Fonctions Gevrey et contrôle frontière de certaines EDP

(Gevrey functions and boundary control of some PDE)

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Outline

A computation due to Holmgren[1]

Take the 1D-heat equation, $\frac{\partial\theta}{\partial t}(x,t) = \frac{\partial^2\theta}{\partial x^2}(x,t)$ for $x \in [0,1]$ and set, formally, $\theta = \sum_{i=0}^{\infty} a_i(t)\frac{x^i}{i!}$. Since,

$$\frac{\partial\theta}{\partial t} = \sum_{i=0}^{\infty} \frac{da_i}{dt}\left(\frac{x^i}{i!}\right), \quad \frac{\partial^2\theta}{\partial x^2} = \sum_{i=0}^{\infty} a_{i+2}\left(\frac{x^i}{i!}\right)$$

the heat equation $\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2}$ reads $\frac{d}{dt}a_i = a_{i+2}$ and thus

$$a_{2i+1} = a_1^{(i)}, \quad a_{2i} = a_0^{(i)}$$

With two arbitrary smooth time-functions $f(t)$ and $g(t)$, playing the role of a_0 and a_1, the general solution reads:

$$\theta(x,t) = \sum_{i=0}^{\infty} f^{(i)}(t)\left(\frac{x^{2i}}{(2i)!}\right) + g^{(i)}(t)\left(\frac{x^{2i+1}}{(2i+1)!}\right).$$

Convergence issues ?

[1] E. Holmgren, Sur l'équation de la propagation de la chaleur. Arkiv für Math. Astr. Physik, t. 4, (1908), p. 1-4

Gevrey functions[2]

- A C^∞-function $[0,T] \ni t \mapsto f(t)$ is of Gevrey-order α when,

$$\exists\, M, A > 0, \quad \forall t \in [0,T], \forall i \geq 0, \quad |f^{(i)}(t)| \leq M A^i \Gamma(1+\alpha i)$$

  where Γ is the gamma function with $n! = \Gamma(n+1)$, $\forall n \in \mathbb{N}$.
- Analytic functions correspond to Gevrey-order ≤ 1.
- When $\alpha > 1$, the set of C^∞-functions with Gevrey-order α contains non-zero functions with compact supports. Prototype of such functions:

$$t \mapsto f(t) = \begin{cases} \exp\left(-\left(\frac{1}{t(1-t)}\right)^{\frac{1}{\alpha-1}}\right) & \text{if } t \in]0,1[\\ 0 & \text{otherwise.} \end{cases}$$

[2] M. Gevrey: La nature analytique des solutions des équations aux dérivées partielles, Ann. Sc. Ecole Norm. Sup., vol.25, pp:129–190, 1918.

Gevrey functions and exponential decay[3]

- Take, in the complex plane, the open bounded sector $\mathcal{S}$ those vertex is the origin. Assume that f is analytic on $\mathcal{S}$ and admits an exponential decay of order $\sigma > 0$ and type A in $\mathcal{S}$:

$$\exists C, \rho > 0, \quad \forall z \in \mathcal{S}, \quad |f(z)| \leq C|z|^{\rho} \exp\left(\frac{-1}{A|z|^{\sigma}}\right)$$

Then in any closed sub-sector $\tilde{\mathcal{S}}$ of $\mathcal{S}$ with origin as vertex, exists $M > 0$ such that

$$\forall z \in \tilde{\mathcal{S}}/\{0\}, \quad |f^{(i)}(z)| \leq MA^{i}\,\Gamma\left(1 + i\left(\frac{1}{\sigma} + 1\right)\right)$$

- Rule of thumb: if a piece-wise analytic f admits an exponential decay of order σ then it is of Gevrey-order $\alpha = \frac{1}{\sigma} + 1$.

[3] J.P. Ramis: Dévissage Gevrey. Astérisque, vol:59-60, pp:173–204, 1978.
See also J.P. Ramis: Séries Divergentes et Théories Asymptotiques; SMF, Panoramas et Synthèses, 1993.

