Flatness and oscillatory control: some theoretical results and case-studies.

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Abstract

The generalization of the notion of flatness [10] to meromorphic systems is roughly sketched. The proposed extension is based on a class of transformations, called invertible endogenous transformations, that were introduced in a different context by Hilbert [16].

Meromorphic without drift systems with four state variables and two control variables, are shown to be flat, if, and only if, they are controllable. An example of a nonflat system, coming from a second order Monges equation, proves that this property does not extend to controllable without drift systems of five state variables and two control variables.

The necessary flatness criterion given in [8] is developed in details and completed for systems of any order. This criterion can also be used to prove that some smooth systems are not linearizable via dynamic feedback in the sense given by Charlet, Lvine and Marino [5].

A method for the control of non flat systems and their approximations by flat systems is presented on two examples. The approximation method is based on an idea of the russian physician Kapitsa [17, 21, 2]. The inverse pendulum, the system considered by Kapitsa, is first considered and reformulated with a control point of view. The classical ball and beam system [15] is treated in details with simulations and robustness analysis.

Introduction

This technical report develops and presents some new results that are directly related to two recent notes ("comptes rendus") for the "Acadmie des Sciences" of Paris [10, 8] (see also [9]). They propose a new point of view on the full linearization problem via dynamic feedback[5] by introducing the notion of flatness.

To summarize, this notion corresponds to a structural property for systems that, generically, up to change of coordinates and addition of integrators, can be transformed into linear controllable ones.

In the absence of singularities, flatness implies that the linearization via dynamic feedback in the sense of Charlet, Lvine and Marino [5] is possible.

In section 1, we recall the feedback linearization problem [5], the notion of endogenous and exogenous feedback due to Martin [18], the concept of flatness introduced, within the differential-algebraic approach, by Fliess, Lvine, Martin and Rouchon [10]. Some hints for extending the notion of flatness to meromorphic systems are provided.

Section 2 is devoted to examples of flat systems. In particular, every controllable system, without drift, with 4 state variables and 2 control variables, is proved to be flat. We give an example of a controllable nonflat systems without drift having 5 state variables and 2 control variables.

In section 3, a necessary flatness criterion is given: it extends the criterion for first order systems given in [8] to systems of arbitrary order . This criterion allows us to prove the non flatness of a physical system with two control variables describing an inverse double pendulum in a vertical plane.

In section 4, we develop, on two examples (inverse pendulum, ball and beam system) a method for approximating, through highly oscillatory control, nonflat systems via flat ones.

Most of the results presented here are directly motivated by many fruitful discussions with Michel Fliess, Jean Lvine and Philippe Martin. These discussions take place during the weekly meeting of the group "tats gnraux et systmes plats" every monday in the office of Jean Lvine at Fontainebleau.

1 Dynamic feedback linearization and flatness

1.1 The dynamic feedback linearization problem

In [5], the dynamic feedback linearization problem is stated as follows. For nonlinear systems of the form,

$$\dot{z} = f(z,u) \ , \quad z \in \mathbb{R}^n \ , \quad u \in \mathbb{R}^m \tag{1}$$

with $f(0,0) = 0$ and rank $\frac{\partial f}{\partial u}(0,0) = m$, the problem consists in finding a regular dynamic compensator of the form

$$\begin{array}{rcll} \dot{w} & = & a(z,w,v) & \qquad w \in \mathbb{R}^q \\ u & = & k(z,w,v) & \qquad v \in \mathbb{R}^m \end{array} \tag{2}$$

where q is the order of the compensator, with $a(0,0,0) = 0$, $k(0,0,0) = 0$, and an extended state diffeomorphism

$$x = \varphi(z,w) \tag{3}$$

such that the system (1) is transformed into a linear controllable one ($n' = n + q$)

$$\dot{x} = Ax + Bv \ , \quad x \in \mathbb{R}^{n'} \ , \quad v \in \mathbb{R}^m \ . \tag{4}$$

Recall that by regular dynamic compensator we mean that the input-output system

$$\left\{ \begin{array}{rcl} \dot{w} & = & a(z,w,v) \\ u & = & k(z,w,v) \end{array} \right.$$

with v as the input and u as the output is invertible.

There is no loss of generality in assuming that the pair (A,B) is in Brunovsky canonical form with controllability indices $\alpha_1, \ldots, \alpha_m$, where $\sum_{i=1}^m \alpha_i = n' = n + q$. Consider then the m output functions, called **linearizing output**,

$$\begin{array}{rcl} y_1 & = & \varphi_1(z,w) \\ y_2 & = & \varphi_{\alpha_1+1}(z,w) \\ & \vdots & \\ y_m & = & \varphi_{\alpha_1+\ldots+\alpha_{m-1}+1}(z,w) \ . \end{array} \tag{5}$$

where φ is defined by (3). It is straightforward to see that

$$\begin{array}{rcl} z & = & \psi(y, \ldots, y^{(\alpha)}) \\ u & = & \omega(y, \ldots, y^{(\alpha+1)}) \end{array} \tag{6}$$

where $(y, \ldots, y^{(\alpha)})$ corresponds symbolically to $(y_1, \ldots, y_1^{(\alpha_1)}, \ldots, y_m, \ldots, y_m^{(\alpha_m)})$ and the same for $(y, \ldots, y^{(\alpha+1)})$.

All the trajectories of the original system (1) can be parameterized by the m-tuple $y_1, \ldots, y_m$ of arbitrary time functions and a finite number of its time derivatives.

Conversely, can we express the linearizing output y as a function of z and its derivatives (or equivalently, since $\dot{z} = f(z, u)$, of z and the derivatives of u)? When the answer is "yes", then the dynamic feedback is **endogenous** [18]: the state extension w can be expressed as a function of z and its derivatives. One does not need the introduction of exogenous variables, such as integral of z ($\int z(t)dt$), for linearizing the system. In this case, the inverse of

$$\left\{ \begin{array}{rcl} \dot{w} & = & a(z, w, v) \\ u & = & k(z, w, v) \end{array} \right.$$

when u is the output and v the input, has no dynamics. Roughly speaking, this system is invertible and its "zero dynamics" are of zero dimension.

A clear and sound analysis of this situation within the differential algebraic setting (i.e., when all the functions considered here above are algebraic) is given in [9, 10]. This analysis leads to the definition of flatness.

In the problem statement, Charlet et al. [5] have mixed regularity conditions (rank condition around an equilibrium to avoid singularity) with the structural aspect. Flatness corresponds exactly to the structural aspect. It does not deal with problems of singularities that are of another nature.

1.2 Flatness for meromorphic systems

This subsection is just an attempt to generalize the notion of flatness to meromorphic systems. We do not pretend to give here, as it has been done for algebraic systems [10], a mathematically sound definition of flatness for meromorphic systems. We just give some hints that can help to understand, for meromorphic systems, the ideas underlying the notion of flatness.

This generalization is motivated by the following reason. The algebraic restriction are too severe: the result here below show that there exist systems that are flat in the meromorphic sense whereas they are not flat in the algebraic sense.

Theorem 1 *Consider the algebraic system*

$$\dot{x}_1 + \frac{\dot{x}_2}{x_2} = x_1 + x_2.$$

It is flat in the meromorphic sense with $x_1 + \log(x_2)$ as linearizing output. It is not flat in the algebraic sense, i.e., there does not exist a linearizing output y that is an algebraic function of (x_1, x_2) and a finite number of its derivatives.

Proof Set $z = x_1 + \log(x_2)$. Assume that there exists a linearizing output y that is an algebraic function of x_1, x_2 and their derivatives. Since z (resp. y) is an linearizing output, we can express y (resp. z) as a function of z (resp. y) and a finite number of its derivatives:

$$y = \varphi(z, \ldots, z^{(r)}), \quad z = \psi(y, \ldots, y^{(q)}).$$

This implies that $r = q = 0$, $y = \varphi(z)$ and $\varphi^{-1} = \psi$. If y depends effectively on the derivatives of x_1 and x_2, then, since $\dot{x}_1 = x_1 + x_2 - \dot{x}_2/x_2$, y can be seen as an algebraic function of x_1, x_2 and the successive derivatives of x_2. But $y = \varphi(x_1 + \log(x_2))$. If y depends effectively on derivative(s) of x_2 then x_1 and x_2 are linked by another differential equation that is independent of $\dot{x}_1 + \dot{x}_2/x_2 = x_1 + x_2$. This is not the case.

Thus, y is an algebraic function of x_1 and x_2. This means that there exist polynomial functions of x_1 and x_2, A_0, …, A_n, such that

$$A_n(x_1, x_2)\ y^n + \ldots + A_0(x_1, x_2) = 0, \quad A_n(x_1, x_2) \neq 0. \tag{7}$$

We can assume that the degree n of this polynomial in y is minimal.

