Deterministic submanifolds and analytic solution of the stochastic differential equation describing a continuously measured qubit

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Outline

Measuring fluorescence of a qubit (slides of Benjamin Huard)

A positivity preserving numerical scheme for quantum filtering

The deterministic surface and integral quantities

When is a qubit confined to a deterministic surface ?

Conclusion

A positivity preserving numerical scheme: diffusive case [1]

With a single imperfect measure $dy_t = \sqrt{\eta}\,\mathrm{Tr}\left((L + L^\dagger)\,\rho_t\right)\,dt + dW_t$ and detection efficiency $\eta \in [0, 1]$, the quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-\frac{i}{\hbar}[H, \rho_t] + L\rho_t L^\dagger - \frac{1}{2}\left(L^\dagger L\rho_t + \rho_t L^\dagger L\right)\right) dt + \sqrt{\eta}\left(L\rho_t + \rho_t L^\dagger - \mathrm{Tr}\left((L + L^\dagger)\rho_t\right)\rho_t\right) dW_t$$

driven by the Wiener process dW_t (Gaussian law of mean 0 and variance dt).

With Itō rules, it can be written as the following "discrete-time" Markov model

$$\rho_{t+dt} = \frac{M_{dy_t}\rho_t M_{dy_t}^\dagger + (1-\eta)L\rho_t L^\dagger dt}{\mathrm{Tr}\left(M_{dy_t}\rho_t M_{dy_t}^\dagger + (1-\eta)L\rho_t L^\dagger dt\right)}$$

with $M_{dy_t} = I + \left(-\frac{i}{\hbar}H - \frac{1}{2}\left(L^\dagger L\right)\right) dt + \sqrt{\eta}\, dy_t L$. The probability to detect dy_t is given by the following density

$$\mathbb{P}\left(dy_t \in [s, s+ds]\right) = \frac{\mathrm{Tr}\left(M_s\rho_t M_s^\dagger + (1-\eta)L\rho_t L^\dagger dt\right)}{\sqrt{2\pi}} e^{-\frac{s^2}{2dt}}\, ds$$

close to a Gaussian law of variance dt and mean $\sqrt{\eta}\,\mathrm{Tr}\left((L + L^\dagger)\,\rho_t\right)\,dt$.

[1] H. Amini, M. Mirrahimi, P.R. IEEE CDC, 2011. P.R., J. Ralph PRA 2015.

A positivity preserving numerical scheme: diffusive/jump case [2]

The quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-i[H, \rho_t] + \sum_\nu L_\nu \rho_t L_\nu^\dagger - \tfrac{1}{2}(L_\nu^\dagger L_\nu \rho_t + \rho_t L_\nu^\dagger L_\nu) + V_\mu \rho_t V_\mu^\dagger - \tfrac{1}{2}(V_\mu^\dagger V_\mu \rho_t + \rho_t V_\mu^\dagger V_\mu) \right) dt$$

$$+ \sum_\nu \sqrt{\eta_\nu} \left(L_\nu \rho_t + \rho_t L_\nu^\dagger - \mathrm{Tr}\left((L_\nu + L_\nu^\dagger)\rho_t \right) \rho_t \right) \boldsymbol{dW_{\nu,t}}$$

$$+ \sum_\mu \left(\frac{\overline{\theta}_\mu \rho_t + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} V_\mu \rho_t V_\mu^\dagger}{\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \mathrm{Tr}\left(V_{\mu'} \rho_t V_{\mu'}^\dagger \right)} - \rho_t \right) \left(\boldsymbol{dN_\mu(t)} - \left(\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \mathrm{Tr}\left(V_{\mu'} \rho_t V_{\mu'}^\dagger \right) \right) dt \right)$$

where $\eta_\nu \in [0, 1]$, $\overline{\theta}_\mu, \overline{\eta}_{\mu,\mu'} \geq 0$ with $\overline{\eta}_{\mu'} = \sum_\mu \overline{\eta}_{\mu,\mu'} \leq 1$ are parameters modelling measurements imperfections.

