ON ARNOL'D STABILITY CRITERION FOR STEADY-STATE FLOWS OF AN IDEAL FLUID

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Abstract

It is proved that, contrarily to the 2-D case, the sufficient Arnol'd stability criterion for steady-state solutions of the incompressible Euler equations is never satisfied when 3-D perturbations are considered.

Suggested running title: On Arnol'd stability criterion

1 Introduction

Arnol'd 1965 proposes a sufficient nonlinear stability criterion for smooth stationary solutions of the Euler equations. In particular, he applies his criterion to parallel shear flows, proves the local nonlinear stability with respect to 2-D perturbations when the velocity profile has no inflexion point and obtains in a more rigorous way the well known Rayleigh theorem.

However, Arnol'd 1966 (page 349, second paragraph) notices that, likely, there does not exist stationary flow satisfying his criterion for 3-D perturbations. In this paper, we prove effectively that, for every smooth stationary flow, this sufficient stability criterion is not satisfied when 3-D perturbations are considered.

All the calculations require a regularity condition that is assumed in this paper : the time and space dependences of the velocity and pressure are smooth (at least, twice continuously differentiable C^2).

Recently, Vallis et al. 1989 propose a method to obtain steady-state flows that are solution of the Euler equations and that satisfy automatically Arnol'd stability criterion. The result below proves that such method cannot be applied to the 3-D case and must be used only for 2-D flows. Otherwise stated, the answer to the open problem concerning the 3-D stability of non zero stationary flows cannot be deduced from this criterion.

We recall first the sufficient stability criterion elaborated by Arnol'd and secondly we show that it is never satisfied for 3-D pertubations.

2 The stability criterion

2.1 The Euler equations

Let D be a domain of the Euclidian space $I\!R^3$ bounded by the fixed surface ∂D as displayed on figure 1; let $\vec{v}$ be the velocity field of an ideal fluid (incompressible, density equal to 1, exterior to a non-potential mass force field) which fills the domain D. Let p be the pressure.

Under such assumptions the fluid motions are described by the Euler equations,

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\nabla p \tag{1}$$

equivalent to the Bernoulli equations,

$$\frac{\partial \vec{v}}{\partial t} = \vec{v}\times\vec{r} - \nabla\lambda, \quad \text{with} \quad \vec{r} = \nabla\times\vec{v} \text{ and } \lambda = p + \vec{v}^2/2. \tag{2}$$

The velocity field $\vec{v}$ satisfies $\nabla\cdot\vec{v} = 0$ in D and $\vec{v}\cdot\vec{n} = 0$ on ∂D where $\vec{n}$ is a vector normal to ∂D.

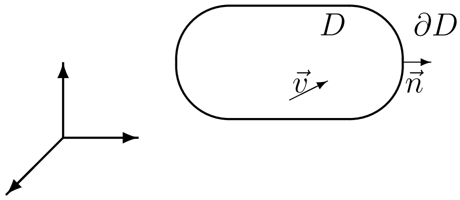

Figure 1: typical 3-D bounded fluid domain.

Using the identity

$$\nabla \times (\vec{a} \times \vec{b}) = [\vec{a}, \vec{b}] + (\nabla \cdot \vec{b})\vec{a} - (\nabla \cdot \vec{a})\vec{b}$$

where $[\vec{a}, \vec{b}]$ denotes the Lie bracket of the vector fields $\vec{a}$ and $\vec{b}$, we obtain the vorticity equation

$$\frac{\partial \vec{r}}{\partial t} = [\vec{v}, \vec{r}]. \tag{3}$$

2.2 Arnol'd's study

A summary of several Arnol'd's results on the hydrodynamics of perfect fluid can be found in the appendix 2 of his book "Mathematical Methods of Classical Mechanics" (1976). Here, we consider only one result, published in 1965, that is very accessible to non-mathematicians.

Arnol'd 1965 proves that a steady-state flow, solution of the above Euler equation (1), is an extremal of the kinetic energy

$$E = 1/2 \iiint_D \vec{v}^2 d\varrho$$

in comparison with equivortical velocity fields[1].

