Dynamical models and feedback issues for super-conducting quantum circuits

Mexican Annual Conference on Automatic Control.
10-12 October, San Luis Potosí.

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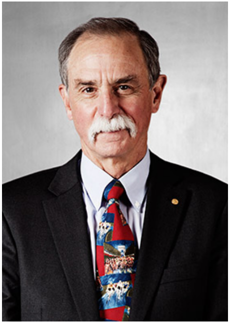
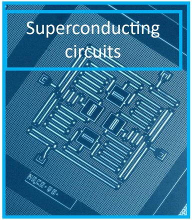


David J. Wineland

" This year's Nobel Prize in Physics honours the experimental inventions and discoveries that have allowed the **measurement and control of individual quantum systems**. *They belong to two separate but related technologies: ions in a harmonic trap and photons in a cavity"*

From the Scientific Background on the Nobel Prize in Physics 2012 compiled by the Class for Physics of the Royal Swedish Academy of Sciences, 9 October 2012.

Technologies for quantum cryptography, metrology, computation [1]



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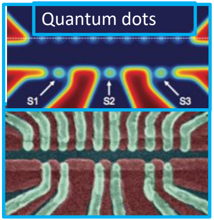

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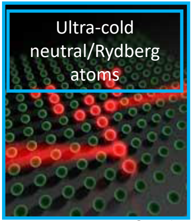

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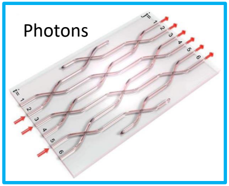

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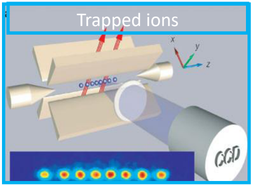

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Requirement:

Scalable modular architecture

Control software from the very beginning.

[1] Courtesy of Walter Riess, IBM Research - Zurich.

Outline

Classical nonlinear LC circuit weakly connected to a transmission line

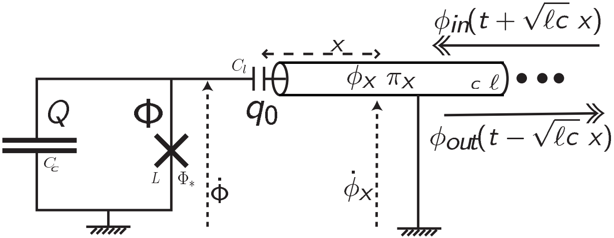
- Classical Hamiltonian dynamics
- Input-state-output dynamics: passivity and measurement back-action

Dynamics of open quantum systems

- LKB photon box: model based on 3 quantum rules
- Discrete-time models
- Continuous-time models driven by Wiener processes
- Operators and decoherence dynamics of qubits and oscillators
- Transmon qubit: typical Josephson super-conducting circuit

Quantum feedback

- Measurement-based feedback
- Coherent feedback (dissipation engineering)
- Quantum feedback engineering



$$\mathcal{H}\Big((\Phi,Q),(\phi_0,q_0),(\phi_x,\pi_x)_{x>0}\Big) = \underbrace{\tfrac{1}{2C_c}Q^2 - \tfrac{\Phi_*^2}{L}\cos\left(\tfrac{1}{\Phi_*}\Phi\right)}_{\mathcal{H}_{sys}}$$

$$+\underbrace{\tfrac{1}{C_c}Q\ q_0}_{\mathcal{H}_{int}} + \underbrace{\tfrac{1}{\tilde{C}_l}q_0^2 + \int_{x>0}\left(\tfrac{1}{2c}\pi_x^2 + \tfrac{1}{2\ell}(\partial_x\phi_x)^2\right)\,dx}_{\mathcal{H}_{line}}$$

where $1/\tilde{C}_l = 1/C_c + 1/C_l$ with weak coupling ($C_l \ll C$).

[2] See e.g. G. M. Bernstein and M. A. Lieberman: A method for obtaining a canonical Hamiltonian for nonlinear LC circuits. IEEE Transactions on Circuits and Systems, 36(3):411-420, 1989.

Hamilton equations: $\frac{d}{dt}\Phi = \partial_Q \mathcal{H}$, $\frac{d}{dt}Q = -\partial_\Phi \mathcal{H}$, ...

$$\frac{d}{dt}\Phi = \tfrac{1}{C_c}(Q + q_0), \quad \frac{d}{dt}Q = -\tfrac{\Phi_*}{L}\sin\left(\tfrac{1}{\Phi_*}\Phi\right) \quad \text{(nonlinear LC)}$$

$$\partial_t\phi_0 = \tfrac{1}{C_c}Q + \tfrac{1}{\tilde{C}_l}q_0, \quad \frac{d}{dt}q_0 = \tfrac{1}{\ell}\partial_x\phi_0 \quad \text{(boundary conditions)}$$

$$\partial_t\phi_x = \tfrac{1}{c}\pi_x, \quad \partial_t\pi_x = \tfrac{1}{\ell}\partial_{xx}\phi_x \text{ for } x > 0 \quad \text{(wave dynamics).}$$

With $\phi(t,x) = \boldsymbol{\phi_{in}(t + \sqrt{\ell c}\ x)} + \boldsymbol{\phi_{out}(t - \sqrt{\ell c}\ x)}$ for $x \geq 0$ we get

$$\frac{d}{dt}\Phi(t) = \tfrac{1}{C_c}(Q(t) + q_0(t))$$

$$\frac{d}{dt}Q(t) = -\tfrac{\Phi_*}{L}\sin\left(\tfrac{1}{\Phi_*}\Phi(t)\right)$$

$$\frac{d}{dt}q_0(t) = \sqrt{\tfrac{c}{\ell}}\left(2\boldsymbol{u(t)} - \tfrac{1}{C_c}Q(t) - \tfrac{1}{\tilde{C}_l}q_0(t)\right) = \sqrt{\tfrac{c}{\ell}}(\boldsymbol{u(t)} - \boldsymbol{y(t)})$$

$$\boldsymbol{y(t)} = -\boldsymbol{u(t)} + \tfrac{1}{C_c}Q(t) + \tfrac{1}{\tilde{C}_l}q_0(t)$$

with input $\boldsymbol{u(t) = \dot{\phi}_{in}(t)}$, output $\boldsymbol{y(t) = \dot{\phi}_{out}(t)}$ and $\boldsymbol{\tilde{C}_l} \ll C_c$.