Gevrey space and ultra-distributions[4]

Denote by $\mathcal{D}_\alpha$ the set of functions $\mathbb{R} \mapsto \mathbb{R}$ of order $\alpha > 1$ and with compact supports. As for the class of C^∞ functions, most of the usual manipulations remain in $\mathcal{D}_\alpha$:

- $\mathcal{D}_\alpha$ is stable by addition, multiplication, derivation, integration, ...
- if $f \in \mathcal{D}_\alpha$ and F is an analytic function on the image of f, then $F(f)$ remains in $\mathcal{D}_\alpha$.
- if $f \in \mathcal{D}_\alpha$ and $F \in L^1_{loc}(\mathbb{R})$ then the convolution $f * F$ is of Gevrey-order α on any compact interval.

As for the construction of $\mathcal{D}'$, the space of distributions (the dual of $\mathcal{D}$ the space of C^∞ functions of compact supports), one can construct $\mathcal{D}'_\alpha \supset \mathcal{D}'$, a space of ultra-distributions, the dual of $\mathcal{D}_\alpha \subset \mathcal{D}$.

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[4] See, e.g., I.M. Guelfand and G.E. Chilov: Les Distributions, tomes 2 et 3. Dunod, Paris,1964.

Symbolic computations: $s := d/dt$, $s \in \mathbb{C}$

The general solution of $\theta'' = s\theta$ reads ($' := d/dx$)

$$\theta = \cosh(x\sqrt{s})\, f(s) + \frac{\sinh(x\sqrt{s})}{\sqrt{s}}\, g(s)$$

where $f(s)$ and $g(s)$ are the two constants of integration. Since cosh and sinh gather the even and odd terms of the series defining exp, we have

$$\cosh(x\sqrt{s}) = \sum_{i\geq 0} s^i \frac{x^{2i}}{(2i)!}, \quad \frac{\sinh(x\sqrt{s})}{\sqrt{s}} = \sum_{i\geq 0} s^i \frac{x^{2i+1}}{(2i+1)!}$$

and we recognize $\theta = \sum_{i=0}^{\infty} f^{(i)}(t)\left(\frac{x^{2i}}{(2i)!}\right) + g^{(i)}(t)\left(\frac{x^{2i+1}}{(2i+1)!}\right)$.
For each x, the operators $\cosh(x\sqrt{s})$ and $\sinh(x\sqrt{s})/\sqrt{s}$ are ultra-distributions of $\mathcal{D}'_{2^-}$:

$$\sum_{i\geq 0} \frac{(-1)^i x^{2i}}{(2i)!}\delta^{(i)}(t), \quad \sum_{i\geq 0} \frac{(-1)^i x^{2i+1}}{(2i+1)!}\delta^{(i)}(t)$$

with δ, the Dirac distribution.

Entire functions of $s = d/dt$ as ultra-distributions

- $\mathbb{C} \ni s \mapsto P(s) = \sum_{i \geq 0} a_i s^i$ is an entire function when the radius of convergence is infinite.
- If its order at infinity is $\sigma > 0$ and its type is finite, i.e., $\exists M, K > 0$ such that $\forall s \in \mathbb{C}$, $|P(s)| \leq M \exp(K|s|^\sigma)$, then

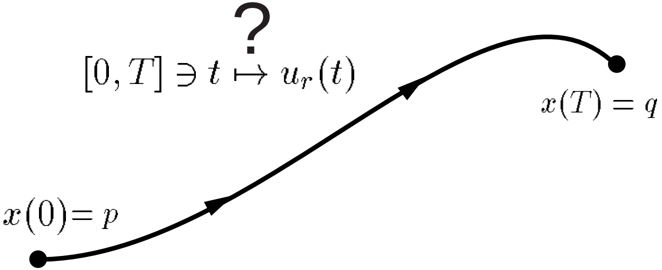
$$\exists A, B > 0 \mid \forall i \geq 0, \quad |a_i| \leq A \frac{B^i}{\Gamma(i/\sigma + 1)}.$$