If, for some μ, $\boldsymbol{N_\mu(t+dt) - N_\mu(t) = 1}$, we have $\rho_{t+dt} = \dfrac{\overline{\theta}_\mu \rho_t + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} V_{\mu'} \rho_t V_{\mu'}^\dagger}{\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \mathrm{Tr}\left(V_{\mu'} \rho_t V_{\mu'}^\dagger \right)}$.

When $\forall \mu$, $\boldsymbol{dN_\mu(t) = 0}$, we have

$$\rho_{t+dt} = \frac{M_{\boldsymbol{dy_t}} \rho_t M_{\boldsymbol{dy_t}}^\dagger + \sum_\nu (1-\eta_\nu) L_\nu \rho_t L_\nu^\dagger dt + \sum_\mu (1-\overline{\eta}_\mu) V_\mu \rho_t V_\mu^\dagger dt}{\mathrm{Tr}\left(M_{\boldsymbol{dy_t}} \rho_t M_{\boldsymbol{dy_t}}^\dagger + \sum_\nu (1-\eta_\nu) L_\nu \rho_t L_\nu^\dagger dt + \sum_\mu (1-\overline{\eta}_\mu) V_\mu \rho_t V_\mu^\dagger dt \right)}$$

with $M_{\boldsymbol{dy_t}} = I + \left(-iH - \frac{1}{2} \sum_\nu L_\nu^\dagger L_\nu + \frac{1}{2} \sum_\mu \left(\overline{\eta}_\mu \mathrm{Tr}\left(V_\mu \rho_t V_\mu^\dagger \right) I - V_\mu^\dagger V_\mu \right) \right) dt + \sum_\nu \sqrt{\eta_\nu} \boldsymbol{dy_{\nu,t}} L_\nu$ and where $\boldsymbol{dy_{\nu,t}} = \sqrt{\eta_\nu} \mathrm{Tr}\left((L_\nu + L_\nu^\dagger) \rho_t \right) dt + \boldsymbol{dW_{\nu,t}}$.

[2] H. Amini, C. Pellegrini, P.R.: Russian Journal of Mathematical Physics, 2014.
P.R:. Proceedings of International Congress of Mathematicians, Seoul 2014 (arXiv:1407.7810).

The deterministic surface[3]

The SME ($\gamma_1 = 1$ here)

$$d\rho_t = \left(\boldsymbol{\sigma_-}\rho\boldsymbol{\sigma_+} - \frac{\boldsymbol{\sigma_+}\boldsymbol{\sigma_-}\rho + \rho\boldsymbol{\sigma_+}\boldsymbol{\sigma_-}}{2}\right) dt$$
$$+ \sqrt{\tfrac{\eta}{2}}\left(\boldsymbol{\sigma_-}\rho + \rho\boldsymbol{\sigma_+} - \operatorname{Tr}\left(\boldsymbol{\sigma_x}\rho\right)\rho\right) dW_I + \sqrt{\tfrac{\eta}{2}}\left(i\boldsymbol{\sigma_-}\rho - i\rho\boldsymbol{\sigma_+} - \operatorname{Tr}\left(\boldsymbol{\sigma_y}\rho\right)\rho\right) dW_Q$$

reads with the Bloch coordinates (x, y, z)

$$dx_t = -\tfrac{1}{2}x_t dt + \sqrt{\tfrac{\eta}{2}}\Big((1 + z_t - x_t^2) dW_I - x_t y_t dW_Q\Big)$$
$$dy_t = -\tfrac{1}{2}y_t dt + \sqrt{\tfrac{\eta}{2}}\Big(-x_t y_t dW_I + (1 + z_t - y_t^2) dW_Q\Big)$$
$$dz_t = -(1 + z_t)\, dt - \sqrt{\tfrac{\eta}{2}}(1 + z_t)\Big(x_t dW_I + y_t dW_Q\Big)$$

For any realization of starting from the same initial point (x_0, y_0, z_0) we have

$$\tfrac{1}{2}(x_t^2 + y_t^2) + c_t(1 + z_t)^2 - (1 + z_t) = 0$$

where $c_t = (c_0 - \frac{\eta}{2})\, e^t + \frac{\eta}{2}$ remains in $[\frac{1}{2}, +\infty)$ and $c_0 = \frac{x_0^2 + y_0^2}{2(1+z_0)^2} + \frac{1}{1+z_0}$.