Adiabatic elimination of q_0 (fast and exponentially stable) exploiting **weak coupling** $\tilde{C}_l = \varepsilon C_c$ with $\varepsilon \ll 1$:

$$\frac{d}{dt}\Phi = \frac{1}{C_c+C_l} Q + 2\varepsilon \boldsymbol{u} - \varepsilon^{\mathbf{2}} \sqrt{\frac{\ell}{c}} \frac{\Phi_*}{L} \sin\left(\frac{1}{\Phi_*}\Phi\right)$$

$$\frac{d}{dt}Q = -\frac{\Phi_*}{L} \sin\left(\frac{1}{\Phi_*}\Phi\right)$$

with output $\boldsymbol{y}$ containing some informations on Φ.

$$\boldsymbol{y} = \boldsymbol{u} - \varepsilon \sqrt{\frac{\ell}{c}} \frac{\Phi_*}{L} \sin\left(\frac{1}{\Phi_*}\Phi\right).$$

Passive system with storage function

$$\mathcal{H}_{sys}(\Phi, Q) = \frac{1}{2(C_c+C_l)} Q^2 - \frac{\Phi_*^2}{L} \cos\left(\frac{1}{\Phi_*}\Phi\right)$$

and ($\sqrt{\frac{\ell}{c}}$ line impedance)

$$\frac{d}{dt}\mathcal{H}_{sys} = \sqrt{\frac{c}{\ell}}\ (\boldsymbol{u}+\boldsymbol{y})(\boldsymbol{u}-\boldsymbol{y}).$$

Remember $\boldsymbol{u} = \dot{\phi}_{in}$ and $\boldsymbol{y} = \dot{\phi}_{out}$ ($\sim$ voltage): classical analogue of measurement back-action.

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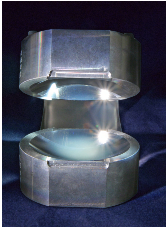
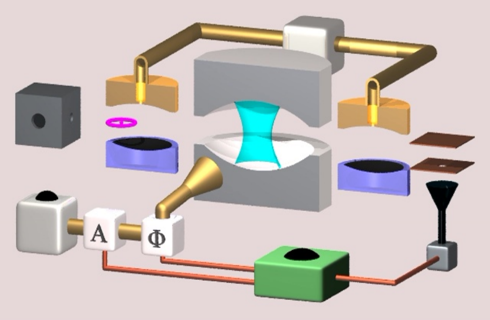
microwave photons
(10 GHz)





Experiment: C. Sayrin, I. Dotsenko, X. Zhou, B. Peaudecerf, T. Rybarczyk, S. Gleyzes, P. Rouchon, M. Mirrahimi, H. Amini, M. Brune, J.M. Raimond, **S. Haroche**: Real-time quantum feedback prepares and stabilizes photon number states. Nature, **2011**, 477, 73-77.

Theory: I. Dotsenko, M. Mirrahimi, M. Brune, **S. Haroche**, J.M. Raimond, P. Rouchon: Quantum feedback by discrete quantum non-demolition measurements: towards on-demand generation of photon-number states. Physical Review A, **2009**, 80: 013805-013813.
H. Amini et al. IEEE Trans. Automatic Control, 57 (8): 1918–1930 **2012**
R. Somaraju et al., Rev. Math. Phys., 25, 1350001, **2013**.
H. Amini et. al., Automatica, 49 (9): 2683-2692, **2013**.

1. Schrödinger: wave funct. $|\psi\rangle \in \mathcal{H}$, density op. $\boldsymbol{\rho} \sim |\psi\rangle\langle\psi|$

$$\frac{d}{dt}|\psi\rangle = -\tfrac{i}{\hbar}\boldsymbol{H}|\psi\rangle, \quad \boldsymbol{H} = \boldsymbol{H}_0 + u\boldsymbol{H}_1, \quad \frac{d}{dt}\boldsymbol{\rho} = -\tfrac{i}{\hbar}[\boldsymbol{H},\boldsymbol{\rho}].$$

2. Origin of dissipation: collapse of the wave packet induced by the measurement of observable $\boldsymbol{O}$ with spectral decomp. $\sum_\mu \lambda_\mu \boldsymbol{P}_\mu$:
   - measurement outcome μ with proba. $\mathbb{P}_\mu = \langle\psi|\boldsymbol{P}_\mu|\psi\rangle = \text{Tr}\left(\boldsymbol{\rho}\boldsymbol{P}_\mu\right)$ depending on $|\psi\rangle$, $\boldsymbol{\rho}$ just before the measurement
   - measurement back-action if outcome $\mu = y$:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\boldsymbol{P}_y|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{P}_y|\psi\rangle}}, \quad \boldsymbol{\rho} \mapsto \boldsymbol{\rho}_+ = \frac{\boldsymbol{P}_y\boldsymbol{\rho}\boldsymbol{P}_y}{\text{Tr}\left(\boldsymbol{\rho}\boldsymbol{P}_y\right)}$$

3. Tensor product for the description of composite systems (S, M):
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $\boldsymbol{H} = \boldsymbol{H}_S \otimes \boldsymbol{I}_M + \boldsymbol{H}_{int} + \boldsymbol{I}_S \otimes \boldsymbol{H}_M$
   - observable on sub-system M only: $\boldsymbol{O} = \boldsymbol{I}_S \otimes \boldsymbol{O}_M$.

[3] S. Haroche and J.M. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts, 2006.

- System S corresponds to a quantized harmonic oscillator:

$$\mathcal{H}_S = \left\{ \sum_{n=0}^{\infty} \psi_n \,|n\rangle \;\middle|\; (\psi_n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

where $|n\rangle$ is the photon-number state with n photons ($\langle n_1|n_2\rangle = \delta_{n_1,n_2}$).

- Meter M is a qubit, a 2-level system:

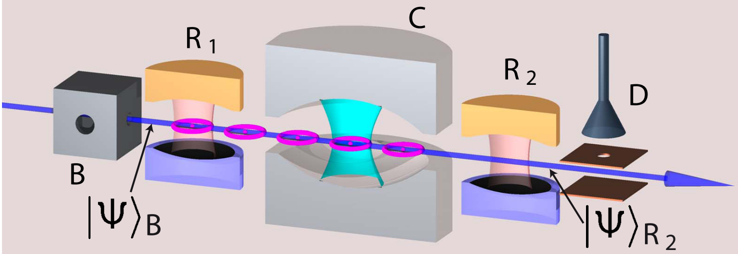
$$\mathcal{H}_M = \left\{ \psi_g \,|g\rangle + \psi_e|e\rangle \;\middle|\; \psi_g, \psi_e \in \mathbb{C} \right\},$$

where $|g\rangle$ (resp. $|e\rangle$) is the ground (resp. excited) state ($\langle g|g\rangle = \langle e|e\rangle = 1$ and $\langle g|e\rangle = 0$)

- **State of the composite system** $|\boldsymbol{\Psi}\rangle \in \mathcal{H}_S \otimes \mathcal{H}_M$:

$$\begin{aligned}|\boldsymbol{\Psi}\rangle &= \sum_{n\geq 0} \left(\Psi_{ng} \,|n\rangle \otimes |g\rangle + \Psi_{ne} \,|n\rangle \otimes |e\rangle \right) \\ &= \left(\sum_{n\geq 0} \Psi_{ng} \,|n\rangle \right) \otimes |g\rangle + \left(\sum_{n\geq 0} \Psi_{ne} \,|n\rangle \right) \otimes |e\rangle, \quad \Psi_{ne}, \Psi_{ng} \in \mathbb{C}.\end{aligned}$$

Ortho-normal basis: $\big(|n\rangle \otimes |g\rangle, |n\rangle \otimes |e\rangle\big)_{n\in\mathbb{N}}$.



- When atom comes out B, the quantum state $|\boldsymbol{\Psi}\rangle_B$ of the composite system is separable: $|\boldsymbol{\Psi}\rangle_B = |\psi\rangle \otimes |g\rangle$.
- Just before the measurement in D, the state is in general entangled (not separable):

$$|\boldsymbol{\Psi}\rangle_{R_2} = \boldsymbol{U}_{SM}\big(|\psi\rangle \otimes |g\rangle\big) = \big(\boldsymbol{M}_g|\psi\rangle\big) \otimes |g\rangle + \big(\boldsymbol{M}_e|\psi\rangle\big) \otimes |e\rangle$$

where $\boldsymbol{U}_{SM}$ is a unitary transformation (Schrödinger propagator) defining the measurement operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ on $\mathcal{H}_S$. Since $\boldsymbol{U}_{SM}$ is unitary, $\boldsymbol{M}_g^\dagger \boldsymbol{M}_g + \boldsymbol{M}_e^\dagger \boldsymbol{M}_e = \boldsymbol{I}$.

Just before the atom detector D the quantum state is **entangled**:

$$|\boldsymbol{\Psi}\rangle_{R_2} = (\boldsymbol{M}_g|\psi\rangle) \otimes |g\rangle + (\boldsymbol{M}_e|\psi\rangle) \otimes |e\rangle$$

Just after outcome y, the state becomes **separable** [4]:

$$|\boldsymbol{\Psi}\rangle_D = \frac{1}{\sqrt{\mathbb{P}_y}} \; (\boldsymbol{M}_y|\psi\rangle) \otimes |y\rangle = \left(\frac{\boldsymbol{M}_y}{\sqrt{\langle\psi|\boldsymbol{M}_y^\dagger \boldsymbol{M}_y|\psi\rangle}}|\psi\rangle\right) \otimes |y\rangle.$$

Outcome y obtained with probability $\langle\psi|\boldsymbol{M}_y^\dagger \boldsymbol{M}_y|\psi\rangle$ depending on $|\psi\rangle$.

Hidden Markov chain:

$$|\psi_{k+1}\rangle = \begin{cases} \frac{\boldsymbol{M}_g}{\sqrt{\langle\psi_k|\boldsymbol{M}_g^\dagger \boldsymbol{M}_g|\psi_k\rangle}}|\psi_k\rangle, & y_k = g \text{ with probability } \langle\psi_k|\boldsymbol{M}_g^\dagger \boldsymbol{M}_g|\psi_k\rangle; \\ \frac{\boldsymbol{M}_e}{\sqrt{\langle\psi_k|\boldsymbol{M}_e^\dagger \boldsymbol{M}_e|\psi_k\rangle}}|\psi_k\rangle, & y_k = e \text{ with probability } \langle\psi_k|\boldsymbol{M}_e^\dagger \boldsymbol{M}_e|\psi_k\rangle; \end{cases}$$

with state $|\psi_k\rangle$ and output $y_k \in \{g, e\}$ at time-step k:

[4] Measurement operator $\boldsymbol{O} = \boldsymbol{I}_S \otimes (|e\rangle\langle e| - |g\rangle\langle g|)$.

Assume known $|\psi_0\rangle$ and **detector out of order**: **what about** $|\psi_1\rangle$ **?**

- Expectation value of $|\psi_1\rangle\langle\psi_1|$ knowing $|\psi_0\rangle$: [5]

$$\mathbb{E}\left(|\psi_1\rangle\langle\psi_1| \;\big|\; |\psi_0\rangle\right) = \boldsymbol{M}_g|\psi_0\rangle\langle\psi_0|\boldsymbol{M}_g^\dagger + \boldsymbol{M}_e|\psi_0\rangle\langle\psi_0|\boldsymbol{M}_e^\dagger.$$

- Set $\boldsymbol{K}(\boldsymbol{\rho}) \triangleq \boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + \boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger$ for any operator $\boldsymbol{\rho}$.
- $\boldsymbol{\rho_k}$ expectation of $|\psi_k\rangle\langle\psi_k|$ knowing $|\psi_0\rangle$:

$$\boldsymbol{\rho}_{k+1} = \boldsymbol{K}(\boldsymbol{\rho}_k) \text{ and } \boldsymbol{\rho}_0 = |\psi_0\rangle\langle\psi_0|.$$

Linear map $\boldsymbol{K}$: trace preserving Kraus map (quantum channel).

Density operators $\boldsymbol{\rho}$: convex space of Hermitian non-negative operators of trace one.

[5] $|\psi\rangle\langle\psi|$: orthogonal projector on line spanned by unitary vector $|\psi\rangle$.

Detector efficiency $\eta \in [0, 1]$. Output $y \in \{g, e, \varnothing\}$:

$$\rho_{k+1} = \begin{cases} \dfrac{K_g(\rho_k)}{\mathrm{Tr}\left(K_g(\rho_k)\right)}, \; y_k = g \text{ with probability } \; \mathrm{Tr}\left(K_g(\rho_k)\right); \\ \dfrac{K_e(\rho_k)}{\mathrm{Tr}\left(K_e(\rho_k)\right)}, \; y_k = e \text{ with probability } \; \mathrm{Tr}\left(K_e(\rho_k)\right); \\ \dfrac{K_\varnothing(\rho_k)}{\mathrm{Tr}\left(K_\varnothing(\rho_k)\right)}, \; y_k = \varnothing \text{ with probability } \; \mathrm{Tr}\left(K_\varnothing(\rho_k)\right); \end{cases}$$

with Kraus maps

$$K_g(\rho) = \eta M_g \rho M_g^\dagger, \qquad K_e(\rho) = \eta M_e \rho M_e^\dagger$$
$$K_\varnothing(\rho) = (1-\eta)\left(M_g \rho M_g^\dagger + M_e \rho M_e^\dagger\right).$$

We still have:

$$\mathbb{E}\left(\rho_{k+1} \mid \rho_k\right) \triangleq K(\rho_k) = M_g \rho_k M_g^\dagger + M_e \rho_k M_e^\dagger = \sum_y K_y(\rho_k).$$

Input u: classical amplitude of a coherent micro-wave pulse
State ρ: the density operator of the photon(s) trapped in the cavity
Output y: measurement of the probe atom

$$\rho_{k+1} = \begin{cases} \dfrac{\boldsymbol{D}_{u_k} \boldsymbol{K}_g(\rho_k) \boldsymbol{D}_{u_k}^\dagger}{\mathrm{Tr}\left(\boldsymbol{K}_g(\rho_k)\right)}, y_k = g \text{ with probability } \mathrm{Tr}\left(\boldsymbol{K}_g(\rho_k)\right); \\ \dfrac{\boldsymbol{D}_{u_k} \boldsymbol{K}_e(\rho_k) \boldsymbol{D}_{u_k}^\dagger}{\mathrm{Tr}\left(\boldsymbol{K}_e(\rho_k)\right)}, y_k = e \text{ with probability } \mathrm{Tr}\left(\boldsymbol{K}_e(\rho_k)\right); \\ \dfrac{\boldsymbol{D}_{u_k} \boldsymbol{K}_\varnothing(\rho_k) \boldsymbol{D}_{u_k}^\dagger}{\mathrm{Tr}\left(\boldsymbol{K}_\varnothing(\rho_k)\right)}, y_k = \varnothing \text{ with probability } \mathrm{Tr}\left(\boldsymbol{K}_\varnothing(\rho_k)\right); \end{cases}$$

Controlled displacement unitary operator ($u \in \mathbb{R}$): $\boldsymbol{D}_u = e^{u\boldsymbol{a}^\dagger - u\boldsymbol{a}}$ with $\boldsymbol{a} = \text{upper diag}(\sqrt{1}, \sqrt{2}, \ldots)$ the photon annihilation operator.
Measurement Kraus operators in the linear dispersive case
$\boldsymbol{M}_g = \cos\left(\frac{\phi_0 \boldsymbol{N} + \phi_R}{2}\right)$ and $\boldsymbol{M}_e = \sin\left(\frac{\phi_0 \boldsymbol{N} + \phi_R}{2}\right)$: $\boldsymbol{M}_g^\dagger \boldsymbol{M}_g + \boldsymbol{M}_e^\dagger \boldsymbol{M}_e = \boldsymbol{I}$
with $\boldsymbol{N} = \boldsymbol{a}^\dagger \boldsymbol{a} = \text{diag}(0, 1, 2, \ldots)$ the photon number operator.

Discrete-time models of open quantum systems

Four features:

1. Bayes law: $\mathbb{P}(\mu/y) = \mathbb{P}(y/\mu)\mathbb{P}(\mu) \;/\; \left(\sum_{\mu'} \mathbb{P}(y/\mu')\mathbb{P}(\mu')\right)$,
2. Schrödinger equations defining unitary transformations.
3. Partial collapse of the wave packet: irreversibility and dissipation are induced by the measurement of observables with degenerate spectra.
4. Tensor product for the description of composite systems.

$\Rightarrow$ **Discrete-time models:** Markov processes of state ρ, (density op.):
$\rho_{k+1} = \frac{\sum_{\mu=1}^{m} \eta_{y,\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger}{\text{Tr}\left(\sum_{\mu=1}^{m} \eta_{y,\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)}$, with proba. $\mathbb{P}_y(\rho_k) = \sum_{\mu=1}^{m} \eta_{y,\mu} \ \text{Tr}\left(\boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)$
associated to Kraus maps [6] (ensemble average, quantum channel)

$$\mathbb{E}\left(\rho_{k+1}|\rho_k\right) = \boldsymbol{K}(\rho_k) = \sum_\mu \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger \quad \text{with} \quad \sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \boldsymbol{I}$$

and left stochastic matrices (imperfections, decoherences) $(\eta_{y,\mu})$.

[6] M.A. Nielsen, I.L. Chuang: Quantum Computation and Quantum Information. Cambridge University Press, 2000.

Discrete-time models: Markov chains

$$\rho_{k+1} = \frac{\sum_{\mu=1}^{m} \eta_{y,\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger}{\text{Tr}\left(\sum_{\mu=1}^{m} \eta_{y,\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)}, \text{ with proba. } \mathbb{P}_y(\rho_k) = \sum_{\mu=1}^{m} \eta_{y,\mu} \text{ Tr}\left(\boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)$$

with ensemble averages corresponding to Kraus linear maps

$$\mathbb{E}\left(\rho_{k+1}|\rho_k\right) = \boldsymbol{K}(\rho_k) = \sum_\mu \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger \quad \text{with} \quad \sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \boldsymbol{I}$$

Continuous-time models: stochastic differential systems [7]

$$d\rho_t = \left(-\tfrac{i}{\hbar}[\boldsymbol{H}, \rho_t] + \sum_\nu \boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)\right) dt + \sum_\nu \sqrt{\eta_\nu}\left(\boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger - \text{ Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\rho_t\right) \rho_t\right) dW_{\nu,t}$$

driven by Wiener processes $dW_{\nu,t}$, with measurements $y_{\nu,t}$,
$dy_{\nu,t} = \sqrt{\eta_\nu} \text{ Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\,\rho_t\right) dt + dW_{\nu,t}$, detection efficiencies $\eta_\nu \in [0,1]$ and Lindblad-Kossakowski master equations ($\eta_\nu \equiv 0$):

$$\frac{d}{dt}\rho = -\tfrac{i}{\hbar}[\boldsymbol{H}, \rho] + \sum_\nu \boldsymbol{L}_\nu \rho \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho + \rho \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)$$

[7] A. Barchielli, M. Gregoratti: Quantum Trajectories and Measurements in Continuous Time: the Diffusive Case. Springer Verlag, 2009.

With a single imperfect measurement $dy_t = \sqrt{\eta}\ \mathrm{Tr}\left(\left(L + L^\dagger\right)\rho_t\right)\ dt + dW_t$ and detection efficiency $\eta \in [0,1]$, the quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-\tfrac{i}{\hbar}[H,\rho_t] + L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L\rho_t + \rho_t L^\dagger L)\right) dt + \sqrt{\eta}\left(L\rho_t + \rho_t L^\dagger - \ \mathrm{Tr}\left((L+L^\dagger)\rho_t\right)\rho_t\right) dW_t$$

driven by the Wiener process dW_t

With Itō rules, it can be written as the following "discrete-time" Markov model

$$\rho_{t+dt} = \frac{M_{dy_t}\rho_t M_{dy_t}^\dagger + (1-\eta) L\rho_t L^\dagger dt}{\mathrm{Tr}\left(M_{dy_t}\rho_t M_{dy_t}^\dagger + (1-\eta) L\rho_t L^\dagger dt\right)}$$

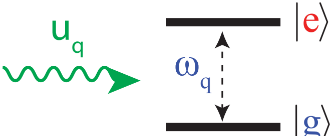
with $M_{dy_t} = I + \left(-\tfrac{i}{\hbar}H - \tfrac{1}{2}\left(L^\dagger L\right)\right) dt + \sqrt{\eta}\, dy_t L$.

ρ_0 density operator $\mapsto$ for all $t > 0$, ρ_t density operator

Positivity preserving numerical scheme.

[8] P. Rouchon: Models and Feedback Stabilization of Open Quantum Systems. Proc. of Int. Congress of Mathematicians, vol. IV, pp 921–946, Seoul 2014 (`http://arxiv.org/abs/1407.7810`).

- Hilbert space:
  $\mathcal{H}_M = \mathbb{C}^2 = \left\{ c_g|g\rangle + c_e|e\rangle,\ c_g, c_e \in \mathbb{C} \right\}.$
- Quantum state space:
  $\mathcal{D} = \{\rho \in \mathcal{L}(\mathcal{H}_M), \rho^\dagger = \rho,\ \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}$.
- Operators and commutations:
  $\boldsymbol{\sigma_-} = |g\rangle\langle e|$, $\boldsymbol{\sigma_+} = \boldsymbol{\sigma_-}^\dagger = |e\rangle\langle g|$
  $\boldsymbol{\sigma_x} = \boldsymbol{\sigma_-} + \boldsymbol{\sigma_+} = |g\rangle\langle e| + |e\rangle\langle g|$;
  $\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_-} - i\boldsymbol{\sigma_+} = i|g\rangle\langle e| - i|e\rangle\langle g|$;
  $\boldsymbol{\sigma_z} = \boldsymbol{\sigma_+}\boldsymbol{\sigma_-} - \boldsymbol{\sigma_-}\boldsymbol{\sigma_+} = |e\rangle\langle e| - |g\rangle\langle g|$;
  ${\boldsymbol{\sigma_x}}^2 = \boldsymbol{I}$, $\boldsymbol{\sigma_x}\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_z}$, $[\boldsymbol{\sigma_x}, \boldsymbol{\sigma_y}] = 2i\boldsymbol{\sigma_z}$, ...

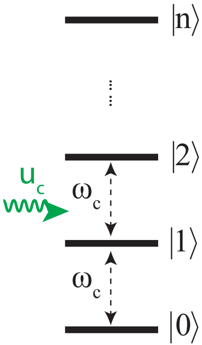


- Hamiltonian: $\boldsymbol{H}_M/\hbar = \omega_q \boldsymbol{\sigma_z}/2 + \boldsymbol{u_q}\boldsymbol{\sigma_x}$.
- Bloch sphere representation:
  $\mathcal{D} = \left\{ \frac{1}{2}\left(\boldsymbol{I} + x\boldsymbol{\sigma_x} + y\boldsymbol{\sigma_y} + z\boldsymbol{\sigma_z}\right) \;\middle|\; (x, y, z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 \leq 1 \right\}$

[9] See S. M. Barnett, P.M. Radmore: Methods in Theoretical Quantum Optics. Oxford University Press, 2003.

Quantum harmonic oscillator (spring system) [9]

- Hilbert space:
  $\mathcal{H}_S = \left\{ \sum_{n\geq 0} \psi_n |n\rangle, \ (\psi_n)_{n\geq 0} \in l^2(\mathbb{C}) \right\} \equiv L^2(\mathbb{R}, \mathbb{C})$
- Quantum state space:
  $\mathcal{D} = \{\rho \in \mathcal{L}(\mathcal{H}_S), \rho^\dagger = \rho, \ \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}$.
- Operators and commutations:
  $\boldsymbol{a}|n\rangle = \sqrt{n}\ |\boldsymbol{n\text{-}1}\rangle$, $\boldsymbol{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$;
  $\boldsymbol{N} = \boldsymbol{a}^\dagger \boldsymbol{a}$, $\boldsymbol{N}|n\rangle = n|n\rangle$;
  $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = \boldsymbol{I}$, $\boldsymbol{a} f(\boldsymbol{N}) = f(\boldsymbol{N} + \boldsymbol{I})\boldsymbol{a}$;
  $\boldsymbol{D}_\alpha = e^{\alpha \boldsymbol{a}^\dagger - \alpha^\dagger \boldsymbol{a}}$.
  $\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$, $[\boldsymbol{X}, \boldsymbol{P}] = \imath \boldsymbol{I}/2$.
- Hamiltonian: $\boldsymbol{H}_S/\hbar = \omega_c \boldsymbol{a}^\dagger \boldsymbol{a} + \boldsymbol{u_c}(\boldsymbol{a} + \boldsymbol{a}^\dagger)$.
  (associated classical dynamics:
  $\frac{dx}{dt} = \omega_c p$, $\frac{dp}{dt} = -\omega_c x - \sqrt{2}u_c$).
- Classical pure state $\equiv$ coherent state $|\alpha\rangle$
  $\alpha \in \mathbb{C}$: $|\alpha\rangle = \sum_{n\geq 0} \left(e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \right) |n\rangle$; $|\alpha\rangle \equiv \frac{1}{\pi^{1/4}} e^{\imath\sqrt{2}x\Im\alpha} e^{-\frac{(x-\sqrt{2}\Re\alpha)^2}{2}}$
  $\boldsymbol{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\boldsymbol{D}_\alpha|0\rangle = |\alpha\rangle$.



Lindbladian super-operator $\mathcal{L}_L(\rho) = L\rho L^\dagger - (L^\dagger L\rho + \rho L^\dagger L)/2$.

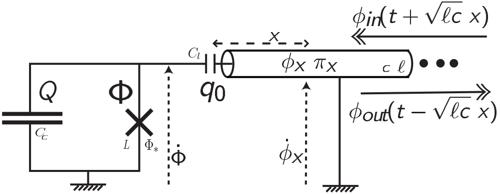
- **Qubit:** u, v are two inputs (drives); $\omega_q/2\pi$ is the qubit frequency; τ_ϕ is dephasing time; τ_1 is life time of $|e\rangle$ (usually $\tau_\phi < \tau_1$ and $\omega_q \tau_\phi \gg 1$).

$$\frac{d}{dt}\rho = -\tfrac{i}{2}\Big[u\boldsymbol{\sigma_x} + v\boldsymbol{\sigma_y} + \omega_q\boldsymbol{\sigma_z} \;,\; \rho\Big] + \tfrac{1}{\tau_\phi}\mathcal{L}_{\boldsymbol{\sigma_z}}(\rho) + \tfrac{1}{\tau_1}\mathcal{L}_{\boldsymbol{\sigma_-}}(\rho)$$

- **Harmonic oscillator:** u, v are two inputs; $\omega_c/2\pi$ is the oscillator frequency; τ_c is the photon life time; n_{th} is the number of thermal photon(s). (usually, $\omega_c\tau_c \gg 1$ and $n_{th} \ll 1$).

$$\frac{d}{dt}\rho = -i\Big[u(\boldsymbol{a}+\boldsymbol{a}^\dagger)+iv(\boldsymbol{a}-\boldsymbol{a}^\dagger)+\omega_c\boldsymbol{a}^\dagger\boldsymbol{a} \;,\; \rho\Big]+\tfrac{1+n_{th}}{\tau_c}\mathcal{L}_{\boldsymbol{a}}(\rho)+\tfrac{n_{th}}{\tau_c}\mathcal{L}_{\boldsymbol{a}^\dagger}(\rho)$$

[10] S. Haroche and J.M. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts, 2006.



$$\frac{d}{dt}\Phi = \frac{1}{C}Q + 2\epsilon u - \epsilon^2\sqrt{\frac{\ell}{c}}\frac{\Phi_*}{L}\sin\left(\frac{1}{\Phi_*}\Phi\right)$$

$$\frac{d}{dt}Q = -\frac{\Phi_*}{L}\sin\left(\frac{1}{\Phi_*}\Phi\right)$$

with $y = u - \epsilon\sqrt{\frac{\ell}{c}}\frac{\Phi_*}{L}\sin\left(\frac{1}{\Phi_*}\Phi\right)$.

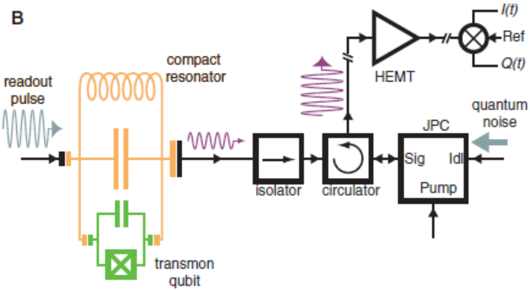
$\mathcal{H}_{sys}(\Phi, Q) = \frac{1}{2C}Q^2 - \frac{\Phi_*^2}{L}\cos\left(\frac{1}{\Phi_*}\Phi\right)$ with nonlinearity ($\Phi_* < (L/C)^{1/4}$):

- anharmonic spectrum with frequency transition between the ground and first excited states larger than frequency transition between first and second excited states.
- qubit model based on restriction to these two slowest energy levels, $|g\rangle$ and $|e\rangle$, with pulsation $\omega_q \sim 1/\sqrt{LC}$.

Two weak coupling regimes:

- resonant, in/out wave pulsation ω_q;
- off-resonant , in/out wave pulsation $\omega_q + \Delta$ with $|\Delta| \ll \omega_q$.

[11] J. Koch et al.: Charge-insensitive qubit design derived from the Cooper pair box. Phys. Rev. A, 76:042319, 2007.



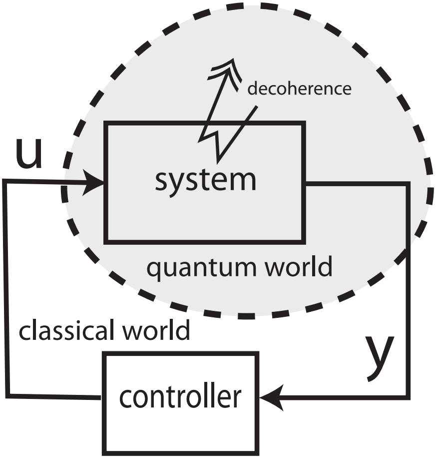
Superconducting qubit dispersively coupled to a cavity traversed by a microwave signal (input/output theory). The back-action on the qubit state of a single measurement of one output field quadrature $\boldsymbol{y}$ is described by a simple SME for the qubit density operator $\boldsymbol{\rho}$.

$$d\rho_t = \Big(-\tfrac{i}{2}[\omega_q \boldsymbol{\sigma_z}, \rho_t] + \gamma(\boldsymbol{\sigma_z}\rho\boldsymbol{\sigma_z} - \rho_t)\Big) dt + \sqrt{\eta\gamma}\Big(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\left(\boldsymbol{\sigma_z}\rho_t\right)\rho_t\Big) \boldsymbol{dW_t}$$

with $\boldsymbol{y_t}$ given by $\boldsymbol{dy_t} = 2\sqrt{\eta\gamma}\,\mathrm{Tr}\left(\boldsymbol{\sigma_z}\rho_t\right) dt + \boldsymbol{dW_t}$ where $\gamma \geq 0$ is related to the measurement strength and $\eta \in [0,1]$ is the detection efficiency.

Quantum Monte Carlo open-loop trajectories with MATLAB : `DiffusiveMeasurementQubit.m`

[12] M. Hatridge et al. Quantum Back-Action of an Individual Variable-Strength Measurement. Science, 2013, 339, 178-181.



P-controller (Markovian feedback [a]):
for $u_t \, dt = k \, dy_t$, average closed-loop dynamics of ρ remains governed by a Lindblad master equation.

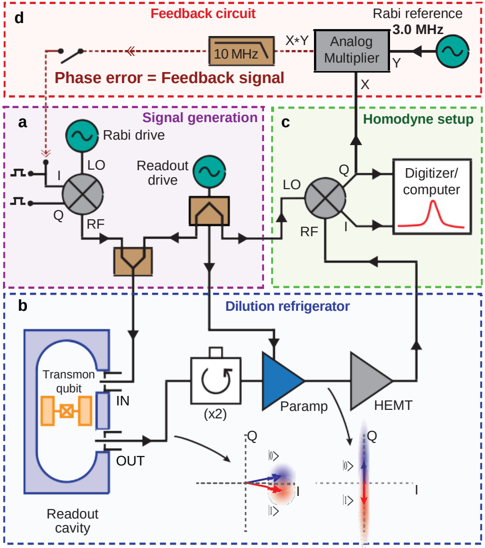
PID controller: no Lindblad master equation in closed-loop;

Nonlinear hidden-state stochastic systems: convergence analysis, Lyapunov exponents, dynamic output feedback, delays, robustness, ...

[a] H.M. Wiseman: Quantum Trajectories and Feedback. PhD Thesis, University of Queensland, 1994.

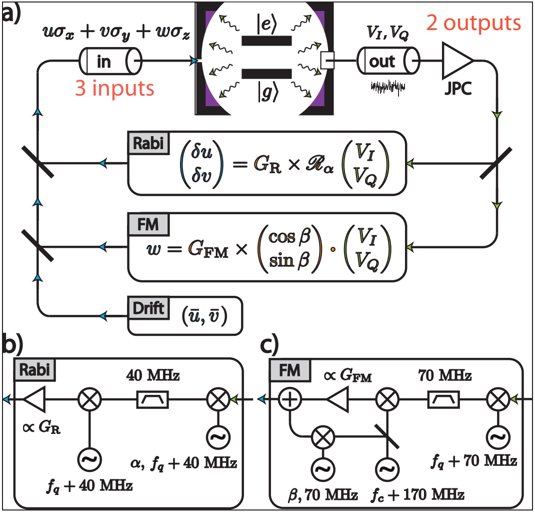
Short sampling times limit feedback complexity

First SISO measurement-based feedback for a superconducting qubit [13]



[13] R. Vijay, . . . , I. Siddiqi. Stabilizing Rabi oscillations in a superconducting qubit using quantum feedback. Nature 490, 77-80, October 2012.

First MIMO measurement-based feedback for a superconducting qubit [14]



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[14] P. Campagne-Ibarcq, . . . , B. Huard: Using Spontaneous Emission of a Qubit as a Resource for Feedback Control. Phys. Rev. Lett. 117(6), 2016.

Theorem [15]**.** *For the stochastic master equation of a qubit*

$$d\rho_t = -iu[\boldsymbol{\sigma_y}, \rho_t]\ dt + \gamma\big(\boldsymbol{\sigma_z}\rho_t\boldsymbol{\sigma_z} - \rho_t\big) + \sqrt{\eta\gamma}\Big(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\ Tr(\boldsymbol{\sigma_z}\rho_t)\,\rho_t\Big)dW_t.$$

with measurement $dy_t = 2\sqrt{\eta\gamma}\ Tr(\boldsymbol{\sigma_z}\rho_t)\ dt + dW_t$*, the feedback*

$$u\ dt = -\eta\gamma\big(1 - \ Tr(\rho_t\boldsymbol{\sigma_z})^2\big)\ dt + \sqrt{\eta\gamma}\big(1 - \ Tr(\rho_t\boldsymbol{\sigma_z})\big)dy_t$$

globally and exponentially stabilizes the system to the excited state $\bar{\rho} = (\boldsymbol{I} + \boldsymbol{\sigma_z})/2$ *with the control Lyapunov function*

$$V(\rho) = \sqrt{1 - \ Tr(\rho\boldsymbol{\sigma_z})}$$

converging exponentially to zero in closed-loop:

$$\mathbb{E}\big(V(\rho_t) \mid \rho_0\big) \leq \exp\left(\tfrac{\eta\gamma}{2}t\right) V(\rho_0), \quad \textit{for any } t \geq 0 \textit{ and density operator } \rho_0.$$

The first mathematical proof of exponential stabilization by measurement-based feedback of a two-level quantum system.

Quantum Monte Carlo closed-loop trajectories with MATLAB :

`ExpoStabilizationQubit.m`

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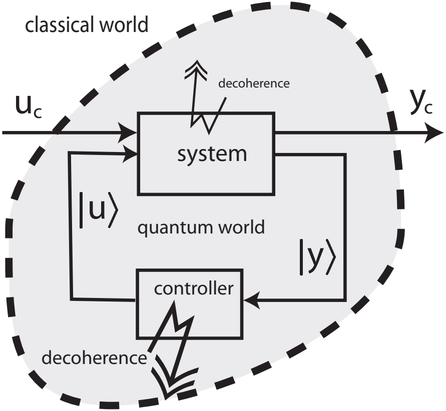
[15] **G. Cardona** et al.: Exponential stochastic stabilization of a two-level quantum system via strict Lyapunov control. CDC 2018 (https://arxiv.org/abs/1803.07542)

Quantum analogue of Watt speed governor: a **dissipative** mechanical system controls another mechanical system [16]



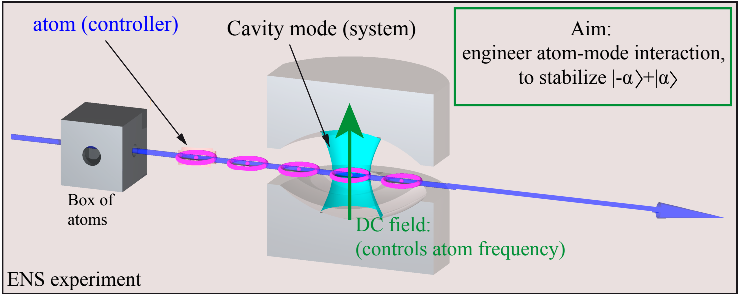
Optical pumping (Kastler 1950), coherent population trapping (Arimondo 1996)

Dissipation engineering, autonomous feedback: (Zoller, Cirac, Wolf, Verstraete, Devoret, Schoelkopf, Siddiqi, Lloyd, Viola, Ticozzi, Leghtas, Mirrahimi, Sarlette, ...)

(S,L,H) theory and linear quantum systems: quantum feedback networks based on stochastic Schrödinger equation, Heisenberg picture (Gardiner, Yurke, Mabuchi, Genoni, Serafini, Milburn, Wiseman, Doherty, Gough, James, Petersen, Nurdin, Yamamoto, Zhang, Dong, . . .)

Stability analysis: Kraus maps and Lindblad propagators are always contractions (non commutative diffusion and consensus).

[16] J.C. Maxwell: On governors. Proc. of the Royal Society, No.100, 1868.



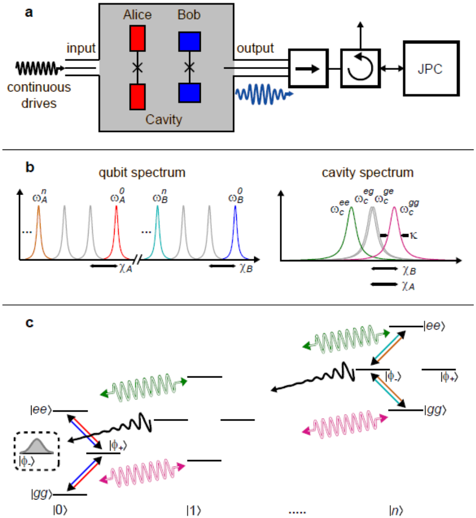
Jaynes-Cumming Hamiltionian

$$\boldsymbol{H}(t)/\hbar = \omega_c \boldsymbol{a}^\dagger \boldsymbol{a} \otimes \boldsymbol{I}_M + \boldsymbol{\omega_q(t)} \boldsymbol{I}_S \otimes \boldsymbol{\sigma_z}/2 + i\Omega(t)\left(\boldsymbol{a}^\dagger \otimes \boldsymbol{\sigma}_- - \boldsymbol{a} \otimes \boldsymbol{\sigma}_+\right)/2$$

with the open-loop control $t \mapsto \boldsymbol{\omega_q(t)}$ combining dispersive $\omega_q \neq \omega_c$ and resonant $\omega_q = \omega_c$ interactions.

Key issues: **convergence** of $\rho_{k+1} = \boldsymbol{K}(\rho_k) = \boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger$.

[17] **A. Sarlette**, et al: Stabilization of nonclassical states of the radiation field in a cavity by reservoir engineering. Phys. Rev. Lett. 107(1), 2011.



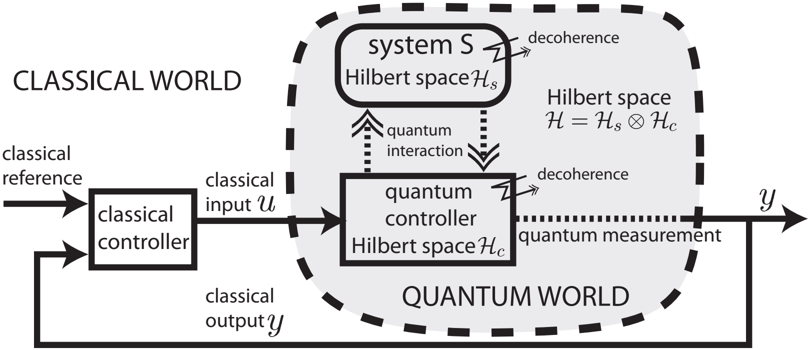
Lindblad master equation:

$$\frac{d}{dt}\boldsymbol{\rho} = -i[\boldsymbol{H}(t), \boldsymbol{\rho}] + \kappa\mathcal{D}_{\boldsymbol{a}}(\boldsymbol{\rho}) + \frac{1}{T_1^A}\mathcal{D}_{\boldsymbol{\sigma}_-^A}(\boldsymbol{\rho}) + \frac{1}{2T_\phi^A}\mathcal{D}_{\boldsymbol{\sigma}_z^A}(\boldsymbol{\rho}) + \frac{1}{T_1^B}\mathcal{D}_{\boldsymbol{\sigma}_-^B}(\boldsymbol{\rho}) + \frac{1}{2T_\phi^B}\mathcal{D}_{\boldsymbol{\sigma}_z^B}(\boldsymbol{\rho})$$

with

$$\boldsymbol{H}(t)/\hbar = \left(\frac{\chi_A}{2}\boldsymbol{\sigma}_z^A + \frac{\chi_B}{2}\boldsymbol{\sigma}_z^B\right)\boldsymbol{a}^\dagger\boldsymbol{a} + 2\epsilon_c \cos\left(\frac{\chi_A+\chi_B}{2}t\right)\left(\boldsymbol{a}+\boldsymbol{a}^\dagger\right) + \Omega_0\left(\boldsymbol{\sigma}_x^A + \boldsymbol{\sigma}_x^B\right) + \Omega_n\left(e^{-in\frac{\chi_A+\chi_B}{2}t}(\boldsymbol{\sigma}_+^A - \boldsymbol{\sigma}_+^B) + \text{h.c.}\right)$$

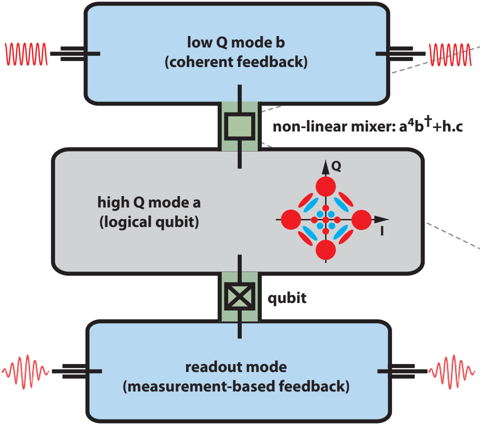
[18] S. Shankar, ..., M.H. Devoret. Autonomously stabilized entanglement between two superconducting quantum bits. Nature, 504: 419-422, 2013.



To stabilize the quantum information localized in system S:

- fast decoherence addressed by a **quantum controller** (coherent feedback);
- slow decoherence and perturbation tackled by a *classical controller* (measurement-based feedback).

Cat-Qubit: feedback stabilization for quantum error correction. [19]



Stochastic master equation :

$$d\rho = -i[\boldsymbol{H},\rho]\ dt + \kappa_a \mathcal{D}_{\boldsymbol{a}}(\rho)\ dt + \kappa_b \mathcal{D}_{\boldsymbol{b}}(\rho)\ dt + \gamma \mathcal{D}_{\boldsymbol{\sigma_z}}(\rho)\ dt + \sqrt{\eta\gamma}\left(\boldsymbol{\sigma_z}\rho + \rho\boldsymbol{\sigma_z} - 2\ \mathrm{Tr}\left(\boldsymbol{\sigma_z}\rho\right)\rho\right)\ dW_t$$

with Hamiltonian

$$\boldsymbol{H} = u_b \boldsymbol{b}^\dagger + u_b^* \boldsymbol{b} + u|e\rangle\langle g| + u^*|g\rangle\langle e| + g\left(\boldsymbol{a}^4\boldsymbol{b}^\dagger + (\boldsymbol{a}^4)^\dagger \boldsymbol{b}\right) + \chi|e\rangle\langle e|\ \boldsymbol{a}^\dagger \boldsymbol{a},$$

measurement output

$$dy_t = 2\sqrt{\eta\gamma}\ \mathrm{Tr}\left(\boldsymbol{\sigma_z}\rho\right) + dW_t,$$

classical control inputs $u_b, u \in \mathbb{C}$,
parameters $\kappa_b, \gamma \gg \kappa_a, g, \chi$.

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[19] **M. Mirrahimi, Z. Leghtas,** . . . , M. Devoret: Dynamically protected cat-qubits: a new paradigm for universal quantum computation. New Journal of Physics, 16: 045014, 2014.
Z. Leghtas, . . . , **M. Mirrahimi,** M. Devoret: Confining the state of light to a quantum manifold by engineered two-photon loss. Science, 347:853–857,2015.