MOTION PLANNING AND NONLINEAR SIMULATIONS FOR A TANK CONTAINING A FLUID

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Abstract

We consider a tank containing a fluid. The tank is subjected to a one-dimensional horizontal move and the motion of the fluid is described by Saint-Venant's equations. We show how to parameterize the trajectories of the linearized system thanks to the horizontal coordinate of a particular point in the system – the "flat output", see figure 2– and a periodic function. The motion planning problem of the linearized model is solved in the general case of joining two steady states. Next we provide an algorithm, based on Godunov scheme, with a dedicated way of dealing with boundary conditions, to numerically simulate the evolution of the nonlinear system. Nonlinear simulations provide a way of checking the accuracy of the motion planning based on the tangent linear system.

1 Introduction

The following problem is derived from an industrial process control problem where tanks filled with liquid are to be moved to different workbenches as fast as possible.

To move such a tank horizontally, one has to take the motion of the liquid into account in order to prevent any overflowing. We show that the corresponding non-linear model has solutions that can be approximately parameterized by using the linearized model.

This allows us to plan moves without overflowing, and gives a full description of the corresponding motion of the fluid. We provide comparisons with Godunov scheme simulations of the non-linear model and conclude that realistic moves can be computed with this method.

This work has been inspired by fruitful discussions with K.J. Åström from the Department of Automatic Control of the Lund University at the time of a visit organized by the Conférence des Grandes Ecoles. We also thank J-M. Coron from the Orsay University for his advice and reference [7].

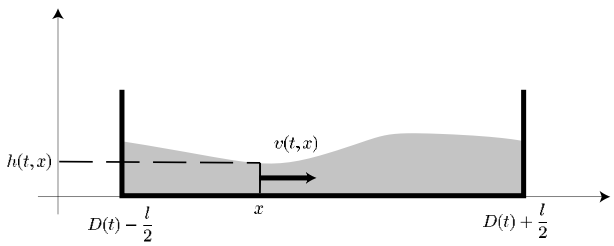

Figure 1: tank of length l containing a perfect fluid.

2 The physical non-linear model

Neglecting superficial tension and viscosity, we get the following system of conservation laws –see [8] for instance:

$$\left.\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hv) &= 0 \\ \frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial x}\left(hv^2 + \frac{g}{2}h^2\right) &= 0 \\ h(0,x) &= h_0(x) \\ v(0,x) &= v_0(x) \\ v(t, D(t) - \frac{l}{2}) &= \dot{D}(t) \\ v(t, D(t) + \frac{l}{2}) &= \dot{D}(t) \end{aligned}\right\} \qquad (1)$$

where $h(t,x)$ denotes the height of the liquid and $v(t,x)$ denotes the horizontal speed of the fluid in the absolute referential.

2.1 Riemann invariants [8]

The preceding PDEs writes:

$$\frac{\partial U}{\partial t} + \frac{\partial f(U)}{\partial x} = 0$$

with $U = \begin{pmatrix} h \\ hv \end{pmatrix}$ and $f(U) = \begin{pmatrix} hv \\ \frac{(hv)^2}{h} + \frac{g}{2}h^2 \end{pmatrix}$. The Jacobian of f is:

$$\mathcal{J}_f = \begin{pmatrix} 0 & 1 \\ gh - v^2 & 2v \end{pmatrix}.$$

Its eigenvalues are $\lambda_1 = v - \sqrt{gh}$, $\lambda_2 = v + \sqrt{gh}$ and its eigenvectors are: $r_1 = \begin{pmatrix} 1 \\ v - \sqrt{gh} \end{pmatrix}$ and $r_2 = \begin{pmatrix} 1 \\ v + \sqrt{gh} \end{pmatrix}$. Let $J_1 = v + 2\sqrt{gh}$, $J_2 = v - 2\sqrt{gh}$. Then J_1 is an 1-Riemann invariant since $\nabla J_1.r_1 = 0$ and J_2 is an 2-Riemann invariant since $\nabla J_2.r_2 = 0$. So:

$$\frac{\partial J_1}{\partial t} + \lambda_2 \frac{\partial J_1}{\partial x} = 0 \ , \ \frac{\partial J_2}{\partial t} + \lambda_1 \frac{\partial J_2}{\partial x} = 0.$$

