Feedback Issues Underlying Quantum Error Correction

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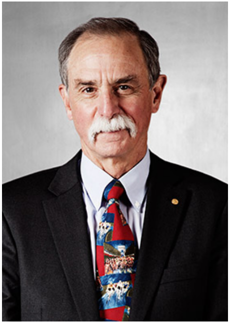
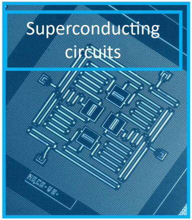


David J. Wineland

" This year's Nobel Prize in Physics honours the experimental inventions and discoveries that have allowed the **measurement and control of individual quantum systems***. They belong to two separate but related technologies: ions in a harmonic trap and photons in a cavity"*

From the Scientific Background on the Nobel Prize in Physics 2012 compiled by the Class for Physics of the Royal Swedish Academy of Sciences, 9 October 2012.

Investigated technologies for quantum computation [1]



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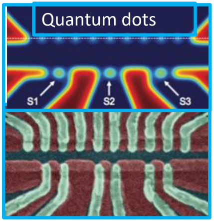

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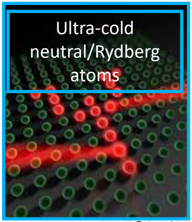

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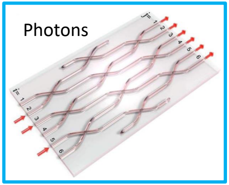

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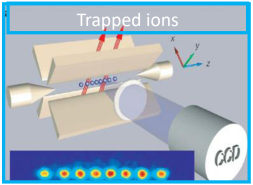

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Requirements:

- scalable modular architecture;
- control software from the very beginning.

[1] Courtesy of Walter Riess, IBM Research - Zurich.

Quantum Error Corrections (QEC) is based on an elementary discrete-time feedback loop: a static-output feedback neglecting the finite bandwidth of the measurement and actuation processes.

- Current experiments: 10^{-2} is the typical error probability during elementary gates (manipulations) involving few physical qubits.
- High-order error-correcting codes with an important overhead; **more than 1000 physical qubits to encode a controllable logical qubit**[2].
- Today, no such controllable logical qubit has been built.
- Key issue: reduction by several magnitude orders such error rates, far below the threshold required by actual QEC, to build a controllable logical qubit encoded in a reasonable number of physical qubits and protected by QEC.

Control engineering can play a crucial role to built a controllable logical qubit protected by much **more elaborated feedback schemes increasing precision and stability.**

[2] A.G. Fowler, M. Mariantoni, J.M. Martinis, A.N. Cleland (2012): Surface codes: Towards practical large-scale quantum computation. Phys. Rev. A,86(3):032324.

Outline

Continuous-time dynamics of open quantum system
- Stochastic Master Equation (SME)
- Key characteristics of SME
- Quantum filtering and state estimation

Feedback schemes
- Measurement-based feedback and classical controller
- Coherent feedback and quantum controller
- Merging measurement-based and coherent feedbacks

Quantum Error Correction (QEC) and feedback
- QEC from scratch
- Storing a logical qubit in a high-quality harmonic oscillator
- Simplified feedback scheme for cat-qubit experiment

1. Schrödinger ($\hbar = 1$): wave funct. $|\psi\rangle \in \mathcal{H}$, density op. $\boldsymbol{\rho} \sim |\psi\rangle\langle\psi|$

$$\frac{d}{dt}|\psi\rangle = -i\boldsymbol{H}|\psi\rangle, \quad \boldsymbol{H} = \boldsymbol{H}_0 + u\boldsymbol{H}_1 = \boldsymbol{H}^\dagger, \quad \frac{d}{dt}\boldsymbol{\rho} = -i[\boldsymbol{H}, \boldsymbol{\rho}].$$

2. Origin of dissipation: collapse of the wave packet induced by the measurement of $\boldsymbol{O} = \boldsymbol{O}^\dagger$ with spectral decomp. $\sum_y \lambda_y \boldsymbol{P}_y$:
   - measurement outcome y with proba. $\mathbb{P}_y = \langle\psi|\boldsymbol{P}_y|\psi\rangle = \text{Tr}\left(\boldsymbol{\rho}\boldsymbol{P}_y\right)$ depending on $|\psi\rangle$, $\boldsymbol{\rho}$ just before the measurement
   - measurement back-action if outcome y:

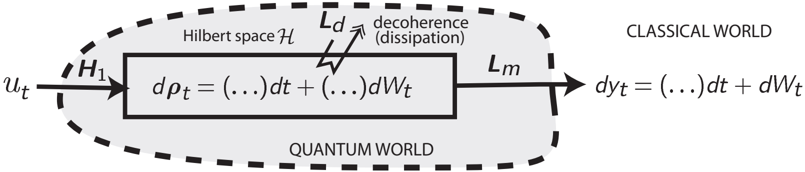
$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\boldsymbol{P}_y|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{P}_y|\psi\rangle}}, \quad \boldsymbol{\rho} \mapsto \boldsymbol{\rho}_+ = \frac{\boldsymbol{P}_y\boldsymbol{\rho}\boldsymbol{P}_y}{\text{Tr}\left(\boldsymbol{\rho}\boldsymbol{P}_y\right)}$$

3. Tensor product for the description of composite systems (S, C):
   - Hilbert space $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_c$
   - Hamiltonian $\boldsymbol{H} = \boldsymbol{H}_s \otimes \boldsymbol{I}_c + \boldsymbol{H}_{sc} + \boldsymbol{I}_s \otimes \boldsymbol{H}_c$
   - observable on sub-system C only: $\boldsymbol{O} = \boldsymbol{I}_s \otimes \boldsymbol{O}_c$.

[3] S. Haroche and J.M. Raimond (2006). *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts.

Outline

Classical I/O dynamics based on Stochastic Master Equation (SME) [4]



Continuous-time models: stochastic differential systems (Itō formulation)
density operator ρ ($\rho^\dagger = \rho$, $\rho \geq 0$, $\text{Tr}(\rho) = 1$) as state ($\hbar \equiv 1$ here):

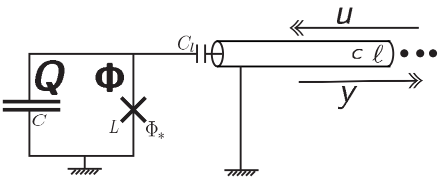
$$d\rho_t = \Big(-i[H_0 + u_t H_1, \rho_t] + \sum_{\nu=d,m} L_\nu \rho_t L_\nu^\dagger - \frac{1}{2}(L_\nu^\dagger L_\nu \rho_t + \rho_t L_\nu^\dagger L_\nu) \Big) dt$$
$$+ \sqrt{\eta_m} \Big(L_m \rho_t + \rho_t L_m^\dagger - \text{Tr}\left((L_m + L_m^\dagger)\rho_t \right) \rho_t \Big) dW_t$$

driven by the Wiener process W_t, with measurement y_t,

$$dy_t = \sqrt{\eta_m}\, \text{Tr}\left((L_m + L_m^\dagger)\, \rho_t \right) dt + dW_t \quad \text{detection efficiencies } \eta_m \in [0, 1].$$

Measurement backaction: **$d\rho$ and dy share the same noises dW**. Very different from the usual Kalman I/O state-space description.