  $\cosh(\sqrt{s})$ and $\sinh(\sqrt{s})/\sqrt{s}$ are entire functions of order $\sigma = 1/2$ and of type 1.
- Take $P(s)$ of order $\sigma < 1$ with $s = d/dt$. Then $P \in \mathcal{D}'_{\frac{1}{\sigma}-}$: $P(s)f(s)$ corresponds, in the time domain, to absolutely convergent series

$$P(s)y(s) \equiv \sum_{i=0}^{\infty} a_i \, f^{(i)}(t)$$

  when $t \mapsto f(t)$ is a C^∞-function of Gevrey-order $\alpha < 1/\sigma$.

Motion planning (trajectory generation)

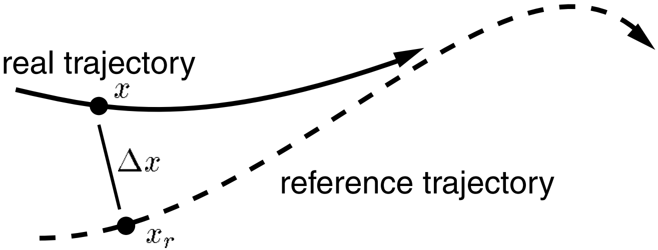


- Difficult problem because it requires, in general, the integration of the open-loop dynamics

$$\frac{d}{dt}x = f(x, u(t)).$$

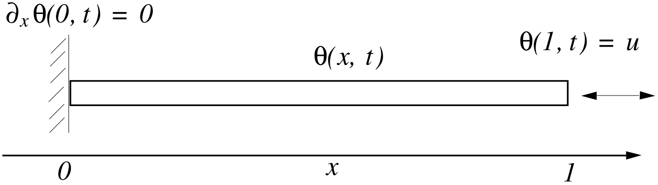
- One fundamental issue in system theory: controllability.

Trajectory tracking (stabilization)



- Compute Δu, $u = u_r + \Delta u$, such that $\Delta x = x - x_r$ converges to 0 at t tends to $+\infty$ (closed-loop stability).
- Another fundamental issue in system theory: feedback.

Motion planning for the 1D heat equation



The data are:

1. the model relating the control input $u(t)$ to the state, $(\theta(x,t))_{x\in[0,1]}$:

$$\frac{\partial \theta}{\partial t}(x,t) = \frac{\partial^2 \theta}{\partial x^2}(x,t), \quad x \in [0,1]$$
$$\frac{\partial \theta}{\partial x}(0,t) = 0 \qquad \theta(1,t) = u(t).$$

2. A transition time $T > 0$, the initial (resp. final) state: $[0,1] \ni x \mapsto p(x)$ (resp. $q(x)$)

The goal is to find the open-loop control $[0,T] \ni t \mapsto u(t)$ steering $\theta(x,t)$ from the initial profile $\theta(x,0) = p(x)$ to the final profile $\theta(x,T) = q(x)$.

Series solutions

Set, formally

$$\theta = \sum_{i=0}^{\infty} a_i(t)\frac{x^i}{i!}, \quad \frac{\partial\theta}{\partial t} = \sum_{i=0}^{\infty} \frac{da_i}{dt}\left(\frac{x^i}{i!}\right), \quad \frac{\partial^2\theta}{\partial x^2} = \sum_{i=0}^{\infty} a_{i+2}\left(\frac{x^i}{i!}\right)$$

and $\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2}$ reads $\frac{d}{dt}a_i = a_{i+2}$. Since $a_1 = \frac{\partial\theta}{\partial x}(0,t) = 0$ and $a_0 = \theta(0,t)$ we have

$$a_{2i+1} = 0, \quad a_{2i} = a_0^{(i)}$$

Set $y := a_0 = \theta(0,t)$ we have, in the time domain,

$$\theta(x,t) = \sum_{i=0}^{\infty}\left(\frac{x^{2i}}{(2i)!}\right) y^{(i)}(t), \quad u(t) = \sum_{i=0}^{\infty}\left(\frac{1}{(2i)!}\right) y^{(i)}(t)$$

