# Flatness, motion planning and trailer systems*

Pierre Rouchon† Michel Fliess‡ Jean Lévine§ Philippe Martin¶

## Abstract

A solution of the motion planning without obstacles for the standard $n$-trailer system is proposed. This solution relies basically on the fact that the system is flat with the Cartesian coordinates of the last trailer as a linearizing output. The Frénet formulae are used to simplify the calculations and permit to deal with angle constraints. The general 1-trailer system, where the trailer is not directly hitched to the car at the center of the rear axle, is also flat. The geometric construction used for the standard 1-trailer system can be extended to this more realistic system. MATLAB simulations illustrate this method.

## 1 Introduction

In [5, 6, 7, 4, 8, 9] a new point of view on the full linearization problem via dynamic feedback [3] is proposed by introducing the notion of flatness and linearizing output. The aim of this paper is to show that such a standpoint can be very useful for motion planning.

Roughly speaking, a control system is said to be (*differentially*) *flat* if the following conditions are satisfied:

1. there exists a finite set $y = (y_1, \dots, y_m)$ of variables which are differentially independent, i.e., not related by any differential equations.

2. the $y_i$'s are differential functions of the system variables, i.e., are functions of the system variables (state, input, ...) and of a finite number of their derivatives.

3. Any system variable is a differential function of the $y_i$'s, i.e., is a function of the $y_i$'s and of a finite number of their derivatives.

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†Centre Automatique et Systèmes, École des Mines de Paris, 60, bd. Saint-Michel, 75272 Paris Cedex 06, France. Tel: (33) (1) 40 51 91 15. E-mail: rouchon@cas.ensmp.fr

‡Laboratoire des Signaux et Systèmes, CNRS-ESE, Plateau de Moulon, 91192 Gif-sur-Yvette Cedex, France. Tel: (33) (1) 69 41 80 40. E-mail: fliess@lss.supelec.fr

§Centre Automatique et Systèmes, École des Mines de Paris, 35, rue Saint-Honoré, 77305 Fontainebleau Cedex, France. Tel: (33) (1) 64 69 48 58. E-mail:levine@cas.ensmp.fr

¶Centre Automatique et Systèmes, École des Mines de Paris, 35, rue Saint-Honoré, 77305 Fontainebleau Cedex, France. Tel: (33) (1) 64 69 48 57. E-mail: martin@cas.ensmp.fr

We call $y = (y_1, \dots, y_m)$ a *flat* or *linearizing* output. Its number of components equals the number of independent input channels.

Notice however that the concept of flatness, which can be made quite precise via the language of *differential algebra* [5, 6, 7, 4] and of *differential geometry* [8, 9], is best defined by not distinguishing between input, state, output and other variables.

For a "classic" dynamics,

$$\dot{x} = f(x, u), \quad x = (x_1, \dots, x_n), \quad u = (u_1, \dots, u_m), \quad (1)$$

flatness implies the existence of a vector-valued function $h$ such that

$$y = h(x, u_1, \dots, u_1^{(\beta_1)}, \dots, u_m, \dots, u_m^{(\beta_m)}),$$

where $y = (y_1, \dots, y_m)$. The components of $x$ and $u$ are, moreover, given without any integration procedure by the vector-valued functions $A$ and $B$:

$$\begin{array}{rcl} x &=& A(y_1, \dots, y_1^{(\alpha_1)}, \dots, y_m, \dots, y_m^{(\alpha_m)}) \\ u &=& B(y_1, \dots, y_1^{(\alpha_1+1)}, \dots, y_m, \dots, y_m^{(\alpha_m+1)}). \end{array} \quad (2)$$

The motion planning problem for (1) consists in finding the control $[0, T] \ni t \to u(t)$ steering the system from the state $x = p$ at $t = 0$ to the state $x = q$ at $t = T$. When the system is flat, this problem is equivalent to finding $[0, T] \ni t \to y(t)$ such that

$$p = A(y_1(0), \dots, y_1^{(\alpha_1)}(0), \dots, y_m(0), \dots, y_m^{(\alpha_m)}(0))$$

and

$$q = A(y_1(T), \dots, y_1^{(\alpha_1)}(T), \dots, y_m(T), \dots, y_m^{(\alpha_m)}(T)).$$