[3] Ph. Campagne-Ibarcq et al. PRX 2016.

The quantum filter is "solvable by elementary quadratures" [4]

The solution of

$$dx_t = -\tfrac{1}{2}x_t dt + \sqrt{\tfrac{\eta}{2}}\Big((1+z_t-x_t^2)dW_I - x_t y_t dW_Q\Big)$$
$$dy_t = -\tfrac{1}{2}y_t dt + \sqrt{\tfrac{\eta}{2}}\Big(-x_t y_t dW_I + (1+z_t-y_t^2)dW_Q\Big)$$
$$dz_t = -(1+z_t)\,dt - \sqrt{\tfrac{\eta}{2}}(1+z_t)\Big(x_t dW_I + y_t dW_Q\Big)$$

can be computed from simple integrals of the signals $dI_t = \sqrt{\frac{\eta}{2}}x_t dt + dW_I$ and $dQ_t = \sqrt{\frac{\eta}{2}}y_t dt + dW_Q$. This results from

$$d\left(\tfrac{x}{1+z}\right) = \tfrac{1}{2}\left(\tfrac{x_t}{1+z_t}\right)dt + \sqrt{\tfrac{\eta}{2}}dI_t, \quad d\left(\tfrac{y}{1+z}\right) = \tfrac{1}{2}\left(\tfrac{y_t}{1+z_t}\right)dt + \sqrt{\tfrac{\eta}{2}}dQ_t$$

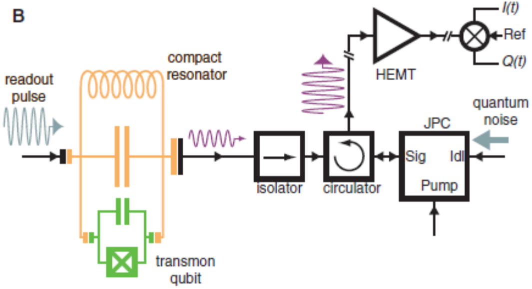
This provides a completion of $\frac{1}{2}(x_t^2+y_t^2) + c_t(1+z_t)^2 - (1+z_t) = 0$ with

$$\tfrac{x_t}{1+z_t} = e^{t/2}\left(\tfrac{x_0}{1+z_0} + \sqrt{\tfrac{\eta}{2}}\int_0^t e^{-\tau/2}dI_\tau\right), \quad \tfrac{y_t}{1+z_t} = e^{t/2}\left(\tfrac{y_0}{1+z_0} + \sqrt{\tfrac{\eta}{2}}\int_0^t e^{-\tau/2}dQ_\tau\right).$$

Related to Picard-Vessiot and Liouvillian extensions of differential fields: the solution of the quantum filter is an algebraic function of some integrals and exponentials of integral of its two inputs I_t and Q_t.

[4] Ph. Campagne-Ibarcq et al. PRX 2016.

The second case where the qubit is confined to a deterministic surface[5]



Superconducting qubit dispersively coupled to a cavity traversed by a microwave signal (input/output theory). The back-action on the qubit state of a single measurement of both output field quadratures I_t and Q_t is described by a simple SME for the qubit density operator. **(M. Hatridge et al.: Science, 2013).**

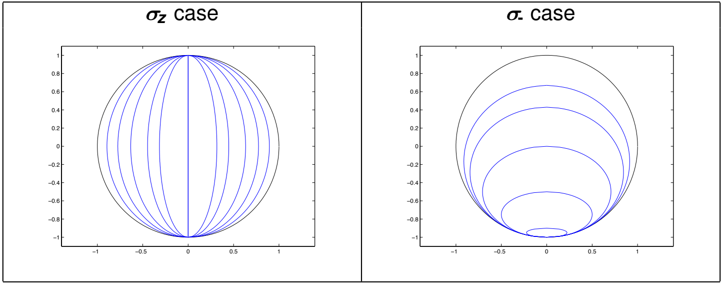
$$d\rho_t = \left(\gamma(\boldsymbol{\sigma_z}\rho\boldsymbol{\sigma_z} - \rho_t)\right)dt + \sqrt{\tfrac{\eta\gamma}{2}}\left(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\left(\boldsymbol{\sigma_z}\rho_t\right)\rho_t\right)\boldsymbol{dW_I} + i\sqrt{\tfrac{\eta\gamma}{2}}[\boldsymbol{\sigma_z}, \rho_t]\boldsymbol{dW_Q}$$

