Lindblad master equation with multi-photon drive and damping.

Nonlinear Partial Differential Equations and Applications
A conference in the honor of Jean-Michel Coron
Institut Henri Poincaré 2016, June 20–24

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Joint work with Rémi Azouit and Alain Sarlette

Outline

Motivation: coherent feedback and reservoir engineering

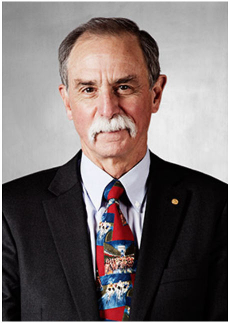
Harmonic oscillator with single-photon drive and damping

Well posedness and convergence for multi-photon drive and damping

Conclusion: many other examples of physical interest

Nobel Prize in Physics 2012

Serge Haroche

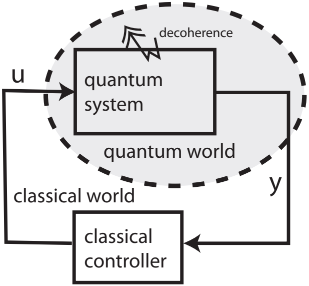


David J. Wineland

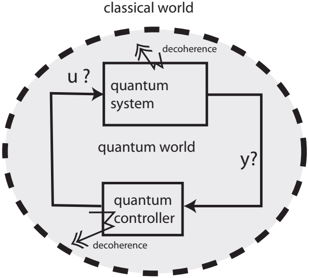
" This year's Nobel Prize in Physics honours the experimental inventions and discoveries that have allowed the measurement and control of individual quantum systems. They belong to two separate but related technologies: ions in a harmonic trap and photons in a cavity . . . "

From the Scientific Background on the Nobel Prize in Physics 2012 compiled by the Class for Physics of the Royal Swedish Academy of Sciences, 9 october 2012.

Two kinds of quantum feedback

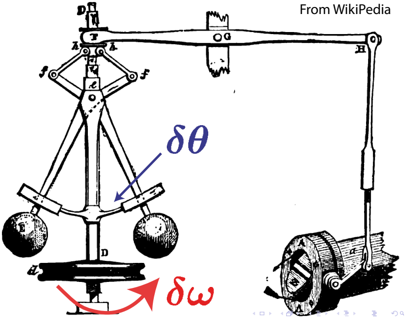


Measurement-based feedback: controller is classical; measurement back-action on the quantum system of Hilbert space $\mathcal{H}$ is stochastic (collapse of the wave-packet); the measured output y is a classical signal; the control input u is a classical variable appearing in some controlled Schrödinger equation; $u(t)$ depends on the past measurements $y(\tau)$, $\tau \leq t$.



Coherent/autonomous feedback and reservoir engineering: the **system of Hilbert space $\mathcal{H}$** is coupled to **the controller, another quantum system**; the composite system of Hilbert space $\mathcal{H}_{controller} \otimes \mathcal{H}$, is an open-quantum system relaxing to some target (separable) state.

Watt regulator: classical analogue of quantum coherent feedback. [1]



The first variations of speed $\delta\omega$ and governor angle $\delta\theta$ obey to

$$\frac{d}{dt}\delta\omega = -a\delta\theta$$

$$\frac{d^2}{dt^2}\delta\theta = -\Lambda\frac{d}{dt}\delta\theta - \Omega^2(\delta\theta - b\delta\omega)$$

with (a, b, Λ, Ω) positive parameters.

Third order system

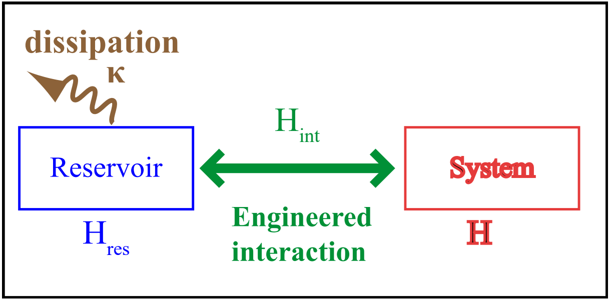
$$\frac{d^3}{dt^3}\delta\omega + \Lambda\frac{d^2}{dt^2}\delta\omega + \Omega^2\frac{d}{dt}\delta\omega + ab\Omega^2\delta\omega = 0.$$

Characteristic polynomial $P(s) = s^3 + \Lambda s^2 + \Omega^2 s + ab\Omega^2$ with roots having negative real parts iff $\Lambda > ab$: governor damping must be strong enough to ensure asymptotic stability.

