INDEX OF AN IMPLICIT TIME-VARYING LINEAR DIFFERENTIAL EQUATION: A NONCOMMUTATIVE LINEAR ALGEBRAIC APPROACH.*

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Abstract

An intrinsic definition of the index for implicit linear time-varying differential systems is proposed. This definition relies on linear algebraic techniques over the noncommutative principal ideal ring of non-constant linear differential operators (noncommutative transfer function or state-variable representation). Accordance with the existing definition for constant coefficient linear systems is demonstrated.

Keywords: implicit linear differential systems, index, noncommutative principal ideal rings, modules, transfer functions.

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1 Introduction

Integrating *implicit* differential equations raises severe difficulties, even in the linear case. A rather abundant literature has been devoted to this subject, which recently culminated in three books [1, 8, 9].

A major ingredient for understanding such implicit equations is their index, which is familiar among numerical analysts and engineers. It measures the number of times the equations have to be differentiated in order to set the system explicitly, or, equivalently, the number of times the entries or sources have to be differentiated.

A clear-cut definition of the index has only been given in the case of linear differential equations with constant coefficients by means of Kronecker canonical form of regular matrix pencil (see [15], e.g.). Our main contribution is a definition of the index for linear equations with time-varying coefficients which bypasses some problems related to the existence of the flow and to some computational procedures. It employs linear algebra over the noncommutative principal ideal ring of non-constant linear differential operators (see [10, 3], e.g.). It is perhaps worthwhile to recall here that this kind of algebra was introduced a long time ago partly in order to provide the appropriate setting for linear differential systems with non constant coefficients (see [14], e.g.).

As often done, we interpret the index problem within the control theoretic framework of input-output differential systems: the inputs are the exogeneous variables, called entries or sources, and the outputs are the unknown variables. The index can then be defined in two ways:

– a noncommutative extension of transfer functions, which was already achieved for other control purposes by some authors (see [11], e.g.);

– a state-variable representation via module theory (see [5]) which proves that there exists an integration procedure where the derivatives of the entries, even if they play a role, do not need to be integrated.

The paper is organized as follows. Section 2 gives the necessary algebraic background on matrices and modules over the noncommutative principal ideal ring of non-constant linear differential operators. Section 3 sets up the linear implicit differential equation. The index is defined in sections 4 and 5 respectively via the noncommutative transfer function and the state variable representation. Section 6 shows the equivalence of those two viewpoints and demonstrates their accordance with the existing work on linear systems with constant coefficients.

A preliminary version of this work was presented in [6]. A nonlinear extension is sketched in [7] and will be developed elsewhere.

2 Algebraic background

2.1 Denote by k a given *differential field* (see [12]), i.e., a commutative field k equipped with a derivation $\frac{d}{dt} =$ "$\cdot$", which satisfies the following rules:

$$\forall a \in k, \ \frac{da}{dt} \in k,$$
$$\forall a, \ b \in k, \ \frac{d}{dt}(a+b) = \frac{da}{dt} + \frac{db}{dt}, \quad \text{and} \quad \frac{d}{dt}(ab) = \frac{da}{dt}b + a\frac{db}{dt}.$$

A *constant* is an element $c \in k$ such that $\frac{dc}{dt} = 0$. A *field of constants* is a differential field which only contains constants.

2.2 Examples

(i) The fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ of rational, real and complex numbers are trivial examples of differential fields of constants.

(ii) The field $\mathbb{R}(t)$ of real rational functions is a differential field with respect to $\frac{d}{dt}$.

(iii) The set of meromorphic functions in the variable t over an open connected domain of $\mathbb{R}$ or $\mathbb{C}$ is a differential field with respect to $\frac{d}{dt}$.

2.3 Assumption Throughout the rest of the paper, we assume that k is a differential field of meromorphic time functions from an open connected subset of $\mathbb{R}$ into $\mathbb{R}$. There would be no problem of dropping this assumption by an extensive use of the formalism of differential algebra [12].

2.4 Denote by $k\left[\frac{d}{dt}\right]$ the set of linear operators

$$\sum_{\text{finite}} a_\mu \frac{d^\mu}{dt^\mu} \quad (a_\mu \in k).$$

Provided with the usual addition and the multiplication defined as composition of operators, $k\left[\frac{d}{dt}\right]$ becomes an entire ring, i.e., without zero divisors. If k is a field of constants, $k\left[\frac{d}{dt}\right]$ is, as well known, a *commutative principal ideal* ring.