Since the mapping

$$\begin{array}{l} (y_1, \dots, y_1^{(\alpha_1)}, \dots, y_m, \dots, y_m^{(\alpha_m)}) \\ \quad \to A(y_1, \dots, y_1^{(\alpha_1)}, \dots, y_m, \dots, y_m^{(\alpha_m)}) \end{array}$$

is onto, in general, the problem consists in finding a smooth function $t \to y(t)$ with prescribed values for some of its derivatives at time 0 and time $T$ and such that

$$[0, T] \ni t \to A(y_1(t), \dots, y_1^{(\alpha_1)}(t), \dots, y_m(t), \dots, y_m^{(\alpha_m)}(t))$$

and

$$\begin{array}{l} [0, T] \ni t \to \\ \quad B(y_1(t), \dots, y_1^{(\alpha_1+1)}(t), \dots, y_m(t), \dots, y_m^{(\alpha_m+1)}(t)) \end{array}$$

are well defined smooth functions.

This paper addresses the standard $n$-trailer system considered by many authors (see, e.g., [22, 24, 17, 21, 16] and the references herein) and the general $n$-trailer system that, as far as we know, has never been considered in details. The general $n$-trailer system is more realistic and differs from the standard one by the fact that trailer $i+1$ is hitched to trailer $i-1$ not directly at the center of its rear axle, but at a certain distance of this point (see, e.g., figure 5).

Section 2 completes [19] where parking simulations were presented. The standard $n$-trailer system is shown to be flat. The Cartesian coordinates of the last trailer give the linearizing output. The natural parameterization of the curve followed by the last trailer and the Frénet formulas are introduced. They lead to a simple geometric construction underlying proposition 2 and provide a global and constructive solution of the motion planning problem.

In section 3, the general 1-trailer system is considered. This system is flat but the construction of the linearizing output is much more involved and relies on [15, 14] where an old result of E. Cartan [2] is used. Nevertheless, the geometric construction used for the standard 1-trailer system can be extended to the general one and proposition 4 gives a global answer to the motion planning. As in [19], detailed calculations and MATLAB simulations are given.

## 2 The standard $n$-trailer system

The notations are summarized on figure 1. The model is the following :

$$
\begin{aligned}
\dot{x}_0 &= \cos(\theta_0)\, u_1 \\
\dot{y}_0 &= \sin(\theta_0)\, u_1 \\
\dot{\varphi} &= u_2 \\
\dot{\theta}_0 &= \frac{1}{d_0} \tan(\varphi)\, u_1 \\
& \text{for } i = 1, \ldots, n \\
\dot{\theta}_i &= \frac{1}{d_i} \left( \prod_{j=1}^{i-1} \cos(\theta_{j-1} - \theta_j) \right) \sin(\theta_{i-1} - \theta_i)\, u_1
\end{aligned}
\tag{3}
$$

where $(x_0, y_0, \varphi, \theta_0, \ldots, \theta_n) \in \mathbb{R}^2 \times (S^1)^{n+2}$ is the state, $(u_1, u_2)$ is the control and $d_0, d_1, \ldots, d_n$ are positive lengths.

In [6],the following proposition was proved.