[4] A. Barchielli, M. Gregoratti (2009): Quantum Trajectories and Measurements in Continuous Time: the Diffusive Case. Springer Verlag.



Classical model ($\frac{C_l}{C+C_l} = \epsilon \ll 1$):

$$\frac{d}{dt}\boldsymbol{\Phi} = \frac{1}{C}\boldsymbol{Q} + 2\epsilon u - \epsilon^2\sqrt{\frac{\ell}{c}}\frac{\Phi_*}{L}\sin\left(\frac{1}{\Phi_*}\boldsymbol{\Phi}\right)$$

$$\frac{d}{dt}\boldsymbol{Q} = -\frac{\Phi_*}{L}\sin\left(\frac{1}{\Phi_*}\boldsymbol{\Phi}\right)$$

with $y = u - \epsilon\sqrt{\frac{\ell}{c}}\frac{\Phi_*}{L}\sin\left(\frac{1}{\Phi_*}\boldsymbol{\Phi}\right)$.

$\boldsymbol{H}_s(\boldsymbol{\Phi}, \boldsymbol{Q}) = \frac{1}{2C}\boldsymbol{Q}^2 - \frac{\Phi_*^2}{L}\cos\left(\frac{1}{\Phi_*}\boldsymbol{\Phi}\right)$ with nonlinearity ($\Phi_* < (L/C)^{1/4}$):

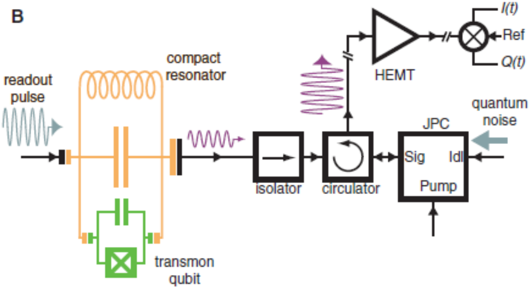
- anharmonic spectrum: frequency transition between the ground and first excited states larger than frequency transition between first and second excited states, ...
- qubit model based on restriction to these two slowest energy levels, $|g\rangle$ and $|e\rangle$, with pulsation $\omega_q \sim 1/\sqrt{LC}$.

Two weak coupling regimes of the transmon qubit[5]:

- resonant, in/out wave pulsation ω_q;
- off-resonant, in/out wave pulsation $\omega_q + \Delta$ with $|\Delta| \ll \omega_q$.

[5] J. Koch et al. (2007): Charge-insensitive qubit design derived from the Cooper pair box. Phys. Rev. A, 76:042319.

A key physical example with super-conducting Josephson circuits [6]



Superconducting qubit dispersively coupled to a cavity traversed by a microwave signal (input/output theory). The back-action on the qubit state of a single measurement of one output field quadrature $\boldsymbol{y}$ is described by a simple SME for the qubit density operator $\boldsymbol{\rho}$, 2×2 Hermitian ≥ 0 matrix.

$$d\boldsymbol{\rho}_t = \Big(-\tfrac{i}{2}[\omega_q \boldsymbol{Z}, \boldsymbol{\rho}_t] + \gamma(\boldsymbol{Z}\boldsymbol{\rho}\boldsymbol{Z} - \boldsymbol{\rho}_t) \Big) dt + \sqrt{\eta\gamma}\Big(\boldsymbol{Z}\boldsymbol{\rho}_t + \boldsymbol{\rho}_t \boldsymbol{Z} - 2\,\mathrm{Tr}\left(\boldsymbol{Z}\boldsymbol{\rho}_t\right)\boldsymbol{\rho}_t \Big) dW_t$$

with y_t given by $dy_t = 2\sqrt{\eta\gamma}\,\mathrm{Tr}\left(\boldsymbol{Z}\boldsymbol{\rho}_t\right) dt + dW_t$ where $\gamma \geq 0$ is related to the measurement strength and $\eta \in [0,1]$ is the detection efficiency.

[6] M. Hatridge et al. (2013): Quantum Back-Action of an Individual Variable-Strength Measurement. Science, 339, 178-181.

Another formulation of diffusive SME [7]

With a single imperfect measurement $dy_t = \sqrt{\eta}\,\mathrm{Tr}\left((\boldsymbol{L}+\boldsymbol{L}^\dagger)\,\boldsymbol{\rho}_t\right)\,dt + dW_t$ and detection efficiency $\eta \in [0,1]$, the quantum state $\boldsymbol{\rho}_t$ obeys to

$$d\boldsymbol{\rho}_t = \left(-i[\boldsymbol{H}_0 + u_t\boldsymbol{H}_1, \boldsymbol{\rho}_t] + \boldsymbol{L}\boldsymbol{\rho}_t\boldsymbol{L}^\dagger - \frac{1}{2}(\boldsymbol{L}^\dagger\boldsymbol{L}\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{L}^\dagger\boldsymbol{L})\right)dt + \sqrt{\eta}\left(\boldsymbol{L}\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{L}^\dagger - \mathrm{Tr}\left((\boldsymbol{L}+\boldsymbol{L}^\dagger)\boldsymbol{\rho}_t\right)\boldsymbol{\rho}_t\right)dW_t$$

driven by the Wiener process dW_t

With Itō rules, it can be written as the following "discrete-time" Markov model

$$\boldsymbol{\rho}_{t+dt} \triangleq \boldsymbol{\rho}_t + d\boldsymbol{\rho}_t = \frac{\boldsymbol{M}_{u_t,dy_t}\boldsymbol{\rho}_t\boldsymbol{M}^\dagger_{u_t,dy_t} + (1-\eta)\boldsymbol{L}\boldsymbol{\rho}_t\boldsymbol{L}^\dagger dt}{\mathrm{Tr}\left(\boldsymbol{M}_{u_t,dy_t}\boldsymbol{\rho}_t\boldsymbol{M}^\dagger_{u_t,dy_t} + (1-\eta)\boldsymbol{L}\boldsymbol{\rho}_t\boldsymbol{L}^\dagger dt\right)}$$

with $\boldsymbol{M}_{u_t,dy_t} = \boldsymbol{I} - \left(i(\boldsymbol{H}_0 + u_t\boldsymbol{H}_1) + \frac{1}{2}\left(\boldsymbol{L}^\dagger\boldsymbol{L}\right)\right)dt + \sqrt{\eta}\boldsymbol{L}dy_t$.

[7] PR (2014): Models and Feedback Stabilization of Open Quantum Systems. Proc. of Int. Congress of Mathematicians, vol. IV, pp 921–946, Seoul. (http://arxiv.org/abs/1407.7810).

