JACOBI EQUATION, RIEMANNIAN CURVATURE AND THE MOTION OF A PERFECT INCOMPRESSIBLE FLUID.

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Abstract

Following some Arnol'd results relative to the geometry underlying the dynamics of a perfect incompressible fluid [3] (geodesics of left-invariant metrics on Lie groups), the linear differential equation relative to a Lagrangian stability analysis are established. This differential equation, called Jacobi equation, describes, for the same fluid element, the time evolution of the difference between the trajectory starting from reference initial position in a reference flow and the trajectory starting from a perturbed initial position in a perturbed flow. Links with the linear differential equation relative to the classical Eulerian stability analysis are given.

The stability of the solution of the Jacobi equation can be investigated through the sign of the Riemannian curvature. we prove here that, for all flows, with the exception of the perfect eddy with constant vorticity (corresponding to a stationnary rotation around a fixed axis), there always exists small perturbations with strictly negative curvature. If one assumes that negative curvature implies instability with exponential divergence of the geodesic flow, the above result proves effectively the instability, from a Lagrangian viewpoint, of all the solutions of the Euler equation, with the exception of the perfect eddy. An estimation of the most negative part of the curvature provides an interesting interpretation of the Kolmogorov time-scale $\sqrt{\nu/\varepsilon}$ as the smallest time-constant relative to exponential divergence of fluid elements that are initially close.

Running title: Jacobi equation and the motion of an incompressible fluid.

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1 Introduction

The stability of fluid motion can be analyzed from two viewpoints:

– the Eulerian viewpoint where one considers, for the same spatial position x, the time evolution of the difference $\vec{f}(t,x)$ between the velocity of the reference flow $\vec{v}(t,x)$ and the velocity of the perturbed flow $\vec{v}(t,x)+\vec{f}(t,x)$;

– the Lagrangian viewpoint where one considers, for the same fluid particle, the time evolution of the difference between the trajectory starting from the reference initial position x in the reference flow $\vec{v}$ and the trajectory starting from a perturbed initial position x' in the perturbed flow $\vec{v}+\vec{f}$ (see figure 2).

In this paper, we investigate the Lagrangian stability of the motion of a perfect incompressible fluid. We use a rather unusual method derived from the geometric approach due to Arnol'd [3]. A summary of this approach can also be found in the appendix 2 of [2] and in [1], page 472.

Arnol'd shows that the solutions of the incompressible Euler equation correspond to the geodesics of an infinite dimensional Riemannian manifold, the Lie group of the volume preserving diffeomorphisms endowed with the right invariant metric derived from the kinetic energy. Arnol'd [3] establishes the general expression of the curvature for a Lie group with a left (or right) invariant metric. He calculates also explicitely the curvature in the case of an ideal fluid filling a plane square domain with periodic boundary conditions (a two dimensional torus with an Euclidian metric) and sketches a nice physical interpretation relative to weather prediction.

This physical interpretation relies on the relation between the divergence time constants and the magnitude of the negative part of the curvature. In Riemannian geometry (see [2], appendix 1 for example), the stability of the geodesics can be studied through their second variation, the Jacobi equation, and is conditionned by the sign of the curvature of plane sections tangent to the geodesics: for compact manifold, negative curvature **in all plane sections** means exponential divergence of two nearby geodesics and implies the ergodicity of the geodesic flow.

In this paper, we complete Arnol'd's results in the case of a perfect incompressible fluid filling a 3-D bounded domain. More precisely,

– in proposition 2, we give the Jacobi equation that, as far as we know, has never been precisely written before; we provide also the link between this linear differential equation with the first order sensitivity equations classically used in linear hydrodynamic stability [5];

– in proposition 3, we provide an estimation of the smallest part of the curvature where dissipation-like terms appear; this gives an interesting interpretation of the Kolmogorov time-scale introduced in turbulence theory .

As in [3], the functional analysis problems (infinite dimension of the system, existence of solutions) are not adressed here. For this aspect, we refer to the fundamental paper of Ebin and Marsden [6] where the existence and uniqueness of the geodesic flow is proved for smooth initial conditions and for small time intervals. Consequently, all along the paper, we assume the existence and the smoothness of all the manipulated objects.

The paper is organized as follows. In section 2, we recall the Euler equation and some classical notations. In section 3, we establish, with very elementary tools, the first variation vector field and the second variation equation (Jacobi equation) relative to the motion of a perfect incompressible fluid filling a 3-D bounded domain. We show how the addition of viscous terms modifies the second variation equation. In section 4, we prove that, for all flows, with the exception of the perfect eddy with constant vorticity, the curvature form has always a negative part and we give an estimation of its most negative part. We conclude by the interpretation of the Kolmogorov time-scale as an exponential divergence time-constant.

In appendix A, we sketch the Lie group and Riemannian structure underlying the motion of a perfect incompressible fluid, and we give the curvature expression obtained by Arnol'd [3]. Appendix B is devoted to the detailed proof of propositions 1 and 3.

We have tried to render accessible this paper to readers who are not familar with Riemannian geometry. We precise through appendix A and several remarks, the correspondance between the objects we introduce and classical notions of Riemannian geometry.

2 The Euler equation

Let Ω be a domain of the Euclidian space $I\!R^3$ bounded by the fixed surface $\partial\Omega$ as displayed on figure 1. Let $\vec{v}$ be the velocity field of an ideal fluid (incompressible, density equal to ϱ exterior to a non-potential mass force field) which fills the domain Ω. Let p be the pressure. Under such assumptions the fluid motions are described by the Euler equation,

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} = -\frac{1}{\varrho} \nabla p \tag{1}$$

equivalent to the Bernoulli equation,

$$\frac{\partial \vec{v}}{\partial t} = \vec{v} \times \vec{r} - \nabla \lambda, \quad \text{with} \quad \vec{r} = \nabla \times \vec{v} \text{ and } \lambda = p/\varrho + \vec{v}^2/2. \tag{2}$$

The velocity field $\vec{v}$ satisfies $\nabla \cdot \vec{v} = 0$ in Ω and $\vec{v} \cdot \vec{n} = 0$ on $\partial\Omega$ where $\vec{n}$ is a vector normal to $\partial\Omega$.

Using the identity

$$\nabla \times (\vec{a} \times \vec{b}) = [\vec{a}, \vec{b}] + (\nabla \cdot \vec{b})\, \vec{a} - (\nabla \cdot \vec{a})\, \vec{b} \tag{3}$$

where $[\vec{a}, \vec{b}]$ denotes the Lie bracket of the vector fields $\vec{a}$ and $\vec{b}$, we obtain the vorticity equation

$$\frac{\partial \vec{r}}{\partial t} = [\vec{v}, \vec{r}]. \tag{4}$$

Notice that

$$(\vec{a} \cdot \nabla)\, \vec{b} = D\vec{b}\, \vec{a}$$

and that

$$[\vec{a}, \vec{b}] = (\vec{b} \cdot \nabla)\, \vec{a} - (\vec{a} \cdot \nabla)\, \vec{b} = D\vec{a}\, \vec{b} - D\vec{b}\, \vec{a}$$

where D denotes the derivation operator with respect to the 3 space coordinates.

We denote by $\mathcal{U}$ the vector space (of infinite dimension) of smooth velocity fields on Ω tangent to $\partial\Omega$ and of zero divergence (see appendix A). $\mathcal{U}$ is equipped with a scalar product, denoted $< \centerdot, \centerdot >$, derived from the kinetic energy :

$$< \vec{a}, \vec{b} >= \int\!\!\int\!\!\int_{\Omega} \vec{a}(x) \cdot \vec{b}(x)\; dx.$$

The length, $\|\vec{a}\|$, of a vector $\vec{a} \in \mathcal{U}$ is then

$$\|\vec{a}\| = \sqrt{< \vec{a}, \vec{a} >}.$$

3 The Jacobi equation

3.1 Linear stability of Lagrangian type

Consider a solution $\vec{v}(t, \centerdot) \in \mathcal{U}$ of the Euler equation (1). We denote by $\phi_t^{\vec{v}}(x)$ the position of a fluid element located at x for $t = 0$. $\phi_t^{\vec{v}}(x)$ is defined by $\phi_t^{\vec{v}}(x) = z(t)$ where $z(t)$ results from the integration over $[0, t]$ of the following ordinary differential system:

$$\left\{ \begin{array}{rcl} \dfrac{dz}{ds} & = & \vec{v}(s, z) \\ z(0) & = & x. \end{array} \right. \tag{5}$$

Because $\vec{v}$ is tangent to $\partial\Omega$, the mapping

$$\begin{array}{rrcl} \phi_t^{\vec{v}} : & \Omega & \longrightarrow & \Omega \\ & x & \longrightarrow & \phi_t^{\vec{v}}(x) \end{array}$$

is a continuously differentiable bijection and its inverse $(\phi_t^{\vec{v}})^{-1}$ too. Since $\nabla \cdot \vec{v} = 0$, $\phi_t^{\vec{v}}$ preserves the Euclidian 3-D volume. $\phi_t^{\vec{v}}$ is called a volume-preserving diffeomorphism.