**Proposition 1** *The car with $n$ trailers described by (3) is a flat system. A linearizing output corresponds to the Cartesian coordinates of the point $P_n$, the middle of the wheels axle of the last trailer :*

$$
y = \begin{pmatrix} x_0 - \sum_{i=1}^{n} \cos(\theta_i) d_i \\ y_0 - \sum_{i=1}^{n} \sin(\theta_i) d_i \end{pmatrix}.
$$

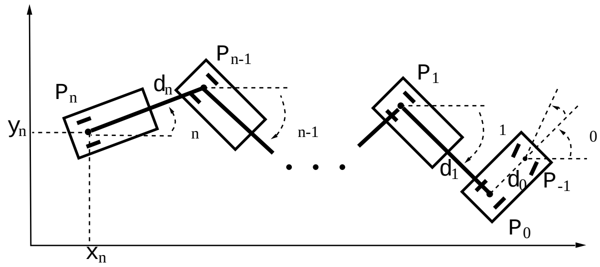


Figure 1: the standard $n$-trailer system is flat with linearizing output $P_n$'s coordinates.

In [19] was also hinted that the computations of the explicit form (2) are strongly simplified by the Frénet formula and a geometric construction. Here we complete proposition 2 of [19] by a more precise one where natural physical constraints (the impossibility of the trailer $i$ to be in front of trailer $i-1$) are explicitly satisfied.

**Proposition 2** *Consider (3) and two different state-space configurations: $\tilde{p} = (\tilde{x}_0, \tilde{y}_0, \tilde{\varphi}, \tilde{\theta}_0, \ldots, \tilde{\theta}_n)$ and $\overline{p} = (\overline{x}_0, \overline{y}_0, \overline{\varphi}, \overline{\theta}_0, \ldots, \overline{\theta}_n)$. Assume that the angles $\tilde{\theta}_{i-1} - \tilde{\theta}_i$, $i = 1, \ldots, n$, $\tilde{\varphi}$, $\overline{\theta}_{i-1} - \overline{\theta}_i$, $i = 1, \ldots, n$, and $\overline{\varphi}$ belong to $]-\pi/2, \pi/2[$. Then, there exists a smooth open-loop control $[0, T] \ni t \to (u_1(t), u_2(t))$ steering the system from $\tilde{p}$ at time $0$ to $\overline{p}$ at time $T > 0$, such that the angles $\theta_{i-1} - \theta_i$, $i = 1, \ldots, n$, and $\varphi$ ($i = 1, \ldots, n$) always remain in $]-\pi/2, \pi/2[$ and such that $(u_1(t), u_2(t)) = 0$ for $t = 0, T$.*

The detailed proof is given here below. It is constructive and gives explicitly $(u_1(t), u_2(t))$. The basic ideas and formula are as follows. Denote by $\mathcal{C}_i$ the curve followed by $P_i$, $i = 0, \ldots, n$. As displayed on figure 2, the point $P_{i-1}$ belongs to the tangent to $\mathcal{C}_i$ at $P_i$ and at the fixed distance $d_i$ from $P_i$:

$$
P_{i-1} = P_i + d_i \tau_i
$$

with $\tau_i$ the unitary tangent vector to $\mathcal{C}_i$. Deriving this relation with respect to $s_i$, the arc length of $\mathcal{C}_i$, leads to

$$
\frac{d}{ds_i} P_{i-1} = \tau_i + d_i \kappa_i \nu_i
$$

where $\nu_i$ is the unitary vector orthogonal to $\tau_i$ and $\kappa_i$ is the curvature of $\mathcal{C}_i$. Since $\frac{d}{ds_i} P_{i-1}$ gives the tangent direction to $\mathcal{C}_{i-1}$, we have

$$
\tan(\theta_{i-1} - \theta_i) = d_i \kappa_i .
$$

Proposition 2 relies on the following technical and constructive lemma.

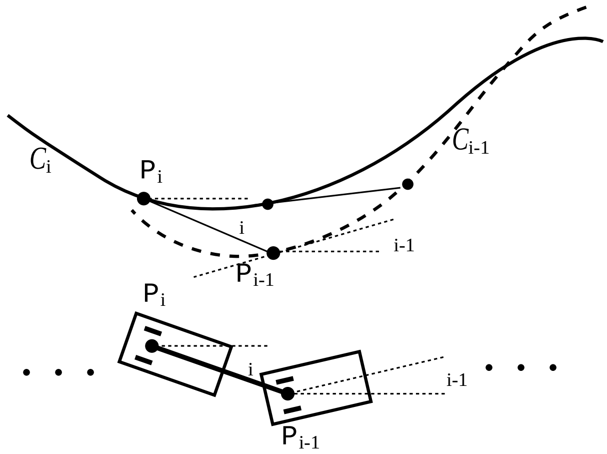


Figure 2: The geometric interpretation of the rolling without slipping conditions for the standard $n$-trailer system.