with $\boldsymbol{dI_t} = \sqrt{\frac{\eta\gamma}{2}}\,\mathrm{Tr}\left(2\boldsymbol{\sigma_z}\rho_t\right)dt + \boldsymbol{dW_I}$ and $\boldsymbol{dQ_t} = \boldsymbol{dW_Q}$, where $\gamma \geq 0$ is related to the measurement strength and $\eta \in [0,1]$ is the detection efficiency.
The deterministic surface is given here by another ellipsoid of revolution axis z:

$$x_t^2 + y_t^2 + b_t\,(z_t^2 - 1) = 0 \text{ where } b_t = b_0\,e^{-2(1-\eta)t} \text{ and } b_0 = \frac{x_0^2 + y_0^2}{1 - z_0^2}.$$

[5] A. Sarlette, P.R.: preprint 2016 (arXiv:1603.05402).

There are only two cases where the qubit is confined on an evolving surface



Theorem [A. Sarlette, P.R.: arXiv:1603.05402] *For any initial state ρ_0, the qubit state ρ_t, solution of the SME ($\eta_\nu \in (0,1)$)*

$$d\rho_t = \left(\sum_\nu L_\nu \rho_t L_\nu^\dagger - \tfrac{1}{2}(L_\nu^\dagger L_\nu \rho_t + \rho_t L_\nu^\dagger L_\nu) \right) dt + \sum_\nu \sqrt{\eta_\nu} \left(L_\nu \rho_t + \rho_t L_\nu^\dagger - Tr\left((L_\nu + L_\nu^\dagger)\rho_t \right) \rho_t \right) \boldsymbol{dW_{\nu,t}},$$

is restricted to a deterministically evolving 2-dimensional manifold if, and only if,

- *either exist $\beta_\nu, \alpha_\nu \in \mathbb{C}$ and $U \in U(2)$ such that $L_\nu = \beta_\nu\, V \boldsymbol{\sigma_z} V^\dagger + \alpha_\nu,\ \forall \nu$*
- *or exist $\beta_\nu \in \mathbb{C}$ and $V \in U(2)$ such that $L_\nu = \beta_\nu\, V \boldsymbol{\sigma_-} V^\dagger,\ \forall \nu$.*

Sketch of the proof (1)

Stroock-Varadhan theorem[6]
Consider a stochastic differential equation

$$dx_t = F(x_t)\,dt + \sum_{j=1}^{m} G_j(x) \circ dW_t^j\,,$$

with $x_t \in \mathbb{R}^N$ the state, $dW_t^1, dW_t^2, ..., dW_t^m$ independent Wiener processes, x_0 fixed and the dynamics to be understood in the Stratonovitch sense (we therefore put the $\circ$ symbol).
The support of the distribution of x_t can be described as the closure, for the natural Banach topology on $C([0,1], \mathbb{R}^N)$, of the set of solutions of the following controlled system:

$$d\tilde{x}_t = F(\tilde{x}_t)\,dt + \sum_{j=1}^{m} G_j(\tilde{x})\,du_t^j\,,$$

with $\tilde{x}_0 = x_0$, for all possible control signals $u_t^1, u_t^2, ..., u_t^m$ in $H^1([0,1], \mathbb{R}^m)$.

[6] D.W. Stroock and S.R.S. Varadhan, "On the support of diffusion processes with applications to the strong maximum principle", Proc. 6th Berkeley Symp. Mathematical Statistics and Probability vol.3, pp.333-359, 1972.