Then we use the following change of coordinates:

$$\begin{pmatrix} t \\ x \end{pmatrix} \mapsto \begin{pmatrix} t \\ z = x - D(t) \end{pmatrix}.$$

Let

$$J_+(t,z) = J_2(t, z + D(t)) \ , \ J_-(t,z) = J_1(t, z + D(t)),$$

while

$$J_2(t,x) = J_+(t, x - D(t)) \ , \ J_1(t,x) = J_-(t, x - D(t)).$$

Then $\frac{\partial J_-}{\partial z} = \frac{\partial J_1}{\partial x}$, $\frac{\partial J_-}{\partial t} = \frac{\partial J_1}{\partial t} + \dot{D} \, \frac{\partial J_1}{\partial x}$. Which means:

$$\frac{\partial J_+}{\partial t} + (\lambda_1 - \dot{D}) \frac{\partial J_+}{\partial z} = 0. \tag{2}$$

Likewise:

$$\frac{\partial J_-}{\partial t} + (\lambda_2 - \dot{D}) \frac{\partial J_-}{\partial z} = 0. \tag{3}$$

The boundary conditions are:

$$\frac{J_+ + J_-}{2}(t, -\frac{l}{2}) = \frac{J_+ + J_-}{2}(t, \frac{l}{2}) = \dot{D}(t), \forall t \in \mathbb{R}. \tag{4}$$

3 Study of the linearized model

3.1 The delay system

In the following we linearize equations (2) and (3) in order to solve them by the method of characteristics. Near $(h, hv) = (\overline{h}, 0)$, we study $\delta J_+ = J_+ - \overline{J_+}$ with $\overline{J_+} = -2\sqrt{g\overline{h}}$ and $\delta J_- = J_- - \overline{J_-}$ with $\overline{J_-} = 2\sqrt{g\overline{h}}$. Assuming that $\|\dot{D} - v\| << \sqrt{g\overline{h}}$, the first order equations derived from (2) and (3) are:

$$\frac{\partial\,(\delta J_+)}{\partial t} - \sqrt{g\overline{h}}\, \frac{\partial\,(\delta J_+)}{\partial x} = 0$$
$$\frac{\partial\,(\delta J_-)}{\partial t} + \sqrt{g\overline{h}}\, \frac{\partial\,(\delta J_-)}{\partial x} = 0.$$

Let $c = \sqrt{g\overline{h}}$, then the characteristics method gives:

$$\delta J_+(t,x) = \varphi_+(t + \frac{x}{c}) \ , \ \delta J_-(t,x) = \varphi_-(t - \frac{x}{c}).$$

The two boundary conditions (4) are equivalent to:

$$\varphi_+(t - \frac{\Delta}{2}) + \varphi_-(t + \frac{\Delta}{2}) = 2\dot{D}(t) \tag{5}$$
$$\varphi_+(t + \frac{\Delta}{2}) + \varphi_-(t - \frac{\Delta}{2}) = 2\dot{D}(t) \tag{6}$$

with $\Delta = \frac{l}{c}$.

3.2 Noncontrollability

Using the module theoretic framework, we consider the last system of delay equations (5,6) as a finitely generated module over the ring $\mathbb{R}[s, \delta = e^{-s\frac{\Delta}{2}}]$ (see [10] or [6] for details). This module has a torsion element, namely $\varphi_+ - \varphi_-$ since:

$$(\delta^2 - 1)(\varphi_+ - \varphi_-) = 0. \tag{7}$$

So the system is not controllable.

Roughly speaking, it is possible to steer the system between two different points provided that their corresponding torsion elements equations are compatible. As we will show next, this is the case between two steady states.