**Lemma** *Consider a trajectory of (3) such that the curve $\mathcal{C}_n$ followed by $P_n$ is smooth with the natural parameterization $[0, L_n] \ni s_n \to P_n(s_n)$: $s_n = 0$ (resp. $s_n = L_n$) corresponds to the starting point (resp. end point); $L_n$ is the length of $\mathcal{C}_n$. Assume also that for $s_n = 0$, $\theta_{i-1} - \theta_i$ ($i = 1, \ldots, n$) and $\varphi$ belong to $]-\pi/2, \pi/2[$. Then,*

- ***(i)*** *for all $s_n \in [0, L_n]$, $\theta_{i-1} - \theta_i$ ($i = 1, \ldots, n$) and $\varphi$ belong to $]-\pi/2, \pi/2[$.*
- ***(ii)*** *the curves $\mathcal{C}_i$ and $\mathcal{C}$ followed by $P_i$ and $Q$ are smooth ($i = 0, 1, \ldots, n$).*
- ***(iii)*** *$\tan(\theta_{i-1} - \theta_i) = d_i \kappa_i$ ($i = 1, \ldots, n$) and $\tan\varphi = d_0 \kappa_0$, where $\kappa_i$ and $\kappa_0$ are the curvatures of $\mathcal{C}_i$ and $\mathcal{C}_0$, respectively;*
- ***(iv)*** *the curvature $\kappa_i$ can be expressed as a smooth function of $\kappa_n$ and of its first $n-i$ derivatives with respect to $s_n$; moreover the mapping (which is independent of $s_n$)*

$$\begin{pmatrix} \kappa_n \\ \dfrac{d\kappa_n}{ds_n} \\ \vdots \\ \dfrac{d^n \kappa_n}{ds_n^n} \end{pmatrix} \longrightarrow \begin{pmatrix} \kappa_n \\ \kappa_{n-1} \\ \vdots \\ \kappa_0 \end{pmatrix}$$

*is a global diffeomorphism from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$.*

**Proof of the lemma** As displayed on figure 2, the point $P_{i-1}$ belongs to the tangent to $C_i$ at $P_i$ and at the fixed distance $d_i$ from $P_i$. By assumption $\tau_n = \frac{dP_n}{ds_n}$ admits the good orientation: $P_{n-1} = P_n + d_n \tau_n$ (we do not have $P_{n-1} = P_n - d_n \tau_n$). Thus $\mathcal{C}_{n-1}$ is given by the parameterization $s_n \to P_n + d_n \tau_n$ which is regular since $\left\| \frac{dP_{n-1}}{ds_n} \right\| = \sqrt{1 + d_n^2 \kappa_n^2}$. A natural parameterization $s_{n-1} \to P_{n-1}$ is given by

$$ds_{n-1} = \sqrt{1 + d_n^2 \kappa_n^2}\, ds_n. \tag{4}$$

The unitary tangent vector, $\tau_{n-1}$, is given by

$$\sqrt{1 + d_n^2 \kappa_n^2}\, \tau_{n-1} = \tau_n + d_n \kappa_n\, \nu_n,$$

where $\nu_n$ is the oriented normal to $\mathcal{C}_n$. The angle $\theta_{n-1} - \theta_n$ is the angle between $\tau_n$ and $\tau_{n-1}$. Thus $\tan(\theta_{n-1} - \theta_n) = d_n \kappa_n$. Since $\kappa_n$ is always finite and $\theta_{n-1} - \theta_n$ belongs $]-\pi/2, \pi/2[$ for $s_n = 0$, $\theta_{n-1} - \theta_n$ cannot escapes from $]-\pi/2, \pi/2[$ for any $s_n \in [0, L_n]$. The oriented normal to $\mathcal{C}_{n-1}$, $\nu_{n-1}$, is given by