Key issues: asymptotic stability and convergence rates.

[1] J.C. Maxwell: On governors. Proc. of the Royal Society, No.100, 1868.

Reservoir Engineering and coherent feedback[2]

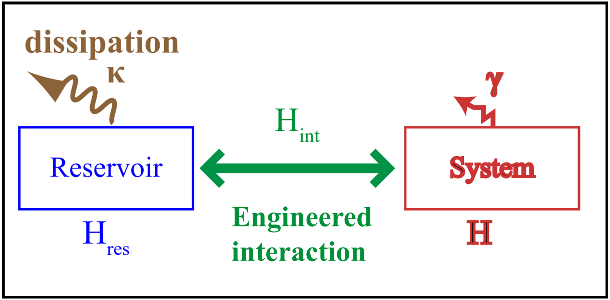


$$H = \boldsymbol{H}_{\text{res}} + \boldsymbol{H}_{\text{int}} + \boldsymbol{H}$$

if $\rho \underset{t\to\infty}{\rightarrow} \rho_{res} \otimes |\bar{\psi}\rangle\langle\bar{\psi}|$ exponentially on a time scale of $\tau \approx 1/\kappa$ then .....

[2] See, e.g., the lectures of H. Mabuchi delivered at the "Ecole de physique des Houches", July 2011.

Reservoir Engineering and coherent feedback[2]



$$H = \boldsymbol{H}_{\text{res}} + \boldsymbol{H}_{\text{int}} + \boldsymbol{H}$$

...... $\rho \underset{t\to\infty}{\rightarrow} \rho_{res} \otimes |\bar{\psi}\rangle\langle\bar{\psi}| + \Delta$, if $\kappa \gg \gamma$ then $\|\Delta\| \ll 1$

[2]See, e.g., the lectures of H. Mabuchi delivered at the "Ecole de physique des Houches", July 2011.

Convergence issues of open-quantum systems

Continuous-time models: Lindbald master eq. (quantum Fokker-Planck eq.):

$$\frac{d}{dt}\rho = -\mathfrak{A}(\rho) \triangleq -\frac{i}{\hbar}[\boldsymbol{H}, \rho] + \sum_{\nu} \left(\boldsymbol{L}_\nu \rho \boldsymbol{L}_\nu^\dagger - (\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho + \rho \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)/2 \right),$$

of state ρ a density operator (Hermitian, non negative, trace-class, trace one) with $\boldsymbol{H}$ Hermitian operator and $\boldsymbol{L}_\nu$ arbitrary operators (usually unbounded).

When $\mathcal{H}$ is of finite dimension, $(e^{-t\mathfrak{A}})_{t\geq 0}$ is a contraction semi-group for many metrics ($\mathrm{Tr}\left(|\rho - \sigma|\right)$, $\mathrm{Tr}\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)$, see the work of D. Petz).
Open issues motivated by robust quantum information processing:

1. characterization of the Ω-limit support of ρ: decoherence free spaces are affine spaces where the dynamics are of Schrödinger types; they can be reduced to a point (pointer-state);
2. Estimation of convergence rate and robustness.
3. Reservoir engineering: design of realistic H and L_ν to achieve rapid convergence towards prescribed affine spaces (protection against decoherence).

Goal of this talk: **well-posedness and convergence for the infinite dimension system with $\boldsymbol{H} = 0$ and $\boldsymbol{L}_\nu = \boldsymbol{a}^k - \alpha^k \boldsymbol{I}$** with $k \in \mathbb{N}$ and $\alpha \in \mathbb{C}$.