Measured output map $dy_t = \sqrt{\eta}\,\mathrm{Tr}\left((\boldsymbol{L} + \boldsymbol{L}^\dagger)\,\boldsymbol{\rho}_t\right)\,dt + dW_t$ and measurement backaction described by

$$\boldsymbol{\rho}_{t+dt} \triangleq \boldsymbol{\rho}_t + d\boldsymbol{\rho}_t = \frac{\boldsymbol{M}_{u_t,dy_t}\boldsymbol{\rho}_t\boldsymbol{M}^\dagger_{u_t,dy_t} + (1-\eta)\boldsymbol{L}\boldsymbol{\rho}_t\boldsymbol{L}^\dagger dt}{\mathrm{Tr}\left(\boldsymbol{M}_{u_t,dy_t}\boldsymbol{\rho}_t\boldsymbol{M}^\dagger_{u_t,dy_t} + (1-\eta)\boldsymbol{L}\boldsymbol{\rho}_t\boldsymbol{L}^\dagger dt\right)}$$

- **if ρ_0 density operator, then, for all $t > 0$, ρ_t remains a density operator**
  The dynamics preserve the cone of non-negative Hermitian operators.
- **Positivity and trace preserving numerical scheme for quantum Monte-Carlo simulations.**
- When $\eta = 1$, $\mathrm{rank}(\boldsymbol{\rho}_t) \leq \mathrm{rank}(\boldsymbol{\rho}_0)$ for all $t \geq 0$. In particular if $\boldsymbol{\rho}_0$ is a rank one projector, then $\boldsymbol{\rho}_t$ remains a rank one projector (pure state).

$$d\boldsymbol{\rho}_t = \left(-i[\boldsymbol{H}_0 + u\boldsymbol{H}_1, \boldsymbol{\rho}_t] + \boldsymbol{L}\boldsymbol{\rho}_t\boldsymbol{L}^\dagger - \frac{1}{2}(\boldsymbol{L}^\dagger\boldsymbol{L}\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{L}^\dagger\boldsymbol{L})\right) dt + \sqrt{\eta}\left(\boldsymbol{L}\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{L}^\dagger - \mathrm{Tr}\left((\boldsymbol{L}+\boldsymbol{L}^\dagger)\boldsymbol{\rho}_t\right)\boldsymbol{\rho}_t\right) dW_t$$

with measured output map $dy_t = \sqrt{\eta}\,\mathrm{Tr}\left((\boldsymbol{L}+\boldsymbol{L}^\dagger)\,\boldsymbol{\rho}_t\right)\,dt + dW_t$

- Invariance of the SME structure under unitary transformations.
  A time-varying change of frame $\widetilde{\boldsymbol{\rho}} = \boldsymbol{U}_t^\dagger \boldsymbol{\rho} \boldsymbol{U}_t$ with $\boldsymbol{U}_t$ unitary.
  The new density operator $\widetilde{\boldsymbol{\rho}}$ obeys to a similar SME where
  $\widetilde{\boldsymbol{H}}_0 + u\widetilde{\boldsymbol{H}}_1 = \boldsymbol{U}_t^\dagger(\boldsymbol{H}_0 + u\boldsymbol{H}_0)\boldsymbol{U}_t + i\boldsymbol{U}_t^\dagger\left(\frac{d}{dt}\boldsymbol{U}_t\right)$ and $\widetilde{\boldsymbol{L}} = \boldsymbol{U}_t^\dagger \boldsymbol{L}\boldsymbol{U}_t$.
- "L^1-contraction" properties. When $\eta = 0$, SME becomes deterministic

$$\frac{d}{dt}\boldsymbol{\rho} = -i[\boldsymbol{H}_0 + u\boldsymbol{H}_1, \boldsymbol{\rho}_t] + \boldsymbol{L}\boldsymbol{\rho}_t\boldsymbol{L}^\dagger - \frac{1}{2}(\boldsymbol{L}^\dagger\boldsymbol{L}\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{L}^\dagger\boldsymbol{L})$$

  generating a contraction semi-group for many distances (nuclear distance[8], Hilbert metric on the cone of non negative operators[9]).
- If the non-negative Hermitian operator $\boldsymbol{A}$ satisfies the operator inequality

$$i[\boldsymbol{H}_0 + u\boldsymbol{H}_1, \boldsymbol{A}] + \boldsymbol{L}^\dagger\boldsymbol{A}\boldsymbol{L} - \frac{1}{2}(\boldsymbol{L}^\dagger\boldsymbol{L}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{L}^\dagger\boldsymbol{L}) \leq 0$$

  then $V(\boldsymbol{\rho}) = \mathrm{Tr}(\boldsymbol{A}\boldsymbol{\rho})$ is a super-martingale (Lyapunov function).

[8] D.Petz (1996). Monotone metrics on matrix spaces. Linear Algebra and its Applications, 244, 81-96.

[9] R. Sepulchre, A. Sarlette, PR (2010). Consensus in non-commutative spaces. IEEE-CDC.

Quantum state $\boldsymbol{\rho}_t$ producing $dy_t = \sqrt{\eta}\ \mathrm{Tr}\left(\boldsymbol{L}\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{L}^\dagger\right) dt + dW_t$ and its estimate $\widehat{\boldsymbol{\rho}}_t$:

$$d\boldsymbol{\rho}_t = \left(-i[\boldsymbol{H}_0 + u\boldsymbol{H}_1, \boldsymbol{\rho}_t] + \boldsymbol{L}\boldsymbol{\rho}_t\boldsymbol{L}^\dagger - \frac{1}{2}(\boldsymbol{L}^\dagger\boldsymbol{L}\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{L}^\dagger\boldsymbol{L})\right) dt$$
$$+ \sqrt{\eta}\left(\boldsymbol{L}\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{L}^\dagger - \ \mathrm{Tr}\left((\boldsymbol{L} + \boldsymbol{L}^\dagger)\boldsymbol{\rho}_t\right)\boldsymbol{\rho}_t\right) dW_t$$
$$d\widehat{\boldsymbol{\rho}}_t = \left(-i[\boldsymbol{H}_0 + u\boldsymbol{H}_1, \widehat{\boldsymbol{\rho}}_t] + \boldsymbol{L}\widehat{\boldsymbol{\rho}}_t\boldsymbol{L}^\dagger - \frac{1}{2}(\boldsymbol{L}^\dagger\boldsymbol{L}\widehat{\boldsymbol{\rho}}_t + \widehat{\boldsymbol{\rho}}_t\boldsymbol{L}^\dagger\boldsymbol{L})\right) dt$$
$$+ \sqrt{\eta}\left(\boldsymbol{L}\widehat{\boldsymbol{\rho}}_t + \widehat{\boldsymbol{\rho}}_t\boldsymbol{L}^\dagger - \ \mathrm{Tr}\left((\boldsymbol{L} + \boldsymbol{L}^\dagger)\widehat{\boldsymbol{\rho}}_t\right)\widehat{\boldsymbol{\rho}}_t\right)\left(dy_t - \sqrt{\eta}\ \mathrm{Tr}\left(\boldsymbol{L}\widehat{\boldsymbol{\rho}}_t + \widehat{\boldsymbol{\rho}}_t\boldsymbol{L}^\dagger\right) dt\right).$$

- Stability[10]: the fidelity $F(\boldsymbol{\rho}_t, \widehat{\boldsymbol{\rho}}_t) = \ \mathrm{Tr}^2\left(\sqrt{\sqrt{\boldsymbol{\rho}_t}\widehat{\boldsymbol{\rho}}_t\sqrt{\boldsymbol{\rho}_t}}\right)$ is always a sub-martingale:

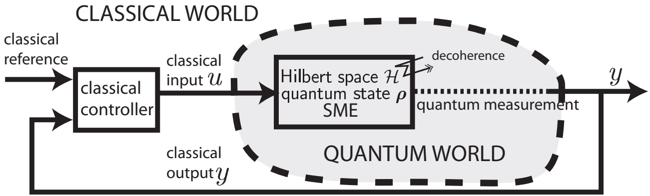
$$\forall t_1 \leq t_2, \quad \mathbb{E}\left(F\left(\boldsymbol{\rho}_{t_2}, \widehat{\boldsymbol{\rho}}_{t_2}\right) \ \Big|\ \boldsymbol{\rho}_{t_1}, \widehat{\boldsymbol{\rho}}_{t_1}\right) \geq F\left(\boldsymbol{\rho}_{t_1}, \widehat{\boldsymbol{\rho}}_{t_1}\right).$$

  Fidelity: $0 \leq F(\boldsymbol{\rho}, \widehat{\boldsymbol{\rho}}) \leq 1$ and $F(\boldsymbol{\rho}, \widehat{\boldsymbol{\rho}}) = 1$ iff $\boldsymbol{\rho} = \widehat{\boldsymbol{\rho}}$.