that also reads in the Laplace domain ($s = d/dt$):

$$\theta(x,s) = \cosh(x\sqrt{s})\, y(s), \quad u(s) = \cosh(\sqrt{s})y(s).$$

An explicit parameterization of trajectories

For any C^∞-function $y(t)$ of Gevrey-order $\alpha < 2$, the time function

$$u(t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!}$$

is well defined and smooth. The (x, t)-function

$$\theta(x, t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!} \; x^{2i}$$

is also well defined (entire versus x and smooth versus t). More over for all t and $x \in [0, 1]$, we have, whatever $t \mapsto y(t)$ is,

$$\frac{\partial \theta}{\partial t}(x, t) = \frac{\partial^2 \theta}{\partial x^2}(x, t), \quad \frac{\partial \theta}{\partial x}(0, t) = 0, \quad \theta(1, t) = u(t)$$

An infinite dimensional analogue of differential flatness.[5]

[5] Fliess et al: Flatness and defect of nonlinear systems: introductory theory and examples, International Journal of Control. vol.61, pp:1327–1361. 1995.

Motion planning of the heat equation[6]

Take $\sum_{i\geq 0} a_i \frac{\xi^i}{i!}$ and $\sum_{i\geq 0} b_i \frac{\xi^i}{i!}$ entire functions of ξ. With $\sigma > 1$

$$y(t) = \left(\sum_{i\geq 0} a_i \frac{t^i}{i!}\right)\left(\frac{e^{\frac{-T^\sigma}{(T-t)^\sigma}}}{e^{\frac{-T^\sigma}{t^\sigma}} + e^{\frac{-T^\sigma}{(T-t)^\sigma}}}\right) + \left(\sum_{i\geq 0} b_i \frac{t^i}{i!}\right)\left(\frac{e^{\frac{-T^\sigma}{t^\sigma}}}{e^{\frac{-T^\sigma}{t^\sigma}} + e^{\frac{-T^\sigma}{(T-t)^\sigma}}}\right)$$

the series

$$\theta(x,t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!} x^{2i}, \quad u(t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!}.$$

are convergent and provide a trajectory from

$$\theta(x,0) = \sum_{i\geq 0} a_i \frac{x^{2i}}{(2i)!} \ \text{ to } \ \theta(x,T) = \sum_{i\geq 0} b_i \frac{x^{2i}}{(2i)!}$$

[6] B. Laroche, Ph. Martin, P. R.: Motion planning for the heat equation. Int. Journal of Robust and Nonlinear Control. Vol.10, pp:629–643, 2000.

Real-time motion planning for the heat equation

Take $\sigma > 1$ and $\epsilon > 0$. Consider the positive function

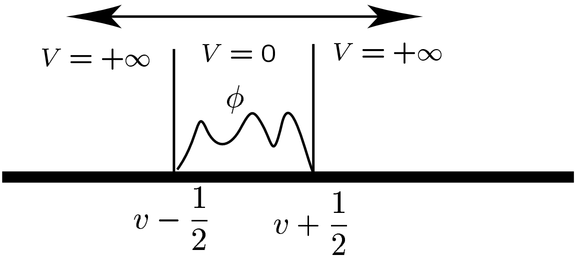
$$\phi_\epsilon(t) = \frac{\exp\left(\frac{-\epsilon^{2\sigma}}{(-t(t+\epsilon))^\sigma}\right)}{A_\epsilon} \quad \text{for} \quad t \in [-\epsilon, 0]$$

prolonged by 0 outside $[-\epsilon, 0]$ and where the normalization constant $A_\epsilon > 0$ is such that $\int \phi_\epsilon = 1$.
For any L^1_{loc} signal $t \mapsto Y(t)$, set $y_r = \phi_\epsilon * Y$: its order $1 + 1/\sigma$ is less than 2. Then $\theta_r = \cosh(x\sqrt{s})y_r$ reads

