Design and Stability of Quantum Filters with Measurement Imperfections: discrete-time and continuous-time cases

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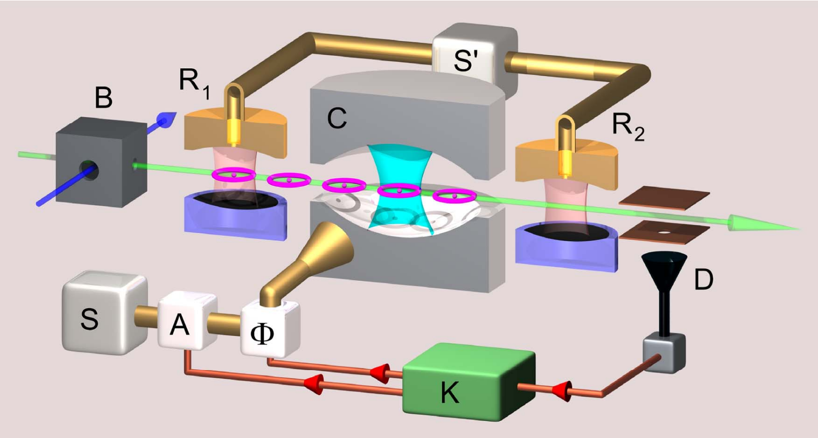
Quantum filtering: some references . . .

- Quantum stochastic equations: R. L. Hudson and K. R. Parthasarathy. Quantum Itô's formula and stochastic evolutions. Commun. Math. Phys., 93:301-323, 1984.
- Infinite dimensional analysis, noncommutative probability, quantum information: V. Belavkin. Quantum stochastic calculus and quantum nonlinear filtering. Journal of Multivariate Analysis, 1992, 42, 171-201.
- Control theory: Bouten, L.; R. van Handel & James, M. R. An introduction to quantum filtering. SIAM J. Control Optim., 2007, 46, 2199-224.
- Discrete-time approximation: Bouten, L. & Van Handel, R. Discrete approximation of quantum stochastic models. J. Math. Phys., AIP, 2008, 49, 102109-19.
- Quantum Monte Carlo trajectories: Dalibard, J.; Castin, Y. & Mølmer, K. Wave-function approach to dissipative processes in quantum optics. Phys. Rev. Lett., 1992, 68, 580-583.

The first experimental realization of a quantum state feedback [2]

[1]



The LKB photon box: sampling time ($\sim 100\ \mu$s) long enough to estimate in real-time the quantum-state ρ and to compute the control $u = Ae^{\imath\Phi}$ as a function of ρ (quantum state feedback).

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[1] Courtesy of Igor Dotsenko

[2] C. Sayrin et al., Nature, 1-September 2011

Experimental data

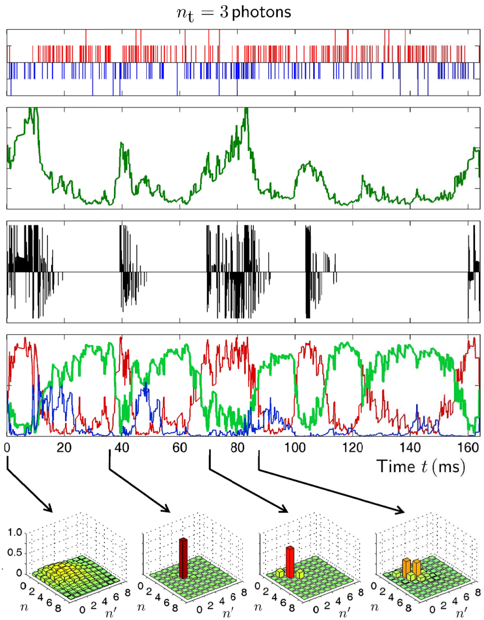
An open-loop trajectory starting from coherent state with an average of 3 photons relaxes towards vacuum (decoherence due to finite photon life time around 70 ms)

Detection efficiency 40%
Detection error rate 10%
Delay 4 sampling periods

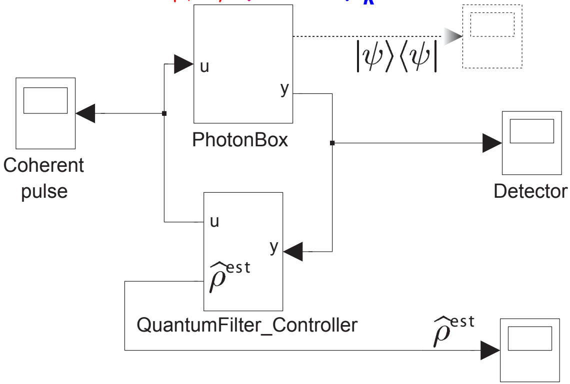
The quantum filter takes into account cavity decoherence, measure imperfections and delays (Bayes law).

Truncation to 9 photons

Stabilization around 3-photon state



Several "quantum states": $|\psi_k\rangle$, $\hat{\rho}_k$ and $\hat{\rho}_k^{est}$.



The state estimation $\hat{\rho}_k^{est}$ used in the feedback law takes into account, measure imperfections, delays and cavity decoherence:

- Derived from Bayes law: depends on past detector outcomes between 0 and k; computed recursively from an initial value $\hat{\rho}_0^{est}$;
- Stable and tends to converge towards $\hat{\rho}_k$, the expectation value of $\rho_k = |\psi_k\rangle\langle\psi_k|$ knowing its initial value $\rho_0 = \hat{\rho}_0$ and the past detector outcomes between 0 and k.