$$\sqrt{1 + d_n^2 \kappa_n^2}\, \nu_{n-1} = -d_n \kappa_n\, \tau_n + \nu_n,$$

and the signed curvature $\kappa_{n-1}$ of $\mathcal{C}_{n-1}$ is, after some calculations,

$$\kappa_{n-1} = \frac{1}{\sqrt{1 + d_n^2 \kappa_n^2}} \left( \kappa_n + \frac{d_n}{1 + d_n^2 \kappa_n^2} \frac{d\kappa_n}{ds_n} \right). \tag{5}$$

Since $\theta_{n-1} - \theta_n$ remains in $]-\pi/2, \pi/2[$, the unitary tangent vector $\tau_{n-1}$ has the good direction, i.e., $P_{n-2} = P_{n-1} + d_{n-1} \tau_{n-1}$. The analysis can be continued for $P_{n-2}, \ldots, P_0$ and $Q$. This proves (i), (ii) and (iii).

Assertion (iv) comes from the following formula derived from (5) and (4) ($i = 1, \ldots, n$):

$$\kappa_{i-1} = \frac{1}{\sqrt{1 + d_i^2 \kappa_i^2}} \left( \kappa_i + \frac{d_i}{1 + d_i^2 \kappa_i^2} \frac{d\kappa_i}{ds_i} \right) \tag{6}$$

where $s_{i-1}$ is the natural parameterization of $\mathcal{C}_{i-1}$ defined by

$$ds_{i-1} = \sqrt{1 + d_i^2 \kappa_i^2}\, ds_i. \tag{7}$$

Consequently, $\kappa_i$ is an algebraic function of $\kappa_n$ and its first $n-i$ derivatives with respect to $s_n$. Moreover, the dependence with respect to $\dfrac{d^{n-i}\kappa_n}{ds_n^{n-i}}$ is linear via the term

$$\frac{d_{i+1}}{(1 + d_{i+1}^2 \kappa_{i+1}^2)^{3/2}} \cdots \frac{d_n}{(1 + d_n^2 \kappa_n^2)^{3/2}} \quad \frac{d^{n-i}\kappa_n}{ds_n^{n-i}}$$

The map of assertion (iv) has a triangular structure with a diagonal dependence that is linear and always invertible: it is a global diffeomorphism. ▪

**Proof of proposition 2** Denote by $(\tilde{x}_n, \tilde{y}_n)$ and $(\overline{x}_n, \overline{y}_n)$ the Cartesian coordinates of $\tilde{P}_n$ and $\overline{P}_n$, the initial and final positions of $P_n$. There always exists a smooth planar curve $\mathcal{C}_n$ with a natural parameterization $s_n \to P_n(s_n)$ satisfying the following constraints:

- $P_n(0) = \tilde{P}_n$ and $P_n(L_n) = \overline{P}_n$ for some $L_n > 0$.

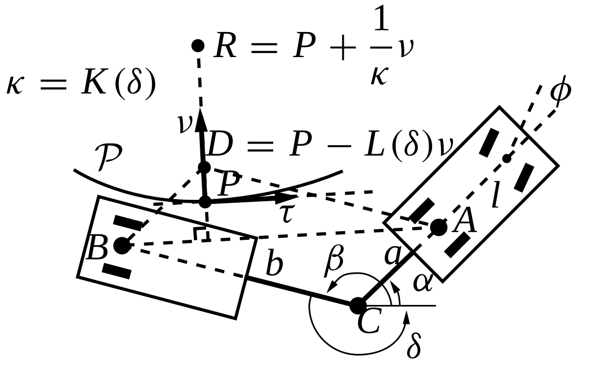


Figure 3: the general 1-trailer system is flat with linearizing point $P$ ($L$ and $K$ are defined by (10) and (12)).