$$\theta_r(x,t) = \Phi_{x,\epsilon} * Y(t), \quad u_r(t) = \Phi_{1,\epsilon} * Y(t),$$

where for each x, $\Phi_{x,\epsilon} = \cosh(x\sqrt{s})\phi_\epsilon$ is a smooth time function with support contained in $[-\epsilon, 0]$. Since $u_r(t)$ and the profile $\theta_r(\cdot, t)$ depend only on the values of Y on $[t-\epsilon, t]$, such computations are well adapted to real-time generation of reference trajectories $t \mapsto (\theta_r, u_r)$ (see matlab code `heat.m`).

Quantum particle inside a moving box[7]



Schrödinger equation in a Galilean frame:

$$\imath \frac{\partial \phi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \phi}{\partial z^2}, \quad z \in [v - \frac{1}{2}, v + \frac{1}{2}],$$

$$\phi(v - \frac{1}{2}, t) = \phi(v + \frac{1}{2}, t) = 0$$

[7] P.R.: Control of a quantum particle in a moving potential well. IFAC 2nd Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control, 2003. See, for the proof of nonlinear controllability, K. Beauchard and J.-M. Coron: Controllability of a quantum particle in a moving potential well; J. of Functional Analysis, vol.232, pp:328–389, 2006.

Particle in a moving box of position v

- In a Galilean frame

$$\imath \frac{\partial \phi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \phi}{\partial z^2}, \quad z \in [v - \frac{1}{2}, v + \frac{1}{2}],$$
$$\phi(v - \frac{1}{2}, t) = \phi(v + \frac{1}{2}, t) = 0$$

  where v is the position of the box and z is an absolute position.

- In the box frame $x = z - v$:

$$\imath \frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} + \ddot{v} x \psi, \quad x \in [-\frac{1}{2}, \frac{1}{2}],$$
$$\psi(-\frac{1}{2}, t) = \psi(\frac{1}{2}, t) = 0$$

Tangent linearization around state $\bar{\psi}$ of energy $\bar{\omega}$

With[8] $-\frac{1}{2}\frac{\partial^2\bar{\psi}}{\partial x^2} = \bar{\omega}\bar{\psi}$, $\bar{\psi}(-\frac{1}{2}) = \bar{\psi}(\frac{1}{2}) = 0$ and with

$$\psi(x,t) = \exp(-\imath\bar{\omega}t)(\bar{\psi}(x) + \Psi(x,t))$$

Ψ satisfies

$$\imath\frac{\partial\Psi}{\partial t} + \bar{\omega}\Psi = -\frac{1}{2}\frac{\partial^2\Psi}{\partial x^2} + \ddot{v}x(\bar{\psi} + \Psi)$$
$$0 = \Psi(-\frac{1}{2}, t) = \Psi(\frac{1}{2}, t).$$

Assume Ψ and $\ddot{v}$ small and neglecte the second order term $\ddot{v}x\Psi$:

$$\imath\frac{\partial\Psi}{\partial t} + \bar{\omega}\Psi = -\frac{1}{2}\frac{\partial^2\Psi}{\partial x^2} + \ddot{v}x\bar{\psi}, \quad \Psi(-\frac{1}{2}, t) = \Psi(\frac{1}{2}, t) = 0.$$

[8] Remember that $\int_{-1/2}^{1/2}\bar{\psi}^2(x)dx = 1$.