Clearly, if k is not a field of constants, i.e., there exists $a \in k$ such that $\frac{da}{dt} \neq 0$, then $k\left[\frac{d}{dt}\right]$ is not commutative:

$$\frac{d}{dt}\left(a\frac{d}{dt}\right) = \dot{a}\frac{d}{dt} + a\frac{d^2}{dt^2} \neq a\frac{d}{dt}\left(\frac{d}{dt}\right) = a\frac{d^2}{dt^2}.$$

However, $k\left[\frac{d}{dt}\right]$ still is a principal ideal ring (see [10, 2], e.g.).

2.5 Even in the noncommutative case, $k\left[\frac{d}{dt}\right]$ can be embedded into a *skew*, i.e., noncommutative, field, or division ring, of fractions $Q\left(k\left[\frac{d}{dt}\right]\right)$, as it satisfies the so called Ore property (see [2]) which implies that any element of $Q\left(k\left[\frac{d}{dt}\right]\right)$ can be written as $u^{-1}v$, where u and v belong to $k\left[\frac{d}{dt}\right]$.

In order to be in accordance with the classical Laplace transform techniques (see section 4), set $s = \dfrac{d}{dt}$. A series of the form

$$\sum_{\nu \geq \nu_0} a_\nu s^{-\nu} \quad \text{where} \quad \nu_0 \in \mathbb{Z}, \quad a_\nu \in k,$$

is called a *Laurent series*. Every element of $k\left[\frac{d}{dt}\right]$ corresponds to a Laurent series with a finite number of nonzero terms. From the Ore property, every element of $Q\left(k\left[\frac{d}{dt}\right]\right)$ can be developed into a Laurent series. The set of all the Laurent series corresponding to elements of $Q\left(k\left[\frac{d}{dt}\right]\right)$ will be denoted by $k(s^{-1})$.

2.6 Take a $p \times q$ matrix A over $k\left[\frac{d}{dt}\right]$. Even in the noncommutative case (see [2]), there exist matrices $P \in GL_p\left(k\left[\frac{d}{dt}\right]\right)$ and $Q \in GL_q\left(k\left[\frac{d}{dt}\right]\right)$ [1] such that PAQ^{-1} is a $p \times q$ matrix with main diagonal of the form $(\varepsilon_1, \ldots, \varepsilon_r, 0, \ldots, 0)$ and 0 elsewhere, where the *invariant factors* $\varepsilon_1, \ldots, \varepsilon_r$ are such that

- $\varepsilon_1, \ldots, \varepsilon_r \neq 0$
- ε_i is a *total divisor* of ε_j for $i \leq j$, i.e., $\exists \delta_{ij} \in k\left[\frac{d}{dt}\right]$ with $\varepsilon_j = \delta_{ij}\varepsilon_i$.

Clearly, the interger r, which is less than or equal to $\min(p, q)$, is the *rank* of A.

Two $p \times q$ matrices A and B are equivalent if, and only if, there exist matrices $P \in GL_p\left(k\left[\frac{d}{dt}\right]\right)$ and $Q \in GL_q\left(k\left[\frac{d}{dt}\right]\right)$ such that $A = PBQ^{-1}$.

If A is square ($p = q$) and if $r = p$, then A is said to be *full*. It is equivalent saying that there exists a $p \times p$ matrix A^{-1} over $k(s^{-1})$ such that $AA^{-1} = A^{-1}A = 1$.

2.7 Remark Solutions ξ of an equation $A\xi = 0$, where A is a $p \times q$ matrix over $k\left[\frac{d}{dt}\right]$, are understood as meromorphic time functions.

2.8 Left (or right) modules over $k\left[\frac{d}{dt}\right]$ behave in much the same manner as modules over commutative principal ideal rings (see [2]). Let M be a finitely generated left $k\left[\frac{d}{dt}\right]$-module. An element $m \in M$ is said to be *torsion* if, and only if, there exists $\pi \in k\left[\frac{d}{dt}\right]$, $\pi \neq 0$, such that $\pi m = 0$. For instance, $\sin t$, which satisfies $\left(\dfrac{d^2}{dt^2} + 1\right) \sin t = 0$, is

[1] $GL_p\left(k\left[\frac{d}{dt}\right]\right)$ is the set of $p \times p$ matrices P over $k\left[\frac{d}{dt}\right]$ such that there exists a $p \times p$ matrix P^{-1} over $k\left[\frac{d}{dt}\right]$, satisfying $PP^{-1} = P^{-1}P = 1$. Matrices in $GL_p\left(k\left[\frac{d}{dt}\right]\right)$ are called *unimodular*.

torsion. A module is said to be torsion if, and only if, all its elements are torsion. The set of all torsion elements of M is called the *torsion submodule* of M. If this torsion submodule is trivial, i.e., equal to $\{0\}$, M is said to be *free*[2]. The next two results, which will play a crucial role, can be found in [2]:

Theorem 1 *For a finitely generated left $k\left[\frac{d}{dt}\right]$-module M, the next two conditions are equivalent:*

(i) *M is torsion;*

(ii) *M is finite dimensional as a k-vector space.*

Theorem 2 *Let M be the left $k\left[\frac{d}{dt}\right]$-module spanned by the solutions ξ of $A\xi = 0$, where A is a $p \times p$ matrix over $k\left[\frac{d}{dt}\right]$. Then M is torsion if, and only if, A is full.*

3 The linear differential implicit equation

3.1 Consider the differential equation

$$A_\alpha y^{(\alpha)} + \ldots + A_1 \dot{y} + A_0 y = e \tag{1}$$

where

- $y = (y_1, \ldots, y_p)^T$ is a set of p unknowns,
- $e = (e_1, \ldots, e_p)^T$ is a set of p independent entries or sources,
- A_0, A_1, $\ldots$, A_α are $p \times p$ matrices over a given differential ground field k.

When writing

$$A = A_\alpha \frac{d^\alpha}{dt^\alpha} + \ldots + A_1 \frac{d}{dt} + A_0,$$

(1) becomes

$$Ay = e. \tag{2}$$

It follows directly from section 2.6 that (2) is solvable in an algebraic sense for any choice of e if, and only if, A is full: for any meromorphic entries $e = (e_1, \ldots, e_p)^T$, there exist p meromorphic functions $y = (y_1, \ldots, y_p)^T$ satisfying (2). We are therefore lead to the following definition

Definition 1 *System (1)-(2) is called solvable if, and only if, the matrix A is full.*

We will make this assumption from now on.

[2]This is not the usual definition of free module, but a characterization which holds for finitely generated modules over principal ideal rings [2].

3.2 Example. It is borrowed from [9] and confirms the necessity of employing noncommutative algebra. Take $k = \mathbb{R}(t)$, $\alpha = 1$, $p = 2$ and

$$A_0 = \begin{pmatrix} 0 & 0 \\ 1 & t \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix}.$$

The matrix $A_1\lambda + A_0$ is singular for all λ in any commutative overfield of k. By setting

$$y = \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \tilde{y},$$

the equation (2) becomes

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{d\tilde{y}}{dt} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{y} = e.$$

This linear differential equation with constant coefficients is obviously solvable: the matrix

$$\tilde{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{d}{dt} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

over $\mathbb{R}\left[\frac{d}{dt}\right]$ is invertible, with inverse being

$$\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}.$$

Thus $A = A_1 \frac{d}{dt} + A_0$ is full as a matrix over $k\left[\frac{d}{dt}\right]$. Notice that the operator A is equivalent to $\tilde{A}$ in the sense of 2.6. This apparent contradiction with the singularity of $A_1\lambda + A_0$ can be traced back to the noncommutativity of $k\left[\frac{d}{dt}\right]$. Notice furthermore that noncommutative determinants have been introduced by Dieudonné [4] (see [3] for a more up to date presentation).

4 The transfer matrix approach to the index

4.1 The inverse matrix A^{-1}, which is over $k(s^{-1})$, can be written as a matrix Laurent series

$$A^{-1} = \sum_{\substack{\nu \geq \nu_0 \\ \nu_0 \in \mathbb{Z}}} \Gamma_\nu s^{-\nu}, \tag{3}$$

where the Γ_ν's are $p \times p$ matrices over k: (3) is the *noncommutative transfer matrix* of (1)-(2). Noncommutativity here means that s^{-1} and coefficients in k do not commute.

4.2 As the index measures the number of the times one needs to differentiate e in order to get the solutions, the following is natural.

Definition 2 *Assume that* $\Gamma_{\nu_0} \neq 0$*. The index of (1)-(2) is* $\max(0, 1 - \nu_0)$.