[1] Definition 3.1 in Arnol'd 1965 : two vector fields $\vec{v}$ and $\vec{v}'$, such that $\nabla \cdot \vec{v} = \nabla \cdot \vec{v}' = 0$ on D and $\vec{v} \cdot \vec{n} = \vec{v}' \cdot \vec{n} = 0$ on ∂D, are equivortical fields if and only if there exists a smooth volume preserving

More precisely, Arnol'd (1965)

– uses perturbation mappings of the form $g_t = \exp(t\vec{f})$ ($\vec{f}$ is a vector field on D of zero divergence and tangent to ∂D) ;

– proves that the resulting equivortical perturbation $\delta\vec{v}$ of the velocity $\vec{v}$ is given by

$$\delta\vec{v} = t\,\vec{f}\times\vec{v} + \frac{t^2}{2}\,\vec{f}\times[\vec{f},\vec{r}] + \nabla\alpha + O(t^3)$$

where the real function α is such that $\delta\vec{v}$ is of zero divergence and is tangent to ∂D ;

– shows that, for stationary flow $\vec{v}$, $\delta E = O(t^2)$ if $\delta\vec{v} = t(\vec{f}\times\vec{v} + \nabla\alpha) + O(t^2)$;

– computes, around a stationary flow $\vec{v}$, the second variation $\delta^2 E$ associated to perturbation mappings of the form $g_t = \exp(t\vec{f})$,

$$2\,\delta^2 E(t\vec{f}) = t^2 \iiint_D \Big((\vec{f}\times\vec{r} + \nabla\alpha)^2 + (\vec{v}\times\vec{f})\cdot[\vec{f},\vec{r}]\Big)\,d\varrho + O(t^3) \qquad (4)$$

where the real function α is chosen such that $\vec{f}\times\vec{r} + \nabla\alpha$ is of zero divergence and tangent to ∂D.

Since for every solution of the Euler equation, the velocity vector fields at two different times are isovortical vector fields (Helmotz's theorem on the conservation of circulation), Arnol'd (1965) uses this extremal property to investigate the nonlinear stability of steady-state solutions: if the extremum is a true minimum or a true maximum then the steady-state flow is stable, i.e. a small change in the initial velocity induces only a small change in the velocity field *for all times*[2].

Arnol'd (1965) notices that he was unable to find a steady-state flow v such that $\delta^2 E(\vec{f})$ is of fixed sign for every *3-D perturbations* parametrized by $\vec{f}$.

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mapping g of the domain D into itself such that

$$\oint_\gamma \vec{v}\cdot\vec{dl} = \oint_{g(\gamma)} \vec{v}'\cdot\vec{dl}$$

for every closed contour γ included in D; this means that the mapping g transforms $\nabla\times\vec{v}$ into $\nabla\times\vec{v}'$, i.e.

$$(\nabla\times\vec{v}')_{g(M)} = Dg_M(\nabla\times\vec{v})_M$$

for all point M in D and where Dg_M denotes the differential of g with respect to the 3 spaces coordinates evaluated at M.

[2]Arnol'd (1976) (appendix 2, page 330, theroem 8) precises that, to be valid, such a sufficient stability criterion requires a regularity condition. This regurality condition tells that every small isovortical perturbation of $\vec{v}$ derives from a perturbation mapping of the form $g_t = \exp(t\vec{f})$. This means that one can parametrize **all** infinitesimal isovortical perturbations $\delta\vec{v}$ by vector fields $\vec{f}$.

3 Limitation of the criterion

Theorem 1 *Consider a non zero steady-state solution $\vec{v}$ of the Euler equation for the 3-D bounded domain D. Then there exists vector fields $\vec{f}^+$ and $\vec{f}^-$ on D such that*

$$\begin{array}{rcl} \nabla \cdot \vec{f}^+ = 0 & \textit{and} & \nabla \cdot \vec{f}^- = 0 \ \textit{on } D, \\ \vec{f}^+ \cdot \vec{n} = 0 & \textit{and} & \vec{f}^- \cdot \vec{n} = 0 \ \textit{on } \partial D, \\ \delta^2 E(\vec{f}^+) > 0 & \textit{and} & \delta^2 E(\vec{f}^-) < 0 \end{array}$$

where $\delta^2 E$ is given by (4). Otherwise stated, the equivortical extremum $\vec{v}$ of E is always a saddle point.