Operational computations $s = d/dt$

The general solution of ($'$ stands for d/dx)

$$(\imath s + \bar{\omega})\Psi = -\frac{1}{2}\Psi'' + s^2 v x \bar{\psi}$$

is

$$\Psi = A(s,x)a(s) + B(s,x)b(s) + C(s,x)v(s)$$

where

$$A(s,x) = \cos\left(x\sqrt{2\imath s + 2\bar{\omega}}\right)$$

$$B(s,x) = \frac{\sin\left(x\sqrt{2\imath s + 2\bar{\omega}}\right)}{\sqrt{2\imath s + 2\bar{\omega}}}$$

$$C(s,x) = (-\imath s x\bar{\psi}(x) + \bar{\psi}'(x)).$$

Case $x \mapsto \bar{\phi}(x)$ even

The boundary conditions imply

$$A(s,1/2)a(s) = 0, \quad B(s,1/2)b(s) = -\psi'(1/2)v(s).$$

$a(s)$ is a torsion element: the system is not controllable. Nevertheless, for steady-state controllability, we have

$$b(s) = -\bar{\psi}'(1/2)\frac{\sin\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right)}{\sqrt{-2\imath s + 2\bar{\omega}}}\ y(s)$$

$$v(s) = \frac{\sin\left(\frac{1}{2}\sqrt{2\imath s + 2\bar{\omega}}\right)}{\sqrt{2\imath s + 2\bar{\omega}}}\frac{\sin\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right)}{\sqrt{-2\imath s + 2\bar{\omega}}} y(s)$$

$$\Psi(s,x) = B(s,x)b(s) + C(s,x)v(s)$$

Series and convergence

$$v(s) = \frac{\sin\left(\frac{1}{2}\sqrt{2\imath s + 2\bar{\omega}}\right)}{\sqrt{2\imath s + 2\bar{\omega}}} \frac{\sin\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right)}{\sqrt{-2\imath s + 2\bar{\omega}}} y(s) = F(s)y(s)$$

where the entire function $s \mapsto F(s)$ is of order $1/2$,

$$\exists K, M > 0, \forall s \in \mathbb{C}, \quad |F(s)| \leq K \exp(M|s|^{1/2}).$$

Set $F(s) = \sum_{n \geq 0} a_n s^n$ where $|a_n| \leq K^n / \Gamma(1 + 2n)$ with $K > 0$ independent of n. Then $F(s)y(s)$ corresponds, in the time domain, to

$$\sum_{n \geq 0} a_n y^{(n)}(t)$$

that is convergent when $t \mapsto y(t)$ is C^∞ of Gevrey-order $\alpha < 2$.

Steady state controllability

Steering from $\Psi = 0$, $v = 0$ at time $t = 0$, to $\Psi = 0$, $v = D$ at $t = T$ is possible with the following C^∞-function of Gevrey-order $\sigma + 1$:

$$[0, T] \ni t \mapsto y(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ \bar{D} \dfrac{\exp\left(-\left(\frac{T}{t}\right)^{\frac{1}{\sigma}}\right)}{\exp\left(-\left(\frac{T}{t}\right)^{\frac{1}{\sigma}}\right) + \exp\left(-\left(\frac{T}{T-t}\right)^{\frac{1}{\sigma}}\right)} & \text{for } 0 < t < T \\ \bar{D} & \text{for } t \geq T \end{cases}$$

with $\bar{D} = \frac{2\bar{\omega} D}{\sin^2(\sqrt{\bar{\omega}}/2)}$. The fact that this C^∞-function is of Gevrey-order $\sigma + 1$ results from its exponential decay of order $1/\sigma$ around 0 and T.

Practical computations via Cauchy formula

Using the "magic" Cauchy formula

$$y^{(n)}(t) = \frac{\Gamma(n+1)}{2\imath\pi} \oint_\gamma \frac{y(t+\xi)}{\xi^{n+1}}\, d\xi$$

where γ is a closed path around zero, $\sum_{n\geq 0} a_n y^{(n)}(t)$ becomes

$$\sum_{n\geq 0} a_n \frac{\Gamma(n+1)}{2\imath\pi} \oint_\gamma \frac{y(t+\xi)}{\xi^{n+1}}\, d\xi = \frac{1}{2\imath\pi} \oint_\gamma \left(\sum_{n\geq 0} a_n \frac{\Gamma(n+1)}{\xi^{n+1}} \right) y(t+\xi)\, d\xi.$$

But

$$\sum_{n\geq 0} a_n \frac{\Gamma(n+1)}{\xi^{n+1}} = \int_{D_\delta} F(s) \exp(-s\xi) ds = B_1(F)(\xi)$$

is the Borel/Laplace transform of F in direction $\delta \in [0, 2\pi]$.