Let us introduce the following derivation operator: $D = \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}$. Clearly $Dz = 0$ and $Dy = D(\varphi(z)) = \left(\frac{d}{dz}\varphi\right) Dz = 0$. The D-derivative of (7) leads to

$$DA_n(x_1, x_2)\ y^n + \ldots + DA_0(x_1, x_2) = 0. \tag{8}$$

Equations (7) and (8) imply

$$([DA_n]A_{n-1} - [DA_{n-1}]A_n)\ y^{n-1} + \ldots + ([DA_n]A_0 - [DA_0]A_n) = 0. \tag{9}$$

Thus y is a root of a polynomial of degree less or equal to $n-1$. Since n is minimal, this polynomial is necessarily equal to zero: for all $i \in \{0, \ldots, n-1\}$, $[DA_n]A_i - [DA_i]A_n = 0$.

Set $A_n = a_0(x_1) + \ldots + a_k(x_1)\ (x_2)^k$ where a_0, …, a_k are polynomial in x_1 and $a_k \neq 0$. Consider $P = p_0(x_1) + \ldots + p_r(x_1)\ (x_2)^r$ a polynomial function of x_1 and x_2 ($p_r \neq 0$) such that $[DA_n]P - [DP]A_n = 0$ then, the zeroing of the highest term in $(x_2)^{k+r}$ leads to

$$\frac{da_k}{dx_1}p_r - \frac{dp_r}{dx_1}a_k = (k-r)a_k p_r.$$

Thus $k = r$ and $p_k = \lambda a_k$ where λ is a constant. The zeroing of the term in $(x_2)^{2k-1}$ leads to

$$\frac{d}{dx_1}\left(\frac{p_{k-1} - \lambda a_{k-1}}{a_k}\right) = -\left(\frac{p_{k-1} - \lambda a_{k-1}}{a_k}\right).$$

Thus $p_{k-1} = \lambda a_{k-1}$. The zeroing of the term in $(x_2)^{2k-2}$ leads to

$$\frac{d}{dx_1}\left(\frac{p_{k-2} - \lambda a_{k-2}}{a_k}\right) = -2\left(\frac{p_{k-2} - \lambda a_{k-2}}{a_k}\right).$$

Thus $p_{k-2} = \lambda a_{k-2}$. By continuing this procedure we obtain the successive differential equations, for $i = 1, \ldots, k$,

$$\frac{d}{dx_1}\left(\frac{p_{k-i} - \lambda a_{k-i}}{a_k}\right) = -i\left(\frac{p_{k-i} - \lambda a_{k-i}}{a_k}\right)$$

that imply $p_{k-i} = \lambda a_{k-i}$. This proves that $P = \lambda A_n$.

Let us return now to (9). Since for all $i \in \{0, \ldots, n-1\}$, $[DA_n]A_i - [DA_i]A_n = 0$, we have $A_i = \lambda_i A_n$ with λ_i constant. The division of (7) by A_n shows that y is constant since it is solution of the constant-coefficient polynomial

$$y^n + \lambda_{n-1}y^{n-1} + \ldots + \lambda_0 = 0.$$

This is impossible because y is a linearizing output. ■

We are aware of the fact that many physical systems are algebraic and moreover, when they are flat, the linearizing outputs can be chosen, generally, as algebraic functions of the system variables and their derivatives. Nevertheless, the system of theorem 1 justifies our interest in the meromorphic case.

In [10], the concept of flatness is based on two thinks: the notion of system and the equivalence via algebraic endogenous feedback. It is thus natural to give a meromorphic generalization of these two thinks.

A system definition in the meromorphic case can be the following. Consider n variables $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $n-m$ differential equations between these variables

$$\left\{\begin{array}{rcl} F_1(x_1, \ldots, x_1^{(\alpha_1)}, \ldots, x_n, \ldots, x_n^{(\alpha_n)})) & = & 0 \\ & \vdots & \\ F_{n-m}(x_1, \ldots, x_1^{(\alpha_1)}, \ldots, x_n, \ldots, x_n^{(\alpha_n)})) & = & 0 \end{array}\right.$$

where $F = (F_1, \dots, F_{n-m})$ are meromorphic functions and $\alpha_1, \dots, \alpha_n$ are nonnegative integers such that $\frac{\partial}{\partial x_i^{(\alpha_i)}} F \neq 0$. It could appear that F does dot depend on a component of x an its derivatives (see definition 1). In this case, some of the α_i are not defined, and straightforward modifications of the sequel are needed.

The couple (x, F) is called a **system** if, and only if, the generic rank of

$$\frac{\partial F}{\partial(x_1^{(\alpha_1)}, \dots, x_n^{(\alpha_n)})}$$

is maximum and equal to $n - m$. This means that the general solution of $F(x, \dots, x^{(\alpha)}) = 0$ depends on m arbitrary functions (in the differential-algebraic setting, m is nothing but the differential transcendent degree of the system).

The generalization of endogenous feedback equivalence can be achieved via the notion of invertible endogenous transformations. They extend the notion of static changes of coordinates. Such transformations were first introduced in a different context by Hilbert[16]: he called them "umkehrbar integrallose Transformations". As far as we know, a systematic study of such transformations has not been done up to now. Here, we just give, without proof, some basic tools for the study of such transformations. These tools are directly related to the structure algorithm [23, 24] and system inversion.

Consider n variables $x = (x_1, \dots, x_n)$ and an application φ, called endogenous transformation, that transforms the variable x into the variable ξ:

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \longrightarrow \quad \xi = \begin{pmatrix} \xi_1 = \varphi_1(x, \dots, x^{(\alpha)}) \\ \vdots \\ \xi_n = \varphi_n(x, \dots, x^{(\alpha)}) \end{pmatrix}, \tag{10}$$

$\varphi = (\varphi_1, \dots, \varphi_n)$ are meromorphic functions, and α is an integer.

The detailed study of such endogenous transformations can be made via the inversion structure algorithm [23, 24, 6, 18]. We just sketch here some basic property.

An endogenous transformation is called **regular** if, and only if,

$$x^{(\alpha)} = u, \quad \xi = \varphi(x, \dots, x^{(\alpha-1)}, u)$$

is invertible (ξ is the output and u the input). Otherwise stated, an endogenous transformation is regular when it does not introduced any differential equations between the components of ξ.

The **defect** of a regular endogenous transformation, corresponds to the number of first order differential equations that are needed to recover x from ξ. A regular endogenous transformation is called invertible if, and only if, its defect is zero.

When the endogenous transformation $\xi = \varphi(x, \dots, x^{(\alpha)})$ is invertible, it inverse is also an endogenous transformation $x = \psi(\xi, \dots, \xi^{(\beta)})$. When $\alpha = 0$, then the notions of regularity and invertibility coincide: the generic rank of $\frac{\partial}{\partial x}\varphi$ is maximum and locally $\xi = \varphi(x)$ corresponds to a classical change of coordinates. When $\alpha > 0$ and φ invertible then necessarily $\beta > 0$. One can prove that $\beta \leq n\alpha$ and conversely that $\beta \leq n\alpha$ (α and β are directly related to the notion of index of square implicit differential systems, notion that has been studied in [20, 12, 11, 13]).

The set of invertible endogenous transformations in n variables is a group for the composition of application. If $x = \psi(\xi, \dots, \xi^{(\beta)})$ is an invertible endogenous transformation and (x, F) a system, then (ξ, φ) is also a system where the meromorphic function φ is obtained by the substitution of $x = \psi(\xi, \dots, \xi_\beta)$ in the equations $F(x, \dots, \dot{x}^r) = 0$.

To this group is directly attached an equivalence relationship: two systems $F(x, \dots, x^{(r)}) = 0$ and $\varphi(\xi, \dots, \xi^{(\rho)}) = 0$ are equivalent, if, and only if, there exists an endogenous invertible transformation $\xi = \varphi(x, \dots, x^{'\alpha)})$ transforming the equations in x, $F(x, \dots, x^{(r)}) = 0$, into the equations in ξ, $\varphi(\xi, \dots, \xi^{(\rho)}) = 0$. Up to an invertible endogenous change of coordinates, the system equations are the same.

Definition 1 (flatness for meromorphic systems) *Consider the meromorphic system* $F(x, \ldots, x^{(r)}) = 0$ *(*$\dim(x) = n$*,* $\dim(F) = n - m$*,* $m > 0$*). This system is said flat if, and only if, there exists an invertible endogenous transformation* $x \rightarrow \varphi(x, \ldots, x^{(\alpha)}) = \xi$ *with inverse* $\xi \rightarrow \psi(\xi, \ldots, \xi^{(\beta)}) = x$ *that transforms the system equations in the variables* x*,* $F(x, \ldots, x^{(r)}) = 0$*, into the following trivial system in the variables* ξ

$$\xi_{m+1} = 0, \ldots, \xi_n = 0.$$

The remaining first m *components of* ξ*,* $(\xi_1, \ldots, \xi_m)$ *are then called linearizing output.*

It suffices to find a linearizing output that can be expressed as function of x and a finite number of its derivatives in order to show that a system is flat, according to the definition above. The invertible endogenous transformation is then obtained by adding to the linearizing output the left hand side of the system equations $F(x, \dot{x})$. The fact that this endogenous transformation is regular and invertible results from a careful analysis of the inversion algorithm and will be given elsewhere.