Outline

Markov chain in the ideal case (1)

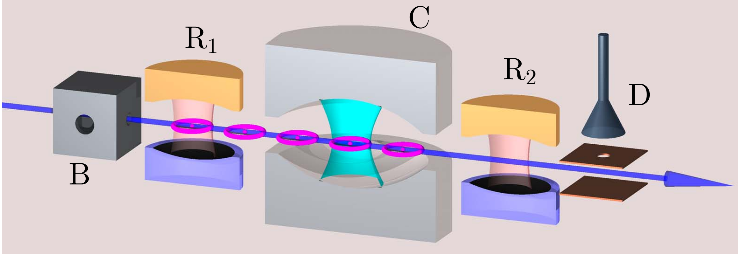
- System S corresponds to a quantized cavity mode:

$$\mathcal{H}_S = \left\{ \sum_{n=0}^{\infty} \psi^n |n\rangle \mid (\psi^n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

  where $|n\rangle$ represents the Fock state associated to exactly n photons inside the cavity
- Meter M is associated to atoms : $\mathcal{H}_M = \mathbb{C}^2$, each atom admits two energy levels and is described by a wave function $c_g|g\rangle + c_e|e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$; atoms leaving B are all in state $|g\rangle$
- When an atom comes out B, the state $|\Psi\rangle_B \in \mathcal{H}_S \otimes \mathcal{H}_M$ of the composite system atom/field is separable

$$|\Psi\rangle_B = |\psi\rangle \otimes |g\rangle.$$

Markov chain in the ideal case (2)



- When an atom comes out B: $|\Psi\rangle_B = |\psi\rangle \otimes |g\rangle$.
- Just before the measurement in D, the state is in general entangled (not separable):

$$|\Psi\rangle_{R_2} = U_{SM}\big(|\psi\rangle \otimes |g\rangle\big) = \big(M_g|\psi\rangle\big) \otimes |g\rangle + \big(M_e|\psi\rangle\big) \otimes |e\rangle$$

where U_{SM} is the total unitary transformation (Schrödinger propagator) defining the linear measurement operators M_g and M_e on $\mathcal{H}_S$. Since U_{SM} is unitary, $M_g^\dagger M_g + M_e^\dagger M_e = \mathbb{I}$.

Markov chain in the ideal case (3)

Just before the measurement in D, the atom/field state is:

$$M_g|\psi\rangle \otimes |g\rangle + M_e|\psi\rangle \otimes |e\rangle$$

Denote by $\mu \in \{g, e\}$ the measurement outcome in detector D: with probability $p_\mu = \langle\psi|M_\mu^\dagger M_\mu|\psi\rangle$ we get μ. Just after the measurement outcome μ, the state becomes separable:

$$|\Psi\rangle_D = \frac{1}{\sqrt{p_\mu}}\,(M_\mu|\psi\rangle) \otimes |\mu\rangle = \frac{(M_\mu|\psi\rangle) \otimes |\mu\rangle}{\sqrt{\langle\psi|M_\mu^\dagger M_\mu|\psi\rangle}}.$$

Markov process (density matrix formulation $\rho \sim |\psi\rangle\langle\psi|$)

$$\rho_+ = \begin{cases} \mathbb{M}_g(\rho) = \frac{M_g \rho M_g^\dagger}{\mathrm{Tr}\left(M_g \rho M_g^\dagger\right)}, & \text{with probability } p_g = \mathrm{Tr}\left(M_g \rho M_g^\dagger\right); \\ \mathbb{M}_e(\rho) = \frac{M_e \rho M_e^\dagger}{\mathrm{Tr}\left(M_e \rho M_e^\dagger\right)}, & \text{with probability } p_e = \mathrm{Tr}\left(M_e \rho M_e^\dagger\right). \end{cases}$$

Kraus map: $\mathbb{E}\left(\rho_+/\rho\right) = \mathbf{K}(\rho) = M_g \rho M_g^\dagger + M_e \rho M_e^\dagger$.

Markov chain with detection errors (1)

- $\rho_+ = \frac{1}{\mathrm{Tr}\left(M_\mu \rho M_\mu^\dagger\right)} M_\mu \rho M_\mu^\dagger$ when the atom collapses in $\mu = g, e$. This happens with probability $\mathrm{Tr}\left(M_\mu \rho M_\mu^\dagger\right)$.
- Detection error rates: $P(y = e/\mu = g) = \eta_g \in [0, 1]$ the probability of erroneous assignation to e when the atom collapses in g; $P(y = g/\mu = e) = \eta_e \in [0, 1]$ (given by the contrast of the Ramsey fringes).

Bayes law gives the probability that the atom collapses in $\mu = g$ knowing the detector outcome $y = g$:

$$P(\mu = g/y = g) = \frac{(1 - \eta_g)\,\mathrm{Tr}\left(M_g \rho M_g^\dagger\right)}{(1 - \eta_g)\,\mathrm{Tr}\left(M_g \rho M_g^\dagger\right) + \eta_e\,\mathrm{Tr}\left(M_e \rho M_e^\dagger\right)}$$

since $P(y = g/\mu = g) = (1 - \eta_g)$ and $P(y = g/\mu = e) = \eta_e$.

