On the Controllability of **spin**-**spring** quantum systems

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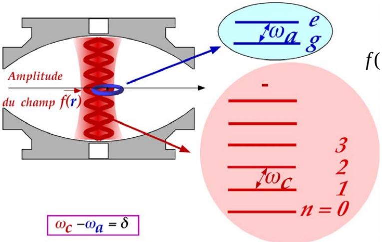
Outline

- Non controllability of the quantum harmonic oscillator (**spring**). (Classical sources generate only quasi-classical light)
- Two-states atom (**spin**) coupled with a resonant electro-magnetic cavity mode: the controlled Jaynes-Cummings Hamiltonian.
- Similar systems: several trapped ions controlled via lasers; qubit and quantum gate.
- Quantum Monte-Carlo trajectories, stochastic control and **feedback**.

Le système atome-cavité

Spin 1/2* couplé à un *oscillateur harmonique



$$f(\vec{r}) \approx e^{-\frac{x^2+y^2}{w^2}} \cos(\frac{2\pi z}{\lambda})$$

$$\omega_c - \omega_a = \delta$$

Hamiltonien: $H = H_a + H_c + V_{a\text{-}c}$

$$H_a = \frac{\hbar\omega_a}{2}\left(|e\rangle\langle e| - |g\rangle\langle g|\right)$$

$$H_c = \frac{\hbar\omega_c}{2}\left(a^\dagger a + a a^\dagger\right) = \hbar\omega_c(a^\dagger a + 1/2)$$

$$V_{a\text{-}c} = -\vec{D}_a \cdot \vec{E}(r)$$

Dipôle électrique

1000 a.u.

$$\vec{D}_a = \vec{d}_{eg}|e\rangle\langle g| + \vec{d}^*_{eg}|g\rangle\langle e|$$

$$\vec{E}(r) = iE_0\left(\vec{f}(\vec{r})\, a - \vec{f}^*(\vec{r})\, a^\dagger\right)$$

$$E_0 = \sqrt{\frac{\hbar\omega_c}{2\varepsilon_0 V_{cav}}}$$ *Champ du vide*

$$V_{cav} = \int |\vec{f}(\vec{r})|^2 d^3r$$ *Volume du mode*

Source: S. Haroche, cours au collège de France.

Rappels sur l 'oscillateur harmonique

$$H = p^2/2m + m\omega^2 x^2/2$$

Opérateurs d 'annihilation et de création de quanta

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + \frac{i}{\sqrt{2m\hbar\omega}}p \qquad \left[a, a^+\right] = 1 \qquad H = \hbar\omega\left(a^+ a + \frac{1}{2}\right)$$

$$a^+ = \sqrt{\frac{m\omega}{2\hbar}}x - \frac{i}{\sqrt{2m\hbar\omega}}p$$

Etat fondamental | 0 > de l 'oscillateur (énergie ħ ω/2)

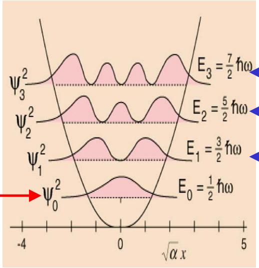
$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

Paquet d 'onde gaussien « minimal »

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}}; \Delta p = \sqrt{\frac{\hbar m\omega}{2}} \rightarrow \Delta x \Delta p = \frac{\hbar}{2}$$

Etat excité à n quanta (énergie : (n+1/2) ħ ω

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle$$



S. Haroche, cours au collège de France.

Quantification of the controlled harmonic oscillator

Classical dynamics: $\frac{d^2}{dt^2}x = -x + u$

$$\frac{d}{dt}x = p = \frac{\partial H}{\partial p}, \quad \frac{d}{dt}p = -x + u = -\frac{\partial H}{\partial x}$$

with $H(x,p,t) = \frac{1}{2}(p^2 + x^2) - u(t)x$.
Quantification: $x \mapsto X$, $p \mapsto P = -\imath\frac{\partial}{\partial x}$, the Hamiltonian becomes an operator

$$H = \frac{1}{2}(P^2 + X^2) - u(t)X = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}x^2 - u(t)x$$

($\hbar = 1$ here) and the Shrödinger equation

$$\imath\frac{d}{dt}\psi = H\psi.$$

describes the time evolution of ψ, the probability amplitude.

Quantification of ... (end)

$$\imath \frac{d}{dt}\psi = H\psi$$

Probability amplitude $\psi(t,x) \in \mathbb{C}$, $\int_{-\infty}^{+\infty} |\psi(t,x)|^2 dx = 1$ obeys the Shrödinger equation:

$$\imath \frac{\partial \psi}{\partial t}(x,t) = H\psi = -\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2}(x,t) + x^2\psi(x,t) - \mathbf{u}(\mathbf{t})x\psi(x,t), \quad x \in \mathbb{R}$$

The averaged position:

$$\bar{X}(t) = \langle \psi | X | \psi \rangle = \int_{-\infty}^{+\infty} x|\psi|^2 dx,$$

The averaged impulsion:

$$\bar{P}(t) = \langle \psi | P | \psi \rangle = -\imath \int_{-\infty}^{+\infty} \psi^* \frac{\partial \psi}{\partial x} dx$$

The language of operators

$$\imath \frac{\partial \psi}{\partial t} = (\frac{1}{2}(P^2 + X^2) - uX)\psi = -\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} + x^2\psi - ux\psi$$

With the **annihilation** and **creation** operators

$$\mathbf{a} = \frac{\mathbf{X} + \imath \mathbf{P}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\mathbf{x} + \frac{\partial}{\partial \mathbf{x}}\right), \quad \mathbf{a}^\dagger = \frac{\mathbf{X} - \imath \mathbf{P}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\mathbf{x} - \frac{\partial}{\partial \mathbf{x}}\right)$$

we get $(u/\sqrt{2} \mapsto u)$

$$[\mathbf{a}, \mathbf{a}^\dagger] = 1, \quad H = \mathbf{a}^\dagger \mathbf{a} + \frac{1}{2} - u(\mathbf{a} + \mathbf{a}^\dagger).$$