– the direction of tangent at $\tilde{P}_n$ (resp. $\overline{P}_n$) is given by the angle $\tilde{\theta}_n$ (resp. $\overline{\theta}_n$);

– the first $n$ derivatives of the signed curvature $\kappa_n$ at points $\tilde{P}_n$ and $\overline{P}_n$ have prescribed values.

According to (iii) and (iv) of the above lemma, the initial and final values of the angles ($i = 1, \ldots, n$) $\theta_{i-1} - \theta_i$ and $\varphi$ define entirely the initial and final first $n$ derivatives of $\kappa_n$. It suffices now to choose a smooth function $[0, T] \ni t \to s_n(t) \in [0, L_n]$ such that $s_n(0) = 0$, $s_n(T) = L_n$ and $\dot{s}_n(0) = \dot{s}_n(L_n) = 0$, to obtain the desired control trajectory via the relations (the notations are those of the above lemma):

$$\begin{cases} \dot{s}_0 = u_1 = \left(\prod_{i=1}^{n} \sqrt{1 + d_i^2 \kappa_i^2}\right) \dot{s}_n \\ u_2 = \left(\prod_{i=1}^{n} \sqrt{1 + d_i^2 \kappa_i^2}\right) \dfrac{d_0}{1 + d_0^2 \kappa_0^2} \dfrac{d\kappa_0}{ds_0} \dot{s}_n. \end{cases} \quad \blacksquare$$

# 3 The general 1-trailer system

This nonholonomic system is displayed on figure 3: here the trailer is not directly hitched to the car at the center of the rear axle, but more realistically at a distance $a$ of this point. The two controls are the driving velocity $u_1$ of the car rear wheels, and the steering velocity $u_2$ of the car front wheels. The kinematic equations are as follows (notations are given on figure 3):

$$\begin{aligned} \dot{x} &= \cos\alpha\, u_1 \\ \dot{y} &= \sin\alpha\, u_1 \\ \dot{\varphi} &= u_2 \\ \dot{\alpha} &= \frac{1}{l} \tan\varphi\, u_1 \\ \dot{\beta} &= \frac{1}{b}\left(\frac{a}{l}\tan\varphi\cos(\alpha - \beta) - \sin(\alpha - \beta)\right) u_1 \end{aligned} \quad (8)$$

where $(x, y)$ are the Cartesian coordinates of point $A$. Parameters $l$, $a$ and $b$ are positive lengths. The case $a < 0$ is similar to $a > 0$ and is not treated here.

In [15], this system was shown to be flat. This result was inspired by [2]. This paper is also connected to [16, 25], where Goursat normal forms and chained systems are constructed: it is not difficult to prove that driftless systems which, up to changes of coordinates and static or dynamic endogenous feedbacks, can be put into chained forms, are necessarily flat.

We just give here the geometric construction and the analog of propositions 1 and 2. Notice that, contrarily to (3), the explicit derivation of the linearizing output is far from being obvious.

**Proposition 3** *System (8) is flat. A possible linearizing output $y = (y_1, y_2)$ is*

$$\begin{aligned} y_1 &= x + b\cos\beta + L(\alpha - \beta)\frac{b\sin\beta - a\sin\alpha}{\sqrt{a^2 + b^2 - 2ab\cos(\alpha - \beta)}} \\ y_2 &= y + b\sin\beta + L(\alpha - \beta)\frac{a\cos\alpha - b\cos\beta}{\sqrt{a^2 + b^2 - 2ab\cos(\alpha - \beta)}} \end{aligned} \quad (9)$$

*with*

$$L(\alpha - \beta) = ab \int_{\pi}^{2\pi + \alpha - \beta} \frac{\cos\sigma}{\sqrt{a^2 + b^2 - 2ab\cos\sigma}}\, d\sigma \quad (10)$$

*Geometrically $(y_1, y_2)$ are the Cartesian coordinates of $P$ (see figure 3).*

Consider now the real function $\Gamma(\delta) = \cos\delta\, \sqrt{a^2 + b^2 - 2ab\cos\delta} - L(\delta)\, \sin\delta$. Routine calculations show that there exists a unique real $\gamma \in [0, \pi/2]$ such that

$$\Gamma(\gamma) = 0 \quad \text{and} \quad \forall \delta \in ]\gamma, 2\pi - \gamma[\ \ \Gamma(\delta) < 0. \quad (11)$$

When $a = 0$, $\gamma = \pi/2$ and $P$ coincides with $B$.