The main idea of the proof

The construction of such vector fields $\vec{f} = \vec{f}^+$ or $\vec{f}^-$ basically relies on a regularity idea: $\vec{f}$ can be small whereas its derivatives may be large. In other words, examples of suitable perturbation $\vec{f}$ are associated to localized eddies of small kinetic energy and high enstrophy.

More precisely, consider the two terms of the integral giving $\delta^2 E(\vec{f})$. The first one $(\vec{f} \times \vec{r} + \nabla\alpha)^2$ is always positive and depends only on $\vec{f}$. Consequently, if $\vec{f} = O(\varepsilon)$ (ε is a small positive scalar), $(\vec{f} \times \vec{r} + \nabla\alpha)^2 = O(\varepsilon^2)$. For the second term $(\vec{v} \times \vec{f}) \cdot [\vec{f}, \vec{r}]$, it is different: if $\vec{f} = O(\varepsilon)$ and $D\vec{f} = O(1)$ then[3] $(\vec{v} \times \vec{f}) \cdot [\vec{f}, \vec{r}] = O(\varepsilon)$. Such a difference of order in ε explains why the second term whose sign may be chosen negative, can dominate the first term which is always non negative.

For the detailed proof, we consider two cases:

1. the velocity field is not always proportional to the vorticity field;

2. the velocity field is always proportional to the vorticity field.

Proof when $\vec{v} \times (\nabla \times \vec{v}) \neq 0$

Consider a point O interior to D such that $\vec{v}$ and $\vec{r} = \nabla \times \vec{v}$ are independent. To build $\vec{f}^+$ and $\vec{f}^-$, we need suitable space coordinates denoted (x, y, z).

They are given by lemma 1. In such local coordinates, the expression of $\delta^2 E(\vec{f})$ becomes more simple for vector fields $\vec{f}$ that are equal to zero outside a small neighbourhood Ω of O in D.

As displayed on figure 2, such vector fields $\vec{f}$ can be written as follows:

$$\vec{f} = \lambda\vec{v} + \mu\vec{r} + \nu\vec{w}$$

[3] $D\vec{f}$ denotes the derivative of $\vec{f}$ with respect to the three space coordinates.

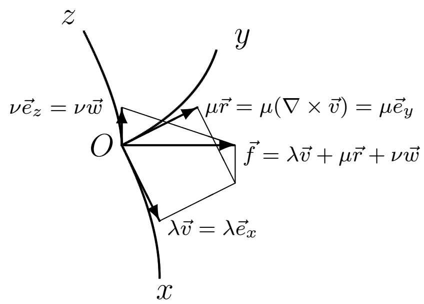

Figure 2: local expression of the perturbation $\vec{f}$ around O.

where λ, μ and ν are functions of the local space coordinates (x, y, z) on Ω, equal to zero outside Ω and where $\vec{v} = \vec{e}_x$, $\vec{r} = \vec{e}_y$ and $\vec{w} = \vec{e}_z$. The condition $\nabla \cdot \vec{f} = 0$ is then equivalent to

$$\frac{\partial \lambda}{\partial x} + \frac{\partial \mu}{\partial y} + \frac{\partial \nu}{\partial z} = 0$$

because the $\vec{v}$, $\vec{r}$, and $\vec{w}$ are zero divergence vector fields. The boundary condition $\vec{f} \cdot \vec{n} = 0$ on ∂D is automatically satisfied since $\vec{f} = 0$ on $D - \Omega$.

We have

$$\begin{aligned} [\vec{f}, \vec{r}] &= \lambda[\vec{v}, \vec{r}] + \mu[\vec{r}, \vec{r}] + \nu[\vec{w}, \vec{r}] \\ &\qquad + (\nabla\lambda \cdot \vec{r})\vec{v} + (\nabla\mu \cdot \vec{r})\vec{r} + (\nabla\nu \cdot \vec{r})\vec{w} \\ &= \frac{\partial \lambda}{\partial y}\,\vec{v} + \frac{\partial \mu}{\partial y}\,\vec{r} + \frac{\partial \nu}{\partial y}\,\vec{w} \end{aligned}$$

and

$$(\vec{v} \times \vec{f}) \cdot [\vec{f}, \vec{r}] = \left(\mu \frac{\partial \nu}{\partial y} - \nu \frac{\partial \mu}{\partial y}\right) \det(\vec{v}, \vec{r}, \vec{w}).$$

Thus[4]

$$2\delta^2 E(\vec{f}) = \iiint_D (\vec{f} \times \vec{r} + \nabla\alpha)^2 d\varrho + \iiint_\Omega \left(\mu \frac{\partial \nu}{\partial y} - \nu \frac{\partial \mu}{\partial y}\right) \det(\vec{v}, \vec{r}, \vec{w})\; dxdydz \tag{5}$$

where α is chosen such that $(\nabla\alpha) \cdot \vec{n} = 0$ on ∂D and $\nabla \cdot \left(\vec{f} \times \vec{r}\right) + \Delta\alpha = 0$ on D.

