Design of trajectory stabilizing feedback for driftless flat systems *

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Abstract A design method for robust stabilization of driftless flat systems around trajectories is proposed. This method is based on time scaling, which results in controlling the clock. It permits to follow with exponential stability arbitrary smooth trajectories. These trajectories, which may be obtained from the the motion planning properties of flatness, may contain and pass through steady-states. We thus obtain a stabilization design around rest points for many nonholonomic systems. The example of the standard n-trailer system is treated in details with simulations for $n = 0$ and $n = 2$.

Key words Nonholonomic system, motion planning, trajectory stabilization, flatness, mobile robots, time scaling.

1 Introduction

After the demonstration in [3] and in [20] of the impossibility of a straightforward nonlinear extension of the linear stabilizing strategies, Coron [6, 7] has recently shown how to utilize time-varying feedback for stabilizing a very large class of nonlinear plants. The practical design of such stabilizing laws is now giving rise to a rapidly growing literature (see, e.g., [2, 4, 16, 19, 22]).

We here attack this problem via a somehow different standpoint. We restrict ourselves to driftless systems which are flat. Remember that last property, which is related to dynamic feedback linearization [5] and is quite often verified in practice [10, 17, 18], permits to tackle in a most efficient way the motion planning problem. We are thus lead to consider stabilization around a given trajectory. This question, despite its importance, has perhaps received less attention (see, nevertheless [22]).

We introduce time-scaling, which may be interpreted as controlling the clock [9]. This tool, which seems perhaps surprising at a first glance, permits to avoid some singularities that are the genuine sources of the mathematical and practical difficulties of nonlinear stabilization.

The paper is organized as follows. In section 2, we briefly recall what is a flat system. Section 3 deals with the general result. In section 4, the control design for the standard n-trailer system is sketched. Simulations for the car without trailer and the car with two trailers are presented.

2 Flat systems

More details can be found in [8, 9, 10]. A control system is said to be (*differentially*) *flat* if the following conditions are satisfied:

1. there exists a finite set $y = (y_1, \ldots, y_m)$ of variables which are differentially independent, i.e., which are not related by any differential equations.

2. the y_i's are differential functions of the system variables, i.e., are functions of the system variables (state and input) and of a finite number of their derivatives.

3. Any system variable is a differential function of the y_i's, i.e., is a function of the y_i's and of a finite number of their derivatives.

We call $y = (y_1, \ldots, y_m)$ a *flat* or *linearizing* output. Its number of components equals the number of independent input channels.

For a "classic" dynamics,

$$\dot{x} = f(x, u), \quad x = (x_1, \ldots, x_n), \quad u = (u_1, \ldots, u_m), \tag{1}$$

flatness implies the existence of a vector-valued function h such that

$$y = h(x, u_1, \ldots, u_1^{(\beta_1)}, \ldots, u_m, \ldots, u_m^{(\beta_m)}),$$

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where $y = (y_1, \dots, y_m)$. The components of x and u are, moreover, given without any integration procedure by the vector-valued functions A and B:

$$\begin{aligned} x &= A(y_1, \dots, y_1^{(\alpha_1)}, \dots, y_m, \dots, y_m^{(\alpha_m)}) \\ u &= B(y_1, \dots, y_1^{(\alpha_1+1)}, \dots, y_m, \dots, y_m^{(\alpha_m+1)}). \end{aligned} \quad (2)$$

The motion planning problem for (1) consists in finding a control trajectory $[0, T] \ni t \to u(t)$ steering the system from state $x = p$ at $t = 0$ to the state $x = q$ at $t = T$. When the system is flat, this problem is equivalent to finding a flat output trajectory $[0, T] \ni t \to y(t)$ such that

$$p = A(y_1(0), \dots, y_1^{(\alpha_1)}(0), \dots, y_m(0), \dots, y_m^{(\alpha_m)}(0))$$

and

$$q = A(y_1(T), \dots, y_1^{(\alpha_1)}(T), \dots, y_m(T), \dots, y_m^{(\alpha_m)}(T)).$$

Since the mapping

$$\begin{aligned} &(y_1, \dots, y_1^{(\alpha_1)}, \dots, y_m, \dots, y_m^{(\alpha_m)}) \\ &\quad \to A(y_1, \dots, y_1^{(\alpha_1)}, \dots, y_m, \dots, y_m^{(\alpha_m)}) \end{aligned}$$

is locally onto, in general, the problem consists in finding a smooth trajectory $t \to y(t)$ with prescribed values for some of its derivatives at time 0 and time T and such that

$$[0, T] \ni t \to A(y_1(t), \dots, y_1^{(\alpha_1)}(t), \dots, y_m(t), \dots, y_m^{(\alpha_m)}(t))$$

and

$$\begin{aligned} &[0, T] \ni t \to \\ &\quad B(y_1(t), \dots, y_1^{(\alpha_1+1)}(t), \dots, y_m(t), \dots, y_m^{(\alpha_m+1)}(t)) \end{aligned}$$

are well defined smooth functions.