3.3 Explicit parameterization

Let us restrict ourselves to the set of moves of the tank such as $\dot{D}$ admits a compact support. Then exists $\mathcal{V}$ such as:

$$2D(t) = \mathcal{V}(t + \frac{\Delta}{2}) + \mathcal{V}(t - \frac{\Delta}{2}). \tag{8}$$

Then:

$$\varphi_+(t + \Delta) - \varphi_+(t - \Delta) = \dot{\mathcal{V}}(t + \Delta) - \dot{\mathcal{V}}(t - \Delta).$$

From this last functional equality one can deduce that:

$$\varphi_+(t) = \dot{\mathcal{V}}(t) + \pi(t) \tag{9}$$

where π is a 2Δ-periodic function. Similarly

$$\varphi_-(t) = \dot{\mathcal{V}}(t) - \pi(t + \Delta). \tag{10}$$

These last two equations allow us to compute $h(t,x)$ and $v(t,x)$. First:

$$\begin{aligned} v(t,x) &= \frac{J_+(t, x - D(t)) + J_-(t, x - D(t))}{2} \\ &= \frac{1}{2}\Big[\dot{\mathcal{V}}(t + \frac{x - D(t)}{c}) + \pi(t + \frac{x - D(t)}{c}) \ldots \\ &\quad + \dot{\mathcal{V}}(t - \frac{x - D(t)}{c}) - \pi(t + \Delta - \frac{x - D(t)}{c})\Big]. \end{aligned} \tag{11}$$

Second:

$$\begin{aligned} h(t,x) &= \frac{1}{16g}\left[J_-(t, x - D(t)) - J_+(t, x - D(t))\right]^2 \\ h(t,x) &= \Big[\sqrt{\overline{h}} + \\ &\quad \frac{1}{\sqrt{4g}}\Big(\dot{\mathcal{V}}(t - \frac{x - D(t)}{c}) - \pi(t + \Delta - \frac{x - D(t)}{c}) \\ &\quad - \dot{\mathcal{V}}(t + \frac{x - D(t)}{c}) - \pi(t + \frac{x - D(t)}{c})\Big)\Big]^2. \end{aligned} \tag{12}$$

Equations (8) (11) and (12) show that all the quantities of the systems can be expressed in terms of $\mathcal{V}$ and π a 2Δ-periodic function. In the following we show that, in particular cases, we can get rid off π.

3.4 Motion planning between two steady states

Let

$$\mathcal{V}(t)=\begin{cases}0 & t<\frac{\Delta}{2}\\ \sigma(t) & \frac{\Delta}{2}\le t\le T+\frac{\Delta}{2}\\ 1 & t>T+\frac{\Delta}{2}.\end{cases}$$

Let us start from the following initial conditions:

$$\left.\begin{array}{r}h(0,x)=\overline{h}\\ v(0,x)=0\end{array}\right\}\forall x\in[-\frac{\Delta}{2},\frac{\Delta}{2}].$$

According to (11) and (12), these initial conditions imply that:

$$\left.\begin{array}{r}\dot{\mathcal{V}}(-s)+\pi(-s)-\dot{\mathcal{V}}(s)+\pi(\Delta+s)=0\\ \dot{\mathcal{V}}(-s)+\pi(-s)+\dot{\mathcal{V}}(s)-\pi(\Delta+s)=0\end{array}\right\}\forall s\in[-\frac{\Delta}{2},\frac{\Delta}{2}].$$

So:

$$\left.\begin{array}{r}\dot{\mathcal{V}}(s)=-\pi(s)\\ \dot{\mathcal{V}}(s)=\pi(\Delta+s)\end{array}\right\}\forall x\in[-\frac{\Delta}{2},\frac{\Delta}{2}].$$

By construction, $\mathcal{V}(s)=0$ for $s<\frac{\Delta}{2}$. So

$$\left.\begin{array}{r}\pi(s)=0\\ \pi(\Delta+s)=0\end{array}\right\}\forall x\in[-\frac{\Delta}{2},\frac{\Delta}{2}].$$