Consider an arbitary constant k (its sign will be chosen in the sequel) and the associated components (F_1, F_2, F_3) of the vector field $\vec{F}$ given by lemma 2 herebelow. Take

[4] We can scale the coordinate z such that, in Ω, the Euclidian volume element $d\varrho$ (that is proportional to $dxdydz$ with a constant coefficient) is identically equal to $dxdydz$.

$\varepsilon > 0$ a small scalar and state

$$\begin{aligned}\lambda &= \varepsilon F_1(x/\varepsilon, y/\varepsilon, z/\varepsilon),\\ \mu &= \varepsilon F_2(x/\varepsilon, y/\varepsilon, z/\varepsilon),\\ \nu &= \varepsilon F_3(x/\varepsilon, y/\varepsilon, z/\varepsilon).\end{aligned}$$

Clearly, the resulting $\vec{f}$ is zero outside an ε neighbourhood Ω_ε of O and satisfies $\nabla\cdot\vec{f} = 0$.[5]

A straightforward computation shows that

$$\iiint_{\Omega_\varepsilon} \left(\mu\frac{\partial\nu}{\partial y} - \nu\frac{\partial\mu}{\partial y}\right) \det(\vec{v},\vec{r},\vec{w})\; dxdydz = k\det(\vec{v}_O,\vec{r}_O,\vec{w}_O)\;\varepsilon^4 + O(\varepsilon^5). \tag{6}$$

We have

$$\iiint_D (\vec{f}\times\vec{r}+\nabla\alpha)^2 d\varrho = \iiint_D (\nabla\alpha)^2 d\varrho + \iiint_{\Omega_\varepsilon} 2\det(\vec{f},\vec{r},\nabla\alpha)\; dxdydz + \iiint_{\Omega_\varepsilon} (\vec{f}\times\vec{r})^2 dxdydz.$$

Since

$$(\nabla\alpha)^2 = \nabla\cdot(\alpha\nabla\alpha) - \alpha\Delta\alpha$$

and, by construction of α,

$$\Delta\alpha = -\nabla\cdot(\vec{f}\times\vec{r}),$$

we obtain

$$(\nabla\alpha)^2 = \nabla\cdot(\alpha\nabla\alpha + \alpha\vec{f}\times\vec{r}) - \det(\vec{f},\vec{r},\nabla\alpha).$$

Using $\nabla\alpha\cdot\vec{n} = 0$ and $\vec{f} = 0$ on ∂D, we have

$$\iiint_D (\nabla\alpha)^2 d\varrho = -\iiint_{\Omega_\varepsilon} \det(\vec{f},\vec{r},\nabla\alpha)\; dxdydz$$

and thus

$$\iiint_D (\vec{f}\times\vec{r}+\nabla\alpha)^2 d\varrho \;=\; \iiint_{\Omega_\varepsilon} (\vec{f}\times\vec{r})^2\; dxdydz - \iiint_D (\nabla\alpha)^2 dxdydz.$$

Consequently

$$\iiint_D (\vec{f}\times\vec{r}+\nabla\alpha)^2 d\varrho \le \iiint_{\Omega_\varepsilon} (\vec{f}\times\vec{r})^2\; dxdydz = O(\varepsilon^5) \tag{7}$$

The substitution of relations (6) and (7) into equation (5) gives

$$2\delta^2 E(\vec{f}) = k\det(\vec{v}_O,\vec{r}_O,\vec{w}_O)\;\varepsilon^4 + O(\varepsilon^5).$$

By choosing the sign of k and ε small enough, $\delta^2 E(\vec{f})$ becomes positive or negative. ■

[5]Notice that the vector field $\vec{F}$ constructed in the proof of lemma 2 and which gives after scaling the perturbation vector field $\vec{f}$, has no obvious symmetry (see figure 4 where $\vec{e}_{x_1}$ is proportional to $\vec{v}$, $\vec{e}_{x_2}$ to $\vec{r}$ and $\vec{e}_{x_3}$ to $\vec{w}$). The resulting equivortical perturbation is three-dimensional.