Let us gives two examples. First, we consider the hopping robot studied in [19] and displayed on figure 1. It admits the following dynamics

$$83mm48mmhopping - robot1000$$

Figure 1: *a simple hopping robot in the space; the leg rotates* u_1 *and extends* u_2*.*

$$\left\{ \begin{array}{rcl} \dot{\psi} &=& u_1 \\ \dot{l} &=& u_2 \\ \dot{\theta} &=& -\dfrac{m(l+1)^2}{1+m(l+1)^2} u_1 \end{array} \right.$$

The associated system is

$$\dot{\theta} = -\frac{m(l+1)^2}{1+m(l+1)^2} \dot{\psi}.$$

An invertible endogenous transformation showing that this system is flat according to the above definition is:

$$x = \begin{pmatrix} \psi \\ \theta \\ l \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} \psi \\ \theta \\ \dot{\theta} + \dfrac{m(l+1)^2}{1+m(l+1)^2} \dot{\psi} \end{pmatrix} = \xi.$$

It transforms the system equation in x into the following equation in ξ: $\xi_3 = 0$. This shows that $(\xi_1, \xi_2) = (\psi, \theta)$ is a linearizing output.

The basic kinematic model of an automobile considered in [19] admits the following dynamics (see figure 2 for the physical meaning of the variables).

$$55mm48mmcar - kinematic1000$$

Figure 2: *kinematic model of an automobile;* u_1 *and* u_2 *are the velocities of the rear wheels and of steering, respectively.*

$$\left\{ \begin{array}{rcl} \dot{x} &=& u_1 \cos\theta \\ \dot{y} &=& u_1 \sin\theta \\ \dot{\varphi} &=& u_2 \\ \dot{\theta} &=& \dfrac{u_1}{l} \tan\varphi \end{array} \right.$$

The associated system is

$$\left\{\begin{array}{rcl} \dot{x}\tan\theta - \dot{y} & = & 0 \\ \dot{x}\tan\varphi - l\dot{\theta}\cos\theta & = & 0 \end{array}\right.$$

A possible invertible endogenous transformation showing the flatness is:

$$\begin{pmatrix} x \\ y \\ \varphi \\ \theta \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \\ \dot{x}\sin\theta - \dot{y}\cos\theta \\ \dot{x}\sin\varphi - l\dot{\theta}\cos\theta\cos\varphi \end{pmatrix} = \xi$$

With these new variables ξ the system equations become:

$$\xi_3 = 0, \quad \xi_4 = 0.$$

This shows that $(\xi_1, \xi_2) = (x, y)$ is a linearizing output.

2 Examples of flat systems

2.1 Satellite

The satellite considered by Byrnes and Isidori [4] admits the following equations

$$\left\{\begin{array}{rcl} \dot{\omega}_1 & = & u_1 \\ \dot{\omega}_2 & = & u_2 \\ \dot{\omega}_3 & = & a\omega_1\omega_2 \\ \dot{\varphi} & = & \omega_1\cos\theta + \omega_3\sin\theta \\ \dot{\theta} & = & \dfrac{\sin\varphi}{\cos\varphi}(\omega_1\sin\theta - \omega_3\cos\theta) + \omega_2 \\ \dot{\psi} & = & -\dfrac{1}{\cos\varphi}(\omega_1\sin\theta - \omega_3\cos\theta) \end{array}\right. \tag{11}$$

where $a = (J_1 - J_2)/J_3$ (J_i are the principal moments of inertia). These equations correspond to the motion around the mass center. The control u_1 and u_2 are associated to the torque around the principal inertia axis 1 and 2, respectively. The Euler angles are (θ, φ, ψ). By taking $\tan(\theta/2)$, $\tan(\varphi/2$ and $\tan(\psi/2)$ instead of θ, φ and ψ, respectively, the previous equations becomes algebraic. Thus for this system, we use the well defined notions of system, dynamics, flatness and defect [7, 10, 8].

Proposition 1 *The dynamics (11) are flat when $a = 1$ with (φ, ψ) as linearizing output.*

Proof (11) is flat if, and only if,

$$\left\{\begin{array}{rcl} \dot{\omega}_3 & = & a\omega_1\omega_2 \\ \dot{\varphi} & = & \omega_1\cos\theta + \omega_3\sin\theta \\ \dot{\theta} & = & \dfrac{\sin\varphi}{\cos\varphi}(\omega_1\sin\theta - \omega_3\cos\theta) + \omega_2 \\ \dot{\psi} & = & -\dfrac{1}{\cos\varphi}(\omega_1\sin\theta - \omega_3\cos\theta) \end{array}\right. \tag{12}$$

is flat (since (u_1, u_2) is "algebraic" over (12)). The system (12) is flat if, and only if,

$$\left\{\begin{array}{rcl} \dot{\omega}_3 & = & a\dfrac{\dot{\varphi} - \omega_3\sin\theta}{\cos\theta}(\dot{\theta} + \dot{\psi}\sin\varphi) \\ \dot{\psi} & = & -\dfrac{1}{\cos\varphi}\left(\dot{\varphi}\tan\theta - \dfrac{1}{\cos\theta}\omega_3\right) \end{array}\right. \tag{13}$$

is flat. This results from the fact that ω_1 and ω_2 are “algebraic” over the system (13):

$$\begin{cases} \omega_1 &=& \dfrac{1}{\cos\theta}(\dot\varphi - \omega_3 \sin\theta) \\ \omega_2 &=& \dot\theta + \dot\psi \sin\varphi \end{cases}$$

System (13) is flat if, and only if,

$$\ddot\psi \cos\varphi \cos\theta + \ddot\varphi \sin\theta + a\dot\psi^2 \sin\theta \sin\varphi \cos\varphi - (1+a)\dot\psi\dot\varphi \cos\theta \sin\varphi + (1-a)\theta(\dot\varphi \cos\theta - \dot\psi \sin\theta \cos\varphi) = 0 \quad (14)$$

is flat (we have substituted $\omega_3 = \dot\psi \cos\varphi \cos\theta + \dot\varphi \sin\theta$ in the first equation of (14)). When $a = 1$, $\cos\theta$ is an algebraic function of φ and ψ and their derivatives. ∎

When $a \neq 1$, the defect of (11) is equal to the defect of the system defined by the single equation (14) (see [8] for the definition of the defect). We do not know if the defect of (14) is zero (i.e. the system is flat) or not.

2.2 Affine systems with n state variables and $n-1$ control variables

This section is essentially a reformulation, via the notion of “meromorphic” flatness, of some previous works of Charlet, Lvine and Marino [5]. We do not care here for regularity conditions around an equilibrium point.

Let us consider the following meromorphic affine system

$$\dot x = f(x) + \sum_{i=1}^{n-1} f_i(x) u_i \quad (15)$$

where $x = (x_1, \ldots, x_n)$ is the state and $u = (u_1, \ldots, u_{n-1})$ is the control. We assume that the generic rank of $(f_1, \ldots, f_{n-1})$ is equal to $n-1$.

Theorem 2 *The dynamics (16) is flat, in the meromorphic sense, if, and only if, the generic rank of the vector space spanned by $(f_i)_{1\le i\le n-1}$, $([f_i, f_j])_{1\le i<j\le n-1}$ and $([f, f_i])_{1\le i\le n-1}$ is maximum and equal to n.*

Proof Assume first that $n = 3$. Up to trivial feedback transformation, the system can be brought, generically, into the following form,

$$\begin{cases} \dot x_1 &=& u_1 \\ \dot x_2 &=& u_2 \\ \dot x_3 &=& g(x_1,x_2,x_3) + g_1(x_1,x_2,x_3)u_1 + g_2(x_1,x_2,x_3)u_2, \end{cases} \quad (16)$$

where the functions g, g_1 and g_2 are meromorphic. Let us introduce the differential form $\alpha = dx_3 - g_1 dx_1 - g_2 dx_2$. As in [3, pages 38], denote by r the rank of $\alpha = 0$: r is defined by $(d\alpha)^r \wedge \alpha \neq 0$ and $(d\alpha)^{r+1} \wedge \alpha \equiv 0$. Since $n = 3$, $r = 0$ or $r = 1$.

If $r = 0$, α is proportional to a closed form: there exists coordinates (z_1, z_2, z_3) such that α is proportional to dz_3. Moreover, the two first coordinates (z_1, z_2) can be forced to be equal to (x_1, x_2). In the coordinates (x_1, x_2, z_3), the system admits the following form

$$\dot x_1 = u_1, \quad \dot x_1 = u_1, \dot z_3 = h(x_1, x_2, z_3)$$

The rank condition of the theorem is independent of feedback and coordinates transformations. Thus, the meromorphic function h depends effectively on (x_1, x_2). Assume, e.g., that $\frac{\partial h}{\partial x_1} \neq 0$, then the output (x_2, z_3) is a linearizing outputs. This shows that the rank condition is sufficient. It is necessary, since, if is is not satisfied, h is independent of x_1 and x_2 and the system contained the uncontrollable equation $\dot z_3 = h(z_3)$. The case $r = 0$ corresponds to classical static feedback linearization.