This means that $\pi(s)=0$ for $s\in[-\frac{\Delta}{2},\frac{3\Delta}{2}]$. Yet π is 2Δ-periodic, so

$$\pi=0.$$

Parameterization of the motion Every quantity of the system writes in terms of $\mathcal{V}$ and $\dot{\mathcal{V}}$. Thus $\mathcal{V}$ is a "flat output" – see [5] and [9] for details. At first order:

$$D(t)=\frac{1}{2}\left[\mathcal{V}(t+\frac{\Delta}{2})+\mathcal{V}(t-\frac{\Delta}{2})\right] \tag{13}$$

$$v(t,x)=\frac{1}{2}\left[\dot{\mathcal{V}}(t+\frac{x-D(t)}{c})+\dot{\mathcal{V}}(t-\frac{x-D(t)}{c})\right] \tag{14}$$

$$h(t,x)=\overline{h}+\frac{\sqrt{\overline{h}}}{2\sqrt{g}}\left(\dot{\mathcal{V}}(t-\frac{x-D(t)}{c})-\dot{\mathcal{V}}(t+\frac{x-D(t)}{c})\right) \tag{15}$$

These last equations arise from the parameterization of the solutions of the linearized model. In fact they are first order approximations to the solutions of the non-linear model.

Physical meaning of the "flat output" An easy calculus leads to:

$$\mathcal{V}(t)=D(t)+\frac{L}{2}\,\frac{M^+-M^-}{M^++M^-} \tag{16}$$

where

$$M^+=\int_{D(t)}^{D(t)+\frac{L}{2}}h(t,s)ds\ ,\ M^-=\int_{D(t)-\frac{L}{2}}^{D(t)}h(t,s)ds.$$

Thus $\mathcal{V}(t)$ is the horizontal coordinate of a point close to the middle of the tank – but slightly different from the

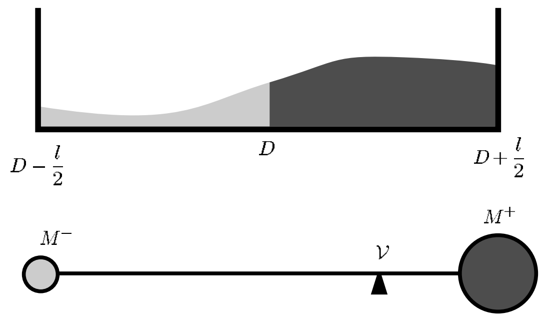

Figure 2: locus of the "flat output" $\mathcal{V}$.

projection of the gravity center of the fluid. As shown on figure 2, this point corresponds to the gravity center two ponctual masses, M^- and M^+, located on the boundaries, $-l/2$ and $+l/2$, respectively. Thus $\mathcal{V}$ coincides with the middle of the tank at the beginning and at the end of the move.

4 Nonlinear numerical scheme

In the following, we describe the techniques we use to simulate the behavior of the tank when subjected to move when $t\mapsto D(t)$ is given by (13).

The set of Saint-Venant (or shallow water) hyperbolic conservation laws

$$\frac{\partial U}{\partial t}+\frac{\partial}{\partial x}f(U)\equiv\frac{\partial}{\partial t}\begin{pmatrix}h\\ hv\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}hv\\ hv^2+\frac{1}{2}gh^2\end{pmatrix}=0 \tag{17}$$

is considered in the following space-time domain :

$$t\ge 0,\ D(t)-\frac{l}{2}\le x\le D(t)+\frac{l}{2}.$$

We introduce an integer N ($N=50$ in the computations) and discretize system (1) with space step $\Delta x=\frac{l}{N}$ and time step Δt with the finite volume method. We denote by D^n the mean value of the tank position at time step $t^n=n\,\Delta t$: $D^n=D(n\,\Delta t)$ and introduce the mean velocity $u^{n+1/2}$ between two time steps