Consider a nearby flow $\vec{v} + \vec{f}$ of the Euler equation where $\vec{f}$ is a small time-dependent element of $\mathcal{U}$. Neglecting second order terms, $\vec{f}$ satisfies the linear time-dependent differential system

$$\frac{\partial \vec{f}}{\partial t} + (\vec{f} \cdot \nabla)\, \vec{v} + (\vec{v} \cdot \nabla)\vec{f} = -\nabla\alpha \tag{6}$$

where the real function α is such that $\vec{f}(t,\centerdot)$ belongs to $\mathcal{U}$ for all t. The Eulerian linear stability theory is based on such linearized equation (see, e.g., [5]) : if $\vec{f}(t,\centerdot)$ remains bounded for $t > 0$ then the flow $\vec{v}$ is said to be linearly stable.

In this paper, we perform a similar linear stability analysis but of Lagrangian type. An adapted mathematical framework is sketched in appendix A and has been developed by Arnol'd (1967) [3]. We have used this framework as a guideline to introduce suitable objects and to obtain some interesting results specific to the motion of a perfect fluid. Since our results can be explained and proved with classical notions of fluid mechanics, we present them independently of their natural mathematical framework. For readers more interested by the mathematical aspect, we have added several remarks refering to appendix A.

The physical question, displayed on figure 2, is the following : what is the difference between the trajectory of a fluid particle, with reference initial position x, in the reference flow $\vec{v}$, and the trajectory of the same particle, with perturbed initial position x', in the perturbed flow $\vec{v}+\vec{f}$?

First of all, we have to quantify and to relate this "difference" to the perturbation $\vec{f}$ and to $x'-x$. In subsection 3.2, we show that this difference is described by a time-dependent element of $\mathcal{U}$, $\vec{\xi}$, that is called, in Riemannian geometry (see, e.g., [2], appendix 1), the first variation along the geodesic. As $\vec{f}$, $\vec{\xi}$ satisfies also a linear time-dependent differential equation, which is called the Jacobi equation or the second variation equation along the geodesic. This equation is established in subsection 3.3. If $\vec{\xi}$ remains bounded in $\mathcal{U}$, then the flow $\vec{v}$ is said to be linearly stable from a Lagrangian point of view : small perturbations of the initial velocities and of the initial particle position induce perturbations of the trajectories that are not linearly amplified.

3.2 The first variation

The difference between the initial positions, $x'-x$, can be described by a small vector field of $\mathcal{U}$, denoted $\vec{\chi}$. As displayed on figure 2, we are looking for a first order estimation of the influence of the velocity perturbation $\vec{f}$ and of the initial position perturbation $\vec{\chi}$. Thus, we introduce the vector field $\vec{\xi}$ corresponding to the first order term of the difference $\phi_t^{\vec{v}+\vec{f}}(x+\vec{\chi}(x)) - \phi_t^{\vec{v}}(x)$. Notice that contrarily to $\vec{f}$, $\vec{\chi}$ does not depend on t.

Proposition 1 *For every t and fluid particle of initial position $x \in \Omega$, the small time-dependent vector field $\vec{\xi}(t,\centerdot)$ defined by the first order approximation*

$$\vec{\xi}(t,\phi_t^{\vec{v}}(x)) \approx \phi_t^{\vec{v}+\vec{f}}(x+\vec{\chi}(x)) - \phi_t^{\vec{v}}(x) \approx D\phi_t^{\vec{v}}(x)\vec{\chi}(x) + \phi_t^{\vec{v}+\vec{f}}(x) - \phi_t^{\vec{v}}(x), \tag{7}$$

belongs to $\mathcal{U}$ and is related to $\vec{f}(t,\centerdot)$ and $\vec{\chi}(\centerdot)$ by the kinematic relation

$$\begin{aligned}\vec{\xi}(t,x) \;=\; & \int_0^t \left(D\left[\phi_s^{\vec{v}} \circ (\phi_t^{\vec{v}})^{-1}\right](x)\right)^{-1} \vec{f}\left(s, \phi_s^{\vec{v}} \circ (\phi_t^{\vec{v}})^{-1}(x)\right)\, ds \\ & + \left(D(\phi_t^{\vec{v}})^{-1}(x)\right)^{-1} \vec{\chi}\left((\phi_t^{\vec{v}})^{-1}(x)\right),\end{aligned} \tag{8}$$

where D denotes the derivation operator with respect to the 3 space variables.

The proof of this proposition is given in appendix B. It is not obvious that the vector field $\xi(t, \boldsymbol{.})$ defined by (7) or (8) is tangent to $\partial\Omega$ and of zero divergence on Ω. Notice also that the proof do not use the fact that $\vec{v}$ and $\vec{f}$ satisfies (1) and (6), respectively. The second member of relation (8) is the sum of two terms:

- the second term is clearly relative, for the flow $\vec{v}$, to the sensitivity of trajectories to initial position; if the trajectories tend to diverge exponentially, then this second terms will grow exponentially; this situation is typical of the so-called Lagrangian chaos and can be observed, e.g., for stationnary flows such as ABC flows [4];
- the first term is a combinaison of two effects:
  - the first effect, $\left(D\left[\phi_s^{\vec{v}} \circ (\phi_t^{\vec{v}})^{-1}\right](x)\right)^{-1}$, is also relative to the behavior of nearby trajectories for the flow $\vec{v}$ and is purely Lagrangian;
  - the second effect, $\vec{f}\left(s, \phi_s^{\vec{v}} \circ (\phi_t^{\vec{v}})^{-1}(x)\right)$, depends essentially on the growth of $\vec{f}$ and is typically Eulerian.

From a stability point of view, it seems that $\vec{\xi}$ remains bounded if no Lagrangian chaos and Eulerian exponential instabilities are present. Otherwise stated, if the particle trajectories are sensitive to initial positions, or if the Euler equations are linearly unstable, ξ will certainly grow exponentially with t. We conjecture that, if ξ grows exponentially with t, then there is, at least, Lagrangian chaos or Eulerian exponential instabilities.

Remark 1 *Since $\vec{v}$ is solution of the Euler equation, the curve $t \longrightarrow \phi_t^{\vec{v}}$ is a geodesic on G (see appendix A). Because $\vec{v} + \vec{f}$ is also solution, in a first approximation, of the Euler equation, and because right translations are isometries of G, the curve $t \longrightarrow \phi_t^{\vec{v}+\vec{f}} \circ \phi_1^{\vec{\chi}}$ is also a geodesic on G which is close to the geodesic $t \longrightarrow \phi_t^{\vec{v}}$.*

Since $\vec{\chi}$ and $\vec{f}$ are small, $\phi_1^{\vec{\chi}}(x) \approx x + \vec{\chi}(x)$ and the volume preserving diffeomorphism

$$\begin{array}{ccc} \Omega & \longrightarrow & \Omega \\ x & \longrightarrow & \phi_t^{\vec{v}+\vec{f}} \circ \phi_1^{\vec{\chi}} \circ (\phi_t^{\vec{v}})^{-1}(x) \end{array}$$

is close to the identity I_d for all t. It can be approximated by a mapping of the form $x \rightarrow x + \vec{\xi}(t, x)$ where $\vec{\xi}$ is a small vector of $\mathcal{U} = TG_{I_d}$.