The spectral decomposition of $\mathbf{a}^\dagger\mathbf{a}$

The Hermitian operator $\mathbf{a}^\dagger\mathbf{a}$ admits $\mathbb{N}$ as non-degenerate spectrum. The eigen-vector associated to $n \in \mathbb{N}$ is denoted $|\mathbf{n}\rangle$ (Fock state, n being the number of vibration quanta):

$$\mathbf{a}|\mathbf{n}\rangle = \sqrt{n}\,|\mathbf{n-1}\rangle, \quad \mathbf{a}^\dagger|\mathbf{n}\rangle = \sqrt{n+1}\,|\mathbf{n+1}\rangle$$

and $|\mathbf{0}\rangle$ ($\mathbf{a}|\mathbf{0}\rangle = 0$) corresponds to a Gaussian distribution:

$$\psi_0(x) = \frac{1}{\pi^{1/4}} \exp(-x^2/2)$$

Remember that

$$\mathbf{a} = \frac{X + \imath P}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right), \quad \mathbf{a}^\dagger = \frac{X - \imath P}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)$$

Modal approximation is controllable (Schirmer et al (2001))

$$\imath \frac{d}{dt}\psi = (a^\dagger a + 1/2) - u(a + a^\dagger)\psi$$

Truncation of $\psi = \sum_{n=0}^{+\infty} c_n(t)\,|n\rangle$ to order N leads to

$$\imath \frac{d}{dt}\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \\ c_N \end{pmatrix} =$$

$$\left[\begin{pmatrix} \frac{1}{2} & 0 & & \dots & 0 \\ 0 & \frac{3}{2} & 0 & & \\ & 0 & \frac{5}{2} & 0 & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & \dots & 0 & \frac{2N+1}{2} \end{pmatrix} - u \begin{pmatrix} 0 & 1 & 0 & \dots & & 0 \\ 1 & 0 & \sqrt{2} & \ddots & & \vdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & & \vdots \\ \vdots & & \sqrt{3} & & \ddots & 0 \\ \vdots & & \ddots & \ddots & & \sqrt{N+1} \\ 0 & \dots & & 0 & \sqrt{N+1} & 0 \end{pmatrix}\right] \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \\ c_N \end{pmatrix}$$

Infinite dimensional system not controllable (MR IEEE-AC 2004)

$$\imath \frac{d}{dt}\psi = (\mathbf{a}^\dagger \mathbf{a} + \mathbf{1/2}) - u(\mathbf{a} + \mathbf{a}^\dagger)\psi$$

This system reads formally $\imath \frac{d}{dt}\psi = (\mathbf{H_0} + u\mathbf{H_1})\psi$. Its controllability is given formally by the Lie algebra spanned by the skew Hermitian operators $\imath H_0$ and $\imath H_1$. Using $[a, a^\dagger] = 1$,

$$[\mathbf{a}^\dagger \mathbf{a}, \mathbf{a} + \mathbf{a}^\dagger] = a^\dagger - a, \quad [\mathbf{a}^\dagger \mathbf{a}, a^\dagger - a] = \mathbf{a} + \mathbf{a}^\dagger, \quad [\mathbf{a}^\dagger + \mathbf{a}, a^\dagger - a] = 2.$$

we get a Lie algebra of dimension 4 containing

$$\imath \mathbf{a}^\dagger \mathbf{a}, \quad \imath(\mathbf{a} + \mathbf{a}^\dagger), \quad a - a^\dagger, \quad \imath I_d$$

Controllable and un-controllable part

$$\imath \frac{d}{dt}\psi = (a^\dagger a + 1/2) - u(a + a^\dagger)\psi$$

Set $\alpha = \langle\psi|a|\psi\rangle \in \mathbb{C}$ then

$$\imath \frac{d}{dt}\alpha = \langle\psi|[a, H]|\psi\rangle = \alpha - u$$

and consider the unitary operator

$$T(t) = \exp\left[\alpha^*(t)\mathbf{a} - \alpha(t)\mathbf{a}^\dagger\right].$$

If A and B commute with $[A, B]$ (Glauber):
$\exp(A + B) = \exp(A)\exp(B)\exp(-[A, B]/2)$, we have

$$T(t)\ \mathbf{a}\ T^\dagger(t) = \mathbf{a} + \alpha(t), \quad T(t)\ \mathbf{a}^\dagger\ T^\dagger(t) = \mathbf{a}^\dagger + \alpha^*(t)$$

Take $\phi = T(t)\ \psi$, then

$$\imath \frac{d}{dt}\phi = \left[a^\dagger a + \frac{1}{2}\right]\phi + \left[|\alpha|^2 - u(\alpha + \alpha^*)\right]\phi$$

Decomposition .... (end)

$$\imath \frac{d}{dt}\phi = \left[a^\dagger a + \frac{1}{2}\right]\phi + \left[|\alpha|^2 - u(\alpha + \alpha^*)\right]\phi$$

With a global phase change $\chi = e^{\left(\imath \int_0^t [|\alpha|^2 - u(\alpha+\alpha^*)]\right)}\phi$ we obtain the following control free Schrödinger equation:

$$\imath \frac{d}{dt}\chi = \left[a^\dagger a + \frac{1}{2}\right]\chi.$$

Two parts, a controllable one $\alpha \in \mathbb{C}$, $\imath \frac{d}{dt}\alpha = \alpha - u$, an uncontrollable one χ, the quantum fluctuations around α. These computations might be extended to an arbitrary number of harmonic oscillators admitting the same control u but with different frequencies.

