Quantum Filtering and Dynamical Parameter Estimation

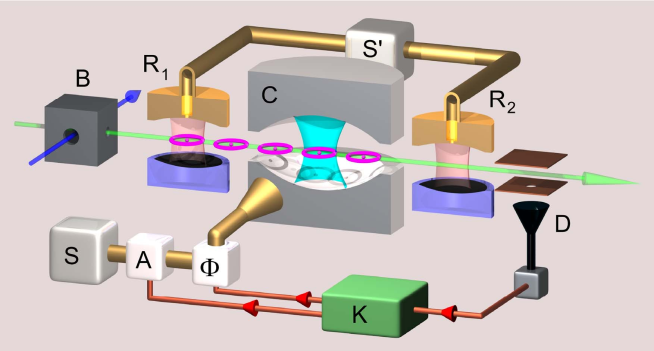
pierre.rouchon@mines-paristech.fr

Isaac Newton Institute, Cambridge, July 2014.

Based on collaborations with
Hadis Amini, Michel Brune, Igor Dotsenko, Serge Haroche,
Zaki Leghtas, Mazyar Mirrahimi, Clément Pellegrini,
Jean-Michel Raimond, Clément Sayrin and Ram Somaraju

The LKB photon Box

Group of Serge Haroche, Jean-Michel Raimond and Michel Brune.[1]



Stabilization by a measurement-based feedback of photon-number states (sampling time 80 μs)

Experiment: C. Sayrin et. al., Nature 477, 73-77, September 2011.

Theory: I. Dotsenko et al., Physical Review A, 2009, 80: 013805-013813.

H. Amini et. al., Automatica, 49 (9): 2683-2692, 2013.

[1] Animation realized by Igor Dotsenko

Stabilization around 3-photon state

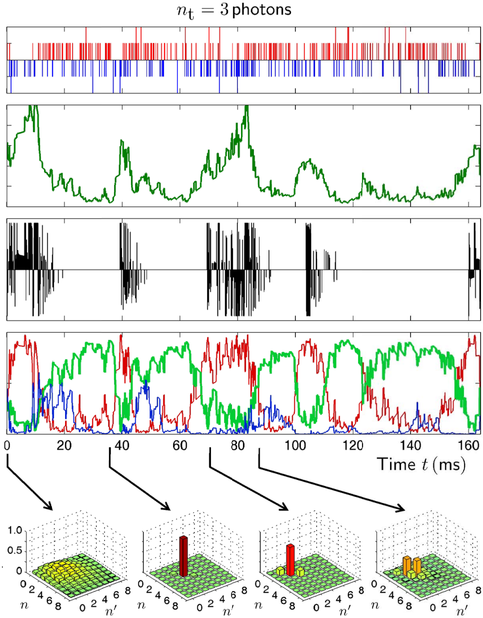
C. Sayrin et. al., Nature 477, 73-77, Sept. 2011.

Decoherence due to finite photon life time around 70 ms)

Detection efficiency 40%
Detection error rate 10%
Delay 4 sampling periods

The quantum filter takes into account cavity decoherence, measurement imperfections and delays (Bayes law).

Truncation to 9 photons



Ideal model: Markov chain with input u, hidden state ρ and output y

Input: control $u = Ae^{\imath\Phi}$ describing the classical EM pulse .
Quantum state : ρ the density operator of the photons .
Output: $y \in \{g, e\}$ measurement of the atom.

$$\rho_{k+1} = \begin{cases} \dfrac{D_{u_k} M_g \rho_k M_g^\dagger D_{u_k}^\dagger}{\mathrm{Tr}\left(M_g \rho_k M_g^\dagger\right)}, & y_k = g \text{ with proba. } \mathbb{P}_{g,k} = \mathrm{Tr}\left(M_g \rho_k M_g^\dagger\right) \\ \dfrac{D_{u_k} M_e \rho_k M_e^\dagger D_{u_k}^\dagger}{\mathrm{Tr}\left(M_e \rho_k M_e^\dagger\right)}, & y_k = e \text{ with proba. } \mathbb{P}_{e,k} = \mathrm{Tr}\left(M_e \rho_k M_e^\dagger\right) \end{cases}$$

QND measurement operators: $M_g = \cos\left(\frac{\phi_0(\mathbf{N}+1/2)+\phi_R}{2}\right)$ et $M_e = \sin\left(\frac{\phi_0(\mathbf{N}+1/2)+\phi_R}{2}\right)$ with $\mathbf{N} = \mathbf{a}^\dagger\mathbf{a} = \mathrm{diag}(0, 1, 2, \ldots)$.
Unitary control operator : $D_u = e^{u\mathbf{a}^\dagger - u^*\mathbf{a}}$ where $\mathbf{a}$ is the photon annihilation operator.
Goal : stabilize state with exactly $\bar{n}$ photon(s), $\bar{\rho} = |\bar{n}\rangle\langle\bar{n}|$, that are open-loop stationary state for $u = 0$.

Structure of this quantum-state feedback (ideal case)[2]

Observer-Controller

- Non linear filtering of the measurements $\boldsymbol{k} \mapsto \boldsymbol{y_k}$ provides an estimate $\boldsymbol{\rho}^{\text{est}}$ of ρ:

$$\boldsymbol{\rho}^{\text{est}}_{\boldsymbol{k+1}} = \frac{D_{u_k} M_{\boldsymbol{y_k}} \boldsymbol{\rho}^{\text{est}}_{\boldsymbol{k}} M^\dagger_{\boldsymbol{y_k}} D^\dagger_{u_k}}{\text{Tr}\left(M_{\boldsymbol{y_k}} \boldsymbol{\rho}^{\text{est}}_{\boldsymbol{k}} M^\dagger_{\boldsymbol{y_k}} \right)}.$$

  Quantum filter in the sense of Belavkin.

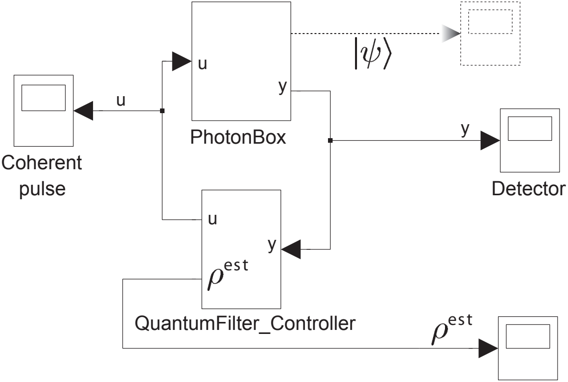
- The stabilizing feedback $u_k = f(\boldsymbol{\rho}^{\text{est}}_{\boldsymbol{k}})$ ensuring convergence towards $\bar{\rho}$ is based on Lyapunov design:

$$u_k = \underset{u}{\text{Argmin}} \quad \mathbb{E}\left(V(\rho_{k+1}) \mid \rho_k = \boldsymbol{\rho}^{\text{est}}_{\boldsymbol{k}},\ u \right)$$

  where V is a well chosen super-martingale constructed with open-loop martingales attached to the QND process.

[2] The global convergence proof of such observer/controller for the realistic case is given in H. Amini et. al., Automatica, 49 (9): 2683-2692, 2013.

In the realistic case: $|\psi_k\rangle$, ρ_k and ρ_k^{est}.



The state estimation ρ_k^{est} used in the feedback law takes into account, measurement imperfections, delays and cavity decoherence:

- Derived from Bayes law: depends on past detector outcomes between 0 and k; computed recursively from an initial value ρ_0^{est};
- Stable and tends to converge towards ρ_k, **the expectation value** of $|\psi_k\rangle\langle\psi_k|$ knowing its initial value $|\psi_0\rangle\langle\psi_0|$ and the past detector outcomes from 0 to k.

Outline

Quantum filtering: discrete-time case

Quantum filtering: continuous-time case

Conclusion

For the photon box, quantum filtering combines

1. Bayes law: $\mathbb{P}(\mu'/\mu) = \mathbb{P}(\mu/\mu')\mathbb{P}(\mu') \,/\, \left(\sum_{\nu'} \mathbb{P}(\mu/\nu')\mathbb{P}(\nu')\right)$.
2. Schrödinger equations defining unitary transformations.
3. Partial collapse of the wave packet: irreversibility and convergence are induced by the measurement of observables $\mathcal{O}$ with degenerate spectra, $\mathcal{O} = \sum_\mu \lambda_\mu P_\mu$:
   - measurement outcome λ_μ with proba. $\mathbb{P}_\mu = \langle\psi|P_\mu|\psi\rangle = \text{Tr}\,(\rho P_\mu)$ depending $|\psi\rangle$, ρ just before the measurement
   - measurement back-action if outcome μ:

$$|\psi\rangle \mapsto |\psi\rangle + = \frac{P_\mu|\psi\rangle}{\sqrt{\langle\psi|P_\mu|\psi\rangle}}, \quad \rho \mapsto \rho_+ = \frac{P_\mu \rho P_\mu}{\text{Tr}\,(\rho P_\mu)}$$

4. Tensor product for the description of composite systems (S, M):
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $H = H_S \otimes \mathbb{I}_M + H_{int} + \mathbb{I}_S \otimes H_M$
   - observable on sub-system M only: $\mathcal{O} = \mathbb{I}_S \otimes \mathcal{O}_M$.

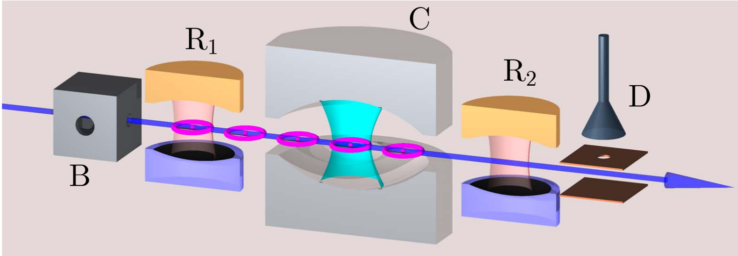
- System S corresponds to a quantized cavity mode:

$$\mathcal{H}_S = \left\{ \sum_{n=0}^{\infty} \psi^n |n\rangle \mid (\psi^n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

  where $|n\rangle$ represents the Fock state associated to exactly n photons inside the cavity
- Meter M is associated to atoms : $\mathcal{H}_M = \mathbb{C}^2$, each atom admits two energy levels and is described by a wave function $c_g|g\rangle + c_e|e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$; atoms leaving B are all in state $|g\rangle$
- When an atom comes out B, the state $|\Psi\rangle_B \in \mathcal{H}_S \otimes \mathcal{H}_M$ of the composite system atom/field is separable

$$|\Psi\rangle_B = |\psi\rangle \otimes |g\rangle.$$

LKB photon-box: Markov chain in the ideal case (2)



- When an atom comes out B: $|\Psi\rangle_B = |\psi\rangle \otimes |g\rangle$.
- Just before the measurement in D, the state is in general entangled (not separable):

$$|\Psi\rangle_{R_2} = U_{SM}\big(|\psi\rangle \otimes |g\rangle\big) = \big(M_g|\psi\rangle\big) \otimes |g\rangle + \big(M_e|\psi\rangle\big) \otimes |e\rangle$$

where U_{SM} is the total unitary transformation (Schrödinger propagator) defining the linear measurement operators M_g and M_e on $\mathcal{H}_S$. Since U_{SM} is unitary, $M_g^\dagger M_g + M_e^\dagger M_e = \mathbb{I}$.

LKB photon-box: Markov chain in the ideal case (3)

Just before the measurement in D, the atom/field state is:

$$M_g|\psi\rangle \otimes |g\rangle + M_e|\psi\rangle \otimes |e\rangle$$

Denote by $\mu \in \{g, e\}$ the measurement outcome in detector D: with probability $\mathbb{P}_\mu = \langle\psi|M_\mu^\dagger M_\mu|\psi\rangle$ we get μ. Just after the measurement outcome μ, the state becomes separable:

$$|\Psi\rangle_D = \frac{1}{\sqrt{\mathbb{P}_\mu}} \left(M_\mu|\psi\rangle\right) \otimes |\mu\rangle = \frac{(M_\mu|\psi\rangle) \otimes |\mu\rangle}{\sqrt{\langle\psi|M_\mu^\dagger M_\mu|\psi\rangle}}.$$

Markov process (density matrix formulation $\rho \sim |\psi\rangle\langle\psi|$)

$$\rho_+ = \begin{cases} \dfrac{M_g \rho M_g^\dagger}{\text{Tr}\left(M_g \rho M_g^\dagger\right)}, & \text{with probability } \mathbb{P}_g = \text{Tr}\left(M_g \rho M_g^\dagger\right); \\ \dfrac{M_e \rho M_e^\dagger}{\text{Tr}\left(M_e \rho M_e^\dagger\right)}, & \text{with probability } \mathbb{P}_e = \text{Tr}\left(M_e \rho M_e^\dagger\right). \end{cases}$$

Kraus map: $\mathbb{E}\left(\rho_+|\rho\right) = \mathbf{K}(\rho) = M_g \rho M_g^\dagger + M_e \rho M_e^\dagger.$

LKB photon-box: Markov process with detection errors (1)

- With pure state $\rho = |\psi\rangle\langle\psi|$, we have
$$\rho_+ = |\psi_+\rangle\langle\psi_+| = \frac{1}{\text{Tr}\left(M_\mu \rho M_\mu^\dagger\right)} M_\mu \rho M_\mu^\dagger$$
when the atom collapses in $\mu = g, e$ with proba. $\text{Tr}\left(M_\mu \rho M_\mu^\dagger\right)$.
- Detection error rates: $\mathbb{P}(y = e/\mu = g) = \eta_g \in [0,1]$ the probability of erroneous assignation to e when the atom collapses in g; $\mathbb{P}(y = g/\mu = e) = \eta_e \in [0,1]$ (given by the contrast of the Ramsey fringes).

Bayes law: expectation ρ_+ of $|\psi_+\rangle\langle\psi_+|$ knowing ρ and the imperfect detection y.

$$\rho_+ = \begin{cases} \frac{(1-\eta_g) M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger}{\text{Tr}\left((1-\eta_g) M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger\right)} \text{if } y = g, \text{ prob. } \text{Tr}\left((1-\eta_g) M_g \rho M_g^\dagger + \eta_e M_e \rho M_e^\dagger\right); \\ \frac{\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger}{\text{Tr}\left(\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger\right)} \text{if } y = e, \text{ prob. } \text{Tr}\left(\eta_g M_g \rho M_g^\dagger + (1-\eta_e) M_e \rho M_e^\dagger\right). \end{cases}$$

ρ_+ does not remain pure: the quantum state ρ_+ becomes a mixed state; $|\psi_+\rangle$ becomes physically irrelevant (not numerically).

Photon-box quantum filter: 6×21 left stochastic matrix $(\eta_{\mu',\mu})$

$$\boldsymbol{\rho}^{est}_{\boldsymbol{k+1}} = \frac{1}{\mathrm{Tr}\left(\sum_\mu \eta_{\mu',\mu} M_\mu \boldsymbol{\rho}^{est}_{\boldsymbol{k}} M_\mu^\dagger\right)} \left(\sum_\mu \eta_{\mu',\mu} M_\mu \boldsymbol{\rho}^{est}_{\boldsymbol{k}} M_\mu^\dagger\right) \text{ with}$$

- we have a total of $m = 3 \times 7 = 21$ Kraus operators M_μ. The "jumps" are labeled by $\mu = (\mu^a, \mu^c)$ with $\mu^a \in \{no, g, e, gg, ge, eg, ee\}$ labeling atom related jumps and $\mu^c \in \{o, +, -\}$ cavity decoherence jumps.
- we have only $m' = 6$ real detection possibilities $\mu' \in \{no, g, e, gg, ge, ee\}$ corresponding respectively to no detection, a single detection in g, a single detection in e, a double detection both in g, a double detection one in g and the other in e, and a double detection both in e.

$\mu' \setminus \mu$	(no, μ^c)	(g, μ^c)	(e, μ^c)	(gg, μ^c)	(ee, μ^c)	(ge, μ^c) (eg, μ^c)
no	1	$1-\epsilon_d$	$1-\epsilon_d$	$(1-\epsilon_d)^2$	$(1-\epsilon_d)^2$	$(1-\epsilon_d)^2$
g	0	$\epsilon_d(1-\eta_g)$	$\epsilon_d\eta_e$	$2\epsilon_d(1-\epsilon_d)(1-\eta_g)$	$2\epsilon_d(1-\epsilon_d)\eta_e$	$\epsilon_d(1-\epsilon_d)(1-\eta_g+\eta_e)$
e	0	$\epsilon_d\eta_g$	$\epsilon_d(1-\eta_e)$	$2\epsilon_d(1-\epsilon_d)\eta_g$	$2\epsilon_d(1-\epsilon_d)(1-\eta_e)$	$\epsilon_d(1-\epsilon_d)(1-\eta_e+\eta_g)$
gg	0	0	0	$\epsilon_d^2(1-\eta_g)^2$	$\epsilon_d^2\eta_e^2$	$\epsilon_d^2\eta_e(1-\eta_g)$
ge	0	0	0	$2\epsilon_d^2\eta_g(1-\eta_g)$	$2\epsilon_d^2\eta_e(1-\eta_e)$	$\epsilon_d^2((1-\eta_g)(1-\eta_e)+\eta_g\eta_e)$
ee	0	0	0	$\epsilon_d^2\eta_g^2$	$\epsilon_d^2(1-\eta_e)^2$	$\epsilon_d^2\eta_g(1-\eta_e)$

Take $|\psi_{k+1}\rangle\langle\psi_{k+1}| = \frac{1}{\mathrm{Tr}\left(M_{\mu_k}|\psi_k\rangle\langle\psi_k|M_{\mu_k}^\dagger\right)}\left(M_{\mu_k}|\psi_k\rangle\langle\psi_k|M_{\mu_k}^\dagger\right)$ with measurement imperfections and decoherence described by the left stochastic matrix η: $\eta_{\mu',\mu} \in [0,1]$ is the probability of having the imperfect outcome $\mu' \in \{1,\ldots,m'\}$ knowing that the perfect one is $\mu \in \{1,\ldots,m\}$.

The optimal quantum filter: $\rho_k = \mathbb{E}\left(|\psi_k\rangle\langle\psi_k| \,\Big|\, |\psi_0\rangle, \mu'_0, \ldots, \mu'_{k-1}\right)$ can be computed efficiently via the following recurrence

$$\rho_{k+1} = \frac{1}{\mathrm{Tr}\left(\sum_{\mu=1}^m \eta_{\mu'_k,\mu} M_\mu \rho_k M_\mu^\dagger\right)}\left(\sum_{\mu=1}^m \eta_{\mu'_k,\mu} M_\mu \rho_k M_\mu^\dagger\right)$$

where the detector outcome μ'_k takes values μ' in $\{1,\cdots,m'\}$ with probability $\mathbb{P}_{\mu',\rho_k} = \mathrm{Tr}\left(\sum_{\mu=1}^m \eta_{\mu'_k,\mu} M_\mu \rho_k M_\mu^\dagger\right)$.

Stability and convergence issues (1)

- The quantum state $\rho_k = \mathbb{E}\left(|\psi_k\rangle\langle\psi_k| \,\middle|\, |\psi_0\rangle, \mu'_0, \ldots, \mu'_{k-1} \right)$ is given by the following optimal Belavkin filtering process

$$\rho_{k+1} = \frac{1}{\mathrm{Tr}\left(\sum_{\mu=1}^{m} \eta_{\mu'_k,\mu} M_\mu \rho_k M_\mu^\dagger\right)} \left(\sum_{\mu=1}^{m} \eta_{\mu'_k,\mu} M_\mu \rho_k M_\mu^\dagger \right)$$

  with the perfect initialization: $\rho_0 = |\psi_0\rangle\langle\psi_0|$.

- Its estimate $\boldsymbol{\rho}^{\mathrm{est}}$ follows the same recurrence

$$\boldsymbol{\rho}^{\mathrm{est}}_{\boldsymbol{k+1}} = \frac{1}{\mathrm{Tr}\left(\sum_{\mu=1}^{m} \eta_{\mu'_k,\mu} M_\mu \boldsymbol{\rho}^{\mathrm{est}}_{\boldsymbol{k}} M_\mu^\dagger\right)} \left(\sum_{\mu=1}^{m} \eta_{\mu'_k,\mu} M_\mu \boldsymbol{\rho}^{\mathrm{est}}_{\boldsymbol{k}} M_\mu^\dagger \right)$$

  but with imperfect initialization $\boldsymbol{\rho}^{\mathrm{est}}_{\boldsymbol{0}} \neq |\psi_0\rangle\langle\psi_0|$.

A natural question : $\boldsymbol{\rho}^{\mathrm{est}}_{\boldsymbol{k}} \mapsto \rho_k$ when $k \mapsto +\infty$?

Stability and convergence issues (2)

Markov chain of state $(\rho_k, \boldsymbol{\rho_k^{\text{est}}})$

$$\rho_{k+1} = \frac{\sum_{\mu=1}^m \eta_{\mu'_k,\mu} M_\mu \rho_k M_\mu^\dagger}{\text{Tr}\left(\sum_{\mu=1}^m \eta_{\mu'_k,\mu} M_\mu \rho_k M_\mu^\dagger\right)}, \quad \boldsymbol{\rho_{k+1}^{\text{est}}} = \frac{\sum_{\mu=1}^m \eta_{\mu'_k,\mu} M_\mu \boldsymbol{\rho_k^{\text{est}}} M_\mu^\dagger}{\text{Tr}\left(\sum_{\mu=1}^m \eta_{\mu'_k,\mu} M_\mu \boldsymbol{\rho_k^{\text{est}}} M_\mu^\dagger\right)}$$

Proba. to get μ'_k at step k, $\text{Tr}\left(\sum_{\mu=1}^m \eta_{\mu'_k,\mu} M_\mu \rho_k M_\mu^\dagger\right)$, depends on ρ_k.

- Convergence of $\boldsymbol{\rho_k^{\text{est}}}$ towards ρ_k when $k \mapsto +\infty$ is an open problem.
  A partial result (continuous-time) due to R. van Handel: The stability of quantum Markov filters. Infin. Dimens. Anal. Quantum Probab. Relat. Top. , 2009, 12, 153-172.
- Stability[3]: the fidelity $F(\rho_k, \boldsymbol{\rho_k^{\text{est}}}) = \text{Tr}^2\left(\sqrt{\sqrt{\rho_k}\boldsymbol{\rho_k^{\text{est}}}\sqrt{\rho_k}}\right)$ is a sub-martingale for any η and M_μ:

$$\mathbb{E}\left(F(\rho_{k+1}, \boldsymbol{\rho_{k+1}^{\text{est}}})/\rho_k, \boldsymbol{\rho_k^{\text{est}}}\right) \geq F(\rho_k, \boldsymbol{\rho_k^{\text{est}}}).$$

[3] Somaraju, A.; Dotsenko, I.; Sayrin, C. & PR. Design and Stability of Discrete-Time Quantum Filters with Measurement Imperfections. American Control Conference, 2012, 5084-5089.

The key inequality underlying $F(\rho, \rho^e)$ is sub-martingale[4]

For

- any set of m matrices M_μ with $\sum_{\mu=1}^{m} M_\mu^\dagger M_\mu = 1$,
- any partition of $\{1, \ldots, m\}$ into $p \geq 1$ sub-sets $\mathcal{P}_\nu$,
- any Hermitian non-negative matrices ρ and σ of trace one,

the following inequality holds

$$\sum_{\nu=1}^{\nu=p} \text{Tr}\left(\sum_{\mu\in\mathcal{P}_\nu} M_\mu \rho M_\mu^\dagger\right) F\left(\frac{\sum_{\mu\in\mathcal{P}_\nu} M_\mu \sigma M_\mu^\dagger}{\text{Tr}\left(\sum_{\mu\in\mathcal{P}_\nu} M_\mu \sigma M_\mu^\dagger\right)}, \frac{\sum_{\mu\in\mathcal{P}_\nu} M_\mu \rho M_\mu^\dagger}{\text{Tr}\left(\sum_{\mu\in\mathcal{P}_\nu} M_\mu \rho M_\mu^\dagger\right)}\right) \geq F(\sigma, \rho)$$

where $F(\sigma, \rho) = \text{Tr}^2\left(\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}\right)$.

Proof combines Cauchy-Schwartz inequalities with a lifting procedure based on Ulhmann's theorem.

[4] PR. Fidelity is a Sub-Martingale for Discrete-Time Quantum Filters. IEEE Transactions on Automatic Control, 2011, 56, 2743-2747.

Bayesian parameter estimations[5]

Consider detector outcomes μ'_k corresponding to a parameter value $\bar{p}$ poorly known. Assume to simplify that either $\bar{p} = a$ or $\bar{p} = b$, with $a \neq b$. We can discriminate between a and b and recover $\bar{p}$ via the following Bayesian scheme using information contained in the μ'_k's:

$$\widehat{\boldsymbol{\rho}}^{\text{est}}_{\boldsymbol{a,k+1}} = \frac{\sum_\mu \eta^a_{\mu'_k,\mu} M^a_\mu \widehat{\boldsymbol{\rho}}^{\text{est}}_{\boldsymbol{a,k}} M^{a\,\dagger}_\mu}{\text{Tr}\left(\sum_{\boldsymbol{p}} \sum_\mu \eta^p_{\mu'_k,\mu} M^p_\mu \widehat{\boldsymbol{\rho}}^{\text{est}}_{\boldsymbol{p,k}} M^{p\,\dagger}_\mu\right)}, \quad \widehat{\boldsymbol{\rho}}^{\text{est}}_{\boldsymbol{b,k+1}} = \frac{\sum_\mu \eta^b_{\mu'_k,\mu} M^b_\mu \widehat{\boldsymbol{\rho}}^{\text{est}}_{\boldsymbol{b,k}} M^{b\,\dagger}_\mu}{\text{Tr}\left(\sum_{\boldsymbol{p}} \sum_\mu \eta^{\boldsymbol{p}}_{\mu'_k,\mu} M^{\boldsymbol{p}}_\mu \widehat{\boldsymbol{\rho}}^{\text{est}}_{\boldsymbol{p,k}} M^{\boldsymbol{p}\,\dagger}_\mu\right)}$$

with initialization $\widehat{\boldsymbol{\rho}}^{\text{est}}_{\boldsymbol{a,k+1}} = \widehat{\boldsymbol{\rho}}^{\text{est}}_{\boldsymbol{b,k+1}} = \widehat{\boldsymbol{\rho}}^{\text{est}}_{\boldsymbol{0}}/2$ where $\widehat{\boldsymbol{\rho}}^{\text{est}}_{\boldsymbol{0}} = \rho_0$ assuming initial probability of $\frac{1}{2}$ to have $\bar{p} = a$ and $\bar{p} = b$. At step k, $\mathbb{P}_{a,k} = \text{Tr}\left(\widehat{\boldsymbol{\rho}}^{\text{est}}_{\boldsymbol{a,k}}\right)$, $\mathbb{P}_{b,k} = \text{Tr}\left(\widehat{\boldsymbol{\rho}}^{\text{est}}_{\boldsymbol{b,k}}\right)$) are the proba. to have $\bar{p} = a$, $\bar{p} = b$, knowing the initial state ρ_0 and the past detection outcomes.

This dynamical parameter estimation process is stable: if the true value of the parameter is a then $\mathbb{P}_{a,k}$ is a sub-martingale.

[5]See Kato, Y. & Yamamoto, N. Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on, 2013, 1904-1909
Discrete-time translation of
Gambetta, J. & Wiseman, H. M., Phys. Rev. A, 2001, 64, 042105
and of Negretti, A. & Mølmer, K. , New Journal of Physics, 2013, 15, 125002.

Discrete-time models of open quantum systems

Four features:

1. Bayes law: $\mathbb{P}(\mu'/\mu) = \mathbb{P}(\mu/\mu')\mathbb{P}(\mu') \;/\; \left(\sum_{\nu'} \mathbb{P}(\mu/\nu')\mathbb{P}(\nu')\right)$,
2. Schrödinger equations defining unitary transformations.
3. Partial collapse of the wave packet: irreversibility and dissipation are induced by the measurement of observables with degenerate spectra.
4. Tensor product for the description of composite systems.

$\Rightarrow$ **Discrete-time models:** Markov processes of state ρ, (density op.):

$\rho_{k+1} = \frac{\sum_{\mu=1}^{m} \eta_{\mu',\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger}{\mathrm{Tr}\left(\sum_{\mu=1}^{m} \eta_{\mu',\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)}$, with proba. $\mathbb{P}_{\mu'}(\rho_k) = \sum_{\mu=1}^{m} \eta_{\mu',\mu} \,\mathrm{Tr}\left(\boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)$

associated to Kraus maps (ensemble average, quantum channel)

$$\mathbb{E}\left(\rho_{k+1}|\rho_k\right) = \boldsymbol{K}(\rho_k) = \sum_{\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger \quad \text{with} \quad \sum_{\mu} \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \boldsymbol{I}$$

and left stochastic matrices (imperfections, decoherences) $(\eta_{\mu',\mu})$.

Discrete-time models: Markov chains

$$\rho_{k+1} = \frac{\sum_{\mu=1}^{m} \eta_{\mu',\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger}{\text{Tr}\left(\sum_{\mu=1}^{m} \eta_{\mu',\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)}, \text{ with proba. } \mathbb{P}_{\mu'}(\rho_k) = \sum_{\mu=1}^{m} \eta_{\mu',\mu} \text{ Tr}\left(\boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)$$

with ensemble averages corresponding to Kraus linear maps

$$\mathbb{E}\left(\rho_{k+1} | \rho_k\right) = \boldsymbol{K}(\rho_k) = \sum_\mu \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger \quad \text{with} \quad \sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \boldsymbol{I}$$

Continuous-time models: stochastic differential systems

$$d\rho_t = \left(-\frac{i}{\hbar}[\mathbf{H}, \rho_t] + \sum_\nu \boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)\right) dt$$
$$+ \sum_\nu \sqrt{\eta_\nu}\left(\boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger - \text{ Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\rho_t\right) \rho_t\right) dW_{\nu,t}$$

driven by Wiener process $dW_{\nu,t} = dy_{\nu,t} - \sqrt{\eta_\nu} \text{ Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\, \rho_t\right) dt$
with measurements $y_{\nu,t}$, detection efficiencies $\eta_\nu \in [0,1]$ and Lindblad-Kossakowski master equations ($\eta_\nu \equiv 0$):

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[\mathbf{H}, \rho] + \sum_\nu \boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)$$

With a single imperfect measurement
$dy_t = \sqrt{\eta}\,\text{Tr}\left((L + L^\dagger)\,\rho_t\right)\,dt + dW_t$ and detection efficiency $\eta \in [0, 1]$, the quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-\frac{i}{\hbar}[\mathbf{H}, \rho_t] + L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L\rho_t + \rho_t L^\dagger L)\right) dt \\ + \sqrt{\eta}\left(L\rho_t + \rho_t L^\dagger - \text{Tr}\left((L + L^\dagger)\rho_t\right)\rho_t\right) dW_t$$

driven by the Wiener process dW_t

With Itō rules, it can be written as the following "discrete-time" Markov model

$$\rho_{t+dt} = \frac{M_{dy_t}\rho_t M_{dy_t}^\dagger + (1-\eta)L\rho_t L^\dagger dt}{\text{Tr}\left(M_{dy_t}\rho_t M_{dy_t}^\dagger + (1-\eta)L\rho_t L^\dagger dt\right)}$$

with $M_{dy_t} = I + \left(-\frac{i}{\hbar}\mathbf{H} - \frac{1}{2}\left(L^\dagger L\right)\right) dt + \sqrt{\eta}\,dy_t L.$

Continuous/discrete-time jump SME

With Poisson process $\boldsymbol{N(t)}$, $\langle \boldsymbol{dN(t)} \rangle = \left(\overline{\theta} + \overline{\eta}\, \mathrm{Tr}\left(V\rho_t V^\dagger\right)\right) dt$, and detection imperfections modeled by $\overline{\theta} \geq 0$ and $\overline{\eta} \in [0,1]$, the quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-i[H,\rho_t] + V\rho_t V^\dagger - \frac{1}{2}(V^\dagger V\rho_t + \rho_t V^\dagger V)\right) dt$$
$$+ \left(\frac{\overline{\theta}\rho_t + \overline{\eta} V\rho_t V^\dagger}{\overline{\theta} + \overline{\eta}\, \mathrm{Tr}\left(V\rho_t V^\dagger\right)} - \rho_t\right)\left(\boldsymbol{dN(t)} - \left(\overline{\theta} + \overline{\eta}\, \mathrm{Tr}\left(V\rho_t V^\dagger\right)\right) dt\right)$$

For $\boldsymbol{N(t+dt) - N(t) = 1}$ we have $\rho_{t+dt} = \dfrac{\overline{\theta}\rho_t + \overline{\eta} V\rho_t V^\dagger}{\overline{\theta} + \overline{\eta}\, \mathrm{Tr}\left(V\rho_t V^\dagger\right)}$.

For $\boldsymbol{dN(t) = 0}$ we have

$$\rho_{t+dt} = \frac{M_{\mathbf{0}}\rho_t M_{\mathbf{0}}^\dagger + (1-\overline{\eta})V\rho_t V^\dagger dt}{\mathrm{Tr}\left(M_{\mathbf{0}}\rho_t M_{\mathbf{0}}^\dagger + (1-\overline{\eta})V\rho_t V^\dagger dt\right)}$$

with $M_{\mathbf{0}} = I + \left(-iH + \frac{1}{2}\left(\overline{\eta}\, \mathrm{Tr}\left(V\rho_t V^\dagger\right) I - V^\dagger V\right)\right) dt$.

Continuous/discrete-time diffusive-jump SME

The quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-i[H,\rho_t] + L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L\rho_t + \rho_t L^\dagger L) + V\rho_t V^\dagger - \frac{1}{2}(V^\dagger V\rho_t + \rho_t V^\dagger V)\right) dt$$
$$+ \sqrt{\eta}\left(L\rho_t + \rho_t L^\dagger - \operatorname{Tr}\left((L+L^\dagger)\rho_t\right)\rho_t\right) \boldsymbol{dW_t}$$
$$+ \left(\frac{\overline{\theta}\rho_t + \overline{\eta} V\rho_t V^\dagger}{\overline{\theta} + \overline{\eta}\operatorname{Tr}(V\rho_t V^\dagger)} - \rho_t\right)\left(\boldsymbol{dN(t)} - \left(\overline{\theta} + \overline{\eta}\operatorname{Tr}\left(V\rho_t V^\dagger\right)\right) dt\right)$$

For $\boldsymbol{N(t+dt) - N(t) = 1}$ we have $\rho_{t+dt} = \dfrac{\overline{\theta}\rho_t + \overline{\eta} V\rho_t V^\dagger}{\overline{\theta} + \overline{\eta}\operatorname{Tr}(V\rho_t V^\dagger)}$.

For $\boldsymbol{dN(t) = 0}$ we have

$$\rho_{t+dt} = \frac{M_{\boldsymbol{dy_t}}\rho_t M_{\boldsymbol{dy_t}}^\dagger + (1-\eta)L\rho_t L^\dagger dt + (1-\overline{\eta})V\rho_t V^\dagger dt}{\operatorname{Tr}\left(M_{\boldsymbol{dy_t}}\rho_t M_{\boldsymbol{dy_t}}^\dagger + (1-\eta)L\rho_t L^\dagger dt + (1-\overline{\eta})V\rho_t V^\dagger dt\right)}$$

with $M_{\boldsymbol{dy_t}} = I + \left(-iH - \frac{1}{2}L^\dagger L + \frac{1}{2}\left(\overline{\eta}\operatorname{Tr}\left(V\rho_t V^\dagger\right) I - V^\dagger V\right)\right) dt + \sqrt{\eta}\,\boldsymbol{dy_t} L$.

Continuous/discrete-time general diffusive-jump SME

The quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \Big(-i[H,\rho_t] + \sum_\nu L_\nu \rho_t L_\nu^\dagger - \tfrac{1}{2}(L_\nu^\dagger L_\nu \rho_t + \rho_t L_\nu^\dagger L_\nu) + V_\mu \rho_t V_\mu^\dagger - \tfrac{1}{2}(V_\mu^\dagger V_\mu \rho_t + \rho_t V_\mu^\dagger V_\mu)\Big)\, dt$$

$$+ \sum_\nu \sqrt{\eta_\nu}\Big(L_\nu \rho_t + \rho_t L_\nu^\dagger - \operatorname{Tr}\Big((L_\nu + L_\nu^\dagger)\rho_t\Big)\,\rho_t\Big)\, \mathbf{dW_{\nu,t}}$$

$$+ \sum_\mu \left(\frac{\overline{\theta}_\mu \rho_t + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} V_\mu \rho_t V_\mu^\dagger}{\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \operatorname{Tr}\left(V_{\mu'}\rho_t V_{\mu'}^\dagger\right)} - \rho_t\right)\left(\mathbf{dN_\mu(t)} - \Big(\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \operatorname{Tr}\left(V_{\mu'}\rho_t V_{\mu'}^\dagger\right)\Big)\, dt\right)$$

where $\eta_\nu \in [0,1], \overline{\theta}_\mu, \overline{\eta}_{\mu,\mu'} \geq 0$ with $\overline{\eta}_{\mu'} = \sum_\mu \overline{\eta}_{\mu,\mu'} \leq 1$ are parameters modelling measurements imperfections.

If, for some μ, $\mathbf{N_\mu(t+dt) - N_\mu(t) = 1}$, we have $\rho_{t+dt} = \dfrac{\overline{\theta}_\mu \rho_t + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} V_{\mu'} \rho_t V_{\mu'}^\dagger}{\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \operatorname{Tr}\left(V_{\mu'}\rho_t V_{\mu'}^\dagger\right)}$.

When $\forall \mu,\ \mathbf{dN_\mu(t) = 0}$, we have

$$\rho_{t+dt} = \frac{M_{\mathbf{dy_t}} \rho_t M_{\mathbf{dy_t}}^\dagger + \sum_\nu (1-\eta_\nu) L_\nu \rho_t L_\nu^\dagger dt + \sum_\mu (1-\overline{\eta}_\mu) V_\mu \rho_t V_\mu^\dagger dt}{\operatorname{Tr}\left(M_{\mathbf{dy_t}} \rho_t M_{\mathbf{dy_t}}^\dagger + \sum_\nu (1-\eta_\nu) L_\nu \rho_t L_\nu^\dagger dt + \sum_\mu (1-\overline{\eta}_\mu) V_\mu \rho_t V_\mu^\dagger dt\right)}$$

with $M_{\mathbf{dy_t}} = I + \left(-iH - \frac{1}{2}\sum_\nu L_\nu^\dagger L_\nu + \frac{1}{2}\sum_\mu \left(\overline{\eta}_\mu \operatorname{Tr}\left(V_\mu \rho_t V_\mu^\dagger\right) I - V_\mu^\dagger V_\mu\right)\right) dt + \sum_\nu \sqrt{\eta_\nu}\, \mathbf{dy_{\nu,t}} L_\nu$ and where $\mathbf{dy_{\nu,t}} = \sqrt{\eta_\nu} \operatorname{Tr}\left((L_\nu + L_\nu^\dagger)\,\rho_t\right) dt + \mathbf{dW_{\nu,t}}$.

Could be used as a numerical integration scheme that preserves the positiveness of ρ.

For clarity'sake, take a single measurement y_t associated to operator L and detection efficiency $\eta \in [0,1]$. Then ρ_t obeys to the following diffusive SME

$$d\rho_t = -i[H,\rho_t]\,dt + \left(L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L\rho_t + \rho_t L^\dagger L)\right)\,dt \\ + \sqrt{\eta}\left(L\rho_t + \rho_t L^\dagger - \operatorname{Tr}\left((L+L^\dagger)\rho_t\right)\rho_t\right)\,\boldsymbol{dW_t}$$

driven by the Wiener processes W_t,

Since $\boldsymbol{dy_t} = \sqrt{\eta}\operatorname{Tr}\left((L+L^\dagger)\,\rho_t\right)\,dt + \boldsymbol{dW_t}$, the estimate $\boldsymbol{\rho_t^{\text{est}}}$ is given by

$$d\boldsymbol{\rho_t^{\text{est}}} = -i[H,\boldsymbol{\rho_t^{\text{est}}}]\,dt + \left(L\boldsymbol{\rho_t^{\text{est}}}L^\dagger - \frac{1}{2}(L^\dagger L\boldsymbol{\rho_t^{\text{est}}} + \boldsymbol{\rho_t^{\text{est}}}L^\dagger L)\right)\,dt \\ +\sqrt{\eta}\left(L\boldsymbol{\rho_t^{\text{est}}} + \boldsymbol{\rho_t^{\text{est}}}L^\dagger - \operatorname{Tr}\left((L+L^\dagger)\boldsymbol{\rho_t^{\text{est}}}\right)\rho_t^{e}\right)\left(\boldsymbol{dy_t} - \sqrt{\eta}\operatorname{Tr}\left((L+L^\dagger)\boldsymbol{\rho_t^{\text{est}}}\right)\,dt\right).$$

initialized to any density matrix $\boldsymbol{\rho_0^{\text{est}}}$.

Stability of diffusive quantum filtering[6]

Assume that $(\rho, \boldsymbol{\rho}^{\text{est}})$ obey to

$$d\rho_t = -i[H,\rho_t]\,dt + \left(L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L\rho_t + \rho_t L^\dagger L)\right)\,dt + \sqrt{\eta}\left(L\rho_t + \rho_t L^\dagger - \text{Tr}\left((L+L^\dagger)\rho_t\right)\rho_t\right)\,\boldsymbol{dW_t}$$

$$d\boldsymbol{\rho_t^{\text{est}}} = -i[H,\boldsymbol{\rho_t^{\text{est}}}]\,dt + \left(L\boldsymbol{\rho_t^{\text{est}}}L^\dagger - \frac{1}{2}(L^\dagger L\boldsymbol{\rho_t^{\text{est}}} + \boldsymbol{\rho_t^{\text{est}}}L^\dagger L)\right)\,dt + \sqrt{\eta}\left(L\boldsymbol{\rho_t^{\text{est}}} + \boldsymbol{\rho_t^{\text{est}}}L^\dagger - \text{Tr}\left((L+L^\dagger)\boldsymbol{\rho_t^{\text{est}}}\right)\boldsymbol{\rho_t^{\text{est}}}\right)\boldsymbol{dW_t} + \underbrace{\eta\left(L\boldsymbol{\rho_t^{\text{est}}} + \boldsymbol{\rho_t^{\text{est}}}L^\dagger - \text{Tr}\left((L+L^\dagger)\boldsymbol{\rho_t^{\text{est}}}\right)\boldsymbol{\rho_t^{\text{est}}}\right)\text{Tr}\left((L+L^\dagger)\boldsymbol{(\rho_t - \rho_t^{\text{est}})}\right)\,dt}_{\text{correction terms vanishing when } \rho_t = \boldsymbol{\rho_t^{\text{est}}}}.$$

Then for any H, L and $\eta \in [0,1]$, $F(\rho_t, \boldsymbol{\rho_t^{\text{est}}}) = \text{Tr}^2\left(\sqrt{\sqrt{\rho_t}\boldsymbol{\rho_t^{\text{est}}}\sqrt{\rho_t}}\right)$ is a sub-martingale:

$$t \mapsto \mathbb{E}\left(F(\rho_t, \boldsymbol{\rho_t^{\text{est}}})\right) \text{ is non-decreasing.}$$

[6] H. Amini, C. Pellegrini, PR: Stability of continuous-time quantum filters with measurement imperfections. `http://arxiv.org/abs/1312.0418`

Metric for the distance between ρ and its estimate $\boldsymbol{\rho}^{\text{est}}$

- $1 - F(\rho_t, \boldsymbol{\rho}_t^{\text{est}})$ remains a super-martingale for all Belavkin SMEs and their associated quantum filters when they are driven simultaneously by several Wiener and Poisson processes.
- Petz has given, via the theory of operator monotone functions, a complete characterization of distance that are contracted for all Lindblad-Kossakovski evolutions[7]:

$$\frac{d}{dt}\rho = -i[H, \rho] + \sum_{\nu} \left(L_\nu \rho L_\nu^\dagger - \tfrac{1}{2}(L_\nu^\dagger L_\nu \rho + \rho L_\nu^\dagger L_\nu) \right).$$

- Could we exploit Petz results to characterize "metrics" $D(\rho, \boldsymbol{\rho}^{\text{est}})$ that are super-martingale for all Belavkin SMEs. and filters ?

[7] D. Petz. Monotone metrics on matrix spaces. *Linear Algebra and its Applications*, 244:81–96, 1996.