Thus, proposition 1 relates the first variation of the geodesic to $\vec{\chi}$ and $\vec{f}$. $\vec{f}(t, \boldsymbol{.}) \in \mathcal{U}$ results from to a difference of initial velocity along the geodesic at I_d: $\vec{v}(0, \boldsymbol{.}) + \vec{f}(0, \boldsymbol{.})$ instead of $\vec{v}(0, \boldsymbol{.})$. $\vec{\chi}(\boldsymbol{.}) \in \mathcal{U}$ corresponds to a difference of initial starting point on G: $\phi_1^{\vec{\chi}} \in G$ instead of I_d.

3.3 The second variation

To derive the time evolution of the vector $\vec{\xi}$ defined by proposition 1, it is natural to derive the relation (7) with respect to the time t. Neglecting second order terms in $\vec{f}$ and $\vec{\chi}$, we obtain

$$\frac{\partial \vec{\xi}}{\partial t}(t, \phi_t^{\vec{v}}(x)) + D\vec{\xi}(t, \phi_t^{\vec{v}}(x))\vec{v}(t, \phi_t^{\vec{v}}(x))$$
$$\approx \frac{\partial}{\partial t}[D\phi_t^{\vec{v}}(x)]\ \vec{\chi}(x) + \vec{v}(t, \phi_t^{\vec{v}+\vec{f}}(x)) + \vec{f}(t, \phi_t^{\vec{v}}(x)) - \vec{v}(t, \phi_t^{\vec{v}}(x)).$$

But

$$\frac{\partial}{\partial t}[D\phi_t^{\vec{v}}(x)] = D\vec{v}(t, \phi_t^{\vec{v}}(x))\ D\phi_t^{\vec{v}}(x)$$

and

$$\vec{v}(t, \phi_t^{\vec{v}+\vec{f}}(x)) - \vec{v}(t, \phi_t^{\vec{v}}(x)) \approx D\vec{v}(t, \phi_t^{\vec{v}}(x))\ \left(\phi_t^{\vec{v}+\vec{f}}(x) - \phi_t^{\vec{v}}(x)\right).$$

Thus, replacing x by $(\phi_t^{\vec{v}})^{-1}(x)$, we conclude that

$$\frac{\partial \vec{\xi}}{\partial t}(t,x) + D\vec{\xi}(t,x)\,\vec{v}(t,x) = D\vec{v}(t,x)\,\vec{\xi}(t,x) + \vec{f}(t,x).$$

Otherwise written:

$$\frac{\partial \vec{\xi}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{\xi} = (\vec{\xi}\cdot\nabla)\,\vec{v} + \vec{f}. \tag{9}$$

It seems interesting to consider here the convective derivation $d/dt = \partial/\partial t + (\vec{v}\cdot\nabla)$. However, this derivation operator has the main drawback that, if $\vec{\xi}(t,\centerdot)$ belongs to $\mathcal{U}$, $d\vec{\xi}/dt\,(t,\centerdot)$ does not, in general, remains in $\mathcal{U}$ for all t. This justifies the introduction of a closely related operator, denoted by D/Dt, called covariant derivation with respect to the flow $\vec{v}$ and defined, for all time-dependent element $\xi(t,\centerdot)$ of $\mathcal{U}$, by

$$\frac{D\vec{\xi}}{Dt} = \frac{d\vec{\xi}}{dt} + \nabla\alpha_{\vec{\xi}} = \frac{\partial \vec{\xi}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{\xi} + \nabla\alpha_{\vec{\xi}} \tag{10}$$

where the real function $\alpha_{\vec{\xi}}$ is such that $D\vec{\xi}/Dt$ belongs to $\mathcal{U}$ for all t. As displayed on figure 3, the addition of a gradient corresponds, geometrically, to the orthogonal projection of $(\vec{v}\cdot\nabla)\,\vec{\xi}$ onto the vector space $\mathcal{U}$ (the scalar product is that derived from the kinetic energy).

It is clear that the operator D/Dt has all the algebraic properties of a differential operator with respect to t . Moreover, as for the convective derivation d/dt or the partial derivation $\partial/\partial t$, we have

$$\frac{d < \vec{\xi}_1, \vec{\xi}_2 >}{dt} =< \frac{D\vec{\xi}_1}{Dt}\ ,\ \vec{\xi}_2 > + < \vec{\xi}_1\ ,\ \frac{D\vec{\xi}_2}{Dt} > .$$

This implies that the time-dependent vector fields $\vec{\xi}$ such that $D\vec{\xi}/Dt = 0$ have constant length $\|\vec{\xi}\|$. Physically, this means that, as the two other derivations $\partial/\partial t$ and d/dt, the covariant derivation D/Dt preserves also the kinetic energy.

Remark 2 *The derivation operator D/Dt was first introduced by Moreau [9]: he defined the least constraint transport of a vector field $\vec{\xi} \in \mathcal{U}$ by $D\vec{\xi}/Dt = 0$. Moreau noticed also a strong analogy between the motion of an hypersolid of infinite dimension around a fixed point and the hydrodynamics of a perfect incompressible fluid. Later on, this analogy has been fully used and deeply understood by Arnold [3].*

The operator D/Dt is nothing but the covariant operator classically used in Riemannian geometry to derive the time evolution of the first variation vector field along a geodesic. Using the notation introduced in the appendix A, we have $D\vec{\xi}/Dt = \nabla_{\vec{v}}\vec{\xi}$. The least constraint transport introduced by Moreau corresponds to parallel transport.

Up to now, we have performed only one derivation (see relation (9)) and we have not used the fact that $\vec{v}$ (resp. $\vec{f}$) satisfies (1) (resp. (6)). Thus, a second time derivation is necessary. We have

$$\begin{aligned}\frac{d^2\vec{\xi}}{dt^2} &= \frac{\partial}{\partial t}\left[D\vec{v}\,\vec{\xi} + \vec{f}\right] + D\left[D\vec{v}\,\vec{\xi} + \vec{f}\right]\,\vec{v} \\ &= D\left[\frac{\partial \vec{v}}{\partial t}\right]\,\vec{\xi} + D^2\vec{v}(\vec{v},\vec{\xi}) + D\vec{v}\left(\frac{\partial \vec{\xi}}{\partial t} + D\vec{\xi}\,\vec{v}\right) + \frac{\partial \vec{f}}{\partial t} + D\vec{f}\,\vec{v}.\end{aligned}$$

(9) leads to

$$\frac{d^2\vec{\xi}}{dt^2} = (\vec{\xi}\cdot\nabla)\ \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v}\right] + \frac{\partial f}{\partial t} + (f\cdot\nabla)\,\vec{v} + (\vec{v}\cdot\nabla)\,f. \tag{11}$$

Since $\vec{v}$ and $\vec{f}$ satisfy (1) and (6) respectively, we obtain the simple relation

$$\frac{d^2\vec{\xi}}{dt^2} = -\frac{1}{\varrho}(\vec{\xi}\cdot\nabla)\ [\nabla p] - \nabla\alpha.$$

In order to remain in $\mathcal{U}$, we consider, instead of $d^2\vec{\xi}/dt^2$, $D^2\vec{\xi}/Dt^2$. Elementary calculations lead to a linear second order equation

$$\frac{D^2\vec{\xi}}{Dt^2} + A_{\vec{v}}(\vec{\xi}) = 0$$

where the linear operator $A_{\vec{v}}$

– sends $\mathcal{U}$ into $\mathcal{U}$;

– depends on $\vec{v}$ and is defined by

$$A_{\vec{v}}(\vec{\xi}) = \frac{1}{\varrho}(\vec{\xi} \cdot \nabla)\ (\nabla p) - (\vec{v} \cdot \nabla)\ (\nabla \alpha_{\vec{\xi}}) + \nabla \gamma \tag{12}$$

where $\nabla \alpha_{\vec{\xi}} = D\vec{\xi}/Dt - d\vec{\xi}/dt$ (see figure 3) and $\nabla \gamma$ is such that $A_{\vec{v}}(\vec{\xi})$ belongs to $\mathcal{U}$.