Classical EM fields generated by classical sources

Bounded smooth domain $\Omega \subset \mathbb{R}^3$, scalar control u associated to time-varying currents inside Ω. The Maxwell equation reads (perfect mirrors on the boundary, $c = 1$):

$$\frac{\partial^2 \vec{E}}{\partial t^2}(r,t) = \Delta \vec{E}(r,t) + u(t)\vec{J}(r) \text{ for } r \in \Omega, \quad \vec{E}(r,t) = 0 \text{ for } r \in \partial\Omega.$$

Modal decomposition $\vec{E}(r,t) = \sum_{j=0}^{+\infty} x_j(t)\vec{E}_j(r)$ leads to an infinite collection of controlled harmonic oscillators:

$$\frac{d^2}{dt^2} x_j = -\omega_j^2 x_j + b_j u, \quad b_j = \int_\Omega \vec{E}_j(r) \cdot \vec{J}(r)\, dr.$$

Quantum EM fields generated by classical sources

$$\frac{d^2}{dt^2}x_j = -\omega_j^2 x_j + b_j u$$

Quantification based on $a_j = \frac{\sqrt{\omega_j}X_j + \frac{\imath}{\sqrt{\omega_j}}P_j}{\sqrt{2}}$ we get the following Hamiltonian ($\hbar = 1$):

$$H = \sum_{j=0}^{\infty}[\omega_j(a_j^\dagger a_j + \frac{1}{2}) - b_j u(a_j + a_j^\dagger)].$$

an operator on the "state space" $\bigotimes_j L^2(\mathbb{R}, \mathbb{C})$.

Controllability decomposition

$$H = \sum_{j=0}^{\infty} [\omega_j (a_j^\dagger a_j + 1/2) - b_j u [a_j + a_j^\dagger].$$

Set $\alpha_j = \langle \psi | a_j | \psi \rangle$ then

$$\imath \frac{d}{dt} \alpha_j = \langle \psi | [a_j, H] | \psi \rangle = \omega_j \alpha_j - b_j u$$

Take $T = \bigotimes_j T_j$ with $T_j = \exp \left[\alpha_j^* a_n - \alpha_j a_j^\dagger \right]$.
We have $T a_j T^\dagger = a_j + \alpha_j$. Taking

$$\phi = e^{\imath \int_0^t \left[\sum_j \omega_j |\alpha_j|^2 - b_j u (\alpha_j + \alpha_j^*) \right]} T(t) \psi$$

we get:

$$\imath \frac{d}{dt} \phi = \left[\sum_j \omega_j \left(a_j^\dagger a_j + \frac{1}{2} \right) \right] \phi.$$

Controllability decomposition (end)

The electro-magnetic field operator

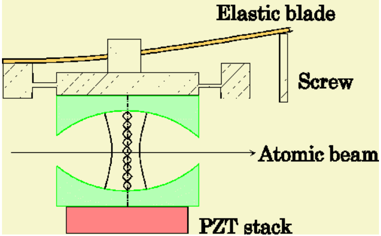
$$\vec{\mathcal{E}}(r,t) = \sum_j \frac{\mathbf{a_j} + \mathbf{a_j}^\dagger}{\sqrt{\mathbf{2}\omega_\mathbf{j}}} \overrightarrow{E}_\mathbf{j}(\mathbf{r}) + \overrightarrow{E}(\mathbf{r},\mathbf{t})$$

is the sum of two terms: the vacuum fluctuation operators a_j that obey the autonomous dynamics $\imath \frac{d}{dt} a_j = \omega_j a_j$; the classical field (scalar operators) $\vec{E}(r,t) = \sum_j \alpha_j(t) \overrightarrow{E}_\mathbf{j}(\mathbf{r})$ solution of the classical wave equation

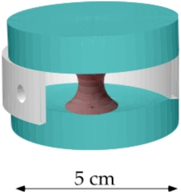
$$\frac{\partial^2 \vec{E}}{\partial t^2}(r,t) = \Delta \vec{E}(r,t) + u(t)\vec{J}(r) \textbf{ for } r \in \Omega, \quad \vec{E}(r,t) = 0 \textbf{ for } r \in \partial\Omega.$$

Control interpretation of a classical result (MR CDC/ECC 05): classical motions of charges generate only quasi-classical (coherent) light (see Glauber, CDG2, ...).

La Cavité Fabry-Perot supraconductrice



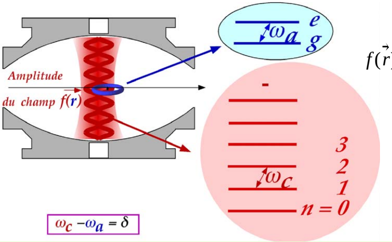
Mode gaussien avec un diamètre de 6mm
Grand champ par photon (1,5mV/m)
Grande durée de vie du champ (1ms) allongée par l 'anneau autour des miroirs
Accord en fréquence facile
Faible champ thermique (< 0,1 photon)



Source: S. Haroche, cours au collège de France.

Le système atome-cavité

Spin 1/2* couplé à un *oscillateur harmonique



$$f(\vec{r}) \approx e^{-\frac{x^2+y^2}{w^2}} \cos\left(\frac{2\pi z}{\lambda}\right)$$

$$\omega_c - \omega_a = \delta$$

Hamiltonien: $H = H_a + H_c + V_{a\text{-}c}$

$$H_a = \frac{\hbar\omega_a}{2}\left(|e\rangle\langle e| - |g\rangle\langle g|\right)$$

$$H_c = \frac{\hbar\omega_c}{2}\left(a^\dagger a + a a^\dagger\right) = \hbar\omega_c\left(a^\dagger a + 1/2\right)$$

$$V_{a\text{-}c} = -\vec{D}_a \cdot \vec{E}(r)$$

Dipôle électrique

1000 a.u.

$$\vec{D}_a = \vec{d}_{eg}|e\rangle\langle g| + \vec{d}^*_{eg}|g\rangle\langle e|$$

$$\vec{E}(r) = iE_0\left(\vec{f}(\vec{r})\, a - \vec{f}^*(\vec{r})\, a^\dagger\right)$$

$$E_0 = \sqrt{\frac{\hbar\omega_c}{2\varepsilon_0 V_{cav}}}$$ *Champ du vide*

$$V_{cav} = \int |\vec{f}(\vec{r})|^2 d^3r$$ *Volume du mode*

Source: S. Haroche, cours au collège de France.

Two-states quantum systems ($\frac{1}{2}$-spin)

Ground state $|g\rangle$ and excited state $|e\rangle$: $\psi = (\psi_g, \psi_e) \in \mathbb{C}^2$ à linear superposition of $|g\rangle$ and $|e\rangle$ (ω_0 the Bohr frequency of the transition).

$$\imath \frac{d}{dt}\begin{pmatrix}\psi_e \\ \psi_g\end{pmatrix} = \frac{\omega_0}{2}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_e \\ \psi_g\end{pmatrix} = H_0\psi$$

with $H_0 = \frac{\omega_0}{2}(|e\rangle\langle e| - |g\rangle\langle g|)$.
Interaction with classical electrical field $u(t) \in \mathbb{R}$ ($\mu \in \mathbb{R}$ dipole coefficient ....):

$$\imath \frac{d}{dt}\begin{pmatrix}\psi_e \\ \psi_g\end{pmatrix} = \left[\frac{\omega_0}{2}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} + \mu u \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\right]\begin{pmatrix}\psi_e \\ \psi_g\end{pmatrix} = (H_0 + u(t)H_1)\psi$$

with $H_1 = \mu(|e\rangle\langle g| + |g\rangle\langle e|)$.

Two-states system coupled to a resonant cavity mode

State space: $L^2(\mathbb{R}, \mathbb{C}) \bigotimes \mathbb{C}^2$.
Cavity Hamiltonian (an isolated resonant mode $\omega \approx \omega_0$ driven by a classical control u):

$$\mathbf{H_c} = \omega a^\dagger a - u(a + a^\dagger)$$

Two-states Hamiltonian

$$\mathbf{H_a} = \frac{\omega_0}{2}(|e\rangle\langle e| - |g\rangle\langle g|) = \frac{\omega_0}{2}\sigma_z$$

Interaction Hamiltonian (Ω Rabi vacuum frequency $\Omega \ll \omega, \omega_0$)

$$\mathbf{H_{int}} = \frac{\Omega}{2}(a + a^\dagger)(|e\rangle\langle g| + |g\rangle\langle e|) = \frac{\Omega}{2}(a + a^\dagger)\sigma_x$$

System Hamiltonian

$$\mathbf{H_c} + \mathbf{H_a} + \mathbf{H_{int}}$$

The rotating wave approximation ($\omega = \omega_0$)

$$\imath \frac{d}{dt}\psi = (H_c + H_a + H_{int})\psi$$

Set $\psi = \exp\left(-\imath\omega t a^\dagger a\right) \exp\left(-\imath\omega t \sigma_z\right) \phi$ (interaction frame). The Hamiltionian becomes

$$-u(e^{-\imath\omega t} a + e^{\imath\omega t} a^\dagger) + \frac{\Omega}{2}(e^{-\imath\omega t} a + e^{\imath\omega t} a^\dagger)(e^{-\imath\omega t} |g\rangle\langle e| + e^{\imath\omega t} |e\rangle\langle g|)$$

because

$$\exp\left(\imath\omega t a^\dagger a\right) a \exp\left(-\imath\omega t a^\dagger a\right) = e^{-\imath\omega t} a$$

$$\exp\left(\imath\omega t \sigma_z\right) \sigma_x \exp\left(-\imath\omega t \sigma_z\right) = e^{-\imath\omega t} |g\rangle\langle e| + e^{\imath\omega t} |e\rangle\langle g|$$

The rotating wave approximation (end)

$$H = -\mathbf{u}(e^{-\imath\omega t}a + e^{\imath\omega t}a^\dagger) + \frac{\Omega}{2}(e^{-\imath\omega t}a + e^{\imath\omega t}a^\dagger)(e^{-\imath\omega t}|g\rangle\langle e| + e^{\imath\omega t}|e\rangle\langle g|)$$

Set $\mathbf{u} = \mathbf{v}e^{-\imath\omega t} + \mathbf{v}^* e^{\imath\omega t}$ with $\mathbf{v}$ a slowly varying complex amplitude (new control $\mathbf{v} \in \mathbb{C}$) we get neglecting oscillating terms $e^{\pm 2\imath\omega t}$, the controlled Jaynes-Cummings Hamiltonian (in the interaction representation)

$$H_{JC} = \frac{\Omega}{2}(\mathbf{a}|\mathbf{e}\rangle\langle\mathbf{g}| + \mathbf{a}^\dagger|\mathbf{g}\rangle\langle\mathbf{e}|) - (\mathbf{v}\mathbf{a}^\dagger + \mathbf{v}^*\mathbf{a})$$

PDE behind the Jaynes-Cummings controlled model

$$\imath \frac{d}{dt}\psi = \frac{\Omega}{2}(a\,|e\rangle\,\langle g| + a^\dagger\,|g\rangle\,\langle e|)\psi - (va^\dagger + v^* a)\psi$$

Since $\psi \in L^2(\mathbb{R},\mathbb{C}) \bigotimes \mathbb{C}^2$ and $L^2(\mathbb{R},\mathbb{C}) \bigotimes \mathbb{C}^2 \sim (L^2(\mathbb{R},\mathbb{C}))^2$ we represent ψ by two components ψ_g and ψ_e elements of $L^2(\mathbb{R},\mathbb{C})$. Up-to some scaling:

$$\imath \frac{\partial \psi_{\mathbf{g}}}{\partial t} = \left(\mathbf{v_1} x + \imath \mathbf{v_2} \frac{\partial}{\partial x}\right)\psi_{\mathbf{g}} + \left(x + \frac{\partial}{\partial x}\right)\psi_{\mathbf{e}}$$
$$\imath \frac{\partial \psi_{\mathbf{e}}}{\partial t} = \left(x - \frac{\partial}{\partial x}\right)\psi_{\mathbf{g}} + \left(\mathbf{v_1} x + \imath \mathbf{v_2} \frac{\partial}{\partial x}\right)\psi_{\mathbf{e}}$$

where $\mathbf{v} = \mathbf{v_1} + \imath \mathbf{v_2} \in \mathbb{C}$ is the control. Remember that

$$a = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)$$

Another form of the control Jaynes-Cummings Hamiltonian

$$H_{JC} = \frac{\Omega}{2}(a\,|e\rangle\,\langle g| + a^\dagger\,|g\rangle\,\langle e|) - (\mathbf{v}a^\dagger + \mathbf{v}^* a)$$

Set $\mathbf{w} \in \mathbb{C}$ such that $\imath \frac{d}{dt}\mathbf{w} = -\mathbf{v}$ and take the unitary transformation $T(t) = \exp\left[\mathbf{w}^* a - \mathbf{w} a^\dagger\right]$. Then H_{JC} becomes $T H_{JC} T^\dagger - \imath T \dot{T}^\dagger$ and up-to a global phase change we get

$$\tilde{H}_{JC} = \frac{\Omega}{2}(a + \mathbf{w})\,|e\rangle\,\langle g| + (a^\dagger + \mathbf{w}^*)\,|g\rangle\,\langle e|)$$

where $\mathbf{w} \in \mathbb{C}$ is the new control corresponding to the integral of the physical control $\mathbf{v}$.

Another PDE formulation

$$\imath \frac{d}{dt}\psi = \frac{\Omega}{2}[(a + w)\left|e\right\rangle\left\langle g\right| + (a^\dagger + w^*)\left|g\right\rangle\left\langle e\right|)]\psi$$

Up-to some scaling, with $\psi = (\psi_{\mathbf{g}}, \psi_{\mathbf{e}})$:

$$\imath \frac{\partial \psi_{\mathbf{g}}}{\partial t} = \left(x + w_1 + \frac{\partial}{\partial x} + \imath w_2\right)\psi_{\mathbf{e}}$$

$$\imath \frac{\partial \psi_{\mathbf{e}}}{\partial t} = \left(x + w_1 - \frac{\partial}{\partial x} - \imath w_2\right)\psi_{\mathbf{g}}$$

where $w = w_1 + \imath w_2 \in \mathbb{C}$ is the control (time integral of the physical control v the amplitude and phase modulations).

Elementary facts relative to controllability

The Jaynes-Cummings systems is equivalent to

$$\imath \frac{d}{dt}\psi = (H_0 + w_1 H_1 + w_2 H_2)\psi$$

with

$$H_0 = X\sigma_x - P\sigma_y, \quad H_1 = \sigma_x, \quad H_2 = \sigma_y$$

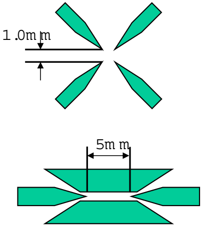
and the commutations are

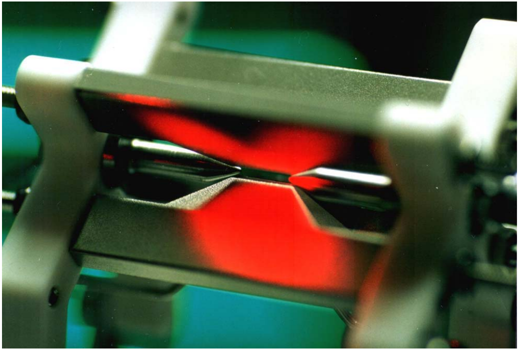
$$[X, P] = \imath, \quad \sigma_x^2 = 1, \quad \sigma_x\sigma_y = \imath\sigma_z \ldots$$

The Lie algebra spanned by $\imath H_0$, $\imath H_1$ and $\imath H_2$ is infinite dimensional now.

But one can prove that the **linear tangent approximation** around any eigen-state of $H_0 + w_1 H_1 + w_2 H_2$ for any control value w_1 and w_2 is **not controllable**.

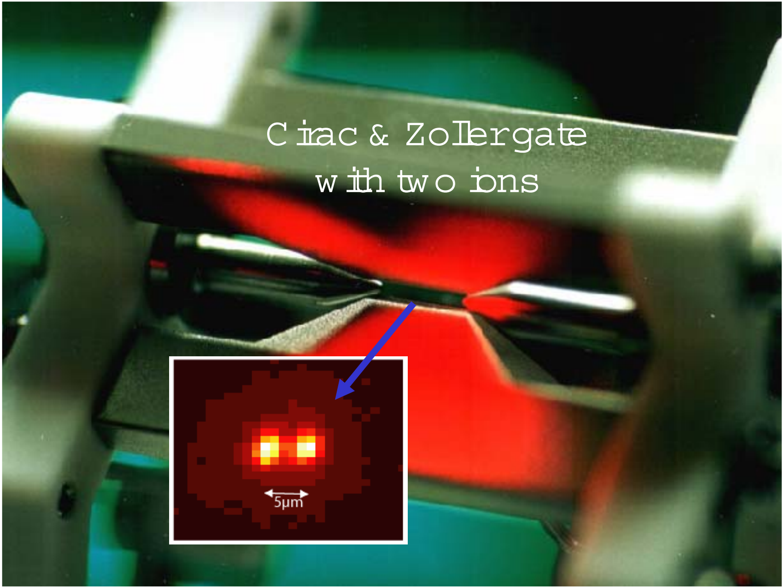
Innsbruck linear ion trap





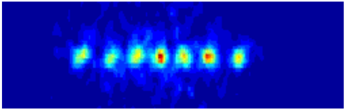
$\omega_{axial} \approx 0.7-2\ \mathrm{MHz}$ $\omega_{radial} \approx 5\ \mathrm{MHz}$

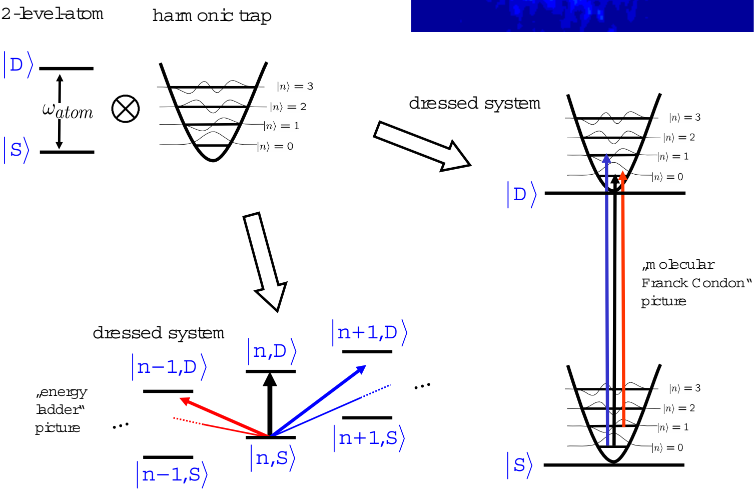
F. Schmidt-Kaler, séminaire au Collège de France en 2004.



F. Schmidt-Kaler, séminaire au Collège de France en 2004.

Laser coupling





F. Schmidt-Kaler, séminaire au Collège de France en 2004.

A single trapped ion controlled via a laser

$$H = \Omega(\mathbf{a}^\dagger \mathbf{a} + 1/2) + \frac{\omega_0}{2}(|e\rangle\langle e| - |g\rangle\langle g|)$$
$$+ \left[\mathbf{u} e^{\imath(\omega t - \mathbf{kX})} + \mathbf{u}^* e^{-\imath(\omega t - \mathbf{kX})}\right](|e\rangle\langle g| + |g\rangle\langle e|)$$

with $kX = \eta(a + a^\dagger)$, $\eta \ll 1$ Lamb-Dicke parameter, $\omega \approx \omega_0$ and the vibration frequency $\Omega \ll \omega$, $u \in \mathbb{C}$ **the control** (amplitude and phase modulations of the laser of frequency ω).
Assume $\omega = \omega_0$. In $\imath \frac{d}{dt}\psi = H\psi$, set $\psi = \exp(-\imath\omega t \sigma_z/2)\phi$. Then the Hamiltonian becomes

$$\Omega(a^\dagger a + 1/2) +$$
$$\left[u e^{\imath(\omega t - \eta(a + a^\dagger))} + u^* e^{-\imath(\omega t - \eta(a + a^\dagger))}\right]\left(e^{-\imath\omega t}|g\rangle\langle e| + e^{\imath\omega t}|e\rangle\langle g|\right)$$

Rotating Wave Approximation and PDE formulation

We neglect highly oscillating terms with $e^{\pm 2\imath\omega t}$ and obtain the averaged Hamiltonian:

$$\tilde{H} = \Omega(a^\dagger a + 1/2) + u e^{-\imath\eta(a+a^\dagger)} |g\rangle \langle e| + u^* e^{\imath\eta(a+a^\dagger)} |e\rangle \langle g|$$

This corresponds to ($\eta \mapsto \eta\sqrt{2}$):

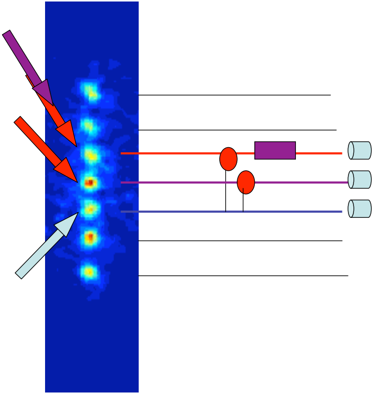
$$\imath \frac{\partial \psi_{\mathbf{g}}}{\partial t} = \frac{\Omega}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_{\mathbf{g}} + u e^{-\imath\eta x}\psi_{\mathbf{e}}$$

$$\imath \frac{\partial \psi_{\mathbf{e}}}{\partial t} = u^* e^{\imath\eta x}\psi_{\mathbf{g}} + \frac{\Omega}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_{\mathbf{e}}$$

where $u \in \mathbb{C}$ is the control $\left|\frac{d}{dt}u\right| \ll \omega|u|$ and $\eta \ll 1$, $\Omega \ll \omega$.

Controllability of this system ?

Logique quantique avec une chaîne d'ions

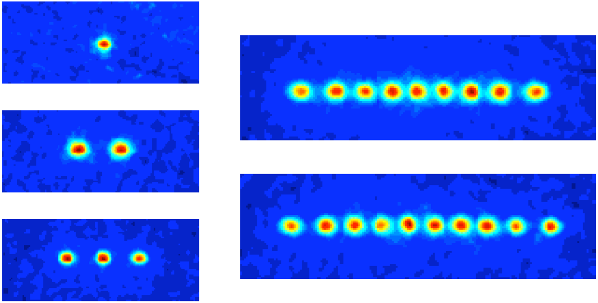


Des impulsions laser appliqués séquentiellement aux ions de la chaîne réalisent des portes à un bit et des portes à deux bits. La détection par fluorescence (éventuellement précédée par une rotation du bit) extrait l'information du système.

Beaucoup de problèmes à résoudre pour réaliser un tel dispositif…..

Source: S. Haroche, CdF 2006.

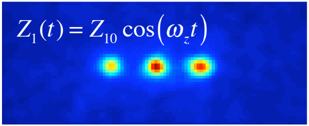
Quelques chaînes d‘ions (Innsbruck)



Fluorescence spatialement résolue. Détection en quelques millisecondes (voir première leçon)

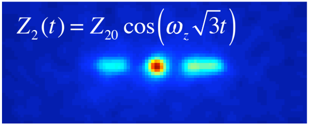
Source: S. Haroche, CdF 2006.

Visualisation des modes (N=3)



Mode du CM

ω_z



Mode «accordéon»

$\omega_z\sqrt{3}$

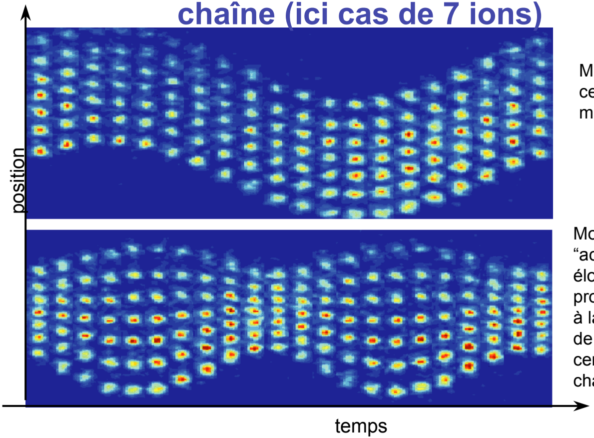
Les deux premiers modes (centre de masse et accordéon) sont pour tout N ceux de fréquences les plus basses. Leurs fréquences sont indépendantes de N. Ce n'est plus vrai pour les modes plus élevés.

$$Z_3(t) = Z_{30}\cos\left(\omega_z\sqrt{29/5}\ t\right)$$

Mode « ciseau »

Source: S. Haroche, CdF 2006.

Les deux premiers modes de vibration sont indépendants du nombre d'ions dans la chaîne (ici cas de 7 ions)



Mode du centre de masse

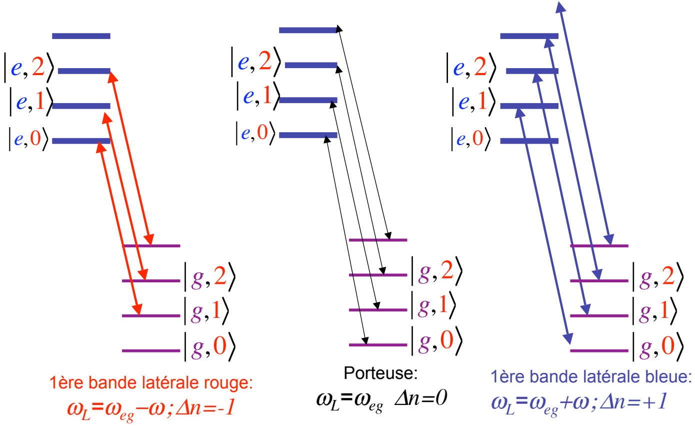
ω_z

Mode “accodéon“: élongation proportionnelle à la distance de l‘ion au centre de la chaîne

$\omega_z \sqrt{3}$

Source: S. Haroche, CdF 2006.

Spectre résolu de l'ion interprété en terme de création/annihilation de phonons $(\Gamma < \omega_z)$



Source: S. Haroche, CdF 2006.

Two trapped ions controlled via two lasers ω (phonons Ω of the center of mass mode only)

The instantaneous Hamiltionian:

$$
\begin{aligned}
H = {} & \Omega(\mathbf{a}^\dagger \mathbf{a} + 1/2) \\
& + \frac{\omega_0}{2}(|e_1\rangle\langle e_1| - |g_1\rangle\langle g_1|) \\
& + \left[\mathbf{u_1} e^{\imath(\omega t - \mathbf{k}(\mathbf{a}+\mathbf{a}^\dagger))} + \mathbf{u_1^*} e^{-\imath(\omega t - \mathbf{k}(\mathbf{a}+\mathbf{a}^\dagger))}\right](|e_1\rangle\langle g_1| + |g_1\rangle\langle e_1|) \\
& + \frac{\omega_0}{2}(|e_2\rangle\langle e_2| - |g_2\rangle\langle g_2|) \\
& + \left[\mathbf{u_2} e^{\imath(\omega t - \mathbf{k}(\mathbf{a}+\mathbf{a}^\dagger))} + \mathbf{u_2^*} e^{-\imath(\omega t - \mathbf{k}(\mathbf{a}+\mathbf{a}^\dagger))}\right](|e_2\rangle\langle g_2| + |g_2\rangle\langle e_2|)
\end{aligned}
$$

Two trapped ions controlled via two lasers ω

The averaged Interaction Hamiltonian (RWA)

$$\tilde{H} = \Omega(\mathbf{a}^\dagger \mathbf{a} + 1/2) + \mathbf{u_1} e^{-\imath\eta(\mathbf{a}+\mathbf{a}^\dagger)} |g_1\rangle\langle e_1| + \mathbf{u_1^*} e^{\imath\eta(\mathbf{a}+\mathbf{a}^\dagger)} |e_1\rangle\langle g_1| + \mathbf{u_2} e^{-\imath\eta(\mathbf{a}+\mathbf{a}^\dagger)} |g_2\rangle\langle e_2| + \mathbf{u_2^*} e^{\imath\eta(\mathbf{a}+\mathbf{a}^\dagger)} |e_2\rangle\langle g_2|$$

with

$$\mathbf{a} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right), \quad \mathbf{a}^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)$$

and the wave function (probability amplitude $\psi_{\mu\nu} \in L^2(\mathbb{R}, \mathbb{C})$, $\mu, \nu = e, g$):

$$|\psi\rangle = \psi_{\mathbf{gg}}(x,t)|g_1 g_2\rangle + \psi_{\mathbf{ge}}(x,t)|g_1 e_2\rangle + \psi_{\mathbf{eg}}(x,t)|e_1 g_2\rangle + \psi_{\mathbf{ee}}(x,t)|e_1 e_2\rangle$$

Qbit notations: $|1\rangle$ corresponds to the ground state $|g\rangle$ and $|0\rangle$ to the excited state $|e\rangle$.

Two trapped ions controlled via two lasers ω

The PDE satisfied by $\psi(x,t) = (\psi_{\mathbf{gg}}, \psi_{\mathbf{eg}}, \psi_{\mathbf{ge}}, \psi_{\mathbf{ee}})$:

$$\imath \frac{\partial \psi_{\mathbf{gg}}}{\partial t} = \frac{\Omega}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_{\mathbf{gg}} + \mathbf{u_1} e^{-\imath\eta x}\psi_{\mathbf{eg}} + \mathbf{u_2} e^{-\imath\eta x}\psi_{\mathbf{ge}}$$

$$\imath \frac{\partial \psi_{\mathbf{ge}}}{\partial t} = \frac{\Omega}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_{\mathbf{ge}} + \mathbf{u_1} e^{-\imath\eta x}\psi_{\mathbf{ee}} + \mathbf{u_2^*} e^{-\imath\eta x}\psi_{\mathbf{gg}}$$

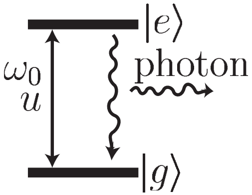
$$\imath \frac{\partial \psi_{\mathbf{eg}}}{\partial t} = \frac{\Omega}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_{\mathbf{eg}} + \mathbf{u_1^*} e^{\imath\eta x}\psi_{\mathbf{gg}} + \mathbf{u_2} e^{-\imath\eta x}\psi_{\mathbf{ee}}$$

$$\imath \frac{\partial \psi_{\mathbf{ee}}}{\partial t} = \frac{\Omega}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_{\mathbf{ee}} + \mathbf{u_1^*} e^{\imath\eta x}\psi_{\mathbf{ge}} + \mathbf{u_2^*} e^{\imath\eta x}\psi_{\mathbf{eg}}$$

where $\mathbf{u_1}, \mathbf{u_2} \in \mathbb{C}$ are the two controls, laser amplitudes for **ion no 1** and **ion no 2** $\left|\frac{d}{dt}u_{1,2}\right| \ll \omega |u_{1,2}|$ and $\eta \ll 1$, $\Omega \ll \omega$.

Controllability of this system ?

Two-levels atom with spontaneous emission from $|e\rangle$



Coherent (conservative) evolution: $\imath \frac{d}{dt} |\psi\rangle = H |\psi\rangle$ with the controlled Hamiltonian H:

$$\frac{\omega_0}{2}(|e\rangle \langle e| - |g\rangle \langle q|) + u(|e\rangle \langle g| + |g\rangle \langle e|)$$

Spontaneous emission (decoherence, dissipation): stochastic jump from $|e\rangle$ to $|g\rangle$ associated to the **jump operator** $L = \sqrt{\Gamma} |g\rangle \langle e|$ with Γ^{-1} the life-time of $|e\rangle$.

Quantum Monte Carlo Trajectories (Dalibard et al. 1992)

Take τ with $\tau\Gamma \ll \tau\omega_0 \ll 1$ (jump operator $\mathbf{L} = \sqrt{\Gamma}\,|g\rangle\langle e|$).
Compute the transition from $|\psi(t)\rangle$ to $|\psi(t+\tau)\rangle$ via the following stochastic jump process:

- Compute jump probability $\mathbf{p} = \tau\,\langle\psi(t)|\,\mathbf{L}^\dagger\mathbf{L}\,|\psi(t)\rangle$ and chose σ randomly between $[0, 1]$.
- if $0 \le \sigma \le 1 - \mathbf{p}$ no jump, no photon, no detector click:

$$|\psi(\mathbf{t}+\tau)\rangle = \frac{1 - \imath\tau H - \frac{\tau}{2}\mathbf{L}^\dagger\mathbf{L}}{\sqrt{1-\mathbf{p}}}\;|\psi(\tau)\rangle$$

- if $\mathbf{p} - 1 < \sigma \le 1$, jump from $|e\rangle$ to $|g\rangle$, spontaneous emission of one photon producing one photo-detector click:

$$|\psi(\mathbf{t}+\tau)\rangle = \frac{\mathbf{L}\,|\psi(t)\rangle}{\sqrt{\mathbf{p}/\tau}}.$$

(**collapse of the wave packet**)

Lindblad master equation

Valid to represent the evolution of the **average value** of the projector $|\psi(t)\rangle\langle\psi(t)|$ known as the **density operator** ρ, a positive symmetric operator of trace one:

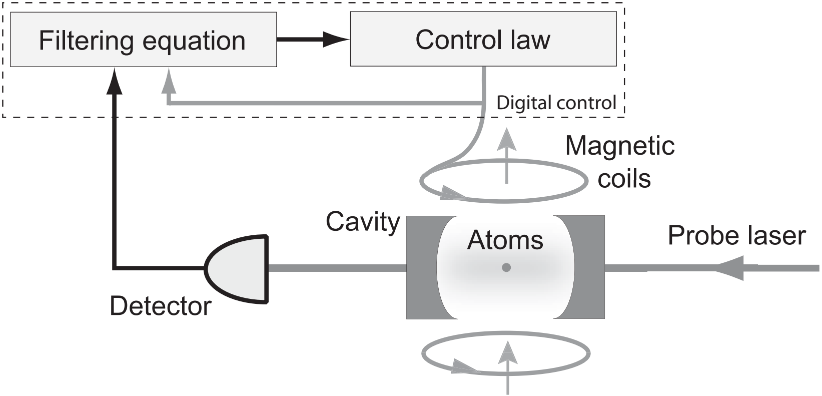
$$\frac{d}{dt}\rho = -\imath[H_0 + \mathbf{u}H_1, \rho] + L\rho L^\dagger - \frac{1}{2}\left(L^\dagger L\rho + \rho L^\dagger L\right)$$

where

$$\mathbf{y(t)} = \text{trace}(\rho L^\dagger L)$$

is the **average number of detector clicks per time-unit** (the measured output).
Notice that the control appears in the Hamiltonian $H = H_0 + \mathbf{u}H_1$ via the coherent laser field u.

Feedback for open quantum systems ...



Such stochastic models are used for feedback on the Caltech experiment of H. Mabuchi (Van Handel et al, IEEE AC 2005)
Stochastic Lyapunov techniques can be used for stabilization: see the work of M. Mirrahimi and R. Van Handel on Stabilizing feedback controls for quantum systems to appear in SIAM J. Cont. Opt. (http://arxiv.org/abs/math-ph/0510066).

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