Proof when $\vec{v} \times (\nabla \times \vec{v}) \equiv 0$

This case differs only slightly from the previous one. Lemma 1 is replaced by lemma 3. A vector field $\vec{f}$ of zero divergence that is zero far from O can be expressed as

$$\vec{f} = \lambda \vec{v} + \nu_1 \vec{w}_1 + \nu_2 \vec{w}_2$$

where

- λ, ν_1 and ν_2 are functions of (x, z_1, z_2) null far from $(x = 0, z_1 = 0, z_2 = 0)$ and satisfying
$$\frac{\partial \lambda}{\partial x} + \frac{\partial \nu_1}{\partial z_1} + \frac{\partial \nu_2}{\partial z_2} = 0;$$
- $\vec{e}_x = \vec{v}$, $\vec{e}_{z_1} = \vec{w}_1$ and $\vec{e}_{z_2} = \vec{w}_2$.

Around O, we can state $\vec{r} = \nabla \times \vec{v} = \xi \vec{v}$ where ξ is a real function. A simple computation gives

$$(\vec{v} \times \vec{f}) \cdot [\vec{f}, \vec{r}] = \xi \left(\nu_1 \frac{\partial \nu_2}{\partial x} - \nu_2 \frac{\partial \nu_1}{\partial x} \right) \det(\vec{v}, \vec{w}_1, \vec{w}_2).$$

Consequently

$$2\, \delta^2 E(\vec{f}) = \iiint_D (\xi\, \vec{f} \times \vec{v} + \nabla \alpha)^2 d\varrho + \iiint_\Omega \xi \left(\nu_1 \frac{\partial \nu_2}{\partial x} - \nu_2 \frac{\partial \nu_1}{\partial x} \right) \det(\vec{v}, \vec{w}_1, \vec{w}_2)\, dx dz_1 dz_2.$$

Lemma 4 provides, after scaling, the desired vector fields $\vec{f}^+$ and $\vec{f}^-$.

Notice that, since the domain D is bounded and closed, there exists a point O interior to D such that $\vec{v}$ and $\vec{r} = \xi \vec{v}$ are non zero at this point. The reason is the following: if such point O does not exist, the velocity field $\vec{v}$ derives necessarily from a scalar potential.Thus $\vec{v}$ must be identically equal to zero since $\vec{v} \cdot \vec{n} = 0$ on ∂D which is an orientable, compact surface of $\mathbb{R}^3$ without boundary. ■

4 Four technical lemmas

During the proof of theorem 1, we use the following lemmas.

Lemma 1 *Denote O a point interior to D where the stationary velocity field $\vec{v}$ and its vorticity field $\vec{r} = \nabla \times \vec{v}$ are independent. Then there exists local coordinates (x, y, z) around O, such that*

$$\vec{e}_x = \vec{v} \quad \textit{and} \quad \vec{e}_y = \vec{r}$$

and such that $\nabla \cdot \vec{e}_z = 0$ where the basis $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ derives from the local coordinates (x, y, z) of a point M near O through the differential relationship $d\vec{M} = \vec{e}_x dx + \vec{e}_y dy + \vec{e}_z dz$.

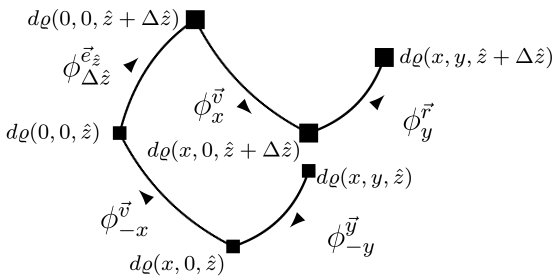

Figure 3: since $\phi^{\vec v}$ and $\phi^{\vec r}$ are volume preserving diffeomorphisms, $\nabla\cdot\vec e_{\hat z}$ depends only on $\hat z$.