Straightforward calculations show that the operator $A_{\vec{v}}$ is symmetric: for all $\vec{\xi}_1$ and $\vec{\xi}_2$ in $\mathcal{U}$,

$$< \vec{\xi}_1\ ,\ A_{\vec{v}}(\vec{\xi}_2) > \quad = \quad < \vec{\xi}_2\ ,\ A_{\vec{v}}(\vec{\xi}_1) > \quad = \quad \int\!\!\int\!\!\int_{\Omega} \left(\frac{1}{\varrho} D^2 p(\vec{\xi}_1, \vec{\xi}_2) - \nabla \alpha_{\vec{\xi}_1} \cdot \nabla \alpha_{\vec{\xi}_2} \right)\, dx \tag{13}$$

where $\nabla \alpha_{\vec{\xi}_i} = D\vec{\xi}_i/Dt - d\vec{\xi}_i/dt,\ \ i = 1, 2$, and where $D^2 p$ denotes the Hessian matrix of the pressure p. To summarize, we have obtained the following result.

Proposition 2 *For a flow $\vec{v}$ solution of the Euler equation (1), the time evolution of the vector $\vec{\xi}$, defined by proposition 1, displayed on figure 2 and where the small vector field f satisfies (6), follows the second order differential equation (corresponding to the second variation of the geodesics and called Jacobi equation)*

$$\frac{D^2 \vec{\xi}}{Dt^2} + A_{\vec{v}}(\vec{\xi}) = 0 \tag{14}$$

where the linear symmetric operator $A_{\vec{v}}$ is given by (12) or (13) and where the covariant derivation D/Dt is defined by (10).

We will see, in the next section, that the stability of the solution of equation (14) can be studied from the sign of the quadratic form attached to $A_{\vec{v}}$, i.e. from the curvature sign.

Remark 3 *By construction, equation (14) corresponds exactly to the Jacobi equation (see [2], appendix 1). One can verify that the general expression, given by Arnol'd and recalled in the appendix A (relation (20)), of the curvature $K(\vec{v}, \vec{\xi})$ of the plane section spanned by $\vec{v}$ and $\vec{\xi}$ is equal to $< A_{\vec{v}}(\vec{\xi})\ ,\ \vec{\xi} >$ when $\vec{v}$ and $\vec{\xi}$ are perpendicular vectors of length equal to* 1.

3.4 Influence of viscosity

It is interesting to see how the Jacobi equation (14) is modified when viscous effects are added. We assume that, instead of satisfying (1), the flow $\vec{v}$ satisfies the Navier equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} = -\frac{1}{\varrho} \nabla p + \nu \Delta \vec{v}$$

(ν is the kinematic viscosity) and that the perturbation $\vec{f}$ satisfies

$$\frac{\partial \vec{f}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{f} + (\vec{f} \cdot \nabla)\, \vec{v} = -\nabla \alpha + \nu \Delta \vec{f}.$$

$\vec{v}$ and $\vec{f}$ remain of zero divergence on Ω but vanish on $\partial\Omega$.

Proposition 1 remains valid. One can still introduce the derivation operator D/Dt by (10). The calculations leading to relations (9) and (11) remain correct. These two relations give the following second order differential equation for $\vec{\xi}$ defined by proposition 1 :

$$\frac{D^2 \vec{\xi}}{Dt^2} - \nu \Delta \left[\frac{D \vec{\xi}}{Dt} \right] + A_{\vec{v},\nu}(\vec{\xi}) = 0 \tag{15}$$

where the linear operator $A_{\vec{v},\nu}$ is defined by

$$\frac{1}{\varrho}(\vec{\xi} \cdot \nabla)(\nabla p) - (\vec{v} \cdot \nabla)\, (\nabla \alpha_{\vec{\xi}}) - \nu(\vec{\xi} \cdot \nabla)\, \Delta \vec{v} + \nu \Delta[(\vec{\xi} \cdot \nabla)\, \vec{v}] + \nabla \gamma$$

with $\nabla \alpha_{\vec{\xi}} = D\vec{\xi}/Dt - d\vec{\xi}/dt$ and with $\nabla\gamma$ such that the $A_{\vec{v},\nu}(\vec{\xi})$ remains of zero divergence on Ω and vanishs on $\partial\Omega$. The influence of viscosity is double:

- it transforms the symmetric operator $A_{\vec{v}} = A_{\vec{v},0}$ into an operator $A_{\vec{v},\nu}$ which is no more symmetric;
- it introduces a new term, $\nu \Delta[D\vec{\xi}/Dt]$, of opposite sign to $D\vec{\xi}/Dt$, since, by the Green formula,

  $$\int\!\!\int\!\!\int_{\Omega} \Delta[D\vec{\xi}/Dt] \cdot (D\vec{\xi}/Dt)\, dx = -\int\!\!\int\!\!\int_{\Omega} \nabla[D\vec{\xi}/Dt] \cdot \nabla[D\vec{\xi}/Dt]\, dx \leq 0;$$

  this term appears to be very similar in nature to the damping term $-k\dot{\theta}$ classically introduced for a pendulum of angle θ described by the second order equation $\ddot{\theta} + k\dot{\theta} + (g/R)\, \theta = 0$.

4 Study of the curvature

4.1 Instability and negative curvature

From equation (13), one can see that $A_{\vec{v}}(\vec{v}) = 0$. With the orthogonal decomposition,

$$\vec{\xi} = \vec{\xi}^{\perp} + \vec{\xi}^{//} \qquad \text{and} \qquad \vec{\xi}^{//} = \frac{< \vec{\xi}, \vec{v} >}{\|\vec{v}\|^2}\, \vec{v},$$

equation (14) is decomposed into

$$\frac{D^2\vec{\xi}^{//}}{Dt^2} = 0 \quad \text{and} \quad \frac{D^2\vec{\xi}^{\perp}}{Dt^2} = -A_{\vec{v}}(\vec{\xi}^{\perp}).$$

The first equation corresponds to the absence of evolution for variations colinear to $\vec{v}$. Since $A_{\vec{v}}$ is symmetric, the second equation is similar to the equation, with the potential energy

$$U_{\vec{v}}(\vec{\xi}^{\perp}) = \frac{1}{2} < A_{\vec{v}}(\vec{\xi}^{\perp}), \vec{\xi}^{\perp} >, \tag{16}$$

of a linear oscillator of infinite degree of freedom:

$$\frac{D^2\vec{\xi}^{\perp}}{Dt^2} = -\text{grad}_{\vec{\xi}^{\perp}}\left[U_{\vec{v}}(\vec{\xi}^{\perp})\right].$$

It is intuitive that, if the potential energy admits a negative part, then the Jacobi equation is unstable around $\vec{\xi}^{\perp} = 0$. Such a conclusion is rigorous when the dimension is finite and when the symmetric operator A is constant ($DA/Dt = 0$): it suffices to consider the orthogonal basis of the eigenvectors of A. In this case, the most unstable direction is relative to the most negative part of U : denoting by $U_{min} < 0$ the minimum of U among the vector of length equal to 1, the smallest divergence time constant τ_{min} is equal to $1/\sqrt{-2U_{min}}$.

In Riemannian geometry, the potential $U_{\vec{v}}(\vec{\xi}^{\perp})$ is related to the curvature, $K(\vec{v}, \vec{\xi})$, of the plane section spanned by $\vec{v}$ and $\vec{\xi} = \vec{\xi}^{\perp} + \vec{\xi}^{//}$:

$$K(\vec{v}, \vec{\xi}) = \frac{2U_{\vec{v}}(\vec{\xi}^{\perp})}{\|\vec{v}\|^2\|\vec{\xi}^{\perp}\|^2}.$$

It can be seen directly on formula (13) that $U_{\vec{v}}(\vec{\xi}^{\perp}) = U_{\vec{\xi}^{\perp}}(\vec{v})$ ($K(\vec{v}, \vec{\xi}) = K(\vec{\xi}, \vec{v})$).

A discussion of the influence of the curvature (or potential sign) on the stability of the geodesics in the finite dimensional and compact case can be found in [2], appendix 1. In particular, if the curvature is always negative, i.e. if $K \leq -b^2 < 0$ for all plane sections, then the geodesic flow (i.e. the dynamics) admits stochastic properties and is sensitive to initial conditions.

When the curvature does not remain strictly negative but still admits, along every geodesic, a negative part, the stochasticity and the sensitivity to initial conditions of the dynamics are not sure. Very few mathematical results appear to be known about this intermediate case. However, it seems reasonable to conjecture that

1. if, along a geodesic, there always exist plane sections with strictly negative curvature, the instability property concerning the sensitivity to initial conditions is still valid;

2. the magnitude of this negative part provides a good estimation (see τ_{min} above) of the exponential divergence time-constant of geodesics closed to the previous one.