Markov chain with detection errors (2)

The expectation value $\widehat{\rho}_+$ of ρ_+ knowing ρ and the imperfect detection $y = g$ is given by

$$\widehat{\rho}_+ = P(\mu = g/y = g)\frac{M_g \rho M_g^\dagger}{\mathrm{Tr}\left(M_g \rho M_g^\dagger\right)} + P(\mu = e/y = g)\frac{M_e \rho M_e^\dagger}{\mathrm{Tr}\left(M_e \rho M_e^\dagger\right)}$$

Since

$$P(\mu = g/y = g) = \frac{(1-\eta_g)\,\mathrm{Tr}\left(M_g \rho M_g^\dagger\right)}{(1-\eta_g)\,\mathrm{Tr}\left(M_g \rho M_g^\dagger\right) + \eta_e\,\mathrm{Tr}\left(M_e \rho M_e^\dagger\right)}$$

and $P(\mu = e/y = g) = 1 - P(\mu = g/y = g)$, this expectation value $\widehat{\rho}_+$ is given by

$$\widehat{\rho}_+ = \frac{1}{\mathrm{Tr}\left((1-\eta_g) M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger\right)}\left((1-\eta_g) M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger\right)$$

Similarly when $y = e$, the expectation value $\widehat{\rho}_+$ is given by

$$\widehat{\rho}_+ = \frac{1}{\mathrm{Tr}\left((1-\eta_e) M_e \rho M_e^\dagger + \eta_g M_g \rho M_g^\dagger\right)}\left((1-\eta_e) M_e \rho M_e^\dagger + \eta_g M_g \rho M_g^\dagger\right)$$

Markov chain with detection errors (3)

We get

$$\widehat{\rho}_+ = \begin{cases} \frac{(1-\eta_g)M_g\rho M_g^\dagger+\eta_e M_e\rho M_e^\dagger}{\text{Tr}\left((1-\eta_g)M_g\rho M_g^\dagger+\eta_e M_e\rho M_e^\dagger\right)}, & \text{with prob. } \text{Tr}\left((1-\eta_g)M_g\rho M_g^\dagger+\eta_e M_e\rho M_e^\dagger\right); \\ \frac{\eta_g M_g\rho M_g^\dagger+(1-\eta_e)M_e\rho M_e^\dagger}{\text{Tr}\left(\eta_g M_g\rho M_g^\dagger+(1-\eta_e)M_e\rho M_e^\dagger\right)} & \text{with prob. } \text{Tr}\left(\eta_g M_g\rho M_g^\dagger+(1-\eta_e)M_e\rho M_e^\dagger\right). \end{cases}$$

Key point:

$$\text{Tr}\left((1-\eta_g)M_g\rho M_g^\dagger+\eta_e M_e\rho M_e^\dagger\right) \quad \text{and} \quad \text{Tr}\left(\eta_g M_g\rho M_g^\dagger+(1-\eta_e)M_e\rho M_e^\dagger\right)$$

are the probabilities to detect $y = g$ and e, knowing ρ.
With $\eta_{\mu',\mu}$ being the probability to detect $y = \mu'$ knowing that the atom collapses in μ, we have

$$\widehat{\rho}_+ = \frac{\sum_\mu \eta_{\mu',\mu} M_\mu \rho M_\mu^\dagger}{\text{Tr}\left(\sum_\mu \eta_{\mu',\mu} M_\mu \rho M_\mu^\dagger\right)} \quad \text{when we detect } y = \mu'.$$

The probability to detect $y = \mu'$ knowing ρ is $\text{Tr}\left(\sum_\mu \eta_{\mu',\mu} M_\mu \rho M_\mu^\dagger\right)$.

The Markov chain with cavity decoherence

When the sampling time ΔT is much smaller than the photon life time T_{cav}, cavity decoherence (at zero temperature) can be described approximatively by the Kraus map

$$\rho \mapsto M_0 \rho M_0^\dagger + M_- \rho M_-^\dagger$$

with $M_0 = (1 - \frac{\Delta T}{2T_{cav}})\mathbf{I} - \frac{\Delta T}{2T_{cav}}\mathbf{N}$ and $M_- = \sqrt{\frac{\Delta T}{T_{cav}}}\mathbf{a}$

M_0 and M_- can be seen as "measurement" operators corresponding to information catched by the "environment", information unknown in the real life but known in "Matlab/Simulink world":

- M_0 corresponds to no photon destruction during the sampling interval ΔT; probability $\operatorname{Tr}\left(M_0 \rho M_0^\dagger\right)$.
- M_- corresponds to one photon destruction during the sampling interval ΔT; probability $\operatorname{Tr}\left(M_- \rho M_-^\dagger\right)$.

The fact that we do not have access to this information can be interpreted as a detection error of 50% for M_0 and M_-. We get

$$\widehat{\rho}_+ = M_0 \rho M_0^\dagger + M_- \rho M_-^\dagger.$$

Photon-box quantum filter parameterized by left stochastic matrix $\eta_{\mu',\mu}$ [3]

$$\widehat{\rho}^{est}_{k+1} = \frac{1}{\mathrm{Tr}\left(\sum_\mu \eta_{\mu',\mu} M_\mu \widehat{\rho}^{est}_k M_\mu^\dagger\right)} \left(\sum_\mu \eta_{\mu',\mu} M_\mu \widehat{\rho}^{est}_k M_\mu^\dagger\right) \text{ with}$$

- we have a total of $m = 3 \times 7 = 21$ Kraus operators M_μ. The "jumps" are labeled by $\mu = (\mu^a, \mu^c)$ with $\mu^a \in \{no, g, e, gg, ge, eg, ee\}$ labeling atom related jumps and $\mu^c \in \{o, +, -\}$ cavity decoherence jumps.
- we have only $m' = 6$ real detection possibilities $\mu' \in \{no, g, e, gg, ge, ee\}$ corresponding respectively to no detection, a single detection in g, a single detection in e, a double detection both in g, a double detection one in g and the other in e, and a double detection both in e.