$$u^{n+1/2}=\frac{D(t^n+\Delta t)-D(t^n)}{\Delta t}.$$

We denote by

$$W_{j+1/2}^{n+1/2}(\xi)\equiv (H_{j+1/2}^{n+1/2},H_{j+1/2}^{n+1/2}V_{j+1/2}^{n+1/2})^T\left(h_j^n,\,v_j^n\,;\xi\,;\,h_{j+1}^n,\,v_{j+1}^n\right) \tag{18}$$

the state $W_{j+1/2}^{n+1/2}(\xi)$, the height $H_{j+1/2}^{n+1/2}(\xi)$ and the velocity $V_{j+1/2}^{n+1/2}(\xi)$ of the self similar solution ($\xi=\frac{x}{t}$) of the Riemann problem for shallow water equations (1) associated with the initial conditions

$$U(0,x)=(h_j^n,h_j^nv_j^n)\text{ for }x<0$$
$$\text{and }U(0,x)=(h_{j+1}^n,h_{j+1}^nv_{j+1}^n)\text{ for }x>0.$$

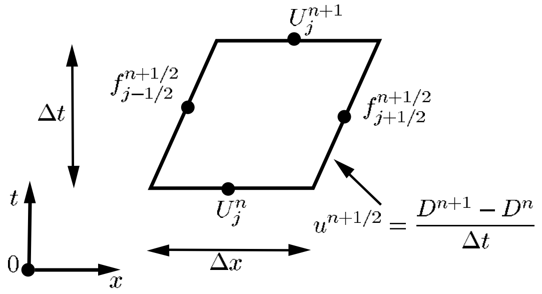

Figure 3: space-time control volume.

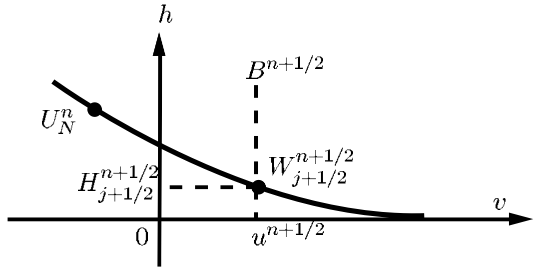

Figure 4: partial Riemann problem at the boundary.

We define the moving flux $f_{j+1/2}^{n+1/2}$ at velocity $u^{n+1/2}$ between two cells by the relation derived e.g. in [2] :

$$f_{j+1/2}^{n+1/2} = f(W_{j+1/2}^{n+1/2}(u^{n+1/2})) - u^{n+1/2} W_{j+1/2}^{n+1/2}(u^{n+1/2}). \tag{19}$$

4.1 Godunov scheme and Galilean invariance

The Godunov scheme can be explicited as follow. We integrate the conservation law (1) in each space-time cell defined by figure 3 or by the following algebraic relations

$$t^n \leq t \leq t^n + \Delta t, \tag{20}$$

$$(j-1)\Delta x \leq x - \left[D^n - \frac{l}{2} + u^{n+1/2}(t - t^n) \right] \leq j \Delta x. \tag{21}$$

The iteration of the scheme is simply a consequence of integrating by parts the conservation law (17) in the space-time domain (21) :

$$\frac{1}{\Delta t}(U_j^{n+1} - U_j^n) + \frac{1}{\Delta x}(f_{j+1/2}^{n+1/2} - f_{j-1/2}^{n+1/2}) = 0, \tag{22}$$

$$\text{for } j = 1, \ldots, N, n \geq 0. \tag{23}$$

This first order scheme is explicit and usual Courant Friedrichs Lewy condition constrains time step Δt. We observe (see Annex or [3]) that due to the Galilean invariance of the model (17), we have the following property :

$$\begin{aligned} H(h_j^n, v_j^n - u; \xi - u; h_{j+1}^n, v_{j+1}^n - u) \\ = H(h_j^n, v_j^n; \xi; h_{j+1}^n, v_{j+1}^n), \end{aligned} \tag{24}$$

$$\begin{aligned} V(h_j^n, v_j^n - u; \xi; h_{j+1}^n, v_{j+1}^n - u) \\ = V(h_j^n, v_j^n; \xi; h_{j+1}^n, v_{j+1}^n) - u. \end{aligned} \tag{25}$$