4.2 Negative curvature and estimation

We are interested in the smallest part of the curvature K, i.e. in the smallest part of the potential $U_{\vec{v}}(\vec{\xi}^{\perp})$ defined by (16).

Proposition 3 *Consider a flow $\vec{v}$ solution of the Euler equation (1), the symmetric operator $A_{\vec{v}}$ defined by (13) and the potential $U_{\vec{v}}(\vec{\xi}^{\perp})$ defined for all element $\vec{\xi}^{\perp} \in \mathcal{U}$ orthogonal to $\vec{v}$. We have*

$$2\, U_{\vec{v}}(\vec{\xi}^{\perp}) \;\geq\; \iiint_{\Omega} \vec{\xi}^{\perp} \cdot M\vec{\xi}^{\perp}\, dx \tag{17}$$

and

$$\min_{x\in\Omega}\left[\lambda_{min}(t,x)\right] \;\leq\; \min_{\|\vec{\xi}^{\perp}\|=1} 2\left[U_{\vec{v}}(\vec{\xi}^{\perp})\right] \;\leq\; \min_{x\in\Omega}\left[\frac{trace[M(t,x)]}{3}\right] \tag{18}$$

where

- *M is the symmetric 3×3 matrix $\frac{1}{\varrho}D^2p - D\vec{v}' D\vec{v}$ depending on $x \in \Omega$ and t; (D is the derivation operator with respect to the 3 space Euclidian coordinates (x_1, x_2, x_3) and $'$ denotes transpose);*
- *$trace[M(t,x)] = -\sum_{i,j=1}^{3}\left(\frac{\partial v_i}{\partial x_j}(t,x) + \frac{\partial v_j}{\partial x_i}(t,x)\right)^2$;*
- *$\lambda_{min}(t,x)$ is the smallest eigenvalue of $M(t,x)$.*

This proposition implies that

- if the flow $\vec{v}$ is a perfect eddy with constant vorticity (i.e. the fluid motion is equivalent to the motion a solid rotating with a constant angular velocity around a fixed axis) the potential and the curvature are always non negative (use (17) where a direct calculation gives $M \equiv 0$); notice that such a fluid motion is possible if, and only if, the domain Ω admits an axis of rotation symetry.
- if the flow $\vec{v}$ is not a perfect eddy, then, for each time t, $\min_{x\in\Omega}[\mathrm{trace}[M(t,x)]] < 0$ and there always exists plane sections containing $\vec{v}$ (the velocity along the geodesic) where the potential and the curvature are strictly negative.

If one suppose that positive curvature means stability (neutral with bounded oscillations) and negative curvature means instability (exponential divergence), one concludes that the perfect eddy is the only stable motion of a perfect incompressible fluid[1].

[1] The stability depends on the topology and thus on the norm used to define this topology. In finite dimension, it is not important since all norms are equivalent. But in infinite dimension, one has to precise the norm. Here, it is the kinetic energy. So, it is natural to conjecture that, with the norm associated to kinetic energy, the only stable solution of the Euler equation is the perfect eddy.

The detailed proof of proposition 3 is rather technical and is given in appendix B. During this proof, it is interesting to remark that vector fields $\vec{\xi}^{\perp}$ having potential $2U_{\vec{v}}(\vec{\xi}^{\perp})$ close to trace$(M(t,x))/3$, corresponds to localized eddy of kinetic energy equal to 1 and very high enstrophy. This indicates that the most negative part of the curvature (or potential) corresponds essentially to small localized eddy of small kinetic energy and high enstrophy. Thus it is reasonable to think that the perturbations $\vec{\xi}$ that are the most rapidly amplificated by the fluid motion correspond to such **localized** eddies.

5 Concluding remark

In proposition 3, it is interesting to remark the appearence of the following fundamental quantity:

$$\sum_{i,j=1}^{3}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)^2$$

which is proportionnal to the local dissipation rate ε and is equal to $2\varepsilon/\nu$ (ν is the kinematic viscosity).

For a real flow $\vec{v}$ with a large Reynolds number, inertial effects are dominant. If one assumes that the most negative curvature part which is, according to proposition 3, of magnitude $\frac{2\varepsilon}{3\nu}$, is related to trajectory instabilities, one obtains an interesting exponential divergence estimation of the positions $x(t)$ and $x'(t)$ of two fluid elements initially close:

$$\|x(t)-x'(t)\| \sim \|x(0)-x'(0))\| \exp\left(\frac{t}{\tau}\right)$$

with $\tau=\sqrt{\frac{3\nu}{2\varepsilon}}$.

τ is nothing but the Kolmogorov time-scale, introduced in turbulence theory on the basis of dimensional analysis arguments, characterizing the viscous time-scale of a turbulent flow (see, e.g., equation 33.19, page 194 of [8] where $\tau=\lambda_0/v_{\lambda_0}$). In this paper τ is interpreted as the shortest exponential divergence time-constant between the trajectory of fluid elements initially close. Otherwise stated, during a time interval of length τ, two initially close fluid elements remain close. This is in accordance with the physical meaning of the Kolmogorov time-scale: during a time interval of magnitude τ , the viscous forces maintain the cohesion of nearby fluid elements.

Acknowledgment The author would like to express his endebtedness to the reviewers for their comments and useful suggestions that have been included into the final version of the paper.

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A Lie group with a right invariant metric and the motion of a perfect incompressible fluid

In [3] and in the appendix 2 of [2], Arnol'd introduces several notions. They are displayed and summarized on figure 4.

The set of the vector fields on Ω of zero divergence and tangent to $\partial\Omega$ is a Lie algebra denoted $\mathcal{U}$ with the Lie bracket $[\vec{a}, \vec{b}]$, $\vec{a}$ and $\vec{b}$ in $\mathcal{U}$, defined by

$$[\vec{a}, \vec{b}] = (\vec{b} \cdot \nabla)\, \vec{a} - (\vec{a} \cdot \nabla)\, \vec{b}.$$

To every solution $\vec{v} \in \mathcal{U}$ of (1) is attached a time-dependent diffeormophism family $\phi_t^{\vec{v}}$ defined by (5). $(\phi_t^{\vec{v}}(x))_{t \geq 0}$ corresponds to the successive positions of the fluid particle x and thus to a **Lagrangian description of the fluid motion**.

Since $\nabla \cdot \vec{v} = 0$, these diffeomorphisms preserve the 3-D Euclidian volume. The set G of the diffeomorphisms on Ω preserving the 3-D Euclidian volume is a non commutative group.

Every element g of G close to the identity, I_d, can be seen as the exponential of a small element $\vec{\xi}$ of $\mathcal{U}$:

$$g = \phi_1^{\vec{\xi}}$$

which is generally denoted by $g = \exp(\vec{\xi})$. Thus the tangent space to G at the identity diffeomorphism I_d is the Lie algebra $\mathcal{U}$: G is a Lie group.

$\mathcal{U}$ is equipped with the scalar product, denoted by the brackets $<\ ,\ >$ and derived from the kinetic energy :

$$\forall \vec{v},\ \vec{v} \in \mathcal{U},\ \ <\vec{v}, \vec{v}>= \iiint_\Omega \vec{v}(x) \cdot \vec{v}(x)\ dx \tag{19}$$

where dx is the element of the 3-D Euclidian volume. Since for every $g \in G$ and every real smooth function α on Ω

$$\iiint_\Omega \alpha(x)\ dx = \iiint_\Omega \alpha(g(x))\ dx,$$

every tangent space to G can be equipped with a metric through right translation. More precisely, for every $g \in G$, the right translation,

$$\begin{array}{rccc} R_g : & G & \longrightarrow & G \\ & h & \longrightarrow & h \circ g \end{array}$$

induces on the tangent spaces $TG_{I_d} = \mathcal{U}$ and TG_g at I_d and g, respectively, a linear bijection, $\tilde{R}_g$, that transports the scalar product from $\mathcal{U}$ to TG_g. Every element of TG_g can be identified to a vector field $\vec{\xi}$ on Ω such that the vector field $x \longrightarrow \vec{\xi}(g^{-1}(x))$ belongs to $\mathcal{U}$. In the sequel, we identify the vector $\xi \in TG_g$ with its image in $\mathcal{U}$ by the linear isometry $[\tilde{R}_g]^{-1}$.