4.3 **Remark** When k is a field of meromorphic functions in the variable t, singularities might happen such that for some values of t, Γ_{ν_0} is not defined or zero. This is related to the difficult analysis of singularities of time-varying linear systems (see [13], e.g.).

5 The state variable approach to the index

5.1 Take the free left $k\left[\frac{d}{dt}\right]$-module $[e^+, y^+]$ spanned by $e^+ = (e_1^+, \ldots, e_p^+)^T$ and $y^+ = (y_1^+, \ldots, y_p^+)^T$. The components of e^+ and y^+ are here arbitary algebraic quantities, as often done in the algebraic setup [2, 12]. In our context they can be viewed as arbitrary meromorphic time functions. Denote by $d^+ = (d_1^+, \ldots, d_p^+)^T$ a set of elements in $[e^+, y^+]$ such that

$$d^+ = Ay^+ - e^+$$

where A comes for (2). Denote by $[d^+]$ the submodule of $[e^+, y^+]$ spanned by d^+. Denote by $e = (e_1, \ldots, e_p)^T$, $y = (y_1, \ldots, y_p)^T$ the *residues*, i.e., the canonical images of e^+ and y^+ in the quotient module $[e^+, y^+]/[d^+]$. By construction e and y satisfy (2). From Theorem 2, we know that the quotient module $[\underline{y}] = [e, y]/[e]$, where $\underline{y} = (\underline{y}_1, \ldots, \underline{y}_p)$ is the residue of y, is torsion as $A\underline{y} = 0$ and A is full.

Following [5], we thus have associated to (1)-(2) a left $k\left[\frac{d}{dt}\right]$-module $[e, y]$ such that the quotient module $[e, y]/[e]$ is torsion.

5.2 The dimension as a k-vector space of the finitely generated torsion module $[e, y]/[e]$ is finite, say n (Theorem 1). Take $\eta = (\eta_1, \ldots, \eta_n)^T$ in $[e, y]$ such that its residue $\underline{\eta} = (\underline{\eta}_1, \ldots, \underline{\eta}_n)^T$ in $[e, y]/[e]$ is a basis of the latter vector space. The components of $\frac{d\underline{\eta}}{dt}$ and of $\underline{y}$ are k-linearly dependent on $\underline{\eta}$. This yields

$$\left\{ \begin{array}{rcl} \dfrac{d\underline{\eta}}{dt} & = & F^+ \underline{\eta} \\ \underline{y} & = & H^+ \underline{\eta} \end{array} \right. \tag{4}$$

where F^+ and H^+ are respectively $n \times n$ and $p \times n$ matrices over k. Pulling back (4) to $[e, y]$ gives

$$\left\{ \begin{array}{l} \dfrac{d\eta}{dt} = F^+\eta + \displaystyle\sum_{\mu=0}^{\tau} G_\mu^+ e^{(\mu)} \\ y = H^+\eta + \displaystyle\sum_{\mu'=0}^{\tau'} J_{\mu'}^+ e^{(\mu')} \end{array} \right. \tag{5}$$

where the G_μ^+'s and $J_{\mu'}^+$'s are respectively $n \times p$ and $p \times p$ matrices over k.

Take another set $\tilde{\eta} = (\tilde{\eta}_1, \ldots, \tilde{\eta}_n)^T$ in $[e, y]$ such that its residue $\underline{\tilde{\eta}}$ in $[e, y]/[e]$ is a basis. Then

$$\underline{\eta} = T\underline{\tilde{\eta}}$$

where T is invertible over k. Thus

$$\eta = T\tilde{\eta} + \sum_{\text{finite}} S_j e^{(j)}$$

where the S_j's are $n \times p$ matrices over k.

If derivatives of e occur in (5) in the dynamics of η, i.e., if $\tau \geq 1$, set

$$\eta = \tilde{\eta} + G_\tau^+ e^{(\tau-1)}.$$

It yields

$$\frac{d\tilde{\eta}}{dt} = F^+\tilde{\eta} + \sum_{\mu=0}^{\tau-1} G_\mu^{++} e^{(\mu)}$$

where the highest derivative of e now is less than or equal to $\tau - 1$. By this elimination procedure we obtain in a finite number of steps

$$\left\{ \begin{array}{rcl} \dfrac{dx}{dt} & = & Fx + Ge \\ y & = & Hx + \displaystyle\sum_{\nu=0}^{\nu_1} J_\nu e^{(\nu)} \end{array} \right. \tag{6}$$

where

- all the matrices are of appropriate sizes over k;
- the dynamics of x is *Kalman* [5], i.e., does not contain any derivatives of e.