If $r = 1$ then there exist coordinates (z_1, z_2, z_3) (the Pfaff normal form [3, theorem 3.1, page 38]) such that α is proportional to $dz_3 - z_2 dz_1$. Thus in these coordinates the system contains the equation $\dot{z}_3 = z_2 \dot{z}_1 + h(z_1, z_2, z_3)$. This proves that the output (z_1, z_3) is a linearizing outputs. The condition $r = 1$ means that the rank of $(f_1, f_2, [f_1, f_2])$ is 3. This case corresponds effectively to dynamic feedback linearization.

The case $n > 3$ is very similar and is left to the reader. ■

The previous result implies that every controllable system without drift of 3 state variables and 2 control variables is flat.

2.3 Systems without drift: 4 states, 2 controls

We show that controllable systems without drift of 4 states and 2 controls are flat systems. The contribution relies here only on the translation into the control language of the Engel's normal form for Pfaffian systems of two equations in four variables.

Theorem 3 *A controllable meromorphic system without drift of 4 state variables and 2 control variables is flat.*

Proof Up to feedback transformation, the system has the form,

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & u_1 \\ \dot{x}_2 & = & u_2 \\ \dot{x}_3 & = & f_3(x)u_1 + g_3(x)u_2 \\ \dot{x}_4 & = & f_4(x)u_1 + g_4(x)u_2, \end{array} \right.$$

where the functions f_3, f_4, g_3 and g_4 are meromorphic ($x = (x_1, x_2, x_3, x_4)$). Denote by f (resp. g) the vector field $\frac{\partial}{\partial x_1} + f_3 \frac{\partial}{\partial x_3} + f_4 \frac{\partial}{\partial x_4}$ (resp. $\frac{\partial}{\partial x_2} + g_3 \frac{\partial}{\partial x_3} + g_4 \frac{\partial}{\partial x_4}$). Let us consider the Pfaffian system

$$\left\{ \begin{array}{l} \alpha = dx_3 - f_3(x)dx_1 - g_3(x)dx_2 = 0 \\ \beta = dx_4 - f_4(x)dx_1 - g_4(x)dx_2 = 0. \end{array} \right. \tag{17}$$

It defines two differential forms, α and β, generating the system ideal I. To the basis $(dx_1, dx_2, \alpha, \beta)$ corresponds, through the duality defined by the relation

$$dh = L_f h\ dx_1 + L_g h\ dx_2 + L_{\frac{\partial}{\partial x_3}} h\ \alpha + L_{\frac{\partial}{\partial x_4}} h\ \beta$$

(h is an arbitrary smooth real function of x), the vector fields basis $(f, g, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4})$. It is then clear that (see [3, page 44]) the first derived system $I^{(1)}$ (resp. the second derived system $I^{(2)}$) is nothing but the orthogonal (with respect with the previous duality) of the vector space generated by $(f, g, [f, g])$ (resp. $(f, g, [f, g], [f, [f, g]], [g, [f, g]])$). The conditions of application of the Engel's normal form theorem [3, theorem 5.1, page 50], namely $\dim I^{(1)} = 1$ and $\dim I^{(2)} = 0$ are strictly equivalent to the controllability rank conditions on the Lie algebra generated by f and g. Thus there exist coordinates (z_1, z_2, z_3, z_4) such that I is generated by the forms $dz_1 - z_2 dz_3$, and $dz_2 - z_4 dz_3$. This shows that the control system contains the equations $\dot{z}_1 = z_2 \dot{z}_3$, $\quad \dot{z}_2 = z_4 \dot{z}_3$. Thus (z_1, z_3) is a linearizing output. ■

Notice that, since Engel's normal form is true in the C^3 case, the above proposition is also true for C^3 systems.

2.4 Controllable systems without drift and flatness

It is tempting to conjecture that every meromorphic controllable system without drift is flat. We will see that this is not the case by giving an example. It derives from a second order Monges equation studied at the beginning of the century by Hilbert [16].

Hilbert has proved that the general solution of

$$\frac{dz}{dx} - \left(\frac{d^2y}{dx^2}\right)^2 = 0$$

cannot be expressed in terms of an arbitrary function and its successive derivatives. More precisely, this equation does not admits solutions of the form:

$$\begin{array}{rcl} x & = & \varphi(s, w(s), \frac{dw}{ds}, \frac{d^2w}{ds^2}, \ldots, \frac{d^rw}{ds^r}) \\ y & = & \psi(s, w(s), \frac{dw}{ds}, \frac{d^2w}{ds^2}, \ldots, \frac{d^rw}{ds^r}) \\ z & = & \chi(s, w(s), \frac{dw}{ds}, \frac{d^2w}{ds^2}, \ldots, \frac{d^rw}{ds^r}) \end{array}$$

where $w(s)$ is an arbitrary function of s, where the functions φ, ψ and χ are fixed functions and where these relations do not imply that any relation, independent of w, between x, y and z.

Via the identities $\frac{dw}{ds} = \frac{\dot{y}}{\dot{x}}$, $\frac{d^2w}{ds^2} = \frac{\ddot{w}\dot{s} - \dot{w}\ddot{s}}{\dot{s}^3}$, $\ldots$, Hilbert has shown that the general solution of

$$\dot{z}(\dot{x})^5 = (\ddot{y}\dot{x} - \dot{y}\ddot{x})^2 \tag{18}$$

cannot be expressed as a function of (s, w), $\frac{\dot{w}}{\dot{s}}$, $\ldots$, i.e. as a function of (s, w) and their time derivatives coming from d^rw/ds^r. This does not prove exactly that (18) is not flat.

Nevertheless, the proof given by Hilbert can be adapted to show that (18) is not flat. To (18) is associated the without drift system of 5 state variables and 2 control variables:

$$\left\{ \begin{array}{rcl} \dot{x} & = & u_1 \\ \dot{\alpha} & = & u_2 \\ \dot{\beta} & = & \alpha u_1 \\ \dot{z} & = & \alpha^2 u_1 \\ \dot{y} & = & \beta u_1. \end{array} \right.$$

The reader can verify that this system is controllable. We will see that it is not flat by proving that (18) is not flat.

Theorem 4 *The system $\dot{z}(\dot{x})^5 = (\ddot{y}\dot{x} - \dot{y}\ddot{x})^2$ is not flat.*

Proof Assume that the general solution can be expressed in the following form

$$\begin{array}{rcl} x & = & \varphi(w_1, \dot{w}_1, \ldots, w_1^{(\alpha_1)}, w_2, \dot{w}_2, \ldots, w_2^{(\alpha_2)}) \\ y & = & \psi(w_1, \dot{w}_1, \ldots, w_1^{(\alpha_1)}, w_2, \dot{w}_2, \ldots, w_2^{(\alpha_2)}) \\ z & = & \chi(w_1, \dot{w}_1, \ldots, w_1^{(\alpha_1)}, w_2, \dot{w}_2, \ldots, w_2^{(\alpha_2)}) \end{array}$$

where w_1 and w_2 are arbitrary functions of t and where α_i is defined by $\frac{\partial}{\partial w_i^{(\alpha_i)}}(\varphi, \psi, \chi) \neq 0$, $i = 1, 2$. We denote by $\varphi_{w_i^{(r)}}$ the partial derivative of φ with respect to $w_i^{(r)}$. and the same for $\psi_{w_i^{(r)}}$ and $\chi_{w_i^{(r)}}$.

Consider $i \in \{1, 2\}$. The substitution of the above relations into the system equation leads to an identity in the jet space of w_1 and w_2. Since $\ddot{y}\dot{x} - \dot{y}\ddot{x}$ is the only term containing $w_i^{(\alpha_i+2)}$, one has $\varphi_{w_i^{(\alpha_i)}}\dot{y} - \psi_{w_i^{(\alpha_i)}}\dot{x} \equiv 0$.

Assume that $\varphi_{w_i^{(\alpha_i)}} = 0$. Then $\psi_{w_i^{(\alpha_i)}} = 0$, since for a general solution $\dot{y} \neq 0$. This means that φ and ψ are independent of $w_i^{(\alpha_i)}$. This implies that the expression $\ddot{y}\dot{x} - \dot{y}\ddot{x}$ depends linearly on $w_i^{(\alpha_i+1)}$. Its square is identically equal to $\dot{z}(\dot{x})^5$ that depends effectively on $w_i^{(\alpha_i+1)}$ since $\psi_{w_i^{(\alpha_i)}} \neq 0$ (definition of α_i, φ and ψ do not depend on $w_i^{(\alpha_i)}$). But $(\ddot{y}\dot{x} - \dot{y}\ddot{x})^2$ cannot be a non constant linear function of $w_i^{(\alpha_i+1)}$. And we are lead to a contradiction.