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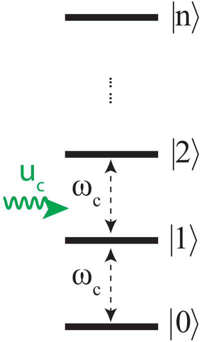
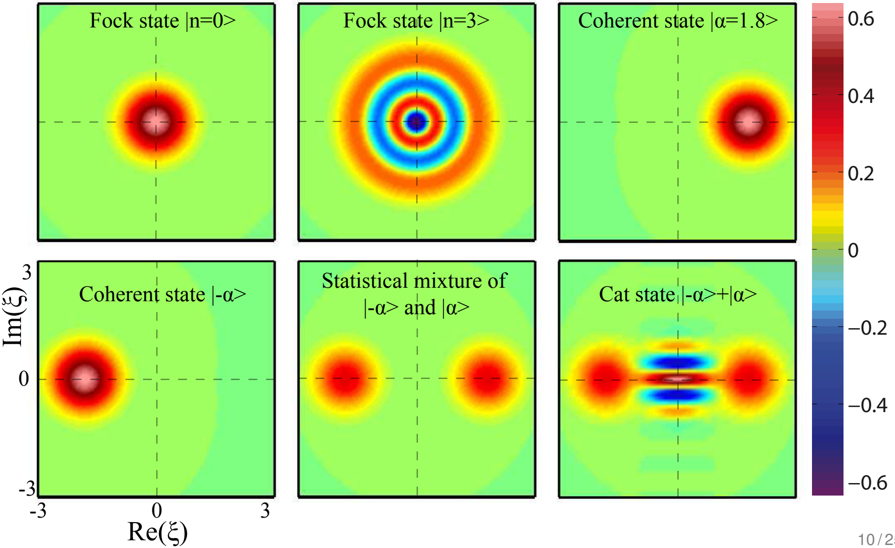
Quantum harmonic oscillator

- Hilbert space:
  $\mathcal{H} = \left\{ \sum_{n\geq 0} \psi_n |n\rangle,\ (\psi_n)_{n\geq 0} \in l^2(\mathbb{C}) \right\} \equiv L^2(\mathbb{R},\mathbb{C})$
- Operators and commutations:
  $\boldsymbol{a}|n\rangle = \sqrt{n}\ |\boldsymbol{n\text{-}1}\rangle$, $\boldsymbol{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$;
  $\boldsymbol{N} = \boldsymbol{a}^\dagger \boldsymbol{a}$, $\boldsymbol{N}|n\rangle = n|n\rangle$;
  $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = \boldsymbol{I}$, $\boldsymbol{a} f(\boldsymbol{N}) = f(\boldsymbol{N}+\boldsymbol{I})\boldsymbol{a}$;
  $\boldsymbol{D}_\alpha = e^{\alpha \boldsymbol{a}^\dagger - \alpha^\dagger \boldsymbol{a}}$.
  $\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$, $[\boldsymbol{X}, \boldsymbol{P}] = \imath \boldsymbol{I}/2$.
- Hamiltonian: $\boldsymbol{H}/\hbar = \omega_c \boldsymbol{a}^\dagger \boldsymbol{a} + \boldsymbol{u_c}(\boldsymbol{a} + \boldsymbol{a}^\dagger)$.
  (associated classical dynamics:
  $\frac{dx}{dt} = \omega_c p,\ \frac{dp}{dt} = -\omega_c x - \sqrt{2} u_c$).
- Classical pure state $\equiv$ coherent state $|\alpha\rangle$
  $\alpha \in \mathbb{C}:\ |\alpha\rangle = \sum_{n\geq 0}\left(e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}\right)|n\rangle$; $|\alpha\rangle \equiv \frac{1}{\pi^{1/4}} e^{\imath\sqrt{2}x\Im\alpha} e^{-\frac{(x-\sqrt{2}\Re\alpha)^2}{2}}$
  $\boldsymbol{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\boldsymbol{D}_\alpha|0\rangle = |\alpha\rangle$.