$\mu' \setminus \mu$	(no, μ^c)	(g, μ^c)	(e, μ^c)	(gg, μ^c)	(ee, μ^c)	(ge, μ^c) or $(eg,$
no	1	$1-\epsilon_d$	$1-\epsilon_d$	$(1-\epsilon_d)^2$	$(1-\epsilon_d)^2$	$(1-\epsilon_d)^2$
g	0	$\epsilon_d(1-\eta_g)$	$\epsilon_d\eta_e$	$2\epsilon_d(1-\epsilon_d)(1-\eta_g)$	$2\epsilon_d(1-\epsilon_d)\eta_e$	$\epsilon_d(1-\epsilon_d)(1-\eta_g$
e	0	$\epsilon_d\eta_g$	$\epsilon_d(1-\eta_e)$	$2\epsilon_d(1-\epsilon_d)\eta_g$	$2\epsilon_d(1-\epsilon_d)(1-\eta_e)$	$\epsilon_d(1-\epsilon_d)(1-\eta_e$
gg	0	0	0	$\epsilon_d^2(1-\eta_g)^2$	$\epsilon_d^2\eta_e^2$	$\epsilon_d^2\eta_e(1-\eta_g$
ge	0	0	0	$2\epsilon_d^2\eta_g(1-\eta_g)$	$2\epsilon_d^2\eta_e(1-\eta_e)$	$\epsilon_d^2((1-\eta_g)(1-\eta_e)$
ee	0	0	0	$\epsilon_d^2\eta_g^2$	$\epsilon_d^2(1-\eta_e)^2$	$\epsilon_d^2\eta_g(1-\eta_e$

[3] Somaraju, A.; Dotsenko, I.; Sayrin, C. & PR. Design and Stability of Discrete-Time Quantum Filters with Measurement Imperfections. American Control Conference, 2012, 5084-5089.

Markov chain in ideal life (e.g. Matlab/Simulink world): pure state $\boldsymbol{\rho_k}$

$$\boldsymbol{\rho_{k+1}} = \mathbb{M}_{\mu_k}(\boldsymbol{\rho_k}) =: \frac{M_{\mu_k}\boldsymbol{\rho_k}M_{\mu_k}^\dagger}{\mathrm{Tr}\left(M_{\mu_k}\boldsymbol{\rho_k}M_{\mu_k}^\dagger\right)}$$

- To each measurement outcome μ is attached the Kraus operator M_μ depending on μ and also on time (not explicitly recalled here, $M_\mu = M_{\mu,k}$ could depend on k).
- μ_k is a random variable taking values μ in $\{1, \cdots, m\}$ with probability $p_{\mu,\boldsymbol{\rho_k}} = \mathrm{Tr}\left(M_\mu\boldsymbol{\rho_k}M_\mu^\dagger\right)$. Conservation of probability ($\sum_\mu p_{\mu,\rho} = 1$ for all ρ) is guarantied by $\sum_{\mu=1}^m M_\mu^\dagger M_\mu = I$.
- The Kraus map $\mathcal{K}(\rho) = \sum_{\mu=1}^m M_\mu \rho M_\mu^\dagger$ provides

$$\mathbb{E}\left(\boldsymbol{\rho_{k+1}}/\boldsymbol{\rho_k}\right) = \mathcal{K}(\boldsymbol{\rho_k})$$

The Markov chain in real life: mixed states, $\widehat{\rho}_k$ and $\widehat{\rho}_k^{est}$ (1) [4]

Take $\rho_{k+1} = \frac{1}{\mathrm{Tr}\left(M_{\mu_k}\rho_k M_{\mu_k}^\dagger\right)}\left(M_{\mu_k}\rho_k M_{\mu_k}^\dagger\right)$ with measure imperfections and decoherence described by the left stochastic matrix η:
$\eta_{\mu',\mu} \in [0,1]$ is the probability of having the imperfect outcome $\mu' \in \{1,\ldots,m'\}$ knowing that the perfect one is $\mu \in \{1,\ldots,m\}$.

- $\widehat{\rho}_k = \mathbb{E}\left(\rho_k | \rho_0, \mu'_0, \ldots, \mu'_{k-1}\right)$ can be computed efficiently via the following recurrence

$$\widehat{\rho}_{k+1} = \frac{1}{\mathrm{Tr}\left(\sum_{\mu=1}^m \eta_{\mu'_k,\mu} M_\mu \widehat{\rho}_k M_\mu^\dagger\right)}\left(\sum_{\mu=1}^m \eta_{\mu'_k,\mu} M_\mu \widehat{\rho}_k M_\mu^\dagger\right)$$

  where the detector outcome μ'_k takes values μ' in $\{1,\cdots,m'\}$ with probability $p_{\mu',\widehat{\rho}_k} = \mathrm{Tr}\left(\sum_{\mu=1}^m \eta_{\mu'_k,\mu} M_\mu \widehat{\rho}_k M_\mu^\dagger\right)$.