In [10, 17, 18], we apply this method and provide a simple solution to the motion planning for systems studied in [11, 21, 14, 15] and describing the nonholonomic motion of a car with n trailers. We also remark that natural parametrizations instead of time parametrizations of the linearizing output curves $\{y(t) | \, t\}$ simplify the calculations (Frénet formula) and bypass singularities in (2) when $\dot{y} = 0$. We significantly prolonge this idea here: time-scaling is not only important for efficient computation of open-loop steering controls but also can be very useful for the design of trajectory stabilizing feedback controllers.

3 Trajectory stabilization

Consider the flat driftless system

$$\dot{x} = \sum_{i=1}^{m} u_i f_i(x), \quad x \in \mathbb{R}^n \quad (3)$$

with $y_i = h_i(x)$, $i = 1, \dots, m$ as flat output (the f_i's and h_i's are smooth functions and the vector fields f_i are linearly independent for all x). Then x and u are given by (2) with A and B defined on open and dense subsets of $\mathbb{R}^{\alpha_1+1} \times \dots \times \mathbb{R}^{\alpha_m+1}$ and $\mathbb{R}^{\alpha_1+2} \times \dots \times \mathbb{R}^{\alpha_m+2}$, respectively.

Consider the change of parametrization $t \mapsto \sigma(t)$ with σ an increasing function. The system equation (3) remains unchanged by replacing $\frac{d}{dt}$ by $\frac{d}{d\sigma}$ and u by $u\dot{\sigma}$ instead of u. Thus, under such transformations, the first equation of (2) giving x remains unchanged whereas the second one, giving u, is multiplied by $\dot{\sigma}$.

Theorem *Consider (3) and assume that there exist (a_i^j), $1 \le i \le m$ and $1 \le j \le \alpha_i$, such that $A((a_i^j)) = 0$ and the map A is a local submersion around (a_i^j).*

Then, for all $z \in \mathbb{R}^n$ close to 0 and $T > 0$, there exists a smooth open-loop control $[0, T] \ni t \mapsto u(t)$ steering (3) from $x(0) = z$, $u(0) = 0$, to $x(T) = 0$, $u(T) = 0$. There also exists a class of smooth time-varying dynamic feedbacks that stabilize the system around this reference trajectory in the following sense: the tracking error $e(t) \in \mathbb{R}^n$ satisfies the estimation for $t \in [0, T]$, $\|e(t)\| \le M \|e(0)\| \exp(-\sigma(t)/d)$ where $e(0)$ close to 0 and M is independent of $e(0)$, $d > 0$ depends on the design control parameters, $[0, t] \ni t \mapsto \sigma(t)$ depends only on the reference trajectory and is a smooth, non negative, strictly increasing function such that $\dot{\sigma}(0) = \dot{\sigma}(T) = 0$.

Sketch of proof. Since A is onto there exists (b_i^j) close to (a_i^j) such that $z = A((b_i^j))$. There exist $S > 0$, m smooth functions, $[0, S] \ni s \mapsto y_{i,c}(s)$ such that $\frac{d^j y_{i,c}}{ds^j}(0) = b_i^j$, $\frac{d^j y_{i,c}}{ds^j}(S) = a_i^j$, and $\frac{d^j y_{i,c}}{ds^j}(s)$ close to (a_i^j) for $i = 1, \dots, m$, $j = 1, \dots, \alpha_i$ and $s \in [0, S]$ (take, e.g., polynomials in s). Take $[0, T] \ni t \mapsto \sigma(t) \in [0, S]$ a smooth increasing function such that $\sigma(0) = 0$, $\sigma(T) = S$ and $\dot{\sigma}(0) = \dot{\sigma}(T) = 0$. then the open-loop control

$$\begin{aligned} u(t) &= \dot{\sigma}(t) \quad B \Big(y_{1,c}(\sigma(t)), \dots, \tfrac{d^{\alpha_1+1} y_{1,c}}{ds^{\alpha_1+1}}(\sigma(t)), \dots \\ &\qquad \dots, \quad y_{m,c}(\sigma(t)), \dots, \tfrac{d^{\alpha_m+1} y_{m,c}}{ds^{\alpha_m+1}}(\sigma(t)) \Big) \end{aligned}$$

steers (3) from $x = z$, $u = 0$ at $t = 0$ to $x = 0$, $u = 0$ at $t = T$. As for the car, the linearizing dynamic feedback is constructed in the s-scale. The method is borrowed from [12]. It relies on the fact that A is a submersion around (a_i^j). This leads to a smooth linearizing control with a linear and asymptotically stable error dynamics in the s-time-scale where d corresponds to the less stable tracking pole. ■