(6) is the *Kalman state variable* representation of (1)-(2).

5.3 The *degree* of the *impulsive polynomial* $I = \sum_{\nu=0}^{\nu_1} J_\nu e^{(\nu)}$ is

$$\deg I = \begin{cases} +\infty & \text{if} \quad I = 0 \\ \nu_1 & \text{if} \quad I \neq 0 \text{ and } J_{\nu_1} \neq 0. \end{cases}$$

Definition 3 *The index of the solvable system (1)-(2) is equal to* $\max(0, 1 + \deg I)$

The previous remark about possible singularities also applies here.

6 Main results

6.1 We first verify the equivalence between our two definitions of the index.

Proposition 1 *The definitions 2 and 3 of the index of (1)- (2) coincide.*

Proof The transfer matrix of (6) is

$$H(s-F)^{-1}G + \sum_{\nu=0}^{\nu_1} J_\nu s^\nu. \tag{7}$$

It is equal to (3). The parts corresponding to non negative powers of s in (3) and in (7) coincide:

$$\sum_{\nu=0}^{\nu_1} J_\nu s^\nu = \sum_{\nu=0}^{-\nu_0} \Gamma_\nu s^\nu.$$

Therefore $\nu_0 = -\nu_1$. ■

6.2 The main advantage of the state variable representation (6) can be summarized as follows.

Proposition 2 *The linear differential equation (1)-(2) can be integrated via (6) where the derivatives of e do not appear in the dynamics of x but act "instantaneously"*[3] *on y.*

6.3 We now proceed to the equivalence of our index with the one already given for constant linear differential equations. Assume that k is a field of constants, $\mathbb{R}$ for instance, and, for simplicity's sake, that $\alpha = 1$ in (1). Definition 1 means that the matrix pencil $A_1\lambda + A_0$ is regular. Its Kronecker canonical form is determined by

$$\begin{aligned} BA_0C^{-1} &= \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \\ BA_1C^{-1} &= \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix} \end{aligned}$$

[3]This terminology, which is usual in engineering, just means that derivatives of e are not needed in any integration procedures.

where

- B, C are invertible $p \times p$ matrices over k;
- R is a $\rho \times \rho$ matrix over k;
- N is a nilpotent $(p-\rho) \times (p-\rho)$ matrix over k.

Sincovec *et al.* [15] define the index of (1) as being the nilpotency index of N, i.e., the least non-negative integer $\imath$ such that $N^\imath = 0$. It is straightforward to verify that the highest non-negative power of s in the Laurent expansion of $(A_1 s + A_0)^{-1}$ is $\nu_1 = -\nu_0 = \imath - 1$.

We have demonstrated the following property:

Proposition 3 *The index of (1), where k is a field of constants and $\alpha = 1$, is the index of nilpotency of N.*

6.4 Suppose that instead of y, we calculate $z = (z_1, \ldots, z_p)^T = Uy$, where U is an invertible $p \times p$ matrix over k. The quantity z also satisfies a type (1)-(2) equation. The transfer matrix of z is UA^{-1}, which implies the following property:

Proposition 4 *The indices of $Ay = 0$ and $AU^{-1}z = 0$ are the same.*

Otherwise stated, the index is invariant under linear changes of the unknown variables.

7 Conclusion

The index of more general implicit linear differential equations can be defined in a completely analogous manner. Take, for instance, the square system

$$A_\alpha y^{(\alpha)} + \ldots + A_1 \dot{y} + A_0 y = Be$$

where e and y do not have necessarily the same dimension : $y = (y_1, \ldots, y_p)^T$ is a set of p unknowns; $e = (e_1, \ldots, e_q)^T$ is a set of q independant entries; A_0, A_1, $\ldots$, A_α are $p \times p$ matrices over a differential field k; B is a $p \times q$ matrix over k.

Calculations with respect to the noncommutative principal ideal ring $k\left[\frac{d}{dt}\right]$ are not much more complicated than in the classic commutative, i.e., constant coefficients, case, since there also exists a very similar division algorithm (see [2]). We therefore feel that our approach could be made quite effective from a computer algebraic standpoint.

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