Then flux $f_{j+1/2}^{n+1/2}$ defined in (19) can also be evaluated according to the expression

$$f_{j+1/2}^{n+1/2} = \begin{pmatrix} HV \\ HV(V + u^{n+1/2}) + \frac{1}{2} g H^2 \end{pmatrix} \begin{pmatrix} h_j^n, v_j^n - u^{n+1/2}; \ldots \\ \ldots 0; h_{j+1}^n, v_{j+1}^n - u^{n+1/2} \end{pmatrix} \tag{26}$$

obtained by taking $\xi = u^{n+1/2}$ inside relations (24) and (25). In this manner, the Godunov flux has been adapted in order to take into account the displacement of the tank.

4.2 Boundary conditions

The treatment of rigid boundary conditions follows the ideas proposed some years ago [4]. It consists to remark that the boundary condition takes the physical form

$$v(D(t) \pm \frac{l}{2}) = u^{n+1/2}, t \geq 0. \tag{27}$$

We introduce the boundary manifold $\mathcal{B}^{n+1/2}$ defined by

$$\mathcal{B}^{n+1/2} = \left\{ U = (H, H u^{n+1/2})^T, H \in \mathbb{R} \right\}$$

and solve the last (respectively the first) interface problem with a partial Riemann problem between state U_N^n and boundary manifold $\mathcal{B}^{n+1/2}$ (respectively boundary manifold $\mathcal{B}^{n+1/2}$ and state U_1^n) : the 1-wave issued from U_N^n intersects the boundary manifold $\mathcal{B}^{n+1/2}$ in a state $W_{N+1/2}^{n+1/2}$ whose velocity is by definition equal to $u^{n+1/2}$ (see figure 4).

The **non trivial** height $H_{N+1/2}^{n+1/2}$ of state $W_{N+1/2}^{n+1/2}$ defines completely the boundary flux and we have from (25), (26) and (27) :

$$f_{N+1/2}^{n+1/2} = \begin{pmatrix} 0 \\ \frac{1}{2} g (H_{N+1/2}^{n+1/2})^2 \end{pmatrix}, n \geq 0$$

and an analogous formula for $f_{1/2}^{n+1/2}$.

5 Linear prediction versus nonlinear simulation

We check the relevance of our approach by imposing a motion planned thanks to linearization and comparing the linear prediction via (13,14,15) to the corresponding behavior obtained by simulation of the nonlinear model.

In the following, Δ, which is the required time for a wave to meet a boundary starting from the opposite one, is equal to 1. The vertical scale of the figures has been enlarged by a factor 3 for the reader to see the details. The Matlab code can be obtained via E-mail at `petit@cas.ensmp.fr` or `rouchon@cas.ensmp.fr`.

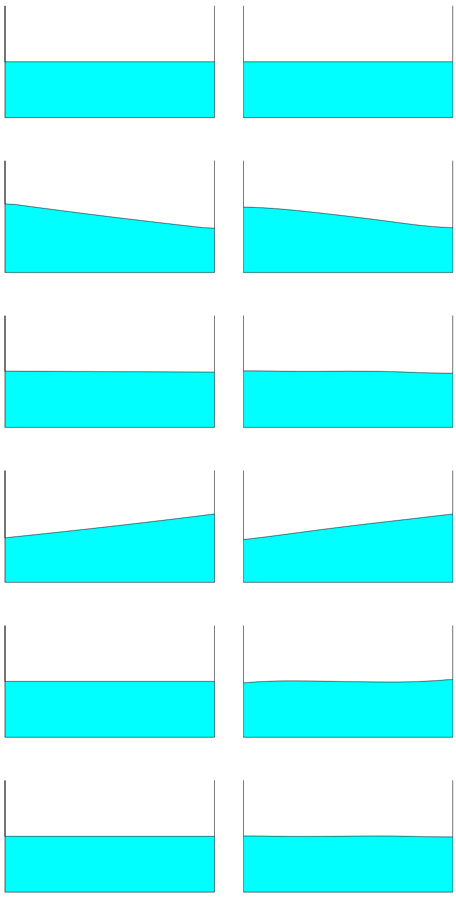

Figure 5: snapshots at t=0, t=T/4, t=T/2, t=3T/4, t=T and t=5T/4. Left: linear prediction. Right: nonlinear simulation.