G is thus a Lie group equipped with a right invariant metric. In particular, G is a Riemannian manifold of infinite dimension. Arnol'd [3] proves that the curves

$$\begin{array}{ccc} I\!\!R & \longrightarrow & G \\ t & \longrightarrow & \phi_t^{\vec{v}} \end{array}$$

where ϕ_t^v is defined by (5) and where $\vec{v}(t, \cdot) \in \mathcal{U}$ satisfies (1), are, formally, geodesics of G. The covariant differentiation, with respect to $\vec{v}$, of $\vec{\xi}(t, \cdot) \in \mathcal{U}$ corresponding to an element of $TG_{\phi_t^{\vec{v}}}$, is given by

$$\boldsymbol{\nabla}_{\vec{v}} \vec{\xi} = \frac{\partial \vec{\xi}}{\partial t} + (\vec{v} \cdot \nabla)\ \vec{\xi} + \nabla \alpha$$

where α is a real function such that $\dfrac{\partial \vec{\xi}}{\partial t} + (\vec{v} \cdot \nabla)\ \vec{\xi} + \nabla \alpha$ belongs to $\mathcal{U}$ ($\Delta \alpha + \nabla \cdot ((\vec{v} \cdot \nabla)\ \vec{\xi}) = 0$ and $\nabla \alpha + (\vec{v} \cdot \nabla)\ \vec{\xi}$ tangent to $\partial \Omega$). In order to avoid any confusion, the large nabla "$\boldsymbol{\nabla}$" is used for the covariant derivation on G and the small nabla "∇" for the gradient operator in the 3-D Euclidian space. Notice that, with these notations, the Euler equation becomes

$$\boldsymbol{\nabla}_{\vec{v}}\vec{v} = 0$$

which definies a geodesic.

In the finite dimensional and compact case, it is well known, since Hadamard [7], that the stability of a geodesic depends on the sign of the curvature in the direction of plane sections containing the velocity along the geodesic: negative curvature **in all plane sections** means exponential divergence of two nearby geodesics. Arnol'd ([2], appendix 2, theorem 10, page 329) gives the expression of the curvature $K(\vec{v},\vec{\xi})$ for every plane section spanned by two orthogonal element of $\mathcal{U}$, $\vec{v}$ and $\vec{\xi}$:

$$\begin{aligned}
&4 < \vec{v},\vec{v} > \quad < \vec{\xi},\vec{\xi} > \quad K(\vec{v},\vec{\xi}) = \\
&\quad < B(\vec{v},\vec{\xi}) + B(\vec{\xi},\vec{v}) \ , \ B(\vec{v},\vec{\xi}) + B(\vec{\xi},\vec{v}) > \ + \ 2 < B(\vec{v},\vec{\xi}) - B(\vec{\xi},\vec{v}) \ , \ [\vec{v},\vec{\xi}] > \\
&\quad - 3 < [\vec{v},\vec{\xi}] \ , \ [\vec{v},\vec{\xi}] > \ - \ 4 < B(\vec{v},\vec{v}) \ , \ B(\vec{\xi},\vec{\xi}) >
\end{aligned} \tag{20}$$

where the bilinear operator B is defined by

$$\begin{array}{rccl}
B: & \mathcal{U}\times\mathcal{U} & \longrightarrow & \mathcal{U} \\
& (\vec{v},\vec{w}) & \longrightarrow & (\nabla\times\vec{v})\times\vec{w} + \nabla\alpha
\end{array}$$

with α such that $(\nabla\times\vec{v})\times\vec{w} + \nabla\alpha$ belongs to $\mathcal{U}$.

B Proofs of propositions 1 and 3

Proof of proposition 1 In a more rigorous setting, $\vec{\xi}$ is given by

$$\vec{\xi}(t,\phi_t^{\vec{v}}(x)) = \frac{d}{d\varepsilon}\left[\phi_t^{\vec{v}+\varepsilon\vec{f}} \circ \phi_1^{\varepsilon\vec{\chi}}\right]_{\varepsilon=0}.$$

Since

$$\frac{d}{d\varepsilon}\left[\phi_t^{\vec{v}+\varepsilon\vec{f}} \circ \phi_1^{\varepsilon\vec{\chi}}\right]_{\varepsilon=0} = \frac{d}{d\varepsilon}\left[\phi_t^{\vec{v}+\varepsilon\vec{f}}\right]_{\varepsilon=0} + D\phi_t^{\vec{v}}\vec{\chi}.$$

it suffices to compute $\frac{d}{d\varepsilon}\left[\phi_t^{\vec{v}+\varepsilon\vec{f}}\right]_{\varepsilon=0}$.

$\phi_t^{\vec{v}+\varepsilon\vec{f}}$ is defined, for all $x\in\Omega$, by

$$\left\{\begin{array}{rcl}
\dfrac{dz}{ds} & = & \vec{v}(s,z) + \varepsilon\vec{f}(s,z) \\
z(0) & = & x
\end{array}\right.$$

with $\phi_t^{\vec{v}+\varepsilon\vec{f}}(x) = z(t)$. The first variation of the above differential equation along the trajectory $(\phi_s^{\vec{v}}(x))_{0\leq s\leq t}$ gives

$$\frac{d\Delta z}{ds} = D\vec{v}(s,\phi_s^{\vec{v}}(x))\;\Delta z + \varepsilon\vec{f}(s,\phi_s^{\vec{v}}(x)) \tag{21}$$

(second order terms in ε and Δz are neglected). Since the Jacobian matrix $D\phi_t^{\vec{v}}(x)$ is solution of

$$\begin{cases} \dfrac{d}{ds}\left[D\phi_s^{\vec{v}}(x)\right] & = & D\vec{v}(s,\phi_s^{\vec{v}}(x))\;D\phi_s^{\vec{v}}(x) \\ D\phi_0^{\vec{v}}(x) & = & I_3, \end{cases}$$

(I_3 is the 3×3 identity matrix) the solution of (21) with initial condition $\Delta z(0) = 0$ is given by

$$\Delta z(t) = \varepsilon D\phi_t^{\vec{v}}(x)\int_0^t \left[D\phi_s^{\vec{v}}(x)\right]^{-1}\vec{f}(s,\phi_s^{\vec{v}}(x))\;ds.$$

It suffices to replace x by $(\phi_t^{\vec{v}})_{-1}(x)$ in the above relations and to remark that

$$I_3 = D\left[\phi_s^{\vec{v}}\circ(\phi_t^{\vec{v}})^{-1}\right](x) = D\phi_s^{\vec{v}}((\phi_t^{\vec{v}})^{-1}(x))\;\;\left(D\phi_t^{\vec{v}}\right)^{-1}((\phi_t^{\vec{v}})^{-1}(x))$$

to obtain (8).

Since $\vec{v}$ is tangent to $\partial\Omega$, $\phi_t^{\vec{v}}\Big|_{\partial\Omega}$ is a diffeomorphism of $\partial\Omega$. Consider $x\in\partial\Omega$. The linear operator $\left(D\left[\phi_s^{\vec{v}}\circ(\phi_t^{\vec{v}})^{-1}\right](x)\right)^{-1}$ sends the tangent plane to $\partial\Omega$ at the point $\phi_s^{\vec{v}}\circ(\phi_t^{\vec{v}})^{-1}(x)\in\partial\Omega$ into the tangent plane to $\partial\Omega$ at the point x. Relation (8) shows that $\vec{\xi}(t,x)$ is the integral and the sum of vectors,

$$\left(D\left[\phi_s^{\vec{v}}\circ(\phi_t^{\vec{v}})^{-1}\right](x)\right)^{-1}\vec{f}\left(s,\phi_s^{\vec{v}}\circ(\phi_t^{\vec{v}})^{-1}(x)\right)$$

and

$$\left(D(\phi_t^{\vec{v}})^{-1}\right)^{-1}\vec{\chi}((\phi_t^{\vec{v}})^{-1})$$

belonging to the tangent plane to $\partial\Omega$ at x. Thus $\vec{\xi}(t,x)$ is tangent to $\partial\Omega$ at x.