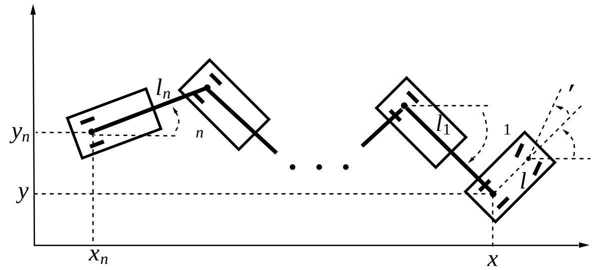

Figure 1: the standard n-trailer system.

4 The standard n-trailer systems

We follow the modeling assumptions of [15]. The notations are summarized on figure 1. A basic model is the following:

$$\begin{array}{rcl} \dot{x} &=& u_1 \cos\theta \\ \dot{y} &=& u_1 \sin\theta \\ \dot{\varphi} &=& u_2 \\ \dot{\theta} &=& \frac{u_1}{l} \tan\phi \\ \dot{\theta}_1 &=& \frac{u_1}{l_1} \sin(\theta-\theta_1) \\ &\vdots& \\ \dot{\theta}_n &=& \frac{u_1}{l_n} \cos(\theta-\theta_1)\cos(\theta_1-\theta_2)\ldots \\ && \ldots\cos(\theta_{n-2}-\theta_{n-1}) \quad \sin(\theta_{n-1}-\theta_n) \end{array} \tag{4}$$

where $(x, y, \phi, \theta, \theta_1 \ldots, \theta_n)$ is the state, (u_1, u_2) is the control and l, l_1, ..., l_n are positive parameters (lengths).

We know from [8] that this system is flat with the cartesian coordinate of the last trailer (x_n, y_n) as flat output. Following the geometric construction of [17], we consider a smooth curve $\mathcal{C}$ defined by the natural parametrization $[0, L] \ni s \mapsto (x_c(s), y_c(s)) \in \mathbb{R}^2$ (s is the arc length). Denote by $\kappa(s)$ the oriented curvature of $\mathcal{C}$, $\theta_c(s)$ the angle of the oriented tangent to $\mathcal{C}$.

For $T > 0$ we consider a smooth real increasing function $[0, T] \ni t \mapsto \sigma(t) \in [0, L]$ such that $\sigma(0) = 0$, $\sigma(T) = L$ and $\dot{\sigma}(0) = \dot{\sigma}(T) = 0$. In [17] open-loop controls $[0, T] \ni t \mapsto u_c(t)$ steering the system from a configuration to another one are explicitly given. They rely on the Frénet relationships of planar curve. They are based on a global diffeomorphism between $(x, y, \varphi, \theta, \theta_1, \ldots, \theta_n)$ and $(x_c, y_c, \theta_c, \kappa, \ldots, \frac{d^n\kappa}{ds^n})$, the contact structure at order $n+2$ of the curve $\mathcal{C}$ followed by (x_n, y_n) (the angles φ, $\theta-\theta_1$, ..., $\theta_{n-1}-\theta_n$ belonging to $]-\pi/2, \pi/2[$).

We just apply the previous theorem to linearize and stabilize the error dynamics with respect to the arc length $s = \sigma(t)$ of $\mathcal{C}$. We just sketch here the main steps of the control design.

Set $u_i = v_i\dot{\sigma}(t)$, $i = 1, 2$ where v_i are new control variables. Since

$$\begin{array}{l} \frac{d}{d\sigma}x_n = v_1\cos\theta_n \;\; \cos(\theta-\theta_1)\cos(\theta_1-\theta_2)\ldots\cos(\theta_{n-1}-\theta_n) \\ \frac{d}{d\sigma}y_n = v_1\sin\theta_n \;\; \cos(\theta-\theta_1)\cos(\theta_1-\theta_2)\ldots\cos(\theta_{n-1}-\theta_n) \end{array}$$

we consider the static feedback (regular since all the angles $\theta-\theta_1$, ..., $\theta_{n-1}-\theta_n$ belong to $]-\pi/2, \pi/2[$)

$$\begin{array}{l} v_1 = \frac{\overline{v}_1}{\cos(\theta-\theta_1)\cos(\theta_1-\theta_2)\ldots\cos(\theta_{n-1}-\theta_n)} \\ v_2 = \overline{v}_2. \end{array} \tag{5}$$