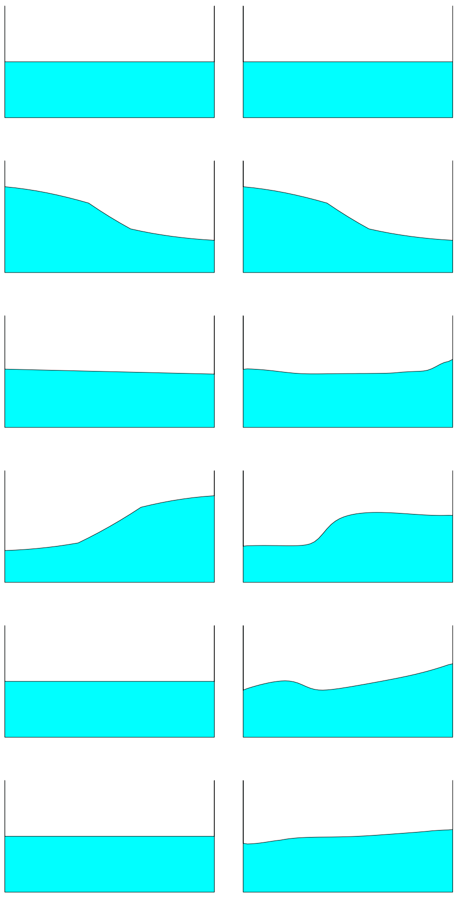

Figure 6: snapshots at t=0, t=T/4, t=T/2, t=3T/4, t=T and t=5T/4. Left: linear prediction. Right: nonlinear simulation.

Transfer time T=4.0 The prediction of a slow move is rather close to the numerical results of a Godunov scheme simulation. Results are shown on figure 5.

Transfer time T=2.5 Yet as the move speeds up the prediction results get more different from the numerical simulation. Results are shown on figure 6.

6 Conclusion

We have proved that the linear approximation of (1) around any steady state is not controllable. Nevertheless, the linear model is "steady state controllable": one can connect, in the sense of [11], any past trajectory passing through a steady state to any future trajectory passing through another steady-state. Is the nonlinear system controllable? If not, is it "steady-state controllable"?

In [7], it is shown that, for D constant, (1) admits a unique entropic solution for $t \geq 0$ when the initial condition is closed to steady state. It is also shown that such solution decays like $1/t$ to the steady state (here the required periodicity is automatically satisfied thanks to the boundary conditions and symmetry arguments). Is it possible, through active control, to improve the convergence rate and to achieve exponential stability despite the lack of controllability of the linear approximation?

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Annex: Galilean invariance and Riemann problem

• We show in this section that, with the notations introduced previously concerning the solution of the Riemann problem, the following relations

$$H(h_l, v_l - u; \xi - u; h_r, v_r - u) = H(h_l, v_l; \xi; h_r, v_r) \tag{A1}$$

$$V(h_l, v_l - u; \xi - u; h_r, v_r - u) = V(h_l, v_l; \xi; h_r, v_r) - u \tag{A2}$$

hold for each set of left parameters h_l and v_l, of right parameters h_r and v_r, of celerity ξ and velocity u.

• With the notations introduced in [1] for the waves of the Riemann problem, we know that the 1-wave issued from state U_l admits a parameterization of the form

$$v_1 = v_l - \Psi(h_1, h_l), h_1 > 0. \tag{A3}$$

On the other hand, the 2-wave of states U_2 arriving to the state U_r have a parameterization given [1] in terms of the same function $\Psi(\bullet, \bullet)$:

$$v_2 = v_r + \Psi(h_2, h_r), h_2 > 0. \tag{A4}$$

The intermediate state U^* between U_l and U_r is defined by imposing $v_1 = v_2$ inside relations (A3) and (A4) ; its height h^* satisfies the equation

$$\Psi(h^*, h_l) - \Psi(h^*, h_r) = v_l - v_r. \tag{A5}$$

Then changing v_l and v_r into $v_l - u$ and $v_r - u$ respectively does not change the variable h^*. In a similar way due to (A3) and (A4), the velocity v^* $(v^* = v_1 = v_2)$ becomes $v^* - u$ when v_l and v_r are changed in the previous manner.