Proof of lemma 1 The existence of local coordinates (x,y,z) such that $\vec v=\vec e_x$ and $\vec r=\vec e_y$ is a classical result of differential geometry and results from $[\vec v,\vec r]=0$ (the vorticity equation (3) at the steady-state). The fact that one can choose the third coordinate z such that $\nabla\cdot\vec e_z=0$ is not so classic and is possible here because $\nabla\cdot\vec v=0$ and $\nabla\cdot\vec r=0$.

(x,y,z) are constructed as follows. Denote $\phi^{\vec v}_s=\exp(s\vec v)$ (resp. $\phi^{\vec r}_s=\exp(s\vec r)$) the flow of the vector field $\vec v$ (resp. $\vec r$). Denote $\vec w_O$ a vector such that $(\vec v_O,\vec r_O,\vec w_O)$ is a basis of $I\!R^3$. Clearly

$$(x,y,\hat z)\longrightarrow \phi^{\vec v}_x(\phi^{\vec r}_y(O+\hat z\vec w_O))$$

is a local diffeomorphism from a neighbourhood of $(0,0,0)$ into a neighbourhood of O in D. Because $[\vec v,\vec r]=0$, $\phi^{\vec v}_x\circ\phi^{\vec r}_y=\phi^{\vec r}_y\circ\phi^{\vec v}_x$. This implies that $\vec v=\vec e_x$ and $\vec r=\vec e_y$. Denoting $\vec e_{\hat z}=\vec e_{\hat z}$, we will see that $\nabla\cdot\vec e_{\hat z}$ which is a function of $(x,y,\hat z)$ depends only on $\hat z$.

Geometrically, $\nabla\cdot\vec e_{\hat z}$ is equal to the relative variation of the volume element under its transport by the flow $\phi^{\vec e_{\hat z}}_s=\exp(s\vec e_{\hat z})$ of $\vec e_{\hat z}$. As displayed on figure 3, the transport of a volume element $d\varrho(x,y,\hat z)$ from the point of local coordinates $(x,y,\hat z)$ to the point of local coordinates $(x,y,\hat z+\Delta\hat z)$ by the mapping $\phi^{\vec e_{\hat z}}_{\Delta\hat z}$ can be decomposed as follows:

$$d\varrho(x,y,\hat z)\xrightarrow{\phi^{\vec v}_{-x}} d\varrho(0,y,\hat z)\xrightarrow{\phi^{\vec r}_{-y}} d\varrho(0,0,\hat z)\ldots$$

$$\ldots\xrightarrow{\phi^{\vec e_{\hat z}}_{\Delta\hat z}} d\varrho(0,0,\hat z+\Delta\hat z)\xrightarrow{\phi^{\vec v}_{x}} d\varrho(x,0,\hat z+\Delta\hat z)\xrightarrow{\phi^{\vec r}_{y}} d\varrho(x,y,\hat z+\Delta\hat z).$$

Because $\nabla\cdot\vec v=0$ and $\nabla\cdot\vec r=0$, $\phi^{\vec v}$ and $\phi^{\vec r}$ induce volume preserving mappings.

Consequently

$$d\varrho(x,y,\hat{z}) = d\varrho(0,y,\hat{z}) = d\varrho(0,0,\hat{z})$$

and

$$d\varrho(x,y,\hat{z}+\Delta\hat{z}) = d\varrho(0,y,\hat{z}+\Delta\hat{z}) = d\varrho(0,0,\hat{z}+\Delta\hat{z}).$$

Thus, $\nabla\cdot\vec{e}_{\hat{z}}$ is only a function of $\hat{z}$ denoted $a(\hat{z})$. If we change the coordinate $\hat{z}$ into

$$z = \int_0^{\hat{z}} \exp\left(\int_0^{\hat{z}_1} a(\hat{z}_2)d\hat{z}_2\right) d\hat{z}_1$$

keeping x and y unchanged, we finally obtain $\nabla\cdot\vec{e}_z = 0$. ∎

Lemma 2 *For every real k, there exists a vector field $\vec{F}$ in the Euclidian space $I\!R^3$ such that*

- $\nabla\cdot\vec{F} = 0,$
- $\vec{F} = 0$ *outside the unit sphere $B(O,1)$,*
- $\iiint_{B(O,1)} \left(F_2\,\dfrac{\partial F_3}{\partial x_2} - F_3\,\dfrac{\partial F_2}{\partial x_2}\right) dx_1dx_2dx_3 = k,$

where (Ox_1, Ox_2, Ox_3) are Euclidian axis and (F_1, F_2, F_3) the components of $\vec{F}$.