- Thus $\mathbb{E}\left(\widehat{\rho}_{k+1} | \widehat{\rho}_k\right) = \mathcal{K}(\widehat{\rho}_k) = \sum_{\mu=1}^m M_\mu \widehat{\rho}_k M_\mu^\dagger$.

[4] Somaraju, A.; Dotsenko, I.; Sayrin, C. & PR. Design and Stability of Discrete-Time Quantum Filters with Measurement Imperfections. American Control Conference, 2012, 5084-5089.

The Markov chain in real life: mixed states, $\widehat{\rho}_k$ and $\widehat{\rho}_k^{\text{est}}$ (2)

$\widehat{\rho}_k = \mathbb{E}\left(\rho_k | \rho_0, \mu'_0, \ldots, \mu'_{k-1}\right)$ is given by

$$\widehat{\rho}_{k+1} = \frac{1}{\operatorname{Tr}\left(\sum_{\mu=1}^{m} \eta_{\mu'_k,\mu} M_\mu \widehat{\rho}_k M_\mu^\dagger\right)} \left(\sum_{\mu=1}^{m} \eta_{\mu'_k,\mu} M_\mu \widehat{\rho}_k M_\mu^\dagger\right)$$

with the perfect initialization: $\widehat{\rho}_0 = \rho_0$.

$\widehat{\rho}_{k+1}^{\text{est}} = \frac{1}{\operatorname{Tr}\left(\sum_{\mu=1}^{m} \eta_{\mu'_k,\mu} M_\mu \widehat{\rho}_k^{\text{est}} M_\mu^\dagger\right)} \left(\sum_{\mu=1}^{m} \eta_{\mu'_k,\mu} M_\mu \widehat{\rho}_k^{\text{est}} M_\mu^\dagger\right)$ but with imperfect initialization $\widehat{\rho}_0^{\text{est}} \neq \rho_0$.

This filtering process is stable[5]: the fidelity $F(\widehat{\rho}_k, \widehat{\rho}_k^{\text{est}})$ is a sub-martingale for any η and M_μ:

$$\mathbb{E}\left(F(\widehat{\rho}_{k+1}, \widehat{\rho}_{k+1}^{\text{est}}) / \widehat{\rho}_k\right) \geq F(\widehat{\rho}_k, \widehat{\rho}_k^{\text{est}})$$

Convergence of $\widehat{\rho}_k^{\text{est}}$ towards $\widehat{\rho}_k$ when $k \mapsto +\infty$ is an open problem.[6]

[5] PR. Fidelity is a Sub-Martingale for Discrete-Time Quantum Filters IEEE Transactions on Automatic Control, 2011, 56, 2743-2747 .

[6] A partial result (continuous-time): R. van Handel. The stability of quantum Markov filters. Infin. Dimens. Anal. Quantum Probab. Relat. Top. , 2009, 12, 153-172.

Bayesian parameter estimations

Consider detector outcomes μ'_k corresponding to a parameter value $\bar{p}$ poorly known. Assume to simplify that either $\bar{p} = a$ or $\bar{p} = b$, with $a \neq b$. We can discriminate between a and b and recover $\bar{p}$ via the following Bayesian scheme using information contained in the μ'_k's:

$$\widehat{\rho}^{est}_{a,k+1} = \frac{\sum_\mu \eta^a_{\mu'_k,\mu} M^a_\mu \widehat{\rho}^{est}_{a,k} M^{a\dagger}_\mu}{\mathrm{Tr}\left(\sum_p \sum_\mu \eta^p_{\mu'_k,\mu} M^p_\mu \widehat{\rho}^{est}_{p,k} M^{p\dagger}_\mu\right)}, \quad \widehat{\rho}^{est}_{b,k+1} = \frac{\sum_\mu \eta^b_{\mu'_k,\mu} M^b_\mu \widehat{\rho}^{est}_{b,k} M^{b\dagger}_\mu}{\mathrm{Tr}\left(\sum_p \sum_\mu \eta^p_{\mu'_k,\mu} M^p_\mu \widehat{\rho}^{est}_{p,k} M^{p\dagger}_\mu\right)}$$

with initialization $\widehat{\rho}^{est}_{a,k+1} = \widehat{\rho}^{est}_{b,k+1} = \widehat{\rho}^{est}_0/2$ where $\widehat{\rho}^{est}_0$ is some guess of $\widehat{\rho}_0$ assuming initial probability of $\frac{1}{2}$ to have $\bar{p} = a$ and $\bar{p} = b$.

This estimation/filtering process is also stable:

- $F(\widehat{\rho}_k, \widehat{\rho}^{est}_{a,k}) + F(\widehat{\rho}_k, \widehat{\rho}^{est}_{b,k})$ is a sub-martingale
- $\mathrm{Tr}\left(\widehat{\rho}^{est}_{a,k}\right)$, $\mathrm{Tr}\left(\widehat{\rho}^{est}_{b,k}\right)$) estim. of proba. to have $\bar{p} = a$, $\bar{p} = b$.