- Convergence[11] of $\widehat{\boldsymbol{\rho}}_t$ towards $\boldsymbol{\rho}_t$ when $t \mapsto +\infty$ is an open problem

[10] H. Amini, C. Pellegrini, C., PR (2014). Stability of continuous-time quantum filters with measurement imperfections. Russian Journal of Mathematical Physics, 21(3), 297-315.

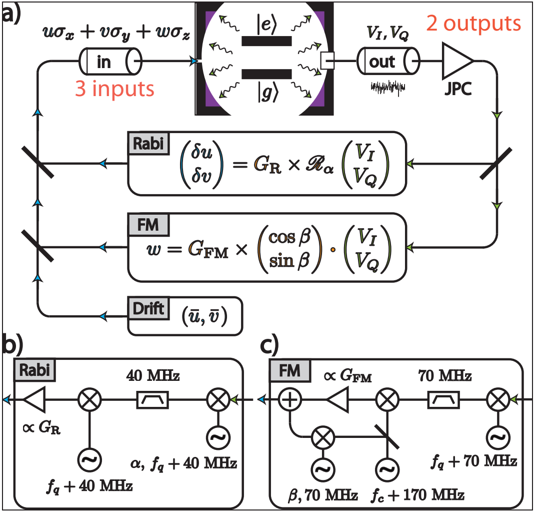
[11] Partial result: R. van Handel (2009): The stability of quantum Markov filters. Infin. Dimens. Anal. Quantum Probab. Relat. Top., 12, 153-172.



- P-controller (Markovian feedback[12]) for $u_t\ dt = k\ dy_t$, the ensemble average closed-loop dynamics of ρ remains governed by a linear Lindblad master equation.
- PID controller: no Lindblad master equation in closed-loop for dynamics output feedback
- Nonlinear hidden-state stochastic systems: Lyapunov state-feedback[13]; many open issues on convergence rates, delays, robustness, . . .
- **Short sampling times limit feedback complexity**

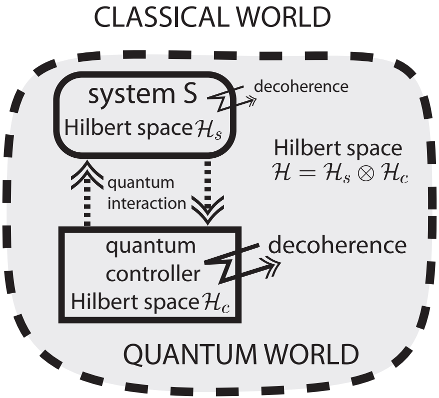
[12] H. Wiseman, G. Milburn (2009). Quantum Measurement and Control. Cambridge University Press.

[13] See e.g.: C. Ahn et. al (2002): Continuous quantum error correction via quantum feedback control. Phys. Rev. A 65;
M. Mirrahimi, R. Handel (2007): Stabilizing feedback controls for quantum systems. SIAM Journal on Control and Optimization, 46(2), 445-467;
G. Cardona, A. Sarlette, PR (2019): Continuous-time quantum error correction with noise-assisted quantum feedback. IFAC Mechatronics & Nolcos Conf.



[14] P. Campagne-Ibarcq, . . . , PR, B. Huard (2016): Using Spontaneous Emission of a Qubit as a Resource for Feedback Control. Phys. Rev. Lett. 117(6).

Quantum analogue of Watt speed governor: a **dissipative** mechanical system controls another mechanical system [15]



Optical pumping (Kastler 1950), coherent population trapping (Arimondo 1996)

Dissipation engineering, autonomous feedback: (Zoller, Cirac, Wolf, Verstraete, Devoret, Schoelkopf, Siddiqi, Martinis, Raimond, Brune,. . . , Lloyd, Viola, Ticozzi, Leghtas, Mirrahimi, Sarlette, PR, ...)

(S,L,H) theory and linear quantum systems: quantum feedback networks based on stochastic Schrödinger equation, Heisenberg picture (Gardiner, Yurke, Mabuchi, Genoni, Serafini, Milburn, Wiseman, Doherty, . . . , Gough, James, Petersen, Nurdin, Yamamoto, Zhang, Dong, . . .)

Stability analysis: Kraus maps and Lindblad propagators are always contractions (non commutative diffusion and consensus).

[15] J.C. Maxwell (1868): On governors. Proc. of the Royal Society, No.100.

The closed-loop Lindblad master equation on $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_c$:

$$\frac{d}{dt}\rho = -i\Big[\boldsymbol{H}_s \otimes \boldsymbol{I}_c + \boldsymbol{I}_s \otimes \boldsymbol{H}_c + \boldsymbol{H}_{sc}\ ,\ \rho\Big] + \sum_{\nu} \mathbb{D}_{\boldsymbol{L}_{s,\nu}\otimes \boldsymbol{I}_c}(\rho) + \sum_{\nu'} \mathbb{D}_{\boldsymbol{I}_s \otimes \boldsymbol{L}_{c,\nu'}}(\rho)$$

with $\mathbb{D}_{\boldsymbol{L}}(\rho) = \boldsymbol{L}\rho\boldsymbol{L}^\dagger - \frac{1}{2}\left(\boldsymbol{L}^\dagger\boldsymbol{L}\rho + \rho\boldsymbol{L}^\dagger\boldsymbol{L}\right)$ and operators made of tensor products.

• Consider a convex subset $\overline{\mathcal{D}}_s$ of steady-states for original system S: each density operator $\overline{\boldsymbol{\rho}}_s$ on $\mathcal{H}_s$ belonging to $\overline{\mathcal{D}}_s$ satisfy $i[\boldsymbol{H}_s, \overline{\boldsymbol{\rho}}_s] = \sum_\nu \mathbb{D}_{\boldsymbol{L}_{s,\nu}}(\overline{\boldsymbol{\rho}}_s)$.

• Designing a **realistic** quantum controller C $(\boldsymbol{H}_c, \boldsymbol{L}_{c,\nu'})$ and coupling Hamiltonian $\boldsymbol{H}_{sc}$ stabilizing $\overline{\mathcal{D}}_s$ is non trivial. **Realistic** means in particular relying on physical time-scales and constraints:

- Fastest time-scales attached to $\boldsymbol{H}_s$ and $\boldsymbol{H}_c$ (Bohr frequencies) and averaging approximations: $\|\boldsymbol{H}_s\|, \|\boldsymbol{H}_c\| \gg \|\boldsymbol{H}_{sc}\|$,
- High-quality oscillations: $\|\boldsymbol{H}_s\| \gg \|\boldsymbol{L}_{s,\nu}^\dagger \boldsymbol{L}_{s,\nu}\|$ and $\|\boldsymbol{H}_c\| \gg \|\boldsymbol{L}_{c,\nu'}^\dagger \boldsymbol{L}_{c,\nu'}\|$.
- Decoherence rates of S much slower than those of C: $\|\boldsymbol{L}_{s,\nu}^\dagger \boldsymbol{L}_{s,\nu}\| \ll \|\boldsymbol{L}_{c,\nu'}^\dagger \boldsymbol{L}_{c,\nu'}\|$: model reduction by quasi-static approximations (adiabatic elimination, singular perturbations).