Proof of lemma 2 Clearly, it suffices, by using scaling arguments (i.e. replacing $F(x_1,x_2,x_3)$ by $\lambda F(\mu x_1, \mu x_2, \mu x_3)$ with λ constant scalar and $\mu =^{+}_{-} 1$), to find a vector field $\vec{F}$ of support $B(O,1)$ having zero divergence and such that

$$\iiint_{B(O,1)} \left(F_2\frac{\partial F_3}{\partial x_2} - F_3\frac{\partial F_2}{\partial x_2}\right) dx_1dx_2dx_3 \neq 0.$$

To obtain a vector field of $I\!R^3$ with a compact support, zero divergence and which is not zero, it suffices to consider the vector field derived from a rotation around an axis ab and to multiply it by a compact support function ϕ of the square Euclidian distance $\vec{PM}^2$ between the current point M and a fixed point P on the axis ab: $\vec{F}(M) = \phi(\vec{PM}^2)(\vec{PM}\times\vec{ab})$ where the vector $\vec{ab}$ is parallel to the rotation axis ab. Clearly $\nabla\cdot\vec{F} = (\nabla(\phi(\vec{PM}^2))\cdot(\vec{PM}\times\vec{ab}) = 0$ because all spheres whose center belongs to the rotation axis ab are invariant sets for the rotation vector field $\vec{PM}\times\vec{ab}$. However, such vector fields $\vec{F}$ are too symmetric to satisfy

$$\iiint_{B(O,1)} \left(F_2\frac{\partial F_3}{\partial x_2} - F_3\frac{\partial F_2}{\partial x_2}\right) dx_1dx_2dx_3 \neq 0.$$

To breakdown such symmetry, the sum of two vector fields is sufficient (the imposed integral property depends nonlinearly on $\vec{F}$ whereas the two others depend linearly).

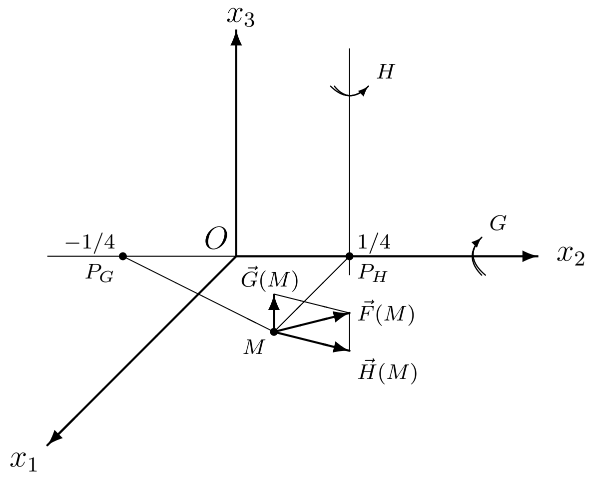

Figure 4: construction of the vector field $\vec{F}$ satisfying lemma 2.

More precisely, we consider the real function $\phi(s) = \exp(4/(4s-1)$ for $0 \leq s < 1/4$ and $\phi(s) = 0$ if $s \geq 1/4$ (ϕ is a smooth function with $\phi' < 0$ on $[0, 1/4[$) and we define the following two vector fields (see figure 4):

–

$$\vec{G}(x_1, x_2, x_3) = \phi(x_1^2 + (x_2 + 1/4)^2 + x_3^2) \begin{pmatrix} -x_3 \\ 0 \\ x_1 \end{pmatrix}$$

which is derived from a rotation around the axis Ox_2 and the point P_G of coordinates $(0, -1/4, 0)$;

–

$$\vec{H}(x_1, x_2, x_3) = \phi(x_1^2 + (x_2 - 1/4)^2 + x_3^2) \begin{pmatrix} -(x_2 - 1/4) \\ x_1 \\ 0 \end{pmatrix}$$

which is derived from a rotation around the axis parallel to Ox_3 passing through the point P_H of coordinates $(0, 1/4, 0)$.

We consider the vector field $\vec{F} = \vec{H} + \vec{G}$ displayed on figure 4.