**Proposition 4** *Consider (8) and two different state-space configurations: $\tilde{p} = (\tilde{x}, \tilde{y}, \tilde{\varphi}, \tilde{\alpha}, \tilde{\beta})$ and $\overline{p} = (\overline{x}, \overline{y}, \overline{\varphi}, \overline{\alpha}, \overline{\beta})$. Assume that the angles $\tilde{\alpha} - \tilde{\beta}$ and $\overline{\alpha} - \overline{\beta}$ (resp. $\tilde{\varphi}$ and $\overline{\varphi}$) belong to $]\gamma - 2\pi, -\gamma[$ (resp. $]-\pi/2, \pi/2[$) ($\gamma$ is defined by (11)). Then, there exists a smooth open-loop control $[0, T] \ni t \to (u_1(t), u_2(t))$ steering the system from $\tilde{p}$ at time 0 to $\overline{p}$ at time $T > 0$, such that the angle $\alpha - \beta$ (resp. $\varphi$) always remains in $]\gamma - 2\pi, -\gamma[$ (resp. $]-\pi/2, \pi/2[$) and such that $(u_1(t), u_2(t)) = 0$ for $t = 0, T$.*

**Sketch of the proof** The arguments are very similar to those used for proposition 2. The computations are slightly more complex. We use the Frénet formulas for the curve $\mathcal{P}$ followed by the linearizing point $P$ and the geometric construction of figure 3.

The unitary tangent vector $\tau$ to $\mathcal{P}$ is colinear to $AB$. The curvature $\kappa$ is a function of $\delta = 2\pi + \alpha - \beta$:

$$\kappa = K(\delta) = \frac{\sin\delta}{\cos\delta\, \sqrt{a^2 + b^2 - 2ab\cos\delta} - L(\delta)\, \sin\delta}. \quad (12)$$