Thus $\varphi_{w_i^{(\alpha_i)}} \neq 0$ and

$$(w_1, \dot{w}_1, \dots, w_1^{(\alpha_1)}, w_1^{(\alpha_1+1)}, \dots w_2, \dot{w}_2, \dots) \to (w_1, \dot{w}_1, \dots, w_1^{(\alpha_1-1)}, x, \dot{x}, \ddot{x}, \dots, w_2, \dot{w}_2, \dots, w_2^{(\alpha_2)}, \dots)$$

is a change of "coordinates" in the jet space associated to w_1 and w_2. We set

$$\begin{array}{l} y = f(w_1, \dot{w}_1, \dots, w_1^{(\alpha_1-1)}, x, w_2, \dot{w}_2, \dots, w_2^{(\alpha_2)}), \\ z = g(w_1, \dot{w}_1, \dots, w_1^{(\alpha_1-1)}, x, w_2, \dot{w}_2, \dots, w_2^{(\alpha_2)}), \\ w_1^{(\alpha_1)} = h(w_1, \dot{w}_1, \dots, w_1^{(\alpha_1-1)}, x, w_2, \dot{w}_2, \dots, w_2^{(\alpha_2)}). \end{array}$$

We have seen that $f_x \neq 0$ and $h_x \neq 0$. With such new coordinates, the substitution of the above relations in the system equation leads to an identity. The term $\ddot{y}\dot{x} - \dot{y}\ddot{x}$ is the only one that depends on $\ddot{x}$ via the expression $(f_x \dot{x} - \dot{y})\ddot{x}$. Thus $\dot{y} = f_x \dot{x}$. This implies that $\ddot{y}\dot{x} - \dot{y}\ddot{x} = \dot{x}\dot{f_x}$. The system equation becomes then $(\dot{f_x})^2 = \dot{x}\dot{z}$ (division by $(\dot{x})^4$). The analysis of the dependence with respect to $\dot{x}$ of each member of this identity implies that $\dot{f_x} = f_{xx}\dot{x}$, $\dot{z} = g_x \dot{x}$ and $(f_{xx})^2 = g_x$.

Clearly $\dot{y} = f_x \dot{x}$ implies that f is independent of $w_2^{(\alpha_2)}$. This implies also that

$$f_{w_1}\dot{w}_1 + \dots + f_{w_1^{(\alpha_1-1)}} h(w_1, \dots, w_1^{(\alpha_1-1)}, x, w_2, \dots, w_2^{(\alpha_2)}) + f_{w_2}\dot{w}_2 + \dots + f_{w_2^{(\alpha_2-1)}} w_2^{(\alpha_2)} = 0.$$

Similarly $\dot{f_x} = f_{xx}\dot{x}$ implies that

$$f_{xw_1}\dot{w}_1 + \dots + f_{xw_1^{(\alpha_1-1)}} h(xw_1, \dots, w_1^{(\alpha_1-1)}, x, w_2, \dots, w_2^{(\alpha_2)}) + f_{xw_2}\dot{w}_2 + \dots + f_{xw_2^{(\alpha_2-1)}} w_2^{(\alpha_2)} = 0.$$

Thus, since $h_x \neq 0$, we have $f_{w_1^{(\alpha_1-1)}} = 0$, i.e. f is independent of $w_1^{(\alpha_1-1)}$. It is then straightforward to see that f is only a function of x. Similarly, $\dot{z} = g_x \dot{x}$ and $g_x = (f_{xx})^2$ show that g depends only on x. This means that every general solution of the system parameterized via φ, ψ and χ by the two arbitrary functions w_1 and w_2, depends in fact on only one arbitrary function since y and z are function of x. This shows that the system is not flat. ■

3 Necessary dynamic linearization criteria

We propose here a necessary flatness criterion that can be easily verified in practice. This criterion remains also valid in the smooth case and also for system that are linearizable via exogenous feedback. However, for clarity sake, we consider here only flat systems.

In this section, we assume that all the functions are meromorphic. We consider generic situation, i.e. when the rank of the functions are maximum.

3.1 Criterion for first order systems

Consider $\dot{x} = f(x, u)$ with $x = (x_1, \dots, x_n)$, $u = (u_1, \dots, u_m)$ and $n \geq m$. Assume that the generic rank of $\frac{\partial}{\partial u} f$ is maximum and equal to m. The elimination of u from $\dot{x} = f(x, u)$ is generically possible and leads to the following system $F(x, \dot{x}) = 0$ where $F = (F_1, \dots, F_{n-m})$ is a meromorphic function and the generic rank of $\frac{\partial}{\partial \dot{x}} F$ is maximum and equal to $n - m$.

Theorem 5 *Consider the meromorphic system $F(x, \dot{x}) = 0$ of codimension m (x is of dimension n and F of dimension $n - m$)) with $\frac{\partial}{\partial \dot{x}} F$ of generic rank $n - m$. Assume that $F(x, \dot{x}) = 0$ is flat.*

Then, generically, for every point (x, p) of the "sub-manifold"[1] defined by $F(x, p) = 0$, there exists a line passing through (x, p), included in this "sub-manifold" and parallel to the x coordinates. In other words, for each generic point (x, p) such that $F(x, p) = 0$, there exists $a = (a_1, \dots, a_n) \neq 0)$ such that $F(x, p + \lambda a) = 0$ for all $\lambda \in \mathbb{R}$.

[1] It is a sub-manifold, locally and for generic point.

Proof Since $F(x, \dot{x}) = 0$ is flat, its general solution can be expressed as a function of m arbitrary time functions $(y_1, \ldots, y_m)$ and their derivatives:

$$x = A(y_1, \ldots, y_1^{(\alpha_1)}, \ldots y_m, \ldots, y_m^{(\alpha_m)})$$

with $\frac{\partial}{\partial y_i^{(\alpha_i)}} A \neq 0$, $i = 1, \ldots, m$. For simplicity sake in the notations, this expression is shortly written

$$x = A(y, \ldots, y^{(\alpha)}).$$

We denote also the partial derivative of A with respect to y, $\dot{y}$, ... by A_y, $A_{\dot{y}}$, ... When we substitute the above expression of x into the system equation, we obtain the identity

$$F(A(y, \ldots, y^{(\alpha)}), A_y \dot{y} + \ldots + A_{y^{(\alpha)}} y^{(\alpha+1)}) = 0. \tag{19}$$

Consider a generic point $(\tilde{x}, \tilde{p}))$ such that $F(\tilde{x}, \tilde{p}) = 0$. Through this point passes a solution corresponding to $\tilde{y}$, $\dot{\tilde{y}}$, ... satisfying $\tilde{x} = A(\tilde{y}, \ldots, \tilde{y}^{(\alpha)})$ and $\tilde{p} = A_y(\tilde{y}, \ldots, \tilde{y}^{(\alpha)})\dot{\tilde{y}} + \ldots + A_{y^{(\alpha)}}(\tilde{y}, \ldots, \tilde{y}^{(\alpha)})\tilde{y}^{(\alpha+1)}$. Taking in (19), $y = \tilde{y}$, $\dot{y} = \dot{\tilde{y}}$, ..., $y^{(\alpha-1)} = \tilde{y}^{(\alpha-1)}$, $y^{(\alpha)} = \tilde{y}^{(\alpha)} + v$, where v is a arbitrary vector of $\mathbb{R}^m$, we are lead to the following identity

$$F(\tilde{x}, \tilde{p} + A_{y^{(\alpha)}}(\tilde{y}, \ldots, \tilde{y}^{(\alpha)})v) = 0$$

for all $v \in \mathbb{R}^m$. Since, by assumption, $A_{y^{(\alpha)}} \neq 0$, every nonzero vector a belonging to the image of the linear operator $A_{y^{(\alpha)}}(\tilde{y}, \ldots, \tilde{y}^{(\alpha)})$ is such that $F(\tilde{x}, \tilde{p} + \lambda a) = 0$ for all $\lambda \in \mathbb{R}$. ■

3.2 Remarks on the linear direction a of theorem 5

The existence of linear direction a for the manifold $F(x, \dot{x}) = 0$ leads to many partial differential relations satisfied by the successive derivatives of F with respect to $\dot{x}$. More precisely, the identity $F(x, \dot{x} + \lambda a) = 0$ for all $\lambda \in \mathbb{R}$ implies (all the derivative are evaluated at $(x, \dot{x})$)

$$\left\{ \begin{array}{rcll} \frac{\partial}{\partial \dot{x}} F \; a & = & 0 & \text{first order condition} \\ \frac{\partial^2}{\partial \dot{x}^2} F \; (a, a) & = & 0 & \text{second order condition} \\ \frac{\partial^3}{\partial \dot{x}^3} F \; (a, a, a) & = & 0 & \text{third order condition} \\ & \vdots & & \end{array} \right. \tag{20}$$

This means that the homogeneous polynomial in a, $\frac{\partial}{\partial \dot{x}} F \; a$, $\frac{\partial^2}{\partial \dot{x}^2} F \; (a, a)$, ..., have a common non zero root. Formally, the theory of elimination initiated by Kronecker and the use of the resultant polynomial leads to an infinite series of partial differential equations of any orders that must be satisfied by the derivatives of F with respect to $\dot{x}$.

When $m = 1$, the intersection of $F(x, \dot{x}) = 0$ with the affine spaces $x =$ constant, defines curves. If the system is flat when, according to the previous theorem, this curves are straight lines. Necessarily, the equations $F(x, \dot{x}) = 0$ are affine with respect to $\dot{x}$. Geometrically, this explains why for mono-input systems there is no difference between dynamic and static feedback linearization.

More generally, if $F(x, \dot{x}) = 0$ is linearizable via static feedback then the relation $x = A(y, \ldots, y^{(\alpha)})$ (see the preceding proof) is in fact a change of coordinates. Thus the rank of $A_{y^{(\alpha)}}$ is maximum. This implies that the intersection of $F(x, \dot{x}) = 0$ with $x =$constant, is an affine subspace: $F(x, \dot{x})$ depends linearly on $\dot{x}$.

Assume that, for every (x, p) such that $F(x, p) = 0$, the kernel of the linear operator $\frac{\partial}{\partial \dot{x}} F(x, p)$ contains a vector a depending on x only, $a = a(x)$. Then $F(x, p + \lambda a(x)) = 0$ for all $\lambda \in \mathbb{R}$. Denote by $\varphi(\lambda)$ the function $\lambda \to F(x, p + \lambda a(x))$. We have $\varphi(0) = 0$ and

$$\frac{d\varphi}{d\lambda} = \frac{\partial F}{\partial p}(x, p + \lambda a(x))a(x)$$

But $a(x)$ belong to the kernel of every $\frac{\partial F}{\partial p}(x,q)$ for all (x,q) such that $F(x,q)=0$. This means that when $\varphi(\lambda)$ is zero then its derivative at λ is also zero. Since $\frac{\partial}{\partial p}F$ is a smooth function, there exists locally in λ a constant $M>0$ such that

$$\left|\frac{d\varphi}{d\lambda}(\lambda)\right| \leq M|\varphi(\lambda)|$$

for λ around 0. Thus, $\varphi \equiv 0$ (Grownwall lemma).

If the linear direction a depends only on x then it is possible to eliminate one variable and, generally, one equation via the following method. Assume, e.g. that the last component of a is not zero. Clearly it can be chosen equal to 1. Taking $\lambda = -p_n$ in the identity $F(x, p+\lambda a(x)) = 0$, one obtains:

$$F(x,(p_1 - p_n a_1(x), \dots, p_{n-1} - p_n a_{n-1}(x), 0)) = 0.$$

Thus, if $x(t)$ is a trajectory of $F(x,\dot{x})=0$, it satisfies

$$F(x,(\dot{x}_1 - \dot{x}_n a_1(x), \dots, \dot{x}_{n-1} - \dot{x}_n a_{n-1}(x), 0)) = 0.$$

Denote by $g_s(z)$ the flow associated to the ordinary differential equation $\frac{d}{ds}z_i = a_i(z_1,\dots,z_{n-1},s)$, $i=1,\dots,n-1$ $(z=(z_1,\dots,z_{n-1}))$. It defines a local diffeomorphism on x via

$$(x_1,\dots,x_{n-1}) = g_s(z), \quad x_n = s.$$

Within such new variables (z,s) the equations of the system become free of $\dot{s}$:

$$G(s,z,\dot{z}) \equiv F\left((g_s(x),s), \left(\frac{\partial g}{\partial z}_s(z)\ \dot{z}, 0\right)\right) = 0.$$

If G is independent of s then we have eliminated one variable. The flatness of $F(x,\dot{x})=0$ is when equivalent to the flatness of $G(z,\dot{z})=0$ whose codimension is decreased by one. If G depends on s, then, formally $G(s,z,\dot{z})=0$ can be written as

$$s = h(z,\dot{z}), \quad H(z,\dot{z}) = 0.$$

We can eliminate one variable and one equation. The codimensions of $F(x,\dot{x})=0$ and of $H(z,\dot{z})=0$ are equal. The system $H(z,\dot{z})=0$ is flat, if, and only if, the extended system $F(x,\dot{x})=0$ is also flat. It can be proved that the static feedback linearization can be interpreted via such elimination and reduction method: at each step there always exists a linear direction that is independent of the variable derivatives.

3.3 Application of theorem 5 on an example

Let us consider the double inverse pendulum of figure 3. It moves in a vertical plane. Denoting by u (resp. v) the horizontal (resp. vertical) exterior force applied to the suspension point (x,y), the equations of motion are (implicit form):

$$\left\{\begin{array}{lcl}
p_1 &=& I_1\dot{\alpha}_1 + I\dot{\alpha}_2\cos(\alpha_1-\alpha_2) + n_1\dot{x}\cos\alpha_1 - n_1\dot{y}\sin\alpha_1\\
p_2 &=& I\dot{\alpha}_1\cos(\alpha_1-\alpha_2) + I_2\dot{\alpha}_2 + n_2\dot{x}\cos\alpha_2 - n_2\dot{v}\sin\alpha_2\\
\dot{p}_1 &=& n_1 g\sin\alpha_1 - n_1\dot{\alpha}_1\dot{x}\sin\alpha_1 - n_1\dot{\alpha}_1\dot{y}\cos\alpha_1\\
\dot{p}_2 &=& n_2 g\sin\alpha_2 - n_2\dot{\alpha}_2\dot{x}\sin\alpha_2 - n_2\dot{\alpha}_2\dot{y}\cos\alpha_2\\
p_x &=& m\dot{x} + n_1\dot{\alpha}_1\cos\alpha_1 + n_2\dot{\alpha}_2\cos\alpha_2\\
p_y &=& m\dot{y} - n_1\dot{\alpha}_1\sin\alpha_1 - n_2\dot{\alpha}_2\sin\alpha_2\\
\dot{p}_x &=& u\\
\dot{p}_y &=& v - mg
\end{array}\right. \tag{21}$$

where p_1, p_2, p_x, and p_y are the generalized impulsions associated to the generalized coordinates α_1, α_2, x and y, respectively. The quantities g m I, I_1, I_2, n_1 and n_2 are constant physical parameters.

$85mm71mmdouble - pendulum1000$

Figure 3: *the inverse double pendulum; the suspension point moves along the horizontal and vertical directions.*

Since u, v, p_x and p_y are functions of $(\alpha_1, \alpha_2, x, y)$ and their derivatives, (21) is flat, if, and only if, the reduced system

$$\left\{\begin{array}{lcl} p_1 &=& I_1\dot\alpha_1 + I\dot\alpha_2\cos(\alpha_1-\alpha_2) + n_1\dot x\cos\alpha_1 - n_1\dot y\sin\alpha_1 \\ p_2 &=& I\dot\alpha_1\cos(\alpha_1-\alpha_2) + I_2\dot\alpha_2 + n_2\dot x\cos\alpha_2 - n_2\dot v\sin\alpha_2 \\ \dot p_1 &=& n_1 g\sin\alpha_1 - n_1\dot\alpha_1\dot x\sin\alpha_1 - n_1\dot\alpha_1\dot y\cos\alpha_1 \\ \dot p_2 &=& n_2 g\sin\alpha_2 - n_2\dot\alpha_2\dot x\sin\alpha_2 - n_2\dot\alpha_2\dot y\cos\alpha_2 \end{array}\right. \tag{22}$$

is flat. This system is not flat. Let us look for a linear direction $a = (a_1, a_2, a_x, a_y, a_{p_1}, a_{p_2})$ associated to $(\alpha_1, \alpha_2, x, y, p_1, p_2)$. The second order conditions (see equation (20)) lead to

$$a_1(a_x\sin\alpha_1 + a_y\cos\alpha_1) = 0, \quad a_2(a_x\sin\alpha_2 + a_y\cos\alpha_2) = 0$$

Two first order conditions are

$$\left\{\begin{array}{lcl} -a_x\cos\alpha_1 + a_y\sin\alpha_1 &=& \frac{I_1}{n_1}a_1 + \frac{I}{n_1}\cos(\alpha_1-\alpha_2)a_2 \\ -a_x\cos\alpha_2 + a_y\sin\alpha_2 &=& \frac{I}{n_2}\cos(\alpha_1-\alpha_2)a_1 + \frac{I_2}{n_2}a_2 \end{array}\right.$$

Simple computations show that, if $\frac{I}{n_1} \neq \frac{I_2}{n_2}$ and $\frac{I_1}{n_1} \neq \frac{I}{n_2}$ [2], then $(a_1, a_2, a_x, a_y) = 0$. The two remaining first order conditions imply that $(a_{p_1}, a_{p_2}) = 0$. Thus $a = 0$ and the inverse double pendulum is not a flat system.

3.4 Criterion for systems of arbitrary order

The necessary linearization criterion of theorem 5 can be generalized when the system equations contain derivatives of arbitrary order.