By construction $\vec{F}$ is zero outside $B(O, 1)$ and $\nabla \cdot \vec{F} = 0$. In the sequel, we denote $\phi_- = \phi(x_1^2 + (x_2 - 1/4)^2 + x_3^2)$ and $\phi_+ = \phi(x_1^2 + (x_2 + 1/4)^2 + x_3^2)$. We have

$$F_2 \frac{\partial F_3}{\partial x_2} - F_3 \frac{\partial F_2}{\partial x_2} = x_1^2 \left(\frac{\partial \phi_+}{\partial x_2}\, \phi_- - \frac{\partial \phi_-}{\partial x_2}\, \phi_+ \right).$$

For x_1 and x_3 fixed, the integration on x_2 gives

$$\int \frac{\partial \phi_+}{\partial x_2}\phi_- dx_2 = [\phi_-\phi_+] - \int \frac{\partial \phi_-}{\partial x_2}\phi_+ dx_2 = -\int \frac{\partial \phi_-}{\partial x_2}\phi_+ dx_2$$

because ϕ_- and ϕ_+ are zero on the boundary. Thus, for fixed x_1 and x_3, we have

$$\int \left(F_2 \frac{\partial F_3}{\partial x_2} - F_3 \frac{\partial F_2}{\partial x_2} \right) dx_2 = 2x_1^2 \int \left(\frac{\partial \phi_+}{\partial x_2}\phi_- \right) dx_2.$$

Since

$$\frac{\partial \phi_+}{\partial x_2}\,\phi_- = 2(x_2 + 1/4)\,\phi'(x_1^2 + (x_2 + 1/4)^2 + x_3^2)\,\phi(x_1^2 + (x_2 - 1/4)^2 + x_3^2)$$

is always non positive, we conclude that

$$\iiint_{B(O,1)} \left(F_2 \frac{\partial F_3}{\partial x_2} - F_3 \frac{\partial F_2}{\partial x_2} \right) dx_1 dx_2 dx_3 < 0.$$

$\blacksquare$

Lemma 3 *Denote O a point interior to D where $v \neq 0$. Then there exists local coordinates (x, z_1, z_2) around O, such that $\vec{e}_x = \vec{v}$ and such that $\nabla \cdot \vec{e}_{z_1} = 0$, $\nabla \cdot \vec{e}_{z_2} = 0$ where the basis $(\vec{e}_x, \vec{e}_{z_1}, \vec{e}_{z_2})$ derives from the local coordinates (x, z_1, z_2) of a point M near O through the differential formula $d\vec{M} = \vec{e}_x dx + \vec{e}_{z_1} dz_1 + \vec{e}_{z_2} dz_2$.*

The proof differs only slightly from the proof of lemma 1 and is left to the reader.

Lemma 4 *For every real k, there exists a vector field $\vec{F}$ in the Euclidian space $\mathbb{R}^3$ such that*

- $\nabla \cdot \vec{F} = 0$,
- $\vec{F} = 0$ *outside the unit sphere* $B(O, 1)$,
- $\displaystyle\iiint_{B(O,1)} \left(F_2 \frac{\partial F_3}{\partial x_1} - F_3 \frac{\partial F_2}{\partial x_1} \right) dx_1 dx_2 dx_3 = k,$

where (Ox_1, Ox_2, Ox_3) are Euclidian axis and (F_1, F_2, F_3) the components of $\vec{F}$.

Proof of lemma 4 It is very similar to the proof of lemma 2. Using the same notation, it suffices to state $\vec{F} = \vec{G} + \vec{H}$ where

$$\vec{G}(x_1, x_2, x_3) = \phi((x_1 + 1/4)^2 + x_2^2 + x_3^2) \begin{pmatrix} -x_3 \\ 0 \\ x_1 + 1/4 \end{pmatrix}$$

is derived from a rotation around the axis parallel to Ox_2 and including the point $(-1/4, 0, 0)$ and where

$$\vec{H}(x_1, x_2, x_3) = \phi((x_1 - 1/4)^2 + x_2^2 + x_3^2) \begin{pmatrix} -x_2 \\ x_1 - 1/4 \\ 0 \end{pmatrix}$$

is derived from a rotation around the axis parallel to Ox_3 including the point of coordinates $(1/4, 0, 0)$. ■

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