Direct generalization to a continuum of choices for $\bar{p} \in [p_{\min}, p^{\max}]$ (see [7] for a first experimental use)

[7] Brakhane, S.; Alt, W.; Kampschulte, T.; Martinez-Dorantes, M.; Reimann, R.; Yoon, S.; Widera, A. & Meschede, D. Bayesian Feedback Control of a Two-Atom Spin-State in an Atom-Cavity System. Phys. Rev. Lett., 2012, 109, 173601-

Dynamical models with a precise structure

Discrete-time models are Markov chains

$$\rho_{k+1} = \frac{1}{p_\mu(\rho_k)} M_\mu \rho_k M_\mu^\dagger \quad \text{with proba.} \quad p_\mu(\rho_k) = \text{Tr}\left(M_\mu \rho_k M_\mu^\dagger\right)$$

associated to Kraus maps (ensemble average, open quantum channels)

$$\mathbb{E}\left(\rho_{k+1}/\rho_k\right) = \mathbf{K}(\rho_k) = \sum_\mu M_\mu \rho_k M_\mu^\dagger \quad \text{with} \quad \sum_\mu M_\mu^\dagger M_\mu = \mathbb{I}$$

Continuous-time models are stochastic differential systems

$$d\rho = \left(-i[H,\rho] + L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)\right) dt + \left(L\rho + \rho L^\dagger - \text{Tr}\left((L+L^\dagger)\rho\right)\rho\right) dw$$

driven by Wiener processes[8] $dw = dy - \text{Tr}\left((L+L^\dagger)\rho\right) dt$ with measure y and associated to Lindbald master equations:

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho] + L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)$$

[8] Another possibility: SDE driven by Poisson processes.

From discrete-time to continuous-time: heuristic connection

For Monte-Carlo simulations of

$$d\rho = \left(-i[H,\rho] + L\rho L^\dagger - \tfrac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)\right) dt + \left(L\rho + \rho L^\dagger - \operatorname{Tr}\left((L+L^\dagger)\rho\right)\rho\right) dw$$

take a small sampling time dt, generate a random number dw_t according to a Gaussian law of standard deviation $\sqrt{dt}$, and set $\rho_{t+dt} = \mathbb{M}_{dy_t}(\rho_t)$ where the jump operator $\mathbb{M}_{dy_t}$ is labelled by the measurement outcome $dy_t = \operatorname{Tr}\left((L+L^\dagger)\rho_t\right) dt + dw_t$:

$$\mathbb{M}_{dy_t}(\rho_t) = \frac{\left(I+(-iH-\frac{1}{2}L^\dagger L)dt + dy_t L\right)\rho_t\left(I+(iH-\frac{1}{2}L^\dagger L)dt + dy_t L^\dagger\right)}{\operatorname{Tr}\left(\left(I+(-iH-\frac{1}{2}L^\dagger L)dt + dy_t L\right)\rho_t\left(I+(iH-\frac{1}{2}L^\dagger L)dt + dy_t L^\dagger\right)\right)}.$$

Then ρ_{t+dt} remains always a density operator and using the Itô rules (dw of order $\sqrt{dt}$ and $dw^2 \equiv dt$) we get the good $d\rho = \rho_{t+dt} - \rho_t$ up to $O((dt)^{3/2})$ terms.

From discrete-time to continuous-time: heuristic connection (end)

For the Lindblad equation

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho] + L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)$$

take a small sampling time ***dt*** and set

$$\rho_{t+dt} = \frac{\big(I + \boldsymbol{dt}(-iH - \frac{1}{2}L^\dagger L)\big)\rho_t\big(I + \boldsymbol{dt}(iH - \frac{1}{2}L^\dagger L)\big) + \boldsymbol{dt}\,L\rho_t L^\dagger}{\mathrm{Tr}\left(\big(I + \boldsymbol{dt}(-iH - \frac{1}{2}L^\dagger L)\big)\rho_t\big(I + \boldsymbol{dt}(iH - \frac{1}{2}L^\dagger L)\big) + \boldsymbol{dt}\,L\rho_t L^\dagger\right)}.$$

Then $\rho_{t+\boldsymbol{dt}}$ remains always a density operator and $\frac{d}{dt}\rho = (\rho_{t+\boldsymbol{dt}} - \rho_t)/\boldsymbol{dt}$ up to $O(\boldsymbol{dt})$ terms.

SDE driven by Poisson and/or Wiener processes

$$d\rho_t = \mathcal{L}(\rho_t)\,dt + \sum_{\nu=1}^{m_w} \Lambda_\nu(\rho_t)\,dw_t^\nu + \sum_{\mu=1}^{m_P} \Upsilon_\mu(\rho_t)\,\left(dN_t^\mu - \operatorname{Tr}\left(C_\mu \rho_t C_\mu^\dagger\right)\,dt\right)$$

where

- $\mathcal{L}(\rho_t) := -i\,[H, \rho_t] + \sum_{\mu=1}^{m_P} \mathcal{L}_\mu^P(\rho_t) + \sum_{\nu=1}^{m_w} \mathcal{L}_\nu^w(\rho_t)$,

$$\mathcal{L}_\mu^P(\rho) := -\tfrac{1}{2}\{C_\mu^\dagger C_\mu, \rho\} + C_\mu \rho C_\mu^\dagger, \quad \mathcal{L}_\nu^w(\rho) := -\tfrac{1}{2}\{L_\nu^\dagger L_\nu, \rho\} + L_\nu \rho L_\nu^\dagger;$$

$$\Upsilon_\mu(\rho) := \frac{C_\mu \rho C_\mu^\dagger}{\operatorname{Tr}\left(C_\mu \rho C_\mu^\dagger\right)} - \rho, \quad \Lambda_\nu(\rho) := L_\nu \rho + \rho L_\nu^\dagger - \operatorname{Tr}\left((L_\nu + L_\nu^\dagger)\rho\right)\rho$$

- Detector click no μ is related to the Poisson process $dN_t^\mu = N^\mu(t+dt) - N^\mu(t) = 1$ and happens with probability $\operatorname{Tr}\left(C_\mu \rho_t C_\mu{}^\dagger\right)\,dt$;
- Continuous detector y_t^ν is related to the Wiener process dw_t^ν by $dy_t^\nu = dw_t^\nu + \operatorname{Tr}\left((L_\nu + L_\nu^\dagger)\rho_t\right)\,dt$.