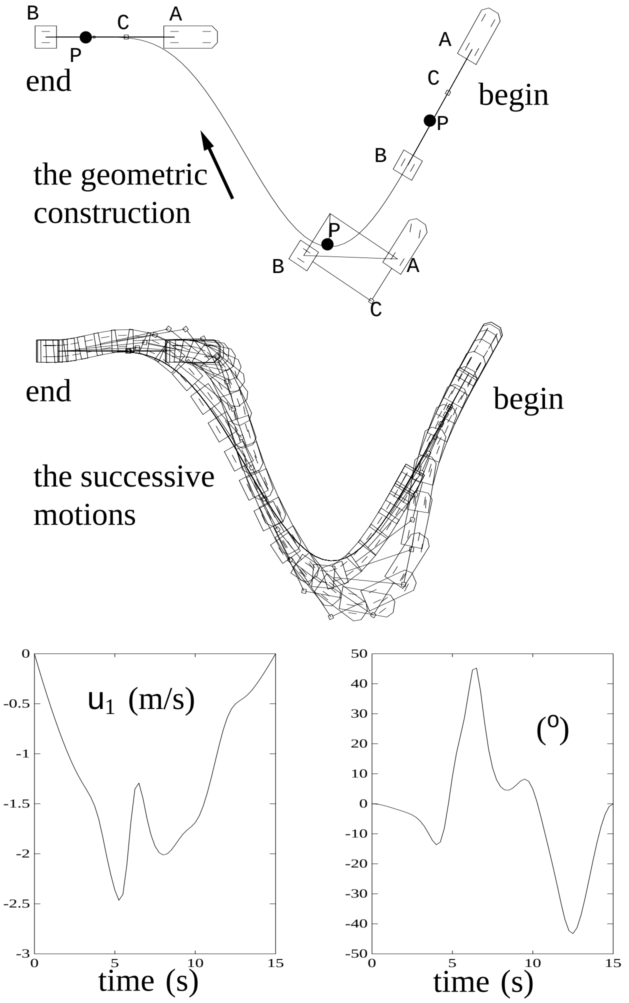


Figure 4: example of backward motions for the general 1-trailer system (8) with $l = 1$ m, $a = 1.5$ m, $b = 2.5$ m and $T = 15$ s.

The function $K$ is an increasing diffeomorphism from $]\gamma, 2\pi - \gamma[$ to $\mathbb{R}$ and $D$ is given by $D = P - L(\delta)\nu$ where $\nu$ is the unitary normal vector to $\mathcal{P}$. This means that $(x, y, \alpha, \beta)$ is a function of $(P, \tau, \kappa)$. The steering angle $\varphi$ depends on $\kappa$ and $d\kappa/ds$ where $s$ is the arc length on $\mathcal{P}$. We have a *global diffeomorphism* from $(x, y, \alpha, 2\pi + \alpha - \beta, \varphi) \in \mathbb{R}^2 \times S^1 \times ]\gamma, 2\pi - \gamma[ \times ]-\pi/2, \pi/2[$ to $(P, \tau, \kappa, d\kappa/ds) \in \mathbb{R}^2 \times S^1 \times \mathbb{R}^2$. ∎

The fact that $(\alpha - \beta)$ and $\varphi$ depend only on $(\kappa, d\kappa/ds)$ results directly from the invariance of the problem with respect to the group of planar Euclidian transformations. Such physical and symmetry considerations are often used here for simplifying the calculations and deriving the coordinates of the linearizing point $P$.

This geometric construction can be easily used for solving the steering problem of the general 1-trailer system. As in [19], a simple steering program can be directly deduced from such developments. The MATLAB simulations of figure 4 illustrate the interest of flatness combined with such geometric constructions.

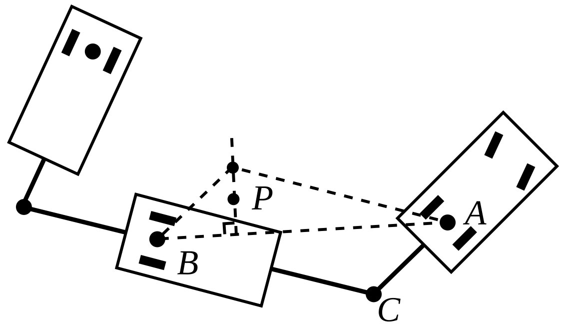


Figure 5: the general 2-trailer system is not flat.

## 4 Concluding remarks

The concept of flatness, which has been illustrated by these nonholonomic systems, may be utilized in many industrial applications, such as the crane [10], aircraft control [13] and chemical reactors [20, 18]. Nevertheless, all systems are not flat. Using the flatness characterization given in [15], one can prove that the general 2-trailer system of figure 5 and the plate-ball system considered in [12] are not flat: their defects [7, 4] are equal to one. These two nonflat systems are closely related to a class of nonlinear second order Monge equations studied in [11].

The multi-steering trailer systems considered in [1, 26, 23] are also flat: the flat output is then obtained by adding to the Cartesian coordinates of the last trailer, the angles of the trailers that are directly steered. This generalization is quite natural in view of the geometric construction of figure 2.

Notice finally that, if we add to the general 2-trailer system of figure 5 a new control that steers directly the last trailer, the system becomes flat: the linearizing output is then formed of the point $P$ with the angle of the last trailer. This fact explains probabily why multi-steering trailer systems are encountered in practice.

## Acknowledgment

The MATLAB programs used for the simulations of figure 4 can be obtained upon request and via electronic mail from the first author (email: rouchon@cas.ensmp.fr).

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