Practical computations via Cauchy formula (end)

(matlab code `Qbox.m`)

In the time domain $F(s)y(s)$ corresponds to

$$\frac{1}{2\imath\pi}\oint_{\gamma} B_1(F)(\xi) y(t+\xi)\, d\xi$$

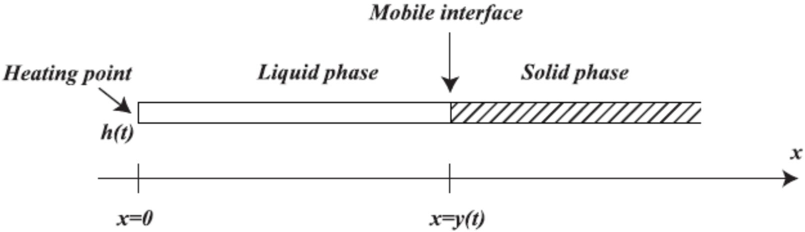
where γ is a closed path around zero. Such integral representation is very useful when y is defined by convolution with a real signal Y,

$$y(\zeta) = \frac{1}{\varepsilon\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-(\zeta - t)^2/2\varepsilon^2) Y(t)\, dt$$

where $\mathbb{R} \ni t \mapsto Y(t) \in \mathbb{R}$ is any measurable and bounded function. Approximate motion planning with:

$$v(t) = \int_{-\infty}^{+\infty} \left[\frac{1}{\imath\varepsilon(2\pi)^{\frac{3}{2}}} \oint_{\gamma} B_1(F)(\xi) \exp(-(\xi-\tau)^2/2\varepsilon^2)\, d\xi \right] Y(t-\tau)\, d\tau.$$

A free-boundary Stefan problem[9]



$$\frac{\partial\theta}{\partial t}(x,t) = \frac{\partial^2\theta}{\partial x^2}(x,t) - \nu\frac{\partial\theta}{\partial x}(x,t) - \rho\theta^2(x,t), \quad x \in [0, y(t)]$$
$$\theta(0,t) = u(t), \quad \theta(y(t),t) = 0$$
$$\frac{\partial\theta}{\partial x}(y(t),t) = -\frac{d}{dt}y(t)$$

with $\nu, \rho \geq 0$ parameters.

[9] W. Dunbar, N. Petit, P. R., Ph. Martin. Motion planning for a non-linear Stefan equation. ESAIM: Control, Optimisation and Calculus of Variations, 9:275–296, 2003.

Series solutions

- Set $\theta(x,t) = \sum_{i=0}^{\infty} a_i(t) \frac{(x-y(t))^i}{i!}$ in

$$\frac{\partial\theta}{\partial t}(x,t) = \frac{\partial^2\theta}{\partial x^2}(x,t) - \nu\frac{\partial\theta}{\partial x}(x,t) - \rho\theta^2(x,t), \quad x \in [0, y(t)]$$

$$\theta(0,t) = u(t), \quad \theta(y(t),t) = 0, \quad \frac{\partial\theta}{\partial x}(y(t),t) = -\frac{d}{dt}y(t)$$