• If the 1-wave between U_l and U^* is a rarefaction (i.e. $h^* < h_l$), the functions $H(\bullet)$ and $V(\bullet)$ satisfy the relations

$$H(h_l, v_l; \xi; h_r, v_r) = h_l$$
$$\text{and } V(h_l, v_l; \xi; h_r, v_r) = v_l \text{ if } \xi < v_l - c_l \tag{A6}$$

$$V(h_l, v_l; \xi; h_r, v_r) - c(H(h_l, v_l; \xi; h_r, v_r)) = \xi,$$
$$v_l - c_l < \xi < v^* - c^* \tag{A7}$$

$$V(h_l, v_l; \xi; h_r, v_r) = v_l - \Psi(H(h_l, v_l; \xi; h_r, v_r),$$
$$h_l, v_l - c_l < \xi < v^* - c^* \tag{A8}$$

$$H(h_l, v_l; \xi; h_r, v_r) = h^*$$
$$\text{and } V(h_l, v_l; \xi; h_r, v_r) = v^* \text{ if } \xi > v^* - c^*, \tag{A9}$$

where $c(h) \equiv \sqrt{g\,h}$ is the sound celerity for shallow waters. When data v_l and v_r are changed into $v_l - u$ and $v_r - u$ respectively, it is clear from the algebraic relations (A6)-(A9) that relations (A1) and (A2) define a solution of system (A6)-(A9) associated with the new data in the domain $\xi < v^* - u$ for celerity of the waves.

• If the 1-wave between U_l and U^* is a shock wave (i.e. $h^* > h_l$), the shock celerity σ_1^* is computed thanks to the Rankine-Hugoniot jump conditions :

$$h^* v^* - h_l v_l = \sigma_1^* (h^* - h_l) \tag{A10}$$

$$h^* {v^*}^2 + \frac{1}{2} g {h^*}^2 - h_l v_l^2 - \frac{1}{2} g h_l^2 = \sigma_1^* (h^* v^* - h_l v_l). \tag{A11}$$

When v_l and v^* are changed into $v_l - u$ and $v^* - u$ respectively, the scalar $\sigma_1^* - u$ is again solution of system (A10)(A11), as shown by the following algebra :

$$\begin{aligned} & h^*(v^* - u)^2 - h_l(v_l - u)^2 + \frac{1}{2} g ({h^*}^2 - h_l^2) \\ & \quad = \sigma_1^*(h^* v^* - h_l v_l) - 2\, u(h^* v^* - h_l v_l) + u^2 (h^* - h_l) \\ & \quad = (\sigma_1^* - u)(h^* v^* - h_l v_l) - u(\sigma_1^* - u)(h - h_l) \\ & \quad = (\sigma_1^* - u)[h^*(v^* - u) - h_l(v_l - u)]. \end{aligned}$$

Then the parameterization of the 1-shock wave

$$H(h_l, v_l; \xi; h_r, v_r) = h_l$$
$$\text{and } V(h_l, v_l; \xi; h_r, v_r) = v_l \text{ if } \xi < \sigma_1^*$$

$$H(h_l, v_l; \xi; h_r, v_r) = h^*$$
$$\text{and } V(h_l, v_l; \xi; h_r, v_r) = v^* \text{ if } \sigma_1^* < \xi < v^*$$

is transformed according to the relations (A1) and (A2) when v_l and v^* are changed into $v_l - u$ and $v^* - u$. The proof is analogous for the 3-wave between U^* and U_r and the proposition is established.