Then we introduce the dynamic compensator of order $n+2$

$$\begin{array}{rcl} \frac{d}{d\sigma}\xi_i &=& \xi_{i+1} \quad i = 1, \ldots, n+1 \\ \frac{d}{d\sigma}\xi_{n+2} &=& w_1 \\ \overline{v}_1 &=& \xi_1 \\ \overline{v}_2 &=& w_2. \end{array} \tag{6}$$

Then the inversion of (4,5,6) in σ-scale, with (x_n, y_n) as output and (w_1, w_2) as input, leads to an invertible 2×2 decoupling matrix and a regular feedback on $\mathcal{X} = (x, y, \varphi, \theta, \theta_1, \ldots, \theta_n, \xi_1, \ldots, \xi_{n+2})$,

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \alpha(\mathcal{X}) + \beta(\mathcal{X}) \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} \tag{7}$$

that linearize the system dynamics with respect to the σ-scale:

$$\frac{d^{n+3}x_n}{d\sigma^{n+3}} = \overline{u}, \quad \frac{d^{n+3}y_n}{d\sigma^{n+3}} = \overline{v}.$$

Standard linear asymptotic tracking methods can be used to ensure the exponential convergence of the tracking error $(x_n - x_c, y_n - y_c)$ to zero in the σ-scale. Expressing this controller in the t-scale reveals no difficulties and yields a smooth time-varying dynamic feedback.

For the backward motions displayed on figure 2, one has $n = 0$, $l = 1$. m and the three tracking poles correspond to the following lengths $l/3.5$, $l/3$ and $l/2.5$. Notice that, after a distance of around 1 m, $e(0)$ is divided by two and that, for $t = T$, $e(T) \approx 0$. Such asymptotic stabilization strategy is interesting when the length L of the reference trajectory $\mathcal{C}$ is much larger than the car length l: typically, we have here $L/l > 3$.

For the backward motions displayed on figure 3, one has $n = 2$, $l = 2$ m, $l_1 = 3$ m, $l_2 = 2$ m and the five tracking poles correspond to the following lengths 2 m, 1.8 m, 1.6 m, 1.4 m, 1.2 m. As for the car, such asymptotic stabilization strategy is interesting when the length L of the reference trajectory $\mathcal{C}$ is much larger than the system length $l + l_1 + l_2$: typically, we have here $L/(l + l_1 + l_2) > 2$.

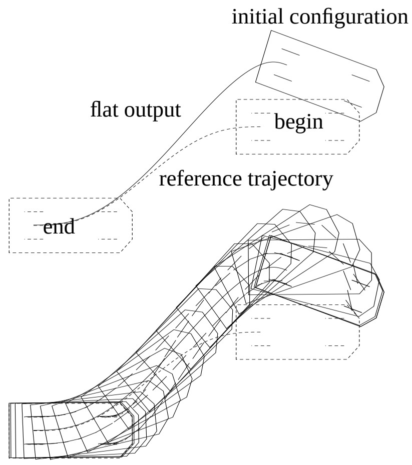

Figure 2: the asymptotic stabilization of a backward trajectory for the car.

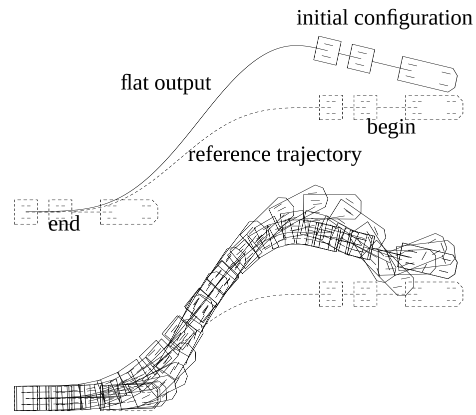

Figure 3: the asymptotic stabilization of a backward trajectory for the standard 2-trailer system.

5 Conclusion

We have described a stabilization method valid for driftless flat systems whose singularities results from the time parametrization. The extension of such method for complex singularities is an open question.

This strategy may be combined with the robust stabilization method proposed in [1]. This leads to approximate motion planning for general trailer systems including non flat ones [13], but close to standard trailer systems.

This method can also be extended to more general systems than driftless ones. The above time scaling is a particular case of clock control introduced in [9].

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