  Then $\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2}$ yields

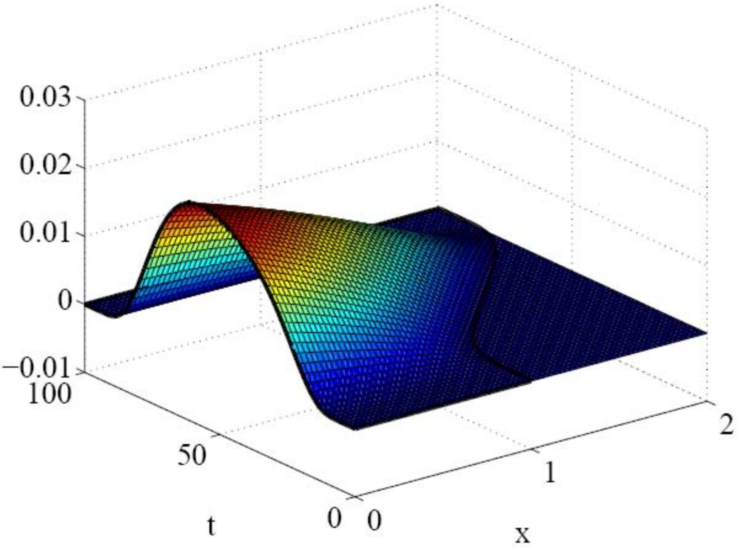
$$a_{i+2} = \frac{d}{dt}a_i - a_{i-1}\frac{d}{dt}y + \nu a_{i+1} + \rho\sum_{k=0}^{i}\left(\begin{array}{c} i \\ k \end{array}\right) a_{i-k}a_k$$

  and the boundary conditions: $a_0 = 0$ and $a_1 = -\frac{d}{dt}y$.

- The series defining θ admits a strictly positive radius of convergence as soon as y is of Gevrey-order α strictly less than 2.

Growth of the liquide zone with $\theta \geq 0$

$\nu = 0.5$, $\rho = 1.5$, y goes from 1 to 2.



Conclusion

- For other 1D PDE of engineering interest where motion planning can be obtained via Gevrey functions, see the book of J. Rudolph: Flatness Based Control of Distributed Parameter Systems (Shaker-Germany, 2003)
- For feedback design on linear 1D parabolic equations, see the book of M. Krstić and A. Smyshlyaev : Boundary Control of PDEs: a Course on Backstepping Designs (SIAM, 2008).
- Open questions:
  - Combine divergent series and smallest-term summation (see the PhD of Th. Meurer: Feedforward and Feedback Tracking Control of Diffusion-Convection-Reaction Systems using Summability Methods (Stuttgart, 2005)).
  - 2D heat equation with a scalar control $u(t)$: with modal decomposition and symbolic computations, we get $u(s) = P(s)y(s)$ with $P(s)$ an entire function (coding the spectrum) of order 1 but infinite type $|P(s)| \leq M \exp(K|s|\log(|s|))$. It yields divergence series for any C^∞ function $y \neq 0$ with compact support.

$u(s) = P(s)y(s)$ for 1D and 2D heat equations

- 1D heat equation: eigenvalue asymptotics $\lambda_n \sim -n^2$:

$$\text{Prototype:} \quad P(s) = \prod_{n=1}^{+\infty}\left(1 - \frac{s}{n^2}\right) = \frac{\sinh(\pi\sqrt{s})}{\pi\sqrt{s}}$$

  entire function of order $1/2$.

- 2D heat equation in a domain Ω with a single scalar control $u(t)$ on the boundary $\partial\Omega_1$ ($\partial\Omega = \partial\Omega_1 \bigcup \partial\Omega_2$):

$$\frac{\partial\theta}{\partial t} = \Delta\theta \text{ on } \Omega, \quad \theta = u(t) \text{ on } \partial\Omega_1, \quad \frac{\partial\theta}{\partial n} = 0 \text{ on } \partial\Omega_2$$

  Eigenvalue asymptotics $\lambda_n \sim -n$

$$\text{Prototype:} \quad P(s) = \prod_{n=1}^{+\infty}\left(1 + \frac{s}{n}\right)\exp(-s/n) = \frac{\exp(-\gamma s)}{s\Gamma(s)}$$

  entire function of order 1 but of infinite type[10]

[10] For the links between the distributions of the zeros and the order at infinity of entire functions see the book of B.Ja Levin: Distribution of Zeros of Entire Functions; AMS, 1972.