Assume additionnaly that $\vec{v}$, $\vec{f}$ and $\vec{\chi}$ are of zero divergence. The curve

$$\varepsilon\longrightarrow\phi_t^{\vec{v}+\varepsilon\vec{f}}\circ\phi_1^{\varepsilon\vec{\chi}}\circ\phi_t^{\vec{v}}$$

lies on G and passes through I_d at $\varepsilon = 0$ (see appendix A). Thus $\vec{\xi}$ belongs to $\mathcal{U} = TG_{I_d}$, i.e., $\vec{\xi}$ is of zero divergence on Ω and tangent to $\partial\Omega$. This can be proved directly with much more calculation via formula (8). ∎

Proof of proposition 3 According to relations (16) and (13), we have

$$U_{\vec{v}}(\vec{\xi^\perp}) = \int\!\!\int\!\!\int_\Omega \left[(\vec{\xi^\perp} \cdot \nabla)(\nabla p/\varrho) \;\cdot\; \vec{\xi^\perp} \quad - \quad \nabla \alpha_{\vec{\xi^\perp}} \cdot \nabla \alpha_{\vec{\xi^\perp}} \right] \, dx$$

where $\nabla \alpha_{\vec{\xi^\perp}} = D\vec{\xi^\perp}/Dt - d\vec{\xi^\perp}/dt$. By Pythagore's theorem (see figure 3), we have

$$\|\nabla \alpha_{\vec{\xi^\perp}}\|^2 \leq \|(\vec{\xi^\perp} \cdot \nabla)\vec{v}\|^2.$$

This directly implies (17). The first inequality of (18) is a straightforward consequence of (17).

The second inequality is less obvious. It suffices to prove that, for all $\overline{x}$ in the interior of Ω,

$$\min_{\|\vec{\xi^\perp}\|=1} U_{\vec{v}}(\vec{\xi^\perp}) \leq \frac{\text{trace}(M(t,\overline{x}))}{3}.$$

According the lemma herebelow, there exists an orthogonal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ of $I\!R^3$ such that

$$\vec{e}_1 \cdot M(t,\overline{x})\vec{e}_1 \leq \text{trace}(M(t,\overline{x})/3 \quad \text{and} \quad \vec{e}_1 \cdot D\vec{v}(t,\overline{x})\vec{e}_1 = 0$$

(take $M = M(t,\overline{x})$ and $N = D\vec{v}(t,\overline{x}) + D\vec{v}'(t,\overline{x})$).

Consider now the Euclidian coordinates (x_1, x_2, x_3) associated to the orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ such that the point $\overline{x}$ corresponds to the origin. We can suppose additionally that $D\vec{v}(t,\overline{x})\vec{e}_1 = a\vec{e}_2$ ($a \geq 0$ constant).

We introduce now $\vec{\xi}_\varepsilon$ depending on a small positive scalar ε and defined by:

$$\vec{\xi}_\varepsilon(x_1,x_2,x_3) = \psi\left((x_1/\varepsilon)^2 + (x_2/\varepsilon^2)^2 + (x_3/\varepsilon)^2\right) \begin{pmatrix} -x_2/\varepsilon \\ \varepsilon x_1 \\ 0 \end{pmatrix}$$

where $\psi(s) = \exp(4/(4s-1))$ for $0 \leq s < 1/4$ and $\psi(s) = 0$ if $s \geq 1/4$. For ε small enough, $\vec{\xi}_\varepsilon$ belongs to $\mathcal{U}$. $\vec{\xi}_\varepsilon$ corresponds to a small localized eddy with low kinetic energy and high enstrophy.

Denote by α_ε the real function such that $\nabla\alpha_\varepsilon + (\vec{\xi}_\varepsilon \cdot \nabla)\,\vec{v}$ belongs to $\mathcal{U}$ and by β_ε the real function

$$(x_1,x_2,x_3) \quad \longrightarrow \quad a\Psi\left((x_1/\varepsilon)^2 + (x_2/\varepsilon^2)^2 + (x_3/\varepsilon)^2\right)$$

with $\Psi = \int \psi$. In fact, β_ε is an approximation of α_ε when ε tends to 0: by construction, $D\vec{v}(t,\overline{x})\vec{e}_1 = a\vec{e}_2$; thus

$$(\vec{\xi}_\varepsilon \cdot \nabla)\vec{v} + \nabla\beta_\varepsilon = O(\varepsilon^2)$$

and

$$< (\vec{\xi}_\varepsilon \cdot \nabla)\vec{v} + \nabla\beta_\varepsilon \;,\; (\vec{\xi}_\varepsilon \cdot \nabla)\vec{v} + \nabla\beta_\varepsilon > \quad = \quad O(\varepsilon^8)$$

because the volume of the support of $\vec{\xi_\varepsilon}$ and $\nabla\beta_\varepsilon$ is proportional to ε^4; since

$$< \nabla\alpha_\varepsilon - \nabla\beta_\varepsilon \ , \ \nabla\alpha_\varepsilon - \nabla\beta_\varepsilon > \quad \leq \quad < (\vec{\xi_\varepsilon}\cdot\nabla)\vec{v} + \nabla\beta_\varepsilon \ , \ (\vec{\xi_\varepsilon}\cdot\nabla)\vec{v} + \nabla\beta_\varepsilon >,$$

we have

$$< \nabla\alpha_\varepsilon - \nabla\beta_\varepsilon, \nabla\alpha_\varepsilon - \nabla\beta_\varepsilon >= O(\varepsilon^8).$$

The following approximation holds true:

$$< A_{\vec{v}}(\vec{\xi_\varepsilon}), \vec{\xi_\varepsilon} > \quad = \quad \frac{1}{\varrho} < (\vec{\xi_\varepsilon}\cdot\nabla)\nabla p \ , \ \vec{\xi_\varepsilon} > \quad - \quad < \nabla\beta_\varepsilon \ , \ \nabla\beta_\varepsilon > \quad + \quad O(\varepsilon^8).$$

But

$$\frac{1}{\varrho}(\vec{\xi_\varepsilon}\cdot\nabla)\nabla p\cdot\vec{\xi_\varepsilon} - \nabla\beta_\varepsilon\cdot\nabla\beta_\varepsilon = \\ (x_2/\varepsilon)^2\psi^2\left[(x_1/\varepsilon)^2 + (x_2/\varepsilon^2)^2 + (x_3/\varepsilon)^2\right] \ \vec{e}_1\cdot M(t,\overline{x})\vec{e}_1 + O(\varepsilon^4).$$

This proves that

$$< A_{\vec{v}}(\vec{\xi_\varepsilon}), \vec{\xi_\varepsilon} >= \vec{e}_1\cdot M(t,\overline{x})\vec{e}_1 \ \ k^2\ \varepsilon^6 + O(\varepsilon^8)$$

with

$$k^2\varepsilon^6 = \iiint_\Omega (x_2/\varepsilon)^2\psi^2\left((x_1/\varepsilon)^2 + (x_2/\varepsilon^2)^2 + (x_3/\varepsilon)^2\right)\ dx_1dx_2dx_3 = \|\vec{\xi_\varepsilon}\|^2.$$

The definition of the potential U, requires $\vec{\xi}^\perp$ orthogonal to $\vec{v}$. Since

$$\vec{\xi}_\varepsilon^\perp = \vec{\xi_\varepsilon} \ - \ \frac{< \vec{\xi_\varepsilon}, \vec{v} >}{\|\vec{v}\|^2}\ \vec{v}$$

and

$$< \vec{\xi_\varepsilon}, \vec{v} >= O(\varepsilon^5),$$

we have $\|\vec{\xi_\varepsilon}\|^2 = \|\vec{\xi}_\varepsilon^\perp\|^2 + O(\varepsilon^{10})$. Thus

$$\frac{U_{\vec{v}}(\vec{\xi}_\varepsilon^\perp)}{\|\vec{\xi}_\varepsilon^\perp\|^2} = \vec{e}_1\cdot M(t,\overline{x})\vec{e}_1 + O(\varepsilon^2).$$

This proves the second inequality of (18).

A direct computation in Euclidian coordinates $x = (x_1, x_2, x_3)$ shows that

$$\text{trace}(M) = -\sum_{i,j=1}^{3}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 . \blacksquare$$

Lemma *Let M and N be two 3×3 symmetric matrices such that trace$(N) = 0$. Then there exists w in $\mathbb{R}^3$ of length equal to 1 ($w'w = 1$) such that*

$$w'Mw \leq \frac{trace(M)}{3} \quad \textit{and} \quad w'Nw = 0$$

($'$ denotes "transpose").