Quantum filter in the ideal case

$$d\rho_t = \mathcal{L}(\rho_t)\,dt + \sum_{\nu=1}^{m_w} \Lambda_\nu(\rho_t)\,dw_t^\nu + \sum_{\mu=1}^{m_P} \Upsilon_\mu(\rho_t)\,\left(dN_t^\mu - \operatorname{Tr}\left(C_\mu \rho_t C_\mu^\dagger\right)\,dt\right),$$

and the associated quantum filter

$$d\widehat{\rho}_t^{est} = \mathcal{L}(\widehat{\rho}_t^{est})\,dt + \sum_{\nu=1}^{m_w} \Lambda_\nu(\widehat{\rho}_t^{est})\left(\boldsymbol{dy_t^\nu} - \operatorname{Tr}\left((L_\nu + L_\nu^\dagger)\widehat{\rho}_t^{est}\right)\,dt\right) + \sum_{\mu=1}^{m_P} \Upsilon_\mu(\widehat{\rho}_t^{est})\,\left(\boldsymbol{dN_t^\mu} - \operatorname{Tr}\left(C_\mu \widehat{\rho}_t^{est} C_\mu^\dagger\right)\,dt\right).$$

It can be rewritten as follows

$$d\widehat{\rho}_t^{est} = \mathcal{L}(\widehat{\rho}_t^{est})\,dt + \sum_{\nu=1}^{m_w} \Lambda_\nu(\widehat{\rho}_t^{est})\left(\operatorname{Tr}\left((L_\nu + L_\nu^\dagger)\rho_t\right) - \operatorname{Tr}\left((L_\nu + L_\nu^\dagger)\widehat{\rho}_t^{est}\right)\right)dt + \sum_{\nu=1}^{m_w} \Lambda_\nu(\widehat{\rho}_t^{est})dw_t^\nu + \sum_{\mu=1}^{m_P} \Upsilon_\mu(\widehat{\rho}_t^{est})\,\left(dN_t^\mu - \operatorname{Tr}\left(C_\mu \widehat{\rho}_t^{est} C_\mu^\dagger\right)\,dt\right).$$

Quantum filters with imperfections and decoherence[9] (1)

- Imperfection model for the Poisson processes dN_t^μ:
  - real outcomes $\mu' \in \{0, 1, \ldots, m'_P\}$
  - ideal outcomes $\mu \in \{0, 1, \ldots, m_P\}$.
  - $(m'_P + 1) \times m_P$ left stochastic matrix $\eta^P = (\eta^P_{\mu',\mu})_{0 \le \mu' \le m'_P, 1 \le \mu \le m_P}$
  - positive vector $\bar{\eta}^P = (\bar{\eta}^P_{\mu'})_{1 \le \mu' \le m'_P}$ in $\mathbb{R}_+^{m'_P}$.
- Imperfection model for the diffusion processes dw_t^ν:
  - m'_w real continuous signals $y_t^{\nu'}$ with $\nu' \in \{1, \ldots, m'_w\}$,
  - m_w ideal continuous signals y_t^ν with $\nu \in \{1, \ldots, m_w\}$
  - correlation $m'_w \times m_w$ matrix $\eta^w = (\eta^w_{\nu',\nu})_{1 \le \nu' \le m'_w, 1 \le \nu \le m_w}$, with $0 \le \eta^w_{\nu',\nu} \le 1$ and $\sum_{\nu'=1}^{mr_w} \eta^w_{\nu',\nu} \le 1$.

[9] see last chapter of H. Amini. Stabilization of discrete-time quantum systems and stability of continuous-time quantum filters. PhD thesis, Mines-ParisTech, September 2012.

Quantum filters with imperfections and decoherence (2)

$$d\widehat{\rho}_t = \mathcal{L}(\widehat{\rho}_t)\,dt + \sum_{\nu'=1}^{m'_w} \sqrt{\bar{\eta}^w_{\nu'}}\widehat{\Lambda}_{\nu'}(\widehat{\rho}_t)\,d\widehat{w}^{\nu'}_t$$

$$+ \sum_{\mu'=1}^{m'_P} \widehat{\Upsilon}_{\mu'}(\widehat{\rho}_t)\left(d\widehat{N}^{\mu'}_t - \bar{\eta}^P_{\mu'}\,dt - \sum_{\mu=1}^{m_P} \eta^P_{\mu',\mu}\,\mathrm{Tr}\left(C_\mu \widehat{\rho}_t C^\dagger_\mu\right)\,dt\right)$$