Sketch of the proof (2)

- **Strong accessibility theorem**[7] *The control system* $\frac{d}{dt}x = F(x) + \sum_{j=1}^{m} \boldsymbol{G_j(x)}\, u_j$ *with analytic vector fields* $F, \boldsymbol{G_1, \dots, G_m}$ *is strongly accessible at* x_0 *if, and only if, the* **drift-preserved Lie algebra** $\mathfrak{G}_F$ [8] *has full dimension N at* x_0.
  Moreover, if $\mathfrak{G}_F$ *has dimension at most* $N - n < N$ *for all* x_0*, then the system stays on a (time-dependent) manifold of dimension* $N - n$*, independently of the control inputs.*
- **Stratonovitch form of the SME:**

$$
\begin{aligned}
d\rho_t = & \sum_\nu (1-\eta_\nu)\left(L_\nu \rho_t L_\nu^\dagger - \tfrac{1}{2}(L_\nu^\dagger L_\nu \rho_t + \rho_t L_\nu^\dagger L_\nu)\right) dt \\
& - \left(\sum_\nu \tfrac{\eta_\nu}{2}\left(L_\nu^\dagger L_\nu \rho + \rho L_\nu^\dagger L_\nu + (L_\nu)^2\rho + \rho (L_\nu^\dagger)^2\right)\right) dt \\
& + \left(\sum_\nu \tfrac{\eta_\nu}{2} \operatorname{Tr}\left(L_\nu^\dagger L_\nu \rho + \rho L_\nu^\dagger L_\nu + (L_\nu)^2\rho + \rho (L_\nu^\dagger)^2\right) \rho\right) dt \\
& + \left(\sum_\nu \eta_\nu \operatorname{Tr}\left(L_\nu \rho + \rho L_\nu^\dagger\right)(L_\nu \rho + \rho L_\nu^\dagger) - \eta_\nu \left(\operatorname{Tr}\left(L_\nu \rho + \rho L_\nu^\dagger\right)\right)^2 \rho\right) dt \\
& + \sum_\nu \sqrt{\eta_\nu}\left(L_\nu \rho_t + \rho_t L_\nu^\dagger - \operatorname{Tr}\left((L_\nu + L_\nu^\dagger)\rho_t\right)\rho_t\right) \circ dW_{\nu,t}.
\end{aligned}
$$

[7] A. Isidori, *Nonlinear Control Systems: An Introduction*, Springer, Berlin, 1985

[8] $\mathfrak{G}_F$ is the smallest Lie algebra containing $\mathfrak{G}$ (the Lie algebra generated by vector fields $\boldsymbol{G_1, G_2, \dots, G_m}$) and closed under Lie brackets with F, i.e. for any $\boldsymbol{G} \in \mathfrak{G}_F$ we have $[F, \boldsymbol{G}] \in \mathfrak{G}_F$.

Concluding remarks

1. Stochastic master equations govern the dynamics of open quantum systems by taking into account measurement back-action and decoherence (unread measurement).
2. Future work could investigate how general confinement of the density operator to submanifolds is in higher-dimensional Hilbert spaces.
3. Interest of continuous fluorescence signals (dI_t, dQ_t) for the characterization of a dephasing noise ξ_t:

$$dx_t = \left(-\tfrac{\gamma_2}{2}x_t + \xi_t y_t - v(t)z_t\right)dt + \sqrt{\tfrac{\eta\gamma_1}{2}}\Big((1+z_t-x_t^2)dW_I - x_t y_t dW_Q\Big)$$

$$dy_t = \left(-\xi_t x_t - \tfrac{\gamma_2}{2}y_t + u(t)z_t\right)dt + \sqrt{\tfrac{\eta\gamma_1}{2}}\Big(-x_t y_t dW_I + (1+z_t-y_t^2)dW_Q\Big)$$

$$dz_t = \big(v(t)x_t - u(t)y_t - \gamma_1(1+z_t)\big)dt - \sqrt{\tfrac{\eta\gamma_1}{2}}(1+z_t)\Big(x_t dW_I + y_t dW_Q\Big)$$

$$dI_t = \sqrt{\tfrac{\eta\gamma_1}{2}}x_t dt + dW_I$$

$$dQ_t == \sqrt{\tfrac{\eta\gamma_1}{2}}y_t dt + dW_Q$$

where $u(t)$ and $v(t)$ are well chosen open-loop controls (see Lorenza Viola presentation and work).