Proof Without loss of generality, we can consider that N is diagonal, $N = \mathrm{diag}(a_i)$ with $a_1 + a_2 + a_3 = 0$. Consider the elements $w_s = (s_1, s_2, s_3)'$ in $I\!R^3$ with $s_i \in \{-1/\sqrt{3}, 1/\sqrt{3}\}$. Clearly $w_s' w_s = 1$ and $w_s' N w_s = 0$. We have

$$\sum_{(s_1,s_2,s_3)\in\{-1/\sqrt{3},1/\sqrt{3}\}^3} w_s' M w_s = 8 \ \mathrm{trace}(M)/3.$$

There exists necessarily $(s_1, s_2, s_3) \in \{-1/\sqrt{3}, 1/\sqrt{3}\}^3$ such that $w_s' M w_s \leq \mathrm{trace}(M)/3.$ ■

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- Figure 2: definition of the vector $\vec{\xi}$ quantifying the difference between the trajectories for two nearby initial condition, x and x', in two nearby flows, $\vec{v}$ and $\vec{v} + \vec{f}$.
- Figure 3: relation between the convective derivation, $d\vec{\xi}/dt = \partial\vec{\xi}/\partial t + (\vec{v} \cdot \nabla)\,\vec{\xi}$, and the covariant derivation $D\vec{\xi}/Dt = \partial\vec{\xi}/\partial t + (\vec{v} \cdot \nabla)\,\vec{\xi} + \nabla\alpha_{\vec{\xi}}$.
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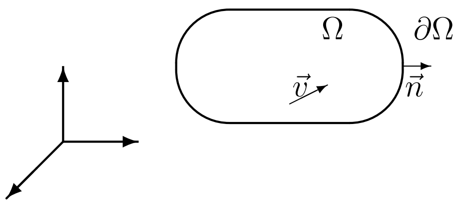


Figure 1:

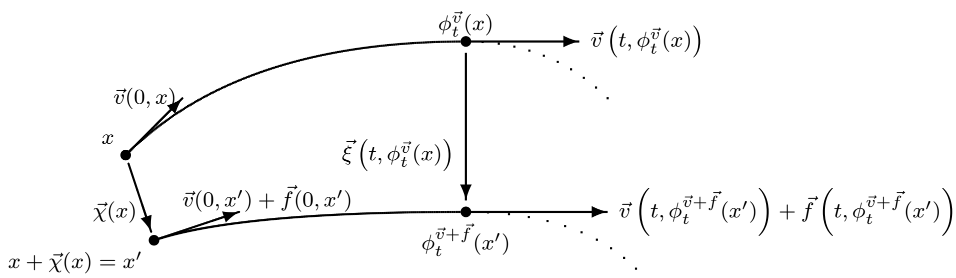

$\phi_t^{\vec{v}}(x)$
$\vec{v}\left(t, \phi_t^{\vec{v}}(x)\right)$
$\vec{v}(0,x)$
x
$\vec{\xi}\left(t, \phi_t^{\vec{v}}(x)\right)$
$\vec{\chi}(x)$
$\vec{v}(0,x') + \vec{f}(0,x')$
$\vec{v}\left(t, \phi_t^{\vec{v}+\vec{f}}(x')\right) + \vec{f}\left(t, \phi_t^{\vec{v}+\vec{f}}(x')\right)$
$\phi_t^{\vec{v}+\vec{f}}(x')$
$x + \vec{\chi}(x) = x'$

Figure 2:

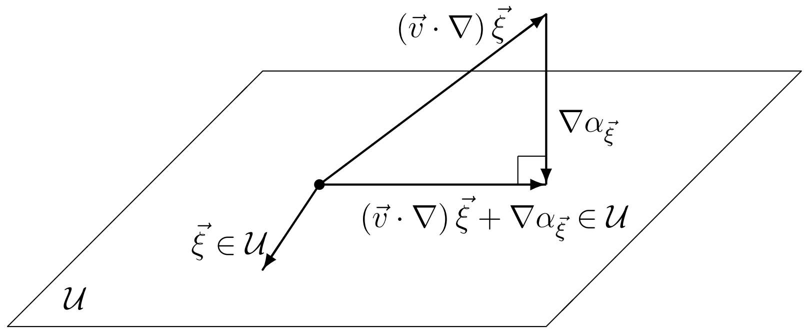
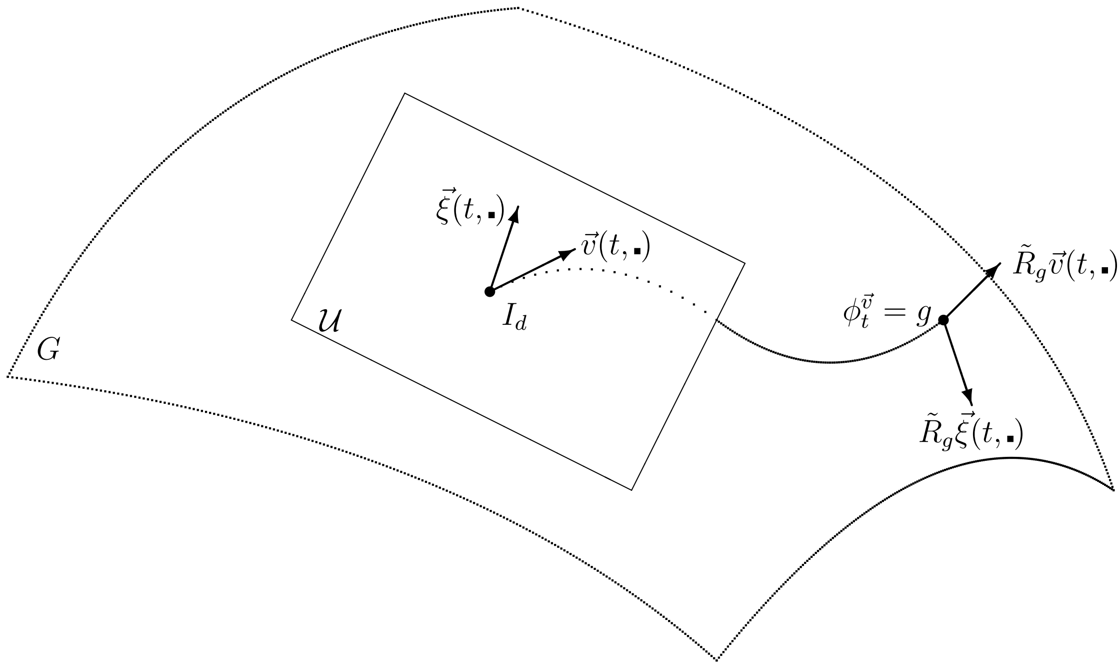

$(\vec{v}\cdot\nabla)\,\vec{\xi}$
$\nabla\alpha_{\vec{\xi}}$
$(\vec{v}\cdot\nabla)\,\vec{\xi}+\nabla\alpha_{\vec{\xi}}\in\mathcal{U}$
$\vec{\xi}\in\mathcal{U}$
$\mathcal{U}$

Figure 3:



■ G is the Lie group of volume preserving diffeomorphisms on Ω.
$\mathcal{U} = T_{I_d}G$ is the Lie algebra of vector fields in Ω of zero divergence and tangent to $\partial\Omega$.

■ G is a Riemannian manifold. The scalar product on $\mathcal{U} = TG_{I_d}$ is derived from the kinetic energy, $< \vec{v}, \vec{\xi} > = \int\int\int_\Omega \vec{v}(x) \cdot \vec{\xi}(x)dx$,
and is invariant through the right translations $R_g : \ h \in G \rightarrow h \circ g \in G \ (g \in G)$.
The covariant derivation is $\nabla_{\vec{v}}\vec{\xi} = \dfrac{\partial\vec{\xi}}{\partial t} + (\vec{v} \cdot \nabla)\vec{\xi} + \nabla\alpha$ with $\vec{v}(t,\centerdot)$ and $\vec{\xi}(t,\centerdot)$ in $\mathcal{U}$.

■ If $\vec{v}(t,\centerdot) \in \mathcal{U}$ is solution of the Euler equation, i.e. $\nabla_{\vec{v}}\vec{v} = 0$, the curve $t \longrightarrow \phi_t^{\vec{v}}$ is a geodesic of G.

Figure 4: