Quantum Gate generation for open quantum systems via a monotonic algorithm with time optimization

A Lighthearted Conference on Control Theory, Celebrating Witold Respondek's (Partial) Retirement!
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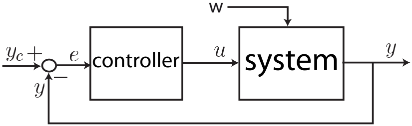
https://arxiv.org/abs/2403.20028

Underlying issues

Quantum Error Correction (QEC) is based on a discrete-time feedback loop

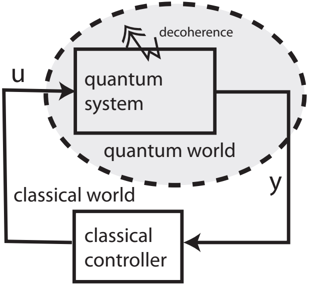
- A typical stabilizing feedback-loop for a classical system



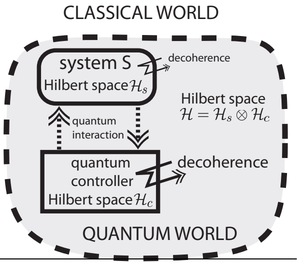
- Current experiments: 10^{-3} is the typical error probability during elementary gates (manipulations) involving few physical qubits.
- High-order error-correcting codes with an important overhead;
- Today, no such controllable logical qubit has been built.
- Key issue: reduction by several magnitude orders of such error rates, far below the threshold required by actual QEC, to build a controllable logical qubit encoded in a reasonable number of physical qubits and protected by QEC.

Control engineering can play a crucial role to build a controllable logical qubit protected by adapted **open-loop** and **closed-loop** control schemes increasing precision and stability.

Two kinds of quantum feedback[1]



Measurement-based feedback: **controller is classical**; measurement back-action on the quantum system of Hilbert space $\mathcal{H}$ is stochastic (collapse of the wave-packet); the measured output y is a classical signal; the control input u is a classical variable appearing in some controlled Schrödinger equation; $u(t)$ depends on the past measurements $y(\tau)$, $\tau \leq t$.



Coherent/autonomous feedback and reservoir/dissipation engineering: the **system of Hilbert space $\mathcal{H}_s$** is coupled to **the controller, another quantum system**; the composite system of Hilbert space $\boldsymbol{\mathcal{H}_s} \otimes \boldsymbol{\mathcal{H}_c}$, is an open-quantum system relaxing to some target (separable) state. Relaxation behaviors in open quantum systems can be exploited: optical pumping of Alfred Kastler.

[1]Wiseman/Milburn: Quantum Measurement and Control, 2009, Cambridge University Press.

Outline

Quantum gate generation for open quantum systems

The monotone/Lyapunov algorithm and optimal control

Numerical case-study: Cnot-gate between two cat-qubits

Quantum dynamics with dissipation (decoherence)

Gorini–Kossakowski –Sudarshan–Lindblad (GKSL) master equation:

$$\frac{d}{dt}\rho = \mathcal{L}_{[u]}(\rho) = \mathcal{L}_0(\rho) + u\ \mathcal{L}_1(\rho) \quad \left(\text{typically } -i[\widehat{H}_0 + \mathrm{u}\widehat{H}_1, \rho] + \sum_\nu \mathcal{D}_{\widehat{L}_\nu}(\rho)\right)$$

with $\mathcal{D}_{\widehat{L}_\nu}(\rho) \triangleq \widehat{L}_\nu \rho \widehat{L}_\nu^\dagger - \frac{1}{2}\left(\widehat{L}_\nu^\dagger \widehat{L}_\nu \rho + \rho \widehat{L}_\nu^\dagger \widehat{L}_\nu\right)$.

- Preservation of trace, hermiticity and positivity: ρ lies in the set of Hermitian and trace-class operators that are non-negative with trace one.
- Invariance under unitary transformations.
  A time-varying change of frame $\rho \mapsto \widehat{U}_t^\dagger \rho \widehat{U}_t$ with $\widehat{U}_t$ unitary.
  The new density operator obeys to a similar master equation where $\widehat{H}_0 + u\widehat{H}_1 \mapsto \widehat{U}_t^\dagger(\widehat{H}_0 + u\widehat{H}_0)\widehat{U}_t + i\widehat{U}_t^\dagger\left(\frac{d}{dt}\widehat{U}_t\right)$ and $\widehat{L}_\nu \mapsto \widehat{U}_t^\dagger \widehat{L}_\nu \widehat{U}_t$.
- "L^1-contraction" properties. Such master equations generate contraction semi-groups for many distances (nuclear distance[2], Hilbert metric on the cone of non negative operators[3]).
- If Hermitian operator $\widehat{A}$ satisfies "adjoint inequality" (Heisenberg view point):

$$i[\widehat{H}_0 + u\widehat{H}_1, \widehat{A}] + \sum_\nu \mathcal{D}^*_{\widehat{L}_\nu}(\rho) \leq 0$$

  then $t \mapsto V(\rho(t)) = \mathrm{Tr}\left(\widehat{A}\rho(t)\right)$ decreases (Lyapunov function if $\widehat{A} \geq 0$).

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[2] D.Petz (1996). Monotone metrics on matrix spaces. Linear Algebra and its Applications

[3] R. Sepulchre, A. Sarlette, PR (2010). Consensus in non-commutative spaces. IEEE-CDC.

Quantum Gate Generation Problem (Unitary Version)

Given the Schrödinger equation

$$\frac{d|\psi(t)\rangle}{dt} = -\imath\Big(H_0 + u(t)H_1\Big)|\psi(t)\rangle$$

with $|\psi(t)\rangle \in \mathbb{C}^n$.
Quantum gate generation (includes state preparation)

- $\{|e_i\rangle, i = 1, \dots, \bar{n}\}$ and $\{|f_i\rangle, i = 1, \dots, \bar{n}\}$ are orthonormal subsets of $\mathbb{C}^n$ with $\bar{n} \leq n$. (Note that $\bar{n} \ll n$ in the case of a cat-qubit)
- Take $T > 0$ and find $u : [0, T] \to \mathbb{R}$ such that $|\psi(t)\rangle$ is steered from $|\psi(0)\rangle = |e_i\rangle$ to $|\psi(T)\rangle = |f_i\rangle$ for $i = 1, \dots, \bar{n}$ up to some admissible error called gate-fidelity.

Quantum Gate Generation for open systems (GKSL Master Equations)

$\frac{d\rho(t)}{dt} = \mathcal{L}_{[u]}(\rho(t)) = \mathcal{L}_0(\rho(t)) + u\mathcal{L}_1(\rho(t))$

- For the "density matrices context" the quantum gate can be defined in an analogous way that appear in quantum Tomographic methods.
- We must steer $\rho(t)$ (at $t = T$):
  $|e_i\rangle\langle e_i| \rightsquigarrow |f_i\rangle\langle f_i|$, $i = 1, \ldots, \bar{n}$
- Let
  $|e_{ijR}\rangle = \frac{1}{\sqrt{2}}(|e_i\rangle + |e_j\rangle)$, $i > j$, and
  $|e_{ijI}\rangle = \frac{1}{\sqrt{2}}(|e_i\rangle + \imath|e_j\rangle)$, $i > j$
  (analogous notation for the f_i, $i = 1, \ldots, \bar{n}$)
- One must also:
  Steer all
  $|e_{ijR}\rangle\langle e_{ijR}| \rightsquigarrow |f_{ijR}\rangle\langle f_{ijR}|$
  $|e_{ijI}\rangle\langle e_{ijI}| \rightsquigarrow |f_{ijI}\rangle\langle f_{ijI}|$
- Remark: all of them are pure states

Numerical methods

Several numerical methods[4] (mainly optimal control, Lyapunov control) have been developed with several packages:

- The Krotov monotone optimal method [Schirmer and de Fouquieres, 2011]
- GRAPE (of first and second orders) [Khaneja et al., 2005, de Fouquieres et al., 2011]
- CRAB [Rach et al., 2015],
- GOAT [Machnes et al., 2018]
- RIGA [Pereira da Silva et al., 2019].
- QDYN [C. Koch et al. since 2007 – today] `https://qdyn-library.net/`

This talk: how control Lyapunov techniques provide a monotone algorithm solving the first order stationary condition of an optimal control problem including time optimization.

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[4] An excellent review of Christiane P Koch: Controlling open quantum systems: tools, achievements, and limitations. Journal of Physics: Condensed Matter, 28(21):213001, may 2016.

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One iteration of the algorithm (fixed time T, single control-input u)

- Initial guess $[0, T] \mapsto u_0(t)$
- $\bar{n}^2$ backward adjoint equations (open-loop):

$$\frac{dJ_\sigma}{dt}(t) = -\mathcal{L}^*_{[u_0(t)]}\left(J_\sigma(t)\right), \qquad J_\sigma(T_f) = \Pi_{|\phi_\sigma\rangle} = |\phi_\sigma\rangle\langle\phi_\sigma|$$

where $|\phi_\sigma\rangle = \begin{cases} |f_i\rangle, \text{ if } \sigma = i \in \{1, \ldots, \bar{n}\} \\ \frac{|f_i\rangle + |f_j\rangle}{\sqrt{2}}, \text{ if } \sigma = ijR, i, j \in \{1, \ldots, \bar{n}\}, i > j \\ \frac{|f_i\rangle + \imath|f_j\rangle}{\sqrt{2}}, \text{ if } \sigma = ijI, i, j \in \{1, \ldots, \bar{n}\}, i > j \end{cases}$

- $\bar{n}^2$ forward equations (closed-loop)

$$\frac{d\rho_\sigma(t)}{dt} = \mathcal{L}_{[u_0 + \Delta u]}\left(\rho_\sigma(t)\right), \qquad \rho_\sigma(0) = \Pi_{|\varepsilon_\sigma\rangle} = |\varepsilon_\sigma\rangle\langle\varepsilon_\sigma|$$

where $|\varepsilon_\sigma\rangle = \begin{cases} |e_i\rangle, \text{ if } \sigma = i \in \{1, \ldots, \bar{n}\} \\ \frac{|e_i\rangle + |e_j\rangle}{\sqrt{2}}, \text{ if } \sigma = ijR, i, j \in \{1, \ldots, \bar{n}\}, i > j \\ \frac{|e_i\rangle + \imath|e_j\rangle}{\sqrt{2}}, \text{ if } \sigma = ijI, i, j \in \{1, \ldots, \bar{n}\}, i > j \end{cases}$

and Δu is given by a time-varying feedback based on the time-varying Lyapunov function

$$\mathcal{V} = \bar{n}^2 - \sum_\sigma \operatorname{Tr}\left(J_\sigma(t)\rho_\sigma\right) \geq 0$$

The forward Lyapunov feedback

Lyapunov function: $\mathcal{V} = \bar{n}^2 - \sum_\sigma \ \mathrm{Tr}\left(J_\sigma(t)\rho_\sigma\right)$

- From $u \mapsto \mathcal{L}_{[u]}(\rho)$ affine and $\mathrm{Tr}\left(\mathcal{L}^*_{[u_0]}(J_\sigma)\rho_\sigma\right) \equiv \mathrm{Tr}\left(J_\sigma \mathcal{L}_{[u_0]}(\rho_\sigma)\right)$:

$$\begin{aligned}
\frac{d\mathcal{V}}{dt} &= -\sum_\sigma \mathrm{Tr}\left(\frac{dJ_\sigma}{dt}\rho_\sigma + J_\sigma \frac{d\rho_\sigma}{dt}\right) \\
&= -\sum_\sigma \mathrm{Tr}\left(-\mathcal{L}^*_{[u_0]}(J_\sigma)\rho_\sigma\right) + \mathrm{Tr}\left(J_\sigma \mathcal{L}_{[u_0+\Delta u]}(\rho_\sigma)\right) \\
&= -\Delta u \left(\sum_\sigma \mathrm{Tr}\left(J_\sigma \ \mathcal{L}_1(\rho_\sigma)\right)\right)
\end{aligned}$$

- Define the Lyapunov-based control with gain $K > 0$:

$$\Delta u = K \left(\sum_\sigma \mathrm{Tr}\left(J_\sigma \ \mathcal{L}_1(\rho_\sigma)\right)\right)$$

  then $\frac{d\mathcal{V}}{dt} = -K\left(\sum_\sigma \ \mathrm{Tr}\left(J_\sigma \ \mathcal{L}_1(\rho_\sigma)\right)\right)^2 \leq 0$.

- Next step: take as initial guess $[0, T] \ni t \mapsto u_1 = u_0 + \Delta u$.
- Since $V_{t=0} \geq V_{t=T}$, $\mathrm{Tr}\left(J_\sigma(0) \ \rho_\sigma(0)\right) = \mathrm{Tr}\left(e^{-T\mathcal{L}^*_{[u_0]}}(J_\sigma(T)) \ \rho_\sigma(0)\right)$ and

$$\mathrm{Tr}\left(J_\sigma(T) \ \rho_\sigma(T)\right) = \mathrm{Tr}\left(J_\sigma(T) \ e^{-T\mathcal{L}_{[u_0+\Delta u]}}(\rho_\sigma(0))\right) = \mathrm{Tr}\left(e^{-T\mathcal{L}^*_{[u_0+\Delta u]}}(J_\sigma(T)) \ \rho_\sigma(0)\right)$$

  the Lyapunov function decreases from step to step.

Including time optimization T

Consider virtual time τ according to $\frac{dt}{d\tau} = (1 + v(\tau))$ where $|v(\tau)| < 1$.
Physical time $t(\tau)$ is given by $t(\tau) = \int_0^\tau (1 + v(\tau'))d\tau'$.

With $\tilde{u} = (1 + v)u$ one gets:

$$\begin{aligned} \frac{d\rho}{dt}\frac{dt}{d\tau} = \frac{d\rho}{d\tau} &= (1 + v(\tau))(\mathcal{L}_0(\rho) + u\mathcal{L}_1(\rho)) \\ &= \mathcal{L}_0(\rho) + v(\tau)\mathcal{L}_0(\rho) + \tilde{u}(\tau)\mathcal{L}_1(\rho) \end{aligned}$$

Algorithm with time-control v:

- With two control-inputs $(v, \tilde{u})$ and initial guess $T = T_0$, $v_0 = 0$ and $\tilde{u}_0$, an algorithm step provides $[0, T_0] \ni \tau \mapsto (v_1(\tau), \tilde{u}_1(\tau))$.
- Update T_1 via $T_1 = \int_0^{T_0} (1 + v_1(\tau'))d\tau'$,
  Compute $u_1(t(\tau)) = \frac{\tilde{u}_1(\tau)}{1+v_1(\tau)}$, for $\tau \in [0, T_0]$ and
  $t(\tau) = \int_0^\tau (1 + v_1(\tau'))d\tau'$.

Optimal control interpretation

Find T and $[0, T] \ni t \mapsto u(t)$ minimizing

$$\bar{n}^2 - \sum_{\sigma} \mathrm{Tr}\left(\Pi_{|\phi_\sigma\rangle}\ \rho_\sigma(T)\right)$$

where $\Pi_{|\phi_\sigma\rangle} = |\phi_\sigma\rangle\langle\phi_\sigma|$

- $|\phi_\sigma\rangle = \begin{cases} |f_i\rangle, \text{ if } \sigma = i \in \{1,\dots,\bar{n}\} \\ \frac{|f_i\rangle + |f_j\rangle}{\sqrt{2}}, \text{ if } \sigma = ijR, i,j \in \{1,\dots,\bar{n}\}, i > j \\ \frac{|f_i\rangle + \imath|f_j\rangle}{\sqrt{2}}, \text{ if } \sigma = ijI, i,j \in \{1,\dots,\bar{n}\}, i > j \end{cases}$
- for each σ, $\frac{d\rho_\sigma(t)}{dt} = \mathcal{L}_{[u]}(\rho_\sigma(t))$ with $\rho_\sigma(0) = \Pi_{|\varepsilon_\sigma\rangle} = |\varepsilon_\sigma\rangle\langle\varepsilon_\sigma|$ and
  $|\varepsilon_\sigma\rangle = \begin{cases} |e_i\rangle, \text{ if } \sigma = i \in \{1,\dots,\bar{n}\} \\ \frac{|e_i\rangle + |e_j\rangle}{\sqrt{2}}, \text{ if } \sigma = ijR, i,j \in \{1,\dots,\bar{n}\}, i > j \\ \frac{|e_i\rangle + \imath|e_j\rangle}{\sqrt{2}}, \text{ if } \sigma = ijI, i,j \in \{1,\dots,\bar{n}\}, i > j \end{cases}$

Lemma: *Consider the above monotone iterative algorithm starting for T_0 and u_0. Assume that the Lyapunov function does not decrease strictly at step ℓ. Then T_ℓ and $[0, T] \ni t \mapsto u_\ell(t)$ satisfy the first-order stationary condition of this optimal control problem.*

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Bosonic code with cat-qubits

- Quantum error corrrection requires redundancy.
- Bosonic code: instead of encoding a logical qubit in N physical qubits living in $\mathbb{C}^{2^N}$, encode a logical qubit in an harmonic oscillator living in Fock space $\text{span}\{|0\rangle, |1\rangle, \ldots, |n\rangle, \ldots\} \sim L^2(\mathbb{R}, \mathbb{C})$ of infinite dimension.
- Cat-qubit [5]: $|\psi_L\rangle \in \text{span}\{|\alpha\rangle, |\text{-}\alpha\rangle\}$ where $|\alpha\rangle$ is the coherent state of real amplitude α: $\widehat{a}|\alpha\rangle = \alpha|\alpha\rangle$ with $\widehat{a} = (\widehat{q} + i\widehat{p})/\sqrt{2}$ and $[\widehat{q}, \widehat{p}] = i$:

$$|\psi\rangle \sim \psi(q) \in L^2(\mathbb{R}, \mathbb{C}),\ \widehat{q}|\psi\rangle \sim q\psi(q),\ \widehat{p}|\psi\rangle \sim -i\frac{d\psi}{dq}(q),\ |\alpha\rangle \sim \frac{\exp\left(-\frac{(q-\alpha\sqrt{2})^2}{2}\right)}{\sqrt{2\pi}}.$$

- Stabilisation of cat-qubit via a single Lindblad dissipator $\widehat{L} = \widehat{a}^2 - \alpha^2$. For any initial density operator $\rho(0)$, the solution $\rho(t)$ of

$$\frac{d}{dt}\rho = \widehat{L}\rho\widehat{L}^\dagger - \frac{1}{2}(\widehat{L}^\dagger\widehat{L}\rho + \rho\widehat{L}^\dagger\widehat{L})$$

  converges exponentially towards a steady-state density operator since

$$\frac{d}{dt}\ \text{Tr}\left(\widehat{L}^\dagger\widehat{L}\rho\right) \leq -2\ \text{Tr}\left(\widehat{L}^\dagger\widehat{L}\rho\right), \quad \ker\widehat{L} = \text{span}\{|\alpha\rangle, |\text{-}\alpha\rangle\}.$$

  Any density operator with support in $\text{span}\{|\alpha\rangle, |\text{-}\alpha\rangle\}$ is a steady-state.

[5] M. Mirrahimi, Z. Leghtas, . . . , M. Devoret: Dynamically protected cat-qubits: a new paradigm for universal quantum computation. 2014, New Journal of Physics.

Asymmetrically Threaded SQUID (ATS) stabilizing a cat-qubit [6]

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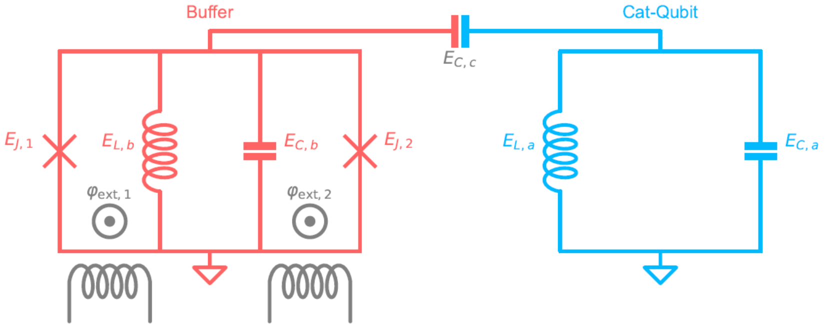


Figure S3. Equivalent circuit diagram. The cat-qubit (blue), a linear resonator, is capacitively coupled to the buffer (red). One recovers the circuit of Fig. 2 by replacing the buffer inductance with a 5-junction array and by setting $\varphi_\Sigma = (\varphi_{\text{ext},1} + \varphi_{\text{ext},2})/2$ and $\varphi_\Delta = (\varphi_{\text{ext},1} - \varphi_{\text{ext},2})/2$. Not shown here: the buffer is capacitively coupled to a transmission line, the cat-qubit resonator is coupled to a transmon qubit

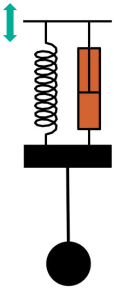
[6] R. Lescanne, . . . , Z. Leghtas: Exponential suppression of bit-flips in a qubit encoded in an oscillator. Nature Physics (2020)
U. Reglade, . . . , Z. Leghtas: Quantum control of a cat-qubit with bit-flip times exceeding ten seconds. Nature (2024)

Mechanical analogue (R. Lescanne/U. Réglade from Alice&Bob))

Both "steady-states" are locally stable

Two ”steady-states" (locally stable) associated to the same motion

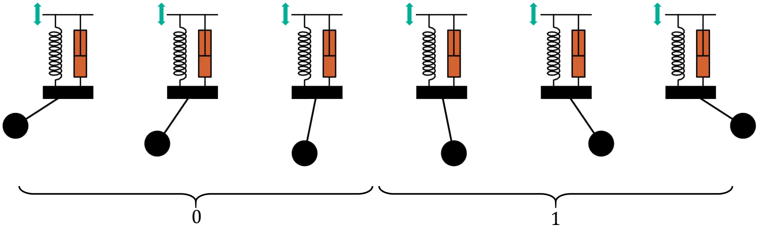
MAIN IDEA IN A CLASSICAL PICTURE



Driven damped oscillator
coupled to a pendulum.

Courtesy of Raphaël Lescanne

A BI-STABLE SYSTEM

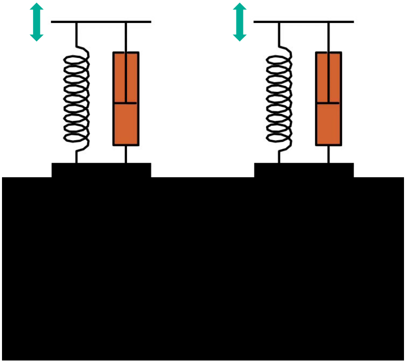


There are **2 steady states** in which we can encode information

Courtesy of Raphaël Lescanne

MAIN IDEA IN A CLASSICAL PICTURE

Stabilization regardless of the state



Neither the **drive** nor the **dissipation** can **distinguish** between 0 and 1

Important to preserve quantum coherence

Courtesy of Raphaël Lescanne

Master equations of the ATS super-conducting circuit

Oscillator $\hat{a}$ with **quantum controller based on a damped oscillator** $\hat{b}$:

$$\frac{d}{dt}\rho = g_2\Big[(\hat{a}^2-\alpha^2)\hat{b}^\dagger - ((\hat{a}^\dagger)^2-\alpha^2)\hat{b}\,,\ \rho\Big] + \kappa_b\Big(\hat{b}\rho\hat{b}^\dagger - (\hat{b}^\dagger\hat{b}\rho+\rho\hat{b}^\dagger\hat{b})/2\Big)$$

with $\alpha \in \mathbb{R}$ such that $\alpha^2 = u/g_2$, the drive amplitude $u \in \mathbb{R}$ applied to mode $\hat{b}$ and $1/\kappa_b > 0$ the life-time of photon in mode $\hat{b}$.
Any density operators $\bar{\rho} = \bar{\rho}_a \otimes |0\rangle\langle 0|_b$ **is a steady-state as soon as the support of** $\bar{\rho}_a$ **belongs to the two dimensional vector space spanned by the quasi-classical wave functions** $|\alpha\rangle$ **and** $|\text{-}\alpha\rangle$ (range$(\bar{\rho}_a) \subset$ span$\{|\alpha\rangle, |\text{-}\alpha\rangle\}$)

Usually $\kappa_b \gg |g_2|$, mode $\hat{b}$ relaxes rapidly to vaccuum $|0\rangle\langle 0|_b$, can be eliminated adiabatically (**singular perturbations**, second order corrections) to provides the slow evolution of mode $\hat{a}$

$$\frac{d}{dt}\rho_a = \tfrac{4|g_2|^2}{\kappa_b}\left(\hat{L}\rho\hat{L}^\dagger - \frac{1}{2}(\hat{L}^\dagger\hat{L}\rho + \rho\hat{L}^\dagger\hat{L})\right) \text{ with } \hat{L} = \hat{a}^2 - \alpha^2.$$

Convergence via the exponential Lyapunov function $V(\rho) = \text{Tr}\left(\hat{L}^\dagger\hat{L}\rho\right)$ [7]

[7] For a mathematical proof of convergence analysis in an adapted Banach space, see :R. Azouit, A. Sarlette, PR: Well-posedness and convergence of the Lindblad master equation for a quantum harmonic oscillator with multi-photon drive and damping. 2016, ESAIM: COCV.

Cat-qubit: exponential suppression of bit-flip for large α.

Since $\langle\alpha|\text{-}\alpha\rangle = e^{-2\alpha^2} \approx 0$:

$$|0_L\rangle \approx |\alpha\rangle,\ |1_L\rangle \approx |\text{-}\alpha\rangle,\ |+_L\rangle \propto \frac{|\alpha\rangle+|\text{-}\alpha\rangle}{\sqrt{2}},\ |-_L\rangle \propto \frac{|\alpha\rangle-|\text{-}\alpha\rangle}{\sqrt{2}}.$$

Photon loss as dominant error channel (dissipator $\widehat{a}$ with $0 < \kappa_1 \ll 1$):

$$\frac{d}{dt}\rho_a = \mathcal{D}_{\widehat{a}^2-\alpha^2}(\rho) + \kappa_1 \mathcal{D}_{\widehat{a}}(\rho)$$

with $\mathcal{D}_{\widehat{L}}(\rho) = \widehat{L}\rho\widehat{L}^\dagger - \frac{1}{2}(\widehat{L}^\dagger\widehat{L}\rho + \rho\widehat{L}^\dagger\widehat{L})$.

- if $\rho(0) = |0_L\rangle\langle 0_L|$ or $|1_L\rangle\langle 1_L|$, $\rho(t)$ converges to a statistical mixture of quasi-classical states close to $\frac{1}{2}|\alpha\rangle\langle\alpha| + \frac{1}{2}|\text{-}\alpha\rangle\langle\text{-}\alpha|$ in a time
$$T_{bit-flip} \sim \frac{e^{2\alpha^2}}{\kappa_1}$$
since $\widehat{a}|0_L\rangle \approx \alpha|0_L\rangle$ and $\widehat{a}|1_L\rangle \approx -\alpha|1_L\rangle$.
- if $\rho(0) = |+_L\rangle\langle +_L|$ or $|-_L\rangle\langle -_L|$, $\rho(t)$ converges also to the same statistical mixture in a time
$$T_{phase-flip} \sim \frac{1}{\kappa_1\alpha^2}$$
since $\widehat{a}|+_L\rangle = \alpha|-L\rangle$ and $\widehat{a}|-_L\rangle = \alpha|+L\rangle$.

Take α large to ignore bit-flip and to correct only the phase-flip with $1D$ code: important overhead reduction.

Cnot-gate between two cat-qubits[8]

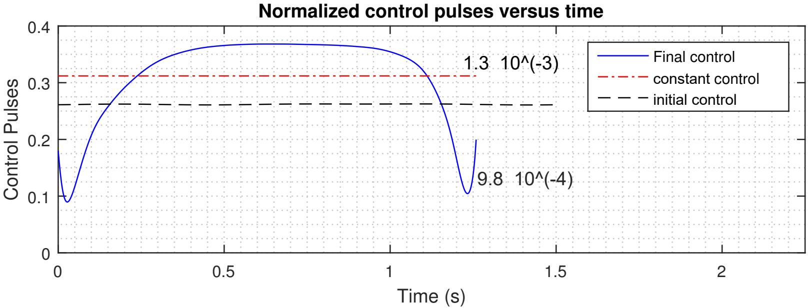
$$\frac{d\rho}{dt} = -iu\Big[(\widehat{a}_{co} + \widehat{a}^{\dagger}_{co} - 2\alpha \widehat{I}_{co}) \otimes (\widehat{a}^{\dagger}_{ta}\widehat{a}_{ta} - \alpha^2 \widehat{I}_{ta}) \otimes \widehat{I}_{qu} \;,\; \rho\Big]$$
$$- ig_2\Big[(\widehat{a}^2_{co} - \alpha^2 \widehat{I}_{co}) \otimes \widehat{I}_{ta} \otimes |e\rangle\langle g| + ((\widehat{a}^{\dagger})^2_{co} - \alpha^2 \widehat{I}_{co}) \otimes \widehat{I}_{ta} \otimes |g\rangle\langle e| \;,\; \rho\Big]$$
$$+ k_2 \mathcal{D}_{(\widehat{a}^2_{co} - \alpha^2 \widehat{I}_{co}) \otimes \widehat{I}_{ta} \otimes \widehat{I}_{qu}}(\rho) + k_1 \mathcal{D}_{\widehat{a}_{co} \otimes \widehat{I}_{ta} \otimes \widehat{I}_{qu}}(\rho) + k_1 \mathcal{D}_{\widehat{I}_{co} \otimes \widehat{a}_{ta} \otimes \widehat{I}_{qu}}(\rho)$$

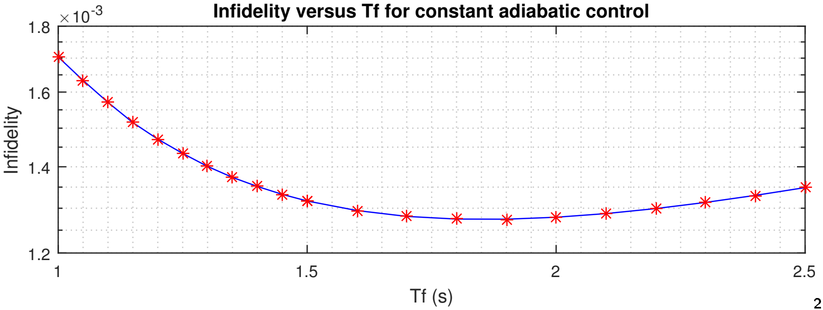
with $\alpha = 2,\ k_2 = 1,\ k_1 = \frac{1}{1000},\ g_2 = 10$.
Cnot-gate in Hilbert space $\mathcal{H}_{co} \otimes \mathcal{H}_{ta} \otimes \mathbb{C}^2$:

- $e_1 = |0_L\rangle_{co} \otimes |0_L\rangle_{ta} \otimes |g\rangle \mapsto f_1 = |0_L\rangle_{co} \otimes |0_L\rangle_{ta} \otimes |g\rangle$
- $e_2 = |0_L\rangle_{co} \otimes |1_L\rangle_{ta} \otimes |g\rangle \mapsto f_2 = |0_L\rangle_{co} \otimes |1_L\rangle_{ta} \otimes |g\rangle$
- $e_3 = |1_L\rangle_{co} \otimes |0_L\rangle_{ta} \otimes |g\rangle \mapsto f_3 = |1_L\rangle_{co} \otimes |1_L\rangle_{ta} \otimes |g\rangle$
- $e_4 = |1_L\rangle_{co} \otimes |1_L\rangle_{ta} \otimes |g\rangle \mapsto f_4 = |1_L\rangle_{co} \otimes |0_L\rangle_{ta} \otimes |g\rangle$.

[8] R. Gautier, A. Sarlette, and M. Mirrahimi: Combined dissipative and hamiltonian confinement of cat qubits. *PRX Quantum*, 3:020339, May 2022.

Cnot-gate between two cat-qubits ($n = 578 \gg 4 = \bar{n}$)

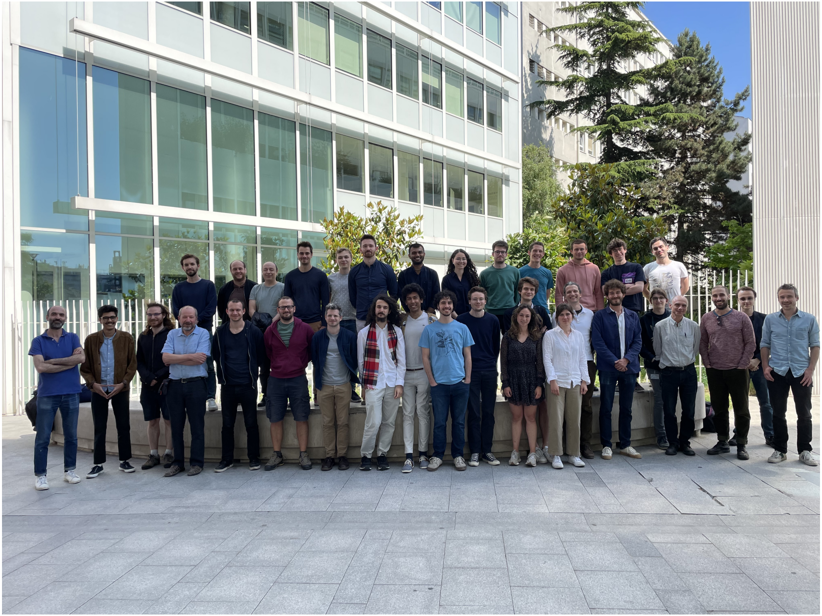




Conclusion

- Key roles of geometric underlying structures: Hilbert space, unitary operators and invariance, convex set of density operators, Schrôdinger/Heisenberg view-points.
- Well-chosen optimization criteria and Lyapunov-control function.

Quantic research group ENS/Inria/Mines/CNRS, June 2023



BON VENT pour la suite Witold !!!!