Theorem 6 *Consider the meromorphic system $F(x, \dot x, \ldots, x^{(r)}) = 0$ of codimension $m > 0$ ($r > 0$, x is of dimension $n > 0$ and F of dimension $n - m > 0$). The generic rank of $\frac{\partial}{\partial x^{(r)}}F$ is maximum and equal to $n - m$. Assume that this system is flat.*

Then, for every generic point $(x, p_1, \ldots, p_r)$ of the sub-manifold[3] $F(x, p_1, \ldots, p_r) = 0$, there exist a vector $a = (a_1, \ldots, a_n)$ and $r - 1$ vectorial polynomials $Q_i(\lambda_1, \ldots, \lambda_i)$, $i = 1, \ldots, r-1$ of dimension n such that

1. *the degree of each $Q_i(\lambda_1, \ldots, \lambda_i)$ with respect to λ_k ($k = 1, \ldots, i$) is less or equal to $i + 1 - k$;*
2. *the vector a is not zero;*
3. *for all $(\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r$ we have the identity*

$$F(\ x\ ,\ p_1 + \lambda_1 a\ ,\ p_2 + \lambda_2 a + Q_1(\lambda_1)\ ,\ \ldots\ ,\ p_i + \lambda_r a + Q_{r-1}(\lambda_1, \ldots, \lambda_{r-1})\ ) = 0.$$

Proof Using the shortcut notation of the proof of theorem 5, we set $x = A(y, \ldots, y^{(\alpha)})$ where y is a set of m arbitrary time functions. It suffices to remark that $x^{(\alpha+i)}$

- is a linear and non constant function of $y^{(\alpha+i)}$ through the term $A_{y^{(\alpha)}}y^{(\alpha+i)}$;
- is a polynomial function of $y^{(\alpha+k)}$ for $k = 1, \ldots, i-1$ of degree less or equal to $i + 1 - k$;

[2]These conditions are always satisfied for homogeneous identical bars.
[3]It is a sub-manifold, locally and generically.

– does not depends on $y^{(\alpha+k)}$ for $k = i+1, \ldots, r$;

– the Jacobian matrix $A_{y^{(\alpha)}}$ is different from zero;

to see that the proof is very similar to the one of theorem 5. ∎

Clearly, theorem 6 includes theorem 5. By adding variables and linear equations every differential system of order exceeding 1 can be seen as a first order system. Thus it remains to show that theorem 6 is really an extension of theorem 5, i.e., to give an example of a second order system such that it does not satisfied the necessary condition of theorem 6, and such that its first order associated system satisfies the necessary flatness condition of theorem 5.

Consider the system $(\dot{x}_1)^4\ddot{x}_1 + \dot{x}_2\ddot{x}_2 = 0$. The linear direction defined by a is $(\dot{x}_2, -(\dot{x}_1)^4)$. Thus the associated first order system satisfies the necessary flatness condition of theorem 5. But it is clear that there does not exist (b_1, b_2, c_1, c_2) such that for every point $(x_1, \dot{x}_1, \ddot{x}_1, x_2, \dot{x}_2, \ddot{x}_2)$ satisfying $(\dot{x}_1)^4\ddot{x}_1 + \dot{x}_2\ddot{x}_2 = 0$, we have the identity

$$(\dot{x}_1 + \lambda\dot{x}_2)^4(\ddot{x}_1 + \lambda b_1 + \lambda^2 c_1) + (\dot{x}_2 - \lambda(\dot{x}_1)^4)(\ddot{x}_2 + \lambda b_2 + \lambda^2 c_2) = 0$$

for all $\lambda \in \mathbb{R}$.

The conditions given in theorems 5 and 6 are necessary but not sufficient in general. It suffices to consider, e.g., the Hilbert system (18) discussed here above.

4 Averaging and High frequency control

We address here a method for the control of non flat systems and their approximations by flat systems. More precisely, we develop on two examples an idea of the russian physician Kapitsa [17, 21, 2]. Kapitsa considers the motion of a particle in a highly oscillating field. He proposes a method for deriving the equations relative to the average motion. He shows that the inverse position of a single pendulum is “stabilized” when the suspension point admits vertical fast oscillations.

We first reformulate and present the Kapitsa pendulum with a control point of view. Then, we treat in details (simulation and robustness test) the classical ball and beam system [15].

4.1 The Kapitsa pendulum

The notation are summarized in the figure 4. We assume that the vertical velocity $\dot{v} = u$ of the suspension point is the control. The equations of motion are:

$$\left\{ \begin{array}{rcl} \dot{\alpha} & = & p + \dfrac{u}{l}\sin\alpha \\ \dot{p} & = & \left(\dfrac{g}{l} - \dfrac{u^2}{l^2}\cos\alpha\right)\sin\alpha - \dfrac{u}{l}p\cos\alpha \\ \dot{v} & = & u \end{array} \right. \tag{23}$$

where p is proportional to the generalized impulsion; g and l are physical constants.

It is clear that this system is not flat. We state

$$u = u_1 + u_2\cos(t/\varepsilon)$$

where u_1 and u_2 are auxiliary control and $\varepsilon \ll \sqrt{l/g}$.

$$54mm38mmkapitsa - pendulum1000$$

Figure 4: *the kapitsa pendulum; the suspension point admits small and fast vertical oscillations.*

It is then natural to consider the following "averaged" system:

$$\left\{\begin{array}{rcl} \dot{\overline{\alpha}} & = & \overline{p} + \dfrac{u_1}{l}\sin\overline{\alpha} \\ \dot{\overline{p}} & = & \left(\dfrac{g}{l} - \dfrac{(u_1)^2}{l^2}\cos\overline{\alpha} - \dfrac{(u_2)^2}{2l^2}\cos\overline{\alpha}\right)\sin\overline{\alpha} - \dfrac{u_1}{l}\overline{p}\cos\overline{\alpha} \\ \dot{\overline{v}} & = & u_1. \end{array}\right. \tag{24}$$

It admits two control variables, u_1 and u_2, whereas the original system (23) admits only one, u. Moreover (24) is flat with $(\overline{\alpha}, \overline{v})$ as linearizing output.

From the averaging method (see [14, 1, 22]), it is clear that, for ε small,

- every open-loop trajectory $(\overline{\alpha}, \overline{p}, \overline{v})$ of (24) can be approximated on every finite time interval by trajectory (α, p, v) of (23).
- if the feedback, $u_1 = k_1(\overline{\alpha}, \overline{p}, \overline{v})$ and $u_2 = k_2(\overline{\alpha}, \overline{p}, \overline{v})$, stabilizes the averaged system (24) around an equilibrium points with poles having strictly negative real part, then, (23) with the time-varying feedback $u = k_1(\alpha, p, v) + k_2(\alpha, p, v)\cos(t/\varepsilon)$ admits a locally asymptotically stable small limit cycle. Its characteristic multipliers are directly related to the exponential of the poles associated to the hyperbolic equilibrium of the closed-loop averaged system. The modules of these characteristic multipliers are strictly less than 1.

These properties justify the name of **averaged control system** given to (24).

4.2 The ball and beam system

As in [15], we consider the control system displayed on figure 5. The equations of motions are the following

$$\left\{\begin{array}{rcl} \ddot{r} & = & -Bg\sin\theta + Br\dot{\theta}^2 \\ (mr^2 + J + J_b)\ddot{\theta} & = & \tau - 2mr\dot{r}\dot{\theta} - mgr\cos\theta \end{array}\right. \tag{25}$$

where $(r, \dot{r}, \theta, \dot{\theta})$ is the state, τ, the torque applied to the beam, is the control. The moment of inertia of the beam is J; the mass, the moment of inertia and the radius of the ball are, respectively, m, J_b and R. The parameter B is equal to $m/(J_b/R^2 + m)$.

$$105mm53mmball - beam1000$$

Figure 5: *the ball and beam system; the ball rolls on the beam without slipping.*

Straightforward calculation shows that (25) is not flat. It suffices to consider the feedback defined by

$$\tau = (mr^2 + J + J_b)u + 2mr\dot{r}\dot{\theta} + mgr\cos\theta \tag{26}$$

to see that the defect of (25) is equal to the defect (see [8] for a definition of the defect) of the reduced system

$$\ddot{r} = -Bg\sin\theta + Br\dot{\theta}^2 \tag{27}$$

since $\ddot{\theta} = u$. In (27), θ plays a similar role as the vertical position v of the suspension point of the inverse pendulum of Kapitsa (see (23)).

The only difference is that $\ddot{\theta} = u$ is the control, instead of $\dot{v} = u$. We state

$$u = u_1 + \frac{1}{\varepsilon}u_2\cos(t/\varepsilon)$$

where u_1 and u_2 are auxiliary control variables and $0 < \varepsilon \ll 1$.