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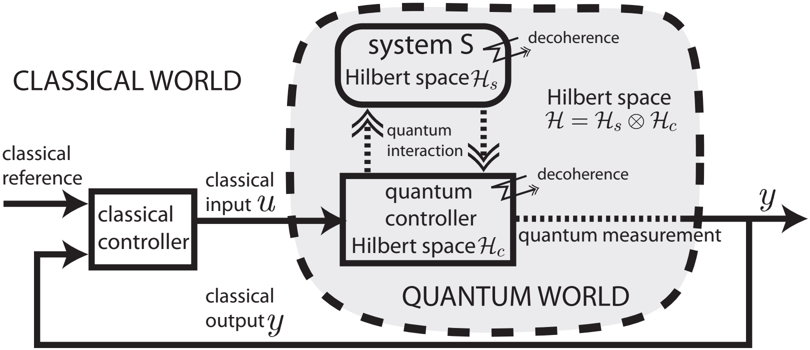

To stabilize the quantum information localized in system S:

- fast decoherence addressed preferentially by quantum controllers (coherent feedback);
- slow decoherence, perturbations and parameter drifts tackled mainly by classical controllers (measurement-based feedback).

• Single bit error model: the bit $b \in \{0,1\}$ flips with probability $p < 1/2$ during Δt (for usual DRAM: $\boldsymbol{p/\Delta t \leq 10^{-14}}$ s^{-1}).
• Multi-bit error model: each bit $b_k \in \{0,1\}$ flips with probability $p < 1/2$ during Δt; no correlation between the bit flips.
•Use redundancy to construct with several physical bits b_k of flip probability p, a logical bit b_L with a flip probability $p_L < p$.
• The simplest solution, the 3-bit code (sampling time Δt):

$t = 0$: $b_L = [bbb]$ with $b \in \{0,1\}$

$t = \Delta t$: measure the three physical bits of $b_L = [b_1 b_2 b_3]$ (instantaneous) :

1. if all 3 bits coincide, nothing to do.
2. if one bit differs from the two other ones, flip this bit (instantaneous);

• Since the flip probability laws of the physical bits are independent, the probability that the logical bit b_L (protected with the above error correction code) flips during Δt is $\boldsymbol{p_L = 3p^2 - 2p^3} < p$ since $p < 1/2$.

The 3-qubit bit-flip code (Peter Shor (1995))[17]

- Local bit-flip errors: each physical qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ becomes $\boldsymbol{X}|\psi\rangle = \alpha|1\rangle + \beta|0\rangle$ [16] with probability $p < 1/2$ during Δt. (for actual super-conducting qubit $\boldsymbol{p/\Delta t > 10^3}$ s^{-1}).
- $t = 0$: $|\psi_L\rangle = \alpha|0_L\rangle + \beta|1_L\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \equiv \mathbb{C}^8$ with $|0_L\rangle = |000\rangle$ and $|1_L\rangle = |111\rangle$.
- $t = \Delta t$: $|\psi_L\rangle$ becomes with

$$\text{1 flip: } \left\{ \begin{array}{l} \alpha|100\rangle + \beta|011\rangle \\ \alpha|010\rangle + \beta|101\rangle \\ \alpha|001\rangle + \beta|110\rangle \end{array} \right. ; \text{ 2 flips: } \left\{ \begin{array}{l} \alpha|110\rangle + \beta|001\rangle \\ \alpha|101\rangle + \beta|010\rangle \\ \alpha|011\rangle + \beta|100\rangle \end{array} \right. ; \text{ 3 flips: } \alpha|111\rangle + \beta|000\rangle.$$

- Key fact: 4 orthogonal planes $\mathcal{P}_c = \text{span}(|000\rangle, |111\rangle)$, $\mathcal{P}_1 = \text{span}(|100\rangle, |011\rangle)$, $\mathcal{P}_2 = \text{span}(|010\rangle, |101\rangle)$ and $\mathcal{P}_3 = \text{span}(|001\rangle, |110\rangle)$.
- Error syndromes: 3 commuting observables $\boldsymbol{S}_1 = \boldsymbol{I} \otimes \boldsymbol{Z} \otimes \boldsymbol{Z}$, $\boldsymbol{S}_2 = \boldsymbol{Z} \otimes \boldsymbol{I} \otimes \boldsymbol{Z}$ and $\boldsymbol{S}_3 = \boldsymbol{Z} \otimes \boldsymbol{Z} \otimes \boldsymbol{I}$ with spectrum $\{-1, +1\}$ and outcomes $(s_1, s_2, s_3) \in \{-1, +1\}$.

-1- $s_1 = s_2 = s_3$: $\mathcal{P}_c \ni |\psi_L\rangle = \left\{ \begin{array}{l} \alpha|000\rangle + \beta|111\rangle \text{ 0 flip} \\ \beta|000\rangle + \alpha|111\rangle \text{ 3 flips} \end{array} \right.$; no correction

-2- $s_1 \neq s_2 = s_3$: $\mathcal{P}_1 \ni |\psi_L\rangle = \left\{ \begin{array}{l} \alpha|100\rangle + \beta|011\rangle \text{ 1 flip} \\ \beta|100\rangle + \alpha|011\rangle \text{ 2 flips} \end{array} \right.$; $(\boldsymbol{X} \otimes \boldsymbol{I} \otimes \boldsymbol{I})|\psi_L\rangle \in \mathcal{P}_c$.

-3- $s_2 \neq s_3 = s_1$: $\mathcal{P}_2 \ni |\psi_L\rangle = \left\{ \begin{array}{l} \alpha|010\rangle + \beta|101\rangle \text{ 1 flip} \\ \beta|010\rangle + \alpha|101\rangle \text{ 2 flips} \end{array} \right.$; $(\boldsymbol{I} \otimes \boldsymbol{X} \otimes \boldsymbol{I})|\psi_L\rangle \in \mathcal{P}_c$.

-4- $s_3 \neq s_1 = s_2$: $\mathcal{P}_3 \ni |\psi_L\rangle = \left\{ \begin{array}{l} \alpha|001\rangle + \beta|110\rangle \text{ 1 flip} \\ \beta|001\rangle + \alpha|110\rangle \text{ 2 flips} \end{array} \right.$; $(\boldsymbol{I} \otimes \boldsymbol{I} \otimes \boldsymbol{X})|\psi_L\rangle \in \mathcal{P}_c$.

[16] $\boldsymbol{X} = |1\rangle\langle 0| + |0\rangle\langle 1|$ and $\boldsymbol{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

[17] M.A Nielsen, I.L. Chuang (2000): Quantum Computation and Quantum Information.Cambridge University Press.

The 3-qubit phase-flip code

- Local phase-flip error: each physical qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ becomes $\boldsymbol{Z}|\psi\rangle = \alpha|0\rangle - \beta|1\rangle$ [18] with probability $p < 1/2$ during Δt.
- Since $\boldsymbol{X} = \boldsymbol{HZH}$ and $\boldsymbol{Z} = \boldsymbol{HXH}$ ($\boldsymbol{H}^2 = \boldsymbol{I}$), use the 3-qubit bit flip code in the frame defined by $\boldsymbol{H}$:

$$|0\rangle \mapsto \frac{|0\rangle + |1\rangle}{\sqrt{2}} \triangleq |+\rangle, \quad |1\rangle \mapsto \frac{|0\rangle - |1\rangle}{\sqrt{2}} \triangleq |-\rangle, \quad \boldsymbol{X} \mapsto \boldsymbol{HXH} = \boldsymbol{Z} = |+\rangle\langle+| + |-\rangle\langle-|.$$

- $t = +$: $|\psi_L\rangle = \alpha|+_L\rangle + \beta|-_L\rangle$ with $|+_L\rangle = |+++\rangle$ and $|-_L\rangle = |---\rangle$.
- $t = \Delta t$: $|\psi_L\rangle$ becomes with

$$\text{1 flip: } \begin{cases} \alpha|-++\rangle + \beta|+--\rangle \\ \alpha|+-+\rangle + \beta|-+-\rangle \\ \alpha|++-\rangle + \beta|--+\rangle \end{cases} ; \text{ 2 flips: } \begin{cases} \alpha|--+\rangle + \beta|++-\rangle \\ \alpha|-+-\rangle + \beta|+-+\rangle \\ \alpha|+--\rangle + \beta|-++\rangle \end{cases} ; \text{ 3 flips: } \alpha|---\rangle + \beta|+++\rangle.$$

- Key fact: 4 orthogonal planes $\mathcal{P}_c = \text{span}(|+++\rangle, |---\rangle)$, $\mathcal{P}_1 = \text{span}(|-++\rangle, |+--\rangle)$, $\mathcal{P}_2 = \text{span}(|+-+\rangle, |-+-\rangle)$ and $\mathcal{P}_3 = \text{span}(|++-\rangle, |--+\rangle)$.
- Error syndromes: 3 commuting observables $\boldsymbol{S}_1 = \boldsymbol{I} \otimes \boldsymbol{X} \otimes \boldsymbol{X}$, $\boldsymbol{S}_2 = \boldsymbol{X} \otimes \boldsymbol{I} \otimes \boldsymbol{X}$ and $\boldsymbol{S}_3 = \boldsymbol{X} \otimes \boldsymbol{X} \otimes \boldsymbol{I}$ with spectrum $\{-1, +1\}$ and outcomes $(s_1, s_2, s_3) \in \{-1, +1\}$.

-1- $s_1 = s_2 = s_3$: $\mathcal{P}_c \ni |\psi_L\rangle = \begin{cases} \alpha|+++\rangle + \beta|---\rangle & \text{0 flip} \\ \beta|+++\rangle + \alpha|---\rangle & \text{3 flips} \end{cases}$; no correction

-2- $s_1 \neq s_2 = s_3$: $\mathcal{P}_1 \ni |\psi_L\rangle = \begin{cases} \alpha|-++\rangle + \beta|+--\rangle & \text{1 flip} \\ \beta|-++\rangle + \alpha|+--\rangle & \text{2 flips} \end{cases}$; $(\boldsymbol{Z} \otimes \boldsymbol{I} \otimes \boldsymbol{I})|\psi_L\rangle \in \mathcal{P}_c$.

-3- $s_2 \neq s_3 = s_1$: $\mathcal{P}_2 \ni |\psi_L\rangle = \begin{cases} \alpha|+-+\rangle + \beta|-+-\rangle & \text{1 flip} \\ \beta|+-+\rangle + \alpha|-+-\rangle & \text{2 flips} \end{cases}$; $(\boldsymbol{I} \otimes \boldsymbol{Z} \otimes \boldsymbol{I})|\psi_L\rangle \in \mathcal{P}_c$.

-4- $s_3 \neq s_1 = s_2$: $\mathcal{P}_3 \ni |\psi_L\rangle = \begin{cases} \alpha|++-\rangle + \beta|--+\rangle & \text{1 flip} \\ \beta|++-\rangle + \alpha|--+\rangle & \text{2 flips} \end{cases}$; $(\boldsymbol{I} \otimes \boldsymbol{I} \otimes \boldsymbol{Z})|\psi_L\rangle \in \mathcal{P}_c$.

[18] $\boldsymbol{X} = |1\rangle\langle 0| + |0\rangle\langle 1|$, $\boldsymbol{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|$ and $\boldsymbol{H} = \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\langle 0| + \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\langle 1|$.

- Take the phase flip code $|+++\rangle$ and $|---\rangle$. Replace each $|+\rangle$ (resp. $|-\rangle$) by $\frac{|000\rangle+|111\rangle}{\sqrt{2}}$ (resp. $\frac{|000\rangle-|111\rangle}{\sqrt{2}}$).

New logical qubit $|\psi_L\rangle = \alpha|0_L\rangle + \beta|1_L\rangle \in \mathbb{C}^{2^9} \equiv \mathbb{C}^{512}$:

$$|0_L\rangle = \frac{\big(|000\rangle+|111\rangle\big)\big(|000\rangle+|111\rangle\big)\big(|000\rangle+|111\rangle\big)}{2\sqrt{2}}, \; |1_L\rangle = \frac{\big(|000\rangle-|111\rangle\big)\big(|000\rangle-|111\rangle\big)\big(|000\rangle-|111\rangle\big)}{2\sqrt{2}}$$

- Local errors: each of the 9 physical qubits can have a bit-flip $\boldsymbol{X}$, a phase flip $\boldsymbol{Z}$ or a bit flip followed by a phase flip $\boldsymbol{ZX} = i\boldsymbol{Y}$ [19] with probability p during Δt.
- Denote by $\boldsymbol{X}_k$ (resp. $\boldsymbol{Y}_k$ and $\boldsymbol{Z}_k$), the local operator $\boldsymbol{X}$ (resp. $\boldsymbol{Y}$ and $\boldsymbol{Z}$) acting on physical qubit no $k \in \{1, \dots, 9\}$. Denote by $\mathcal{P}_c = \text{span}(|0_L\rangle, |1_L\rangle)$ the code space. One get a family of the $1 + 3 \times 9 =$ **28 orthogonal planes:**

$$\mathcal{P}_c, \quad \Big(\boldsymbol{X}_k\mathcal{P}_c\Big)_{k=1,\dots,9}, \quad \Big(\boldsymbol{Y}_k\mathcal{P}_c\Big)_{k=1,\dots,9}, \quad \Big(\boldsymbol{Z}_k\mathcal{P}_c\Big)_{k=1,\dots,9}.$$

- One can always construct error syndromes to obtain, when there is only one error among the 9 qubits during Δt, the number k of the qubit and the error type it has undergone ($\boldsymbol{X}$, $\boldsymbol{Y}$ or $\boldsymbol{Z}$). These 28 planes are then eigen-planes by the syndromes.
- If the physical qubit k is subject to **any kind of local errors** associated to arbitrary operator $\boldsymbol{M}_k = g\boldsymbol{I} + a\boldsymbol{X}_k + b\boldsymbol{Y}_k + c\boldsymbol{Z}_k$ ($g, a, b, c \in \mathbb{C}$), $|\psi_L\rangle \mapsto \frac{\boldsymbol{M}_k|\psi_L\rangle}{\sqrt{\langle\psi_L|\boldsymbol{M}_k^\dagger\boldsymbol{M}_k|\psi_L\rangle}}$, the syndrome measurements will project the corrupted logical qubit on one of the 4 planes $\mathcal{P}_c$, $\boldsymbol{X}_k\mathcal{P}_c$, $\boldsymbol{Y}_k\mathcal{P}_c$ or $\boldsymbol{Z}_k\mathcal{P}_c$. It is then simple by using either $\boldsymbol{I}$, $\boldsymbol{X}_k$, $\boldsymbol{Y}_k$ or $\boldsymbol{Z}_k$, to recover up to a global phase the original logical qubit $|\psi_L\rangle$.

[19] $\boldsymbol{X} = |1\rangle\langle 0| + |0\rangle\langle 1|$, $\boldsymbol{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|$ and $\boldsymbol{Y} = i|1\rangle|0\rangle - i|0\rangle|1\rangle$.

Practical open issues with usual QEC

• For a logical qubit relying on n physical qubits, the dimension of the Hilbert has to be larger than $2(1+3n)$ to recover a single but arbitrary qubit error: $2^n \geq 2(1+3n)$ imposing $n \geq 5$ ($\mathcal{H} = \mathbb{C}^{2^5} = \mathbb{C}^{32}$)

• Efficient constructions of quantum error-correcting codes: stabilizer codes, surface codes where the physical qubits are located on a 2D-lattice, topological codes, ...

• Fault tolerant computations: computing on encoded quantum states; fault-tolerant operations to avoid propagations of errors during encoding, gates and measurement; concatenation and threshold theorem, ...

• Actual experiments: 10^{-2} is the typical error probability during elementary gates involving few physical qubits.

• High-order error-correcting codes with an important overhead; **more than 1000 physical qubits to encode a logical qubit** [20] $\mathcal{H} \sim \mathbb{C}^{2^{1000}}$.

[20] A.G. Fowler, M. Mariantoni, J.M. Martinis, A.N. Cleland (2012): Surface codes: Towards practical large-scale quantum computation. Phys. Rev. A,86(3):032324.

- System S (LC circuit): a high-quality harmonic oscillator ($1 \ll \omega_s/\kappa_s \sim 10^6$):

$$\frac{d}{dt}\widetilde{\boldsymbol{\rho}}_s = -i\omega_s[\boldsymbol{N}_s, \widetilde{\boldsymbol{\rho}}_s] + \kappa_s \mathbb{D}_{\boldsymbol{a}_s}(\widetilde{\boldsymbol{\rho}}_s)$$

with $\mathcal{H}_s = \text{span}\{|0\rangle, |1\rangle, \ldots, |n\rangle, \ldots\}$, photon-number operator $\boldsymbol{N}_s$ and annihilation operator $\boldsymbol{a}_s$ ($\boldsymbol{N}_s = \boldsymbol{a}_s^\dagger \boldsymbol{a}_s$, $\boldsymbol{N}_s|n\rangle = n|n\rangle$, $\boldsymbol{a}_s|n\rangle = \sqrt{n}|n-1\rangle$).

- In the rotation frame ($\widetilde{\boldsymbol{\rho}}_s = e^{-i\omega_s t \boldsymbol{N}_s} \boldsymbol{\rho}_s e^{+i\omega_s t \boldsymbol{N}_s}$):

$$\frac{d}{dt}\boldsymbol{\rho}_s = \kappa_s \mathbb{D}_{\boldsymbol{a}_s}(\boldsymbol{\rho}_s) = \kappa_s \left(\boldsymbol{a}_s \boldsymbol{\rho} \boldsymbol{a}_s^\dagger - \tfrac{1}{2}(\boldsymbol{a}_s^\dagger \boldsymbol{a}_s \boldsymbol{\rho} + \boldsymbol{\rho} \boldsymbol{a}_s^\dagger \boldsymbol{a}_s) \right).$$

- Goal: engineer a logical qubit with superpositions of coherent states

$$|\beta\rangle = e^{-\frac{|\beta|^2}{2}} \sum_{n \geq 0} \frac{\beta^n}{\sqrt{n!}} |n\rangle$$

of complex amplitudes $\beta \in \{\alpha, i\alpha, -\alpha, -i\alpha\}$ with $\alpha \gg 1$ (typically $\alpha \geq 3$).

- Coherent states are robust to decoherence: $\boldsymbol{a}_s|\beta\rangle = \beta|\beta\rangle$: if $\boldsymbol{\rho}_s(0) = |\beta_0\rangle\langle\beta_0|$, then $\boldsymbol{\rho}_s(t) = |\beta_t\rangle\langle\beta_t|$ with $\beta_t = \beta_0 e^{-\kappa_s t/2}$.

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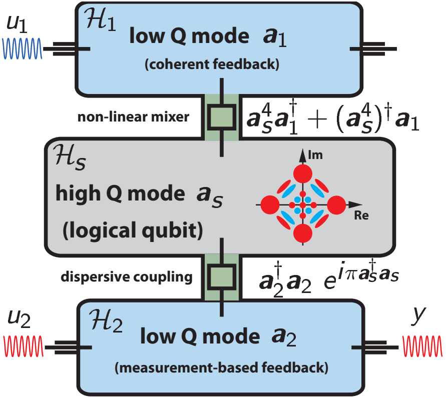
[21] M.Mirrahimi, Z. Leghtas . . . , M. Devoret. (2014). Dynamically protected cat-qubits: a new paradigm for universal quantum computation. New Journal of Physics, 16, 045014.

Simplified feedback scheme for a possible cat-qubit experiment[22]



Stochastic master equation on $\mathcal{H}_s \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$:

$$d\rho = \kappa_s \mathbb{D}_{a_s}(\rho)\, dt - i[\boldsymbol{H}, \rho]\, dt + \kappa_1 \mathbb{D}_{a_1}(\rho)\, dt + \kappa_2 \mathbb{D}_{a_2}(\rho)\, dt + \sqrt{\eta\kappa_2}\left(i a_2 \rho - i \rho a_2^\dagger - \operatorname{Tr}\left(i a_2 \rho - i \rho a_2^\dagger\right) \rho\right) dW_t$$

Engineered Hamiltonian

$$\boldsymbol{H} = i(u_1 a_1^\dagger - u_1^* a_1) + i(u_2 a_2^\dagger - u_2^* a_2) + g_1\left(a_s^4 a_1^\dagger + (a_s^4)^\dagger a_1\right) + g_2 a_2^\dagger a_2 \; e^{i\pi a_s^\dagger a_s},$$

Classical control inputs $u_1, u_2 \in \mathbb{C}$

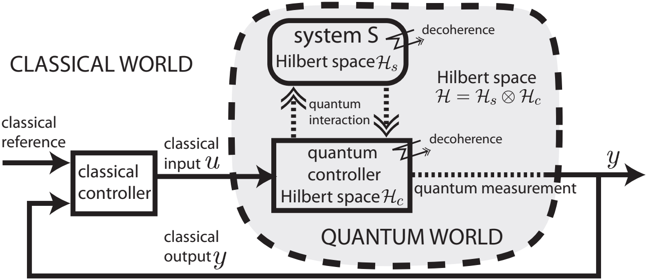
Measurement output classical signal:

$$dy_t = \sqrt{\eta\kappa_2}\operatorname{Tr}\left(i a_2 \rho - i \rho a_2^\dagger\right) dt + dW_t$$

Many time-scales, $\kappa_s \ll g_1, g_2 \ll \kappa_1, \kappa_2$ and $\kappa_s \ll \frac{g_1^2}{\kappa_1}, \frac{g_2^2}{\kappa_2}$, providing a reduced slow SME on span$\{|\alpha\rangle, |i\alpha\rangle, |\text{-}\alpha\rangle, |\text{-}i\alpha\rangle\}$ with $\alpha = \sqrt[4]{u_1/g_1}$

[22] Inspired by
- N. Ofek, ..., M. Mirrahimi, M. Devoret, R. Schoelkopf (2016). Extending the lifetime of a quantum bit with error correction in superconducting circuits. Nature, 536(7617),441-445.
- R. Lescanne, ..., M. Mirrahimi, Z. Leghtas (2019): Exponential suppression of bit-flips in a qubit encoded in an oscillator. arXiv:1907.11729 [quant-ph].

Quantum feedback engineering for robust quantum information processing



To protect quantum information stored in system S (alternative to usual QEC):

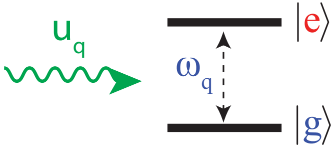
- fast stabilization and protection mainly achieved by a **quantum controller** (coherent feedback stabilizing decoherence-free sub-spaces);
- slow decoherence and perturbations mainly tackled by a **classical controller** (measurement-based feedback "finishing the job")

Underlying mathematical methods for high-precision dynamical modeling and control based on stochastic master equations (SME):

- High-order averaging methods and geometric singular perturbations for coherent feedback.
- Stochastic control Lyapunov methods for exponential stabilization via measurement-based feedback.

Qubit (2-level system, half-spin) [23]

- Hilbert space:
  $\mathcal{H}_M = \mathbb{C}^2 = \left\{ c_g |g\rangle + c_e |e\rangle,\ c_g, c_e \in \mathbb{C} \right\}.$
- Quantum state space:
  $\mathcal{D} = \{\boldsymbol{\rho} \in \mathcal{L}(\mathcal{H}_M), \boldsymbol{\rho}^\dagger = \boldsymbol{\rho},\ \mathrm{Tr}(\boldsymbol{\rho}) = 1, \boldsymbol{\rho} \geq 0\}$.
- Operators and commutations:
  $\boldsymbol{\sigma}_- = |g\rangle\langle e|$, $\boldsymbol{\sigma}_+ = \boldsymbol{\sigma}_-{}^\dagger = |e\rangle\langle g|$
  $\boldsymbol{X} \equiv \boldsymbol{\sigma_x} = \boldsymbol{\sigma}_- + \boldsymbol{\sigma}_+ = |g\rangle\langle e| + |e\rangle\langle g|;$
  $\boldsymbol{Y} \equiv \boldsymbol{\sigma_y} = i\boldsymbol{\sigma}_- - i\boldsymbol{\sigma}_+ = i|g\rangle\langle e| - i|e\rangle\langle g|;$
  $\boldsymbol{Z} \equiv \boldsymbol{\sigma_z} = \boldsymbol{\sigma}_+\boldsymbol{\sigma}_- - \boldsymbol{\sigma}_-\boldsymbol{\sigma}_+ = |e\rangle\langle e| - |g\rangle\langle g|;$
  $\boldsymbol{\sigma_x}^2 = \boldsymbol{I}$, $\boldsymbol{\sigma_x}\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_z}$, $[\boldsymbol{\sigma_x}, \boldsymbol{\sigma_y}] = 2i\boldsymbol{\sigma_z}$, ...

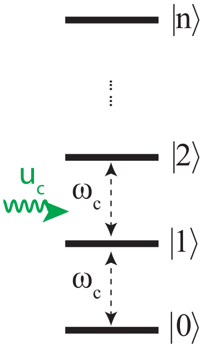


- Hamiltonian: $\boldsymbol{H}_M = \omega_q \boldsymbol{\sigma_z}/2 + \boldsymbol{u_q}\boldsymbol{\sigma_x}.$
- Bloch sphere representation:
  $\mathcal{D} = \left\{ \frac{1}{2}\left(\boldsymbol{I} + x\boldsymbol{\sigma_x} + y\boldsymbol{\sigma_y} + z\boldsymbol{\sigma_z}\right) \ \middle|\ (x, y, z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 \leq 1 \right\}$

[23] See S. M. Barnett, P.M. Radmore (2003): Methods in Theoretical Quantum Optics. Oxford University Press.

Quantum harmonic oscillator (spring system) [23]

- Hilbert space:
  $\mathcal{H}_S = \left\{ \sum_{n\geq 0} \psi_n |n\rangle, \; (\psi_n)_{n\geq 0} \in l^2(\mathbb{C}) \right\} \equiv L^2(\mathbb{R}, \mathbb{C})$
- Quantum state space:
  $\mathcal{D} = \{\boldsymbol{\rho} \in \mathcal{L}(\mathcal{H}_S), \boldsymbol{\rho}^\dagger = \boldsymbol{\rho}, \; \text{Tr}\left(\boldsymbol{\rho}\right) = 1, \boldsymbol{\rho} \geq 0\}$.
- Operators and commutations:
  $\boldsymbol{a}|n\rangle = \sqrt{n} \; |\boldsymbol{n\text{-}1}\rangle$, $\boldsymbol{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$;
  $\boldsymbol{N} = \boldsymbol{a}^\dagger \boldsymbol{a}$, $\boldsymbol{N}|n\rangle = n|n\rangle$;
  $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = \boldsymbol{I}$, $\boldsymbol{a} f(\boldsymbol{N}) = f(\boldsymbol{N}+\boldsymbol{I})\boldsymbol{a}$;
  $\boldsymbol{D}_\alpha = e^{\alpha \boldsymbol{a}^\dagger - \alpha^\dagger \boldsymbol{a}}$.
  $\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$, $[\boldsymbol{X}, \boldsymbol{P}] = \imath \boldsymbol{I}/2$.



- Hamiltonian: $\boldsymbol{H}_S = \omega_c \boldsymbol{a}^\dagger \boldsymbol{a} + \boldsymbol{u_c}(\boldsymbol{a} + \boldsymbol{a}^\dagger)$.
  (associated classical dynamics:
  $\frac{dx}{dt} = \omega_c p$, $\frac{dp}{dt} = -\omega_c x - \sqrt{2} u_c$).
- Classical pure state $\equiv$ coherent state $|\alpha\rangle$
  $\alpha \in \mathbb{C}$: $|\alpha\rangle = \sum_{n\geq 0}\left(e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}\right)|n\rangle$; $|\alpha\rangle \equiv \frac{1}{\pi^{1/4}} e^{\imath\sqrt{2}x\Im\alpha} e^{-\frac{(x-\sqrt{2}\Re\alpha)^2}{2}}$
  $\boldsymbol{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\boldsymbol{D}_\alpha|0\rangle = |\alpha\rangle$.