I_2'(z,w) \leq \sqrt{w}\frac{bc}{6}\gamma^{3/2}\left(1 + \sup_{|\xi|\leq\gamma^2} |\mu(\xi)|\right), \tag{2.24}$$

where $b > 0$ is such that $\left|h(t^2)\right| \leq b/(4t^2)$ for all $t \in]0,\pi]$. On the other hand, it is easy to check that

$$I_2''(z,w) \leq \frac{bw}{4\gamma}M(z,w). \tag{2.25}$$

Gathering (2.24) and (2.25), we get

$$I_2(z,w) \leq \sqrt{w}\frac{bc}{6}\gamma^{3/2}\left(1 + \sup_{|\xi|\leq\gamma^2} |\mu(\xi)|\right) + \frac{bw}{4\gamma}M(z,w). \tag{2.26}$$

Thanks to the majorations (2.23) and(2.26), we get

$$KwM(z,w) \leq K_1 + \sqrt{w}\frac{bc}{6}\gamma^{3/2}\left(1 + \sup_{|\xi|\leq\gamma^2} |\mu(\xi)|\right) + \frac{bw}{4\gamma}M(z,w).$$

This majoration is valid for $z \in]0,\pi]$, $w > 0$, and $\gamma > 0$ such that $\gamma/w \leq z$. Now we take

$$\gamma = \frac{b}{2K}.$$

Thus for each $z > 0$ and each $w > \gamma/z$, we have

$$(K - b/4\gamma)wM(z,w) \leq K_1 + \sqrt{w}\frac{bc}{6}\gamma^{3/2}\left(1 + \sup_{|\xi|\leq\gamma^2} |\mu(\xi)|\right).$$

Since $K - b/4\gamma = K/2$, we have

$$\frac{1}{2}\; KwM(z,w) \leq K_1 + \sqrt{w}\frac{bc}{6}\gamma^{3/2}\left(1 + \sup_{|\xi|\leq\gamma^2} |\mu(\xi)|\right). \tag{2.27}$$

Thus there exists $C_0 > 0$ such that for each $z \in]0,\pi]$ and for every $w > \gamma/z$,

$$|D(z,iw)| \leq \frac{C_0}{\sqrt{|w|}}. \tag{2.28}$$

Since $D(z,0) \equiv 0$, we deduce for each $z > 0$ that $s \mapsto D(z,s)/s$ remains an entire function of s (of exponential type), and the above majoration says that $i\mathbb{R} \ni s \mapsto D(z,s)/s$ belongs to L^2.

Using the Paley–Wiener theorem. The Paley–Wiener theorem [17, p. 375] ensures that, for any $z \in [0,\pi]$, there exists $[-\frac{G^{-1}(z)}{\sqrt{a}}, \frac{G^{-1}(z)}{\sqrt{a}}] \ni t \mapsto \mathcal{K}(z,t)$ in L^2 such that

$$D(z,s)/s = \int_{-\frac{G^{-1}(z)}{\sqrt{a}}}^{\frac{G^{-1}(z)}{\sqrt{a}}} \mathcal{K}(z,\xi)\exp(s\xi)d\xi. \tag{2.29}$$

The integral bounds results from the following facts.

1. Via (2.16), $2\sqrt{x} = G^{-1}(z)$, and (2.13), we have

$$\forall s \in \mathbb{C}, \quad |(u(z,s)| \leq N(z)\exp\left(|s|\,\frac{G^{-1}(z)}{\sqrt{a}}\right)$$

   for some $N(z) > 0$.
2. A well-known property on J_0 implies that

$$\forall s \in \mathbb{C}, \quad |(u_0(z,s)| \leq N_0(z)\exp(|s|\,zK)$$

   for some $N_0(z) > 0$.
3. Since $\tau_1 x \geq ax$, (2.15) implies that $zK < \frac{G^{-1}(z)}{\sqrt{a}}$.
4. Thus

$$\forall s \in \mathbb{C}, \quad |D(z,s)| = |u(z,s) - u_0(z,s)| \leq (N(z)+N_0(z))\exp\left(|s|\,\frac{G^{-1}(z)}{\sqrt{a}}\right).$$

Conclusion.