With such choice of u, (25) is not in standard form for averaging. This is due to the presence of $1/\varepsilon$ in the equations. To eliminate $1/\varepsilon$, we introduce the following change of coordinates

$$\begin{pmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{pmatrix} \longrightarrow \begin{pmatrix} r \\ \dot{r} \\ \theta \\ z = \dot{\theta} - u_2 \sin(t/\varepsilon) \end{pmatrix}.$$

Notice that this change of coordinates depends on the time and on the control u_2. With $(r, \dot{r}, \theta, z)$, the system becomes

$$\left\{ \begin{array}{lcl} \frac{d}{dt}(r) & = & \dot{r} \\ \frac{d}{dt}(\dot{r}) & = & -Bg \sin\theta + Br(z + u_2 \sin(t/\varepsilon))^2 \\ \frac{d}{dt}(\theta) & = & z + u_2 \sin(t/\varepsilon) \\ \frac{d}{dt}(z) & = & u_1 - \dot{u}_2 \sin(t/\varepsilon). \end{array} \right.$$

Adding u_2 in the state and setting $\dot{u}_2 = w_2$, we are lead to the following **averaged control system** associated to (25)

$$\left\{ \begin{array}{lcl} \frac{d}{dt}(\overline{r}) & = & \overline{\dot{r}} \\ \frac{d}{dt}(\overline{\dot{r}}) & = & -Bg \sin\overline{\theta} + B\overline{r}(\overline{z}^2 + (\overline{u}_2)^2/2) \\ \frac{d}{dt}(\overline{\theta}) & = & \overline{z} \\ \frac{d}{dt}(\overline{z}) & = & u_1 \\ \frac{d}{dt}\overline{u}_2 & = & w_2. \end{array} \right. \tag{28}$$

The averaged state $\overline{x} = (\overline{r}, \overline{\dot{r}}, \overline{\theta}, \overline{z}, \overline{u}_2)$ is of dimension 5. The control variables are u_1 and w_2. It is clear that (28) is a flat system with $(\overline{r}, \overline{\theta})$ as linearizing output.

From the averaging method (see [14, 22]), it results that, for ε small enough, the trajectories of (28) can be approximated by the trajectories of the original system (25). More precisely, for every $\eta, T > 0$, every smooth functions $[0, T] \ni t \rightarrow (f_1(t), f_2(t))$ and initial condition $\overline{x}^0 = (r^0, \dot{r}^0, \theta^0, \dot{\theta}^0, u_2^0)$, there exists $\varepsilon > 0$ such that, for all $t \in [0, T]$

$$|r(t) - \overline{r}(t)| \leq \eta, \quad |\dot{r}(t) - \overline{\dot{r}}(t)| \leq \eta, \quad |\theta(t) - \overline{\theta}(t)| \leq \eta, \quad |\dot{\theta}(t) - \overline{u}_2 \sin(t/\varepsilon) - \overline{z}(t)| \leq \eta$$

where

$$t \rightarrow (\overline{r}(t), \overline{\dot{r}}(t), \overline{\theta}(t), \overline{z}(t), \overline{u}_2(t))$$

is the trajectory of (28) starting from $\overline{x}^0$ at time 0 with $u_1(t) = f(t)$ and $w_2(t) = f_2(t)$, and where

$$t \rightarrow (r(t), \dot{r}(t), \theta(t), \dot{\theta}(t))$$

is the trajectory of (25) starting from $(r^0, \dot{r}^0, \theta^0, \dot{\theta}^0)$ with $u = f_1(t) + u_2(t)\cos(t/\varepsilon)$. This means that, when ε tends to zero $(r, \dot{r}, \theta)$ tends strongly to $(\overline{r}, \overline{\dot{r}}, \overline{\theta})$ and that $\dot{\theta}$ tends, in general, only weakly to $\overline{z}$.

Assume that the feedback $u_1 = k_1(\overline{r}, \overline{\dot{r}}, \overline{\theta}, \overline{z}, \overline{u}_2)$ and $w_2 = k_2(\overline{r}, \overline{\dot{r}}, \overline{\theta}, \overline{z}, \overline{u}_2)$ stabilizes the averaged control system (28) around an hyperbolic equilibrium. Then, the original system (25) with the following time-varying dynamic feedback

$$\left\{ \begin{array}{rcl} \frac{d}{dt}(u_2) & = & k_2(r, \dot{r}, \theta, \dot{\theta} - u_2 \sin(t/\varepsilon)) \\ u & = & k_1(r, \dot{r}, \theta, \dot{\theta} - u_2 \sin(t/\varepsilon)) + u_2 \cos(t/\varepsilon)/\varepsilon, \end{array} \right.$$

admits an asymptotically stable limit cycle that is hyperbolic. With such control r, $\dot{r}$ and θ remain in an ε neighborhood of fixed values corresponding to the equilibrium values of $(\overline{r}, \overline{\dot{r}}, \overline{\theta})$ for the averaged closed-loop system (28).

The simulations below use such control strategy. More precisely, the functions k_1 and k_2 are design on the averaged system (28) in order to satisfied the following equations (model matching for r and θ):

$$\left\{\begin{array}{rcl} e_r^{(3)} & = & -\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}+\frac{1}{\tau_3}+\frac{1}{\tau_4}\right)e_r^{(2)} - \left(\frac{1}{\tau_1\tau_2}+\frac{1}{\tau_1\tau_3}+\frac{1}{\tau_1\tau_4}\frac{1}{\tau_2\tau_3}+\frac{1}{\tau_2\tau_4}+\frac{1}{\tau_3\tau_4}\right)e_r^{(1)} \\ & & -\left(\frac{1}{\tau_1\tau_2\tau_3}+\frac{1}{\tau_1\tau_2\tau_4}+\frac{1}{\tau_2\tau_3\tau_4}\right)e_r - \frac{1}{\tau_1\tau_2\tau_3\tau_4}\int e_r \\ e_\theta^{(2)} & = & -\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}+\frac{1}{\tau_3}\right)e_\theta^{(1)} - \left(\frac{1}{\tau_1\tau_2}+\frac{1}{\tau_1\tau_3}+\frac{1}{\tau_2\tau_3}\right)e_\theta - \frac{1}{\tau_1\tau_2\tau_3}\int e_\theta \end{array}\right.$$

where $e_r = \overline{r} - r_c(t)$ and $e_\theta = \overline{\theta} - \theta_c(t)$ with $r_c(t)$ and $\theta_c(t)$ are reference trajectories for r and θ.

For the simulations, the system parameters are

$$m = 0.05 \text{ kg}, \quad J = 0.02 \text{ kg m}^2, \quad J_b = 2. \times 10^{-6} \text{ kg m}^2, \quad R = 0.01 \text{ m}, \quad g = 9.81 \text{ m/s}^2,$$

and the controller parameters are

$$\varepsilon = \frac{0.1}{2\pi} \text{ s}, \quad \tau_1 = \tau_2 = \tau_3 = \tau_4 = 0.1 \text{ s}.$$

The choice of ε is made such that $|\varepsilon u_2| \ll 1$: $r \approx 1$ m, $\theta \approx \pi/4$, $u_2 \approx \sqrt{2g\sin\theta/r} \approx 4.\ \text{s}^{-1}$, $\varepsilon u_2 \approx 0.1 \ll 1$.

For the simulation displayed in figure 6, page 19, the parameters (m, J, J_b, R, g) used in the expressions of k_1 and k_2 are the same as those used for the integration of (25). The values of the torque τ are not very large and can be realized with classical motors. For such oscillatory control, the ball weight is not sufficient to keep the contact with the beam. Some elementary computations show that the normal force of the beam on the ball is not always positive . Thus, the ball must be maintained by a mechanical device in contact with the beam.

For the simulation displayed in figure 7, page 19, the values of parameters used in the expression of k_1 and k_2 differ from the ones used for (25). For the controller, we have

$$(m, J, J_b, R, g) = (0.04, 0.016, 1.6 \times 10^{-6}, 0.008, 8.2).$$

This is the only difference with the simulation of figure 6. The reference trajectories r_c and θ_c remain unchanged. A comparison with the simulation of figure 6 shows that the control performance are only slightly modified by such parameter differences. This illustrates the robustness that can be achieved with such control method.

The above method can be used to control the inverse double pendulum of figure 3: the cartesian coordinates of the suspension point, x and y, play the same role as the beam angle θ.

The appearance of new control variable for the approximated averaged system has clearly something to do with some works of Sussmann and co-workers [25, 26] on the limit of highly oscillatory control for system without drift.

The problem of singularity when using the linearizing feedback on the averaged system is not addressed here. For (28), singularity in the control appears when $r = 0$. This problem is important and will be addressed in forthcoming works.

80. mm by 80. mm (bb0-r-theta) 80. mm by 80. mm (bb0-tau)
80. mm by 80. mm (bb0-ecart-r) 80. mm by 80. mm (bb0-ecart-theta)

Figure 6: *ball and beam; parameters* (m, J, J_b, R, g) *without any error.*

80. mm by 80. mm (bb1-r-theta) 80. mm by 80. mm (bb1-tau)
80. mm by 80. mm (bb1-ecart-r) 80. mm by 80. mm (bb1-ecart-theta)

Figure 7: *ball and beam; parameters* (m, J, J_b, R, g) *with* -20% *of errors.*

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