- $\bar{\eta}^w_{\nu'} = \sum_{\nu=1}^{m_w} \eta^w_{\nu',\nu}$, $\widehat{\Upsilon}_{\mu'}(\rho) := \frac{\bar{\eta}^P_{\mu'}\rho + \sum_{\mu=1}^{m_P} \eta^P_{\mu',\mu} C_\mu \rho C^\dagger_\mu}{\bar{\eta}^P_{\mu'} + \sum_{\mu=1}^{m_P} \eta^P_{\mu',\mu}\,\mathrm{Tr}\left(C_\mu \rho C^\dagger_\mu\right)} - \rho$,
  $\widehat{\Lambda}_{\nu'}(\rho) = \widehat{L}_{\nu'}\rho + \rho\widehat{L}^\dagger_{\nu'} - \mathrm{Tr}\left((\widehat{L}_{\nu'} + \widehat{L}^\dagger_{\nu'})\rho\right)\rho$, $\widehat{L}_{\nu'} := (\sum_{\nu=1}^{m_w} \eta^w_{\nu',\nu} L_\nu)/\bar{\eta}^w_{\nu'}$;
- the jump detector μ' corresponds to $\widehat{N}^{\mu'}(t)$:
  $d\widehat{N}^{\mu'}_t = \widehat{N}^{\mu'}(t+dt) - \widehat{N}^{\mu'}(t) = 1$ happens with probability
  $\widehat{P}_{\mu'}(\widehat{\rho}_t) = \bar{\eta}^P_{\mu'}\,dt + \sum_{\mu=1}^{m_P} \eta^P_{\mu',\mu}\,\mathrm{Tr}\left(C_\mu \widehat{\rho}_t C^\dagger_\mu\right)\,dt$;
- the continuous detector ν' refers to $\widehat{y}^{\nu'}_t$ and $d\widehat{w}^{\nu'}_t$:

$$d\widehat{y}^{\nu'}_t = d\widehat{w}^{\nu'}_t + \sqrt{\bar{\eta}^w_{\nu'}}\,\mathrm{Tr}\left((\widehat{L}_{\nu'} + \widehat{L}^\dagger_{\nu'})\widehat{\rho}_t\right)\,dt.$$

Quantum filters with imperfections and decoherence (3)

$$d\widehat{\rho}_t = \mathcal{L}(\widehat{\rho}_t)\,dt + \sum_{\nu'=1}^{m'_w} \sqrt{\bar{\eta}^w_{\nu'}}\widehat{\Lambda}_{\nu'}(\widehat{\rho}_t)\,d\widehat{w}^{\nu'}_t$$

$$+ \sum_{\mu'=1}^{m'_P} \widehat{\Upsilon}_{\mu'}(\widehat{\rho}_t)\left(d\widehat{N}^{\mu'}_t - \bar{\eta}^P_{\mu'}\,dt - \sum_{\mu=1}^{m_P} \eta^P_{\mu',\mu}\,\mathrm{Tr}\left(C_\mu \widehat{\rho}_t C^\dagger_\mu\right)\,dt\right)$$

and the associated quantum filter

$$d\widehat{\rho}^{est}_t = \mathcal{L}(\widehat{\rho}^{est}_t)\,dt + \sum_{\nu'=1}^{m'_w} \sqrt{\bar{\eta}^w_{\nu'}}\widehat{\Lambda}_{\nu'}(\widehat{\rho}^{est}_t)\left(d\widehat{\boldsymbol{y}}^{\boldsymbol{\nu'}}_t - \sqrt{\bar{\eta}^w_{\nu'}}\,\mathrm{Tr}\left((\widehat{L}_{\nu'} + \widehat{L}^\dagger_{\nu'})\widehat{\rho}^{est}_t\right)\,dt\right)$$

$$+ \sum_{\mu'=1}^{m'_P} \widehat{\Upsilon}_{\mu'}(\widehat{\rho}^{est}_t)\left(d\widehat{\boldsymbol{N}}^{\boldsymbol{\mu'}}_t - \bar{\eta}^P_{\mu'}\,dt - \sum_{\mu=1}^{m_P} \eta^P_{\mu',\mu}\,\mathrm{Tr}\left(C_\mu \widehat{\rho}^{est}_t C^\dagger_\mu\right)\,dt\right)$$

Quantum filtering combines the following key points

1. Bayes law: $P(\mu'/\mu) = P(\mu/\mu')P(\mu') \,/\, \left(\sum_{\nu'} P(\mu/\nu')P(\nu')\right)$.
2. Schrödinger equations defining unitary transformations.
3. Partial collapse of the wave packet: irreversibility and convergence are induced by the measure of observables $\mathcal{O}$ with degenerate spectra, $\mathcal{O} = \sum_\mu \lambda_\mu P_\mu$:
   - measure outcome λ_μ with proba. $p_\mu = \langle\psi|P_\mu|\psi\rangle = \text{Tr}\left(\rho P_\mu\right)$ depending $|\psi\rangle$, ρ just before the measurement
   - measure back-action if outcome μ:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{P_\mu|\psi\rangle}{\sqrt{\langle\psi|P_\mu|\psi\rangle}}, \quad \rho \mapsto \rho_+ = \frac{P_\mu \rho P_\mu}{\text{Tr}\left(\rho P_\mu\right)}$$

4. Tensor product for the description of composite systems (S, M):
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $H = H_S \otimes \mathbb{I}_M + H_{int} + \mathbb{I}_S \otimes H_M$
   - observable on sub-system M only: $\mathcal{O} = \mathbb{I}_S \otimes \mathcal{O}_M$.