$$(u(z,s) - u_0(z,s))/s = \int_{-\frac{G^{-1}(z)}{\sqrt{a}}}^{\frac{G^{-1}(z)}{\sqrt{a}}} \mathcal{K}(z,\xi)\exp(s\xi)d\xi.$$

This gives

$$u(z,s) = \frac{(Lg)^{1/4}}{\sqrt{\pi}}\sqrt{z}J_0(iKsz) + \int_{-\frac{G^{-1}(z)}{\sqrt{a}}}^{\frac{G^{-1}(z)}{\sqrt{a}}} s\mathcal{K}(z,\xi)\exp(s\xi)d\xi.$$

Pulling back this relation in the (x,A) coordinates, we deduce using (2.16) that

$$\begin{aligned} A(x,s) = &\frac{(Lg)^{1/4}}{\sqrt{\pi}}\frac{1}{(\tau_1(x))^{1/4}}\sqrt{G(2\sqrt{x})}J_0(iKsG(2\sqrt{x})) \\ &+ \frac{1}{(\tau_1(x))^{1/4}}\int_{-2\sqrt{\frac{x}{a}}}^{2\sqrt{\frac{x}{a}}} s\mathcal{K}(G(2\sqrt{x}),\xi)\exp(s\xi)d\xi. \end{aligned}$$

Then we quickly get $Y(x,s) = Y(0,s)A(x,s)$. This gives in the time domain

$$\begin{aligned} Y(x,t) =& \frac{(Lg)^{1/4}}{\sqrt{\pi}} \frac{1}{(\tau_1(x))^{1/4}} \sqrt{G(2\sqrt{x})}\ \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(0, t + KG(2\sqrt{x})\sin\theta) d\theta \\ &+ \frac{1}{(\tau_1(x))^{1/4}} \int_{-2\sqrt{\frac{x}{a}}}^{2\sqrt{\frac{x}{a}}} \mathcal{K}(G(2\sqrt{x}),\xi) \left[\frac{\partial}{\partial t} Y(0,t+\xi)\right] d\xi. \end{aligned}$$

Then substituting

$$\begin{aligned} X(x,t) &= Y\left(\tau(x)/g, t\right), \\ Y(0,t) &= X(0,t), \\ \frac{\partial Y}{\partial t}(0,t) &= \frac{\partial X}{\partial t}(0,t), \end{aligned}$$

we get

(2.30)

$$\begin{aligned} X(x,t) =& \frac{L^{1/4}\sqrt{g}}{2\pi^{3/2}(\tau(x)\tau'(x))^{1/4}} \sqrt{G(2\sqrt{\tau(x)/g})}\ \int_{-\pi}^{\pi} y(t + KG(2\sqrt{\tau(x)/g})\sin\theta) d\theta \\ &+ \frac{1}{(\tau(x)\tau'(x)/g)^{1/4}} \int_{-2\sqrt{\frac{\tau(x)}{ag}}}^{2\sqrt{\frac{\tau(x)}{ag}}} \mathcal{K}(G(2\sqrt{\tau(x)/g}),\xi)\ \dot{y}\left(t+\xi\right)\ d\xi \end{aligned}$$

with $y(t) = X(0,t)$. □

Remark. In the case of a homogeneous chain, we can substitute

$$\begin{aligned} &\tau(x) = gx, \quad \tau'(x) = g, \quad \tau_1(x) = gx = \tau(x), \\ &K = \frac{2}{\pi}\sqrt{\frac{L}{g}}, \quad z = G(2\sqrt{x}) = \pi\sqrt{\frac{x}{L}}, \mathcal{K} = 0, \end{aligned}$$

and (2.30) reads

$$X(x,t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} y\left(t + 2\sqrt{\frac{x}{g}}\sin\theta\right) d\theta,$$

which is indeed identical to (1.4).

3. The inhomogeneous chain with punctual load. The system of Figure 3.1 consists of a heavy chain with a variable section carrying a punctual load m. Small deviations $X(x,t) - u(t)$ from the vertical position are described by the partial differential system

$$\text{(3.1)} \qquad \begin{cases} \dfrac{\partial}{\partial x}\left(\tau(x)\dfrac{\partial X}{\partial x}\right) - \dfrac{\tau'(x)}{g}\dfrac{\partial^2 X}{\partial t^2} = 0, \\ \dfrac{\partial^2 X}{\partial t^2}(0,t) = g\dfrac{\partial X}{\partial x}(0,t), \\ X(L,t) = u(t), \end{cases}$$

where u is the control. The tension in the chain writes $\tau(x)$: $\tau(0) = mg$, and $\tau'(x)/g > 0$ is the mass distribution along the chain.

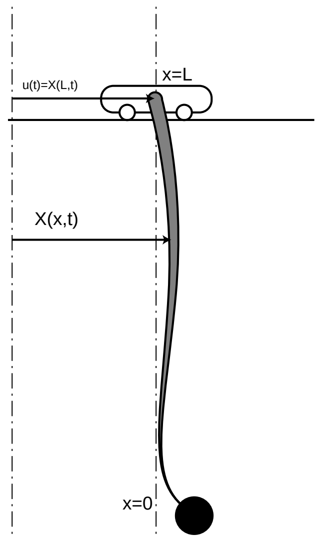

FIG. 3.1. *The inhomogeneous (variable section) chain with punctual load.*

THEOREM 2. *Consider* (3.1) *with* $[0, L] \ni x \mapsto \tau(x)$ *a smooth increasing function with* $\tau(0) = m$*. There is a one-to-one correspondence between the solutions* $[0, L] \times \mathbb{R} \ni (x, t) \mapsto (X(x, t), u(t))$ *that are* C^3 *in* t *and the* C^3 *functions* $\mathbb{R} \ni t \mapsto y(t)$ *via the following formulas:*

$$\left\{ \begin{aligned} X(x,t) &= \phi(x)\ [y(t+\theta(x)) + y(t-\theta(x))] + \psi(x)\ [\dot{y}(t+\theta(x)) - \dot{y}(t-\theta(x))] \\ &\qquad + \int_0^x \mathcal{B}(x,\xi)[y(t+\theta(\xi)) + y(t-\theta(\xi))]\ d\xi, \\ u(t) &= X(L,t) \end{aligned} \right. \tag{3.2}$$

with

$$\begin{aligned} y(t) &= X(0,t), \\ \theta(x) &= \int_0^x \sqrt{\frac{\tau'}{g\tau}}, \\ \psi(x) &= \left(\frac{\tau(0)\tau'(0)}{\tau(x)\tau'(x)}\right)^{\frac{1}{4}} \frac{1}{2}\sqrt{\frac{\tau(0)}{g\tau'(0)}}, \\ \phi(x) &= \left(\frac{\tau(0)\tau'(0)}{\tau(x)\tau'(x)}\right)^{\frac{1}{4}} \cdots \\ &\quad \times \left[1 + \frac{1}{8}\sqrt{\frac{\tau(0)}{\tau'(0)}} \left(\left(\sqrt{\frac{\tau'}{\tau}} + \frac{\tau''}{\tau'}\sqrt{\frac{\tau}{\tau'}}\right)(x) - \left(\sqrt{\frac{\tau'}{\tau}} + \frac{\tau''}{\tau'}\sqrt{\frac{\tau}{\tau'}}\right)(0) \right.\right. \\ &\qquad\qquad \left.\left. + \cdots + \frac{1}{4}\int_0^x \left(\sqrt{\frac{\tau'}{\tau}} + \frac{\tau''}{\tau'}\sqrt{\frac{\tau}{\tau'}}\right)^2 \sqrt{\frac{\tau'}{\tau}} \right)\right], \end{aligned}$$

$B(x,\xi)$ a smooth function of x, and ξ defined by the function τ via formula (3.15).

Correspondence (3.2) defines a family of linear operators $\mathcal{A}_x$ with compact support such that, for any C^3 time function, $X(x,t) = \mathcal{A}_x y|_t$ is automatically the solution of (3.1) with $u(t) = X(L,t)$ and $X(0,t) = y(t)$.

The proof relies on the following points.

1. Symbolic computations where the time derivation is replaced by the Laplace variable s are performed. This yields a second order differential equation with nonconstant coefficients in the space variable x.
2. The solution $X(x,s)$ is factorized as $X(x,s) = X(0,s)A(x,s)$. A partial differential system is derived for $A(x,s)$.
3. The study of $s \mapsto A(x,s)$ is simplified by a Liouville transformation $(x,A) \mapsto (z,u)$.
4. The solution $A(x,s)$ of this differential equation is proven to be an entire function of s and of exponential type. (Volterra expansion and majoring series arguments are used.)
5. A careful study of the Volterra equation of the second kind satisfied by A shows that modulo some functions (exponentials of s, depending on x and explicitly calculated), for each x, the restriction of $A(x,s)$ to the imaginary axis is in L^2.
6. Thanks to the Paley–Wiener theorem and the last two properties of A, we prove that, for each x, A can be represented as a compact sum (discrete and continuous) of exponentials in s. This gives (3.2).

Proof. Symbolic computation. Replacing the time derivation by s gives

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\tau(x)\dfrac{\partial X}{\partial x}\right) - \dfrac{\tau'(x)}{g}s^2 X = 0, \\ s^2 X(0,s) = gX'(0,s). \end{cases} \tag{3.3}$$

We do not consider the other boundary condition since u is the control and can be obtained explicitly from X via $u(t) = X(L,t)$.

Factorization. It is very easy to check that $X(x,s) = X(0,s)A(x,s)$ is the solution of (3.3), provided that $A(x,s)$ is the solution of the following partial differential system:

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\tau(x)\dfrac{\partial A}{\partial x}\right) - \dfrac{\tau'(x)}{g}s^2 A = 0, \\ A(0,s) = 1, \\ gA'(0,s) = s^2. \end{cases} \tag{3.4}$$

Liouville transformation. This time we perform a Liouville transformation (already used in section 2)

$$(x,A) \mapsto (z,u)$$

with

$$p(x) = \tau(x), \quad \lambda = -\frac{s^2}{g}, \quad r(x) = \tau'(x), \quad q = 0, \quad x \in [0,L].$$

The new variables (z, u) are defined by the following formulas:

$$z = \frac{1}{K}\int_0^x \sqrt{\frac{\tau'}{\tau}}, \quad 0 \le z \le \pi, \quad K = \frac{1}{\pi}\int_0^L \sqrt{\frac{\tau'}{\tau}}, \tag{3.5}$$

$$u(z, s) = (\tau(x)\tau'(x))^{1/4} A(x, s). \tag{3.6}$$

System (3.4) is turned into

$$\frac{d^2 u}{dz^2} + (\rho^2 - h(z))u = 0 \quad \text{with } \frac{du}{dz}(0) = (a + b\rho^2), \quad u(0) = 1, \tag{3.7}$$

where

$$\rho = \imath \, \frac{K}{\sqrt{g}} s, \quad \imath = \sqrt{-1},$$

$$h(z) = \frac{f''(z)}{f(z)} \quad \text{with } f(z) = (\tau(x)\tau'(x))^{1/4},$$

$$a = \frac{f'(0)}{f(0)}, \quad b = \frac{1}{K}\sqrt{\frac{\tau(0)}{\tau'(0)}}.$$

Proving that $\mathbb{C} \ni \rho \mapsto u(z, \rho)$ is an entire function of exponential type. We claim that, for each z, $\rho \mapsto u(z, \rho)$ is an entire function of exponential type.

Denote by $W(z, \rho)$ the 2×2 matrix solution of

$$\frac{dW}{dz} = \begin{pmatrix} 0 & 1 \\ h(z) - \rho^2 & 0 \end{pmatrix} W$$

with $W(0, \rho) = I$. Since

$$u(z, \rho) = \begin{pmatrix} 1 & 0 \end{pmatrix} W(z, \rho) \begin{pmatrix} 1 \\ a + b\rho^2 \end{pmatrix},$$

it suffices to prove that W is entire in ρ and of exponential type. Using the classical fixed point technique, W can be expressed as an absolutely convergent series of iterated integrals (Volterra expansion)

$$W(z, \rho) = \sum_{i \ge 0} W_i(z, \rho)$$

with

$$W_0(z, \rho) = I, \quad W_{i+1}(z) = \int_0^z \begin{pmatrix} 0 & 1 \\ h(\sigma) - \rho^2 & 0 \end{pmatrix} W_i(\sigma, \rho) \, d\sigma. \tag{3.8}$$

For each $i > 0$, $W_i(z, \rho)$ is a polynomial in ρ^2 of degree i with coefficients depending on z. Thus we have

$$\sum_{0 \le i \le k} W_i(z, \rho) = \sum_{0 \le j \le k} W^{j,k}(z) \, \rho^{2j}.$$

From step k to $k + 1$, we add to $W^{j,k}(z)$ the coefficient of ρ^{2j} in W_{k+1}, say, $\mathcal{W}^{j,k+1}$, to obtain $W^{j,k+1}(z)$:

$$W^{j,k+1}(z) = W^{j,k}(z) + \mathcal{W}^{j,k+1}(z).$$

Let $\alpha = \sup_{[0,\pi]} |h|$. Then the absolute value of each entry of $W_i(z,\rho)$ is bounded by the corresponding entries in the following *majoring series* $M_i(z,\rho)$ defined by the induction (to be compared to (3.8)):

$$M_0(z,\rho) = I, \quad M_{i+1}(z) = \int_0^z \begin{pmatrix} 0 & 1 \\ \alpha+\rho^2 & 0 \end{pmatrix} M_i(\sigma,\rho)\, d\sigma. \tag{3.9}$$

As for W, we can define $M = \sum_{i\geq 0} M_i$ and, for each $k > 0$, the matrices $M^{j,k}$ and $\mathcal{M}^{j,k+1}$ satisfying

$$\sum_{0\leq i\leq k} M_i(z,\rho) = \sum_{0\leq j\leq k} M^{j,k}(z)\ \rho^{2j}, \quad M^{j,k+1}(z) = M^{j,k}(z) + \mathcal{M}^{j,k+1}(z).$$

Standard matrix computations show that

$$\begin{aligned} M(z,\rho) = I &+ \sum_{i>0} \frac{z^{2i}}{(2i)!} \begin{pmatrix} (\rho^2+\alpha)^i & 0 \\ 0 & (\rho^2+\alpha)^i \end{pmatrix} \\ &+ \sum_{i>0} \frac{z^{2i+1}}{(2i+1)!} \begin{pmatrix} 0 & (\rho^2+\alpha)^i \\ (\rho^2+\alpha)^{i+1} & 0 \end{pmatrix}. \end{aligned}$$

That is,

$$M(z,\rho) = \begin{pmatrix} \cosh(z\sqrt{\rho^2+\alpha}) & \sinh(z\sqrt{\rho^2+\alpha})/\sqrt{\rho^2+\alpha} \\ \sinh(z\sqrt{\rho^2+\alpha})\sqrt{\rho^2+\alpha} & \cosh(z\sqrt{\rho^2+\alpha}) \end{pmatrix}. \tag{3.10}$$

For each j, the matrices $M^{j,k} = \sum_{j\leq l\leq k-1} \mathcal{M}^{j,l}$ converge as k tends to ∞. Denote by M^j the limit. By construction, $M = \sum_{j\geq 0} M^j(z)\ \rho^{2j}$, and this series has an infinite radius of convergence in ρ, since, for each z, the functions $\rho \mapsto \cosh(z\sqrt{\rho^2+\alpha})$, $\rho \mapsto \sinh(z\sqrt{\rho^2+\alpha})/\sqrt{\rho^2+\alpha}$, and $\rho \mapsto \sinh(z\sqrt{\rho^2+\alpha})\sqrt{\rho^2+\alpha}$ are entire functions of ρ^2.

But, for each i, j, and k, the matrices $M^{j,k}$ and $\mathcal{M}^{j,k+1}$, whose entries are always nonnegative, dominate the absolute values of the entries of $W^{j,k}$ and $\mathcal{W}^{j,k+1}$, respectively. Thus for each j, the matrices $W^{j,k} = \sum_{j\leq l\leq k-1} \mathcal{W}^{j,l}$ converge as k tends to ∞. Denote by W^j the limit. By construction, $W = \sum_{j\geq 0} W^j(z)\rho^{2j}$, and this series has an infinite radius of convergence in ρ, since M has one. In other words, W is an entire function of ρ. Moreover, the entries of M are upper bounds of the entries of W. Thus W is of exponential type in ρ: for each $z \in [0,\pi]$, there exists $E > 0$ such that

$$\forall \rho \in \mathbb{C}, \quad |W(z,\rho)| \leq E \exp(z|\rho|).$$

We have proven that, for each $z \in [0,\pi]$, $u(z,\rho)$ is an entire function of ρ of exponential type with

$$\forall \rho \in \mathbb{C}, \quad |u(z,\rho)| \leq b(z) \exp(z|\rho|)$$

for some $b(z) > 0$ well-chosen.

Proving that "a part" of $\mathbb{R} \ni \rho \mapsto u(z,\rho)$ belongs to L^2. In general, $\mathbb{R} \ni \rho \mapsto u(z,\rho)$ does not belong to L^2. Thus the Paley–Wiener theorem does not apply directly. Removing some appropriate terms, the remaining is in L^2.

Let

$$v(z,\rho) = u(z,\rho) + b\rho \sin(\rho z) - \left(1 + \frac{b\int_0^z h}{2}\right) \cos(\rho z). \tag{3.11}$$

In the following we prove that this entire function of exponential type is such that $\mathbb{R} \ni \rho \mapsto v(z,\rho)$ belongs to L^2.

From the Volterra equation of the second kind satisfied by u (see [19, p. 111]),

$$u(z,\rho) = \left(\cos(\rho z) + (a - b\rho^2)\frac{\sin(\rho z)}{\rho}\right) + \frac{1}{\rho}\int_0^z \sin(\rho(z-\zeta))\ h(\zeta)\ u(\zeta,\rho)\ d\zeta,$$

we quickly derive a similar equation satisfied by v,

$$v(z,\rho) = \phi(z,\rho) + \frac{1}{\rho}\int_0^z \sin(\rho(z-\zeta))\ h(\zeta)\ v(\zeta,\rho)\ d\zeta,$$

where $\phi = \phi_1 - b\phi_2$ with

$$\begin{aligned} \phi_1(z,\rho) =& a\frac{\sin(\rho z)}{\rho} + \frac{1}{\rho}\int_0^z \sin(\rho(z-\zeta))h(\zeta)\cos(\rho\zeta)\left(1 + (b/2)\int_0^\zeta h\right)\ d\zeta, \\ \phi_2(z,\rho) =& \cos(\rho z)\int_0^z h/2 + \int_0^z \sin(\rho(z-\zeta))h(\zeta)\sin(\zeta)\ d\zeta. \end{aligned}$$

Clearly, there exists $D_1 > 0$ such that for all $z \in [0,\pi]$ and $\rho \in \mathbb{R}$,

$$|\phi_1(z,\rho)| \leq \frac{D_1}{1+|\rho|}$$

(h is bounded). With $2\sin(\rho(z-\zeta))\sin(\zeta) = \cos(\rho(z-2\zeta)) - \cos(\rho z)$, we have

$$\phi_2(z,\rho) = \int_0^z \cos(\rho(z-2\zeta))h(\zeta)\ d\zeta.$$

The integration by part (by assumption τ is C^4 thus h is C^1)

$$\int_0^z \cos(\rho(z-2\zeta))h(\zeta)\ d\zeta = \frac{h(0)+h(z)}{2\rho}\sin(\rho z) + \frac{1}{2\rho}\ \int_0^z \sin(\rho(z-2\zeta))h'(\zeta)\ d\zeta$$

shows that for large $|\rho|$, ϕ_2 tends to zero at least as $1/|\rho|$. Thus there exists $D_2 > 0$ such that for all $z \in [0,\pi]$ and $\rho \in \mathbb{R}$,

$$|\phi_2(z,\rho)| \leq \frac{D_2}{1+|\rho|}.$$

This proves that v satisfies

$$v(z,\rho) = \phi(z,\rho) + \frac{1}{\rho}\int_0^z \sin(\rho(z-\zeta))h(\zeta)v(\zeta,\rho)\ d\zeta \tag{3.12}$$

with $|\phi(z,\rho)| \leq D/(1+|\rho|)$ for all $z \in [0,\pi]$ and $\rho \in \mathbb{R}$. ($D > 0$ is a well-chosen constant independent of z and ρ.)

This last inequality gives the desired conclusion by the following classical computation (see [19, p. 112], for instance).

Let $\beta(z,\rho) = \sup_{0\leq\zeta\leq z} |v(\zeta,\rho)|$. By (3.12) we have for each z_1 and z_2 in $[0,\pi]$, $z_1 \leq z_2$

$$|v(z_1,\rho)| \leq \frac{D}{1+|\rho|} + \frac{\alpha z_1 \beta(z_2,\rho)}{|\rho|} \leq \frac{D}{1+|\rho|} + \frac{\alpha\pi}{|\rho|}\ \beta(z_2,\rho).$$

(Remember that $\alpha = \sup_{[0,\pi]} |h|$.) In particular, when $z_1 = z_2 = z$, we have

$$\beta(z,\rho)\left(1 - \frac{\alpha\pi}{|\rho|}\right) \leq \frac{D}{1+|\rho|}. \tag{3.13}$$

Finally, for $|\rho| \geq 2\alpha\pi$, $\beta(z,\rho) \leq 2D/(1+|\rho|)$. This proves that $\mathbb{R} \ni \rho \mapsto v(z,\rho)$ belongs to L^2.

Using the Paley–Wiener theorem. At last, the Paley–Wiener theorem ensures that the Fourier transform of $\rho \mapsto v(z,\rho)$ has a compact support included in $[-z,z]$ since for all $\rho \in \mathbb{C}, |v(z,\rho)| \leq N \exp(z|\rho|)$ for some constant $N > 0$. This means that, for each $z \in [0,\pi]$, there exists $[-z,z] \ni \zeta \mapsto \mathcal{K}(z,\zeta)$ in $L^2([-z,z])$ such that

$$v(z,\rho) = \int_{-z}^{+z} \mathcal{K}(z,\zeta) \exp(\imath\zeta\rho)\ d\zeta.$$

Since v is an even function of ρ, $\mathcal{K}$ is also an even function of ζ. Thus we have, finally,

$$v(z,\rho) = \int_0^{+z} \mathcal{K}(z,\zeta)(\exp(\imath\zeta\rho) + \exp(-\imath\zeta\rho))\ d\zeta. \tag{3.14}$$

Conclusion. Pulling back this last relation in the (x, A) coordinates, noticing that $\rho = \imath K s/\sqrt{g}$, that $\exp(-\theta s)$ is the Laplace transform of the θ-delay operator, and that $u(0,\rho)$ is, up to a constant, the Laplace transform of $X(0,t)$, we deduce after some standard but tedious computations formulae (3.2). The new function $\mathcal{B}(x,\xi)$ is related to $\mathcal{K}(z,\zeta)$ via

$$K\sqrt{\frac{\tau(\xi)}{\tau'(\xi)}}\ \mathcal{B}(x,\xi) = \left(\frac{\tau(0)\tau'(0)}{\tau(x)\tau'(x)}\right)^{\frac{1}{4}}\ \mathcal{K}\left(\frac{\sqrt{g}}{K}\theta(x), \frac{\sqrt{g}}{K}\theta(\xi)\right). \tag{3.15}$$

At last,

$$\begin{aligned} A(x,s) = \varphi(x)\left(\exp\theta(x)s + \exp\theta(x)s\right) + \psi(x)s\left(\exp\theta(x)s - \exp\theta(x)s\right) \\ + \int_0^x \mathcal{K}(x,\zeta)(\exp(\theta(\zeta)s) + \exp(-\theta(\zeta)s))\ d\zeta, \end{aligned}$$

so $X(x,s) = X(0,s)A(x,s)$ when turned back into the time-domain does give formulae (3.2). $\square$

4. Conclusion. We have shown that, around the stable vertical position, heavy chain systems with or without load, with constant or variable section, are "flat": the trajectories of these systems are parameterizable by the trajectories of their free ends. Relations (1.4), (2.2), and (3.2) show that such parameterizations involve operators of compact supports.

It is surprising that such parameterizations can also be applied around the inverse and unstable vertical position. For the homogenous heavy chain, we have only to replace g by $-g$ to obtain a family of smooth solutions to the elliptic equation (singular at $x = 0$)

$$\frac{\partial}{\partial x}\left(gx\frac{\partial X}{\partial x}\right) + \frac{\partial^2 X}{\partial t^2} = 0$$

by the integral

$$X(x,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} y(t + 2\imath\sqrt{x/g}\sin\theta)\ d\theta,$$

where y is now a holomorphic function in $\mathbb{R} \times [-2\sqrt{L/g}, +2\sqrt{L/g}]$ that is real on the real axis. This parameterization can still be used to solve the motion planning problem in spite of the fact that the Cauchy problem associated to this elliptic equation is not well-posed in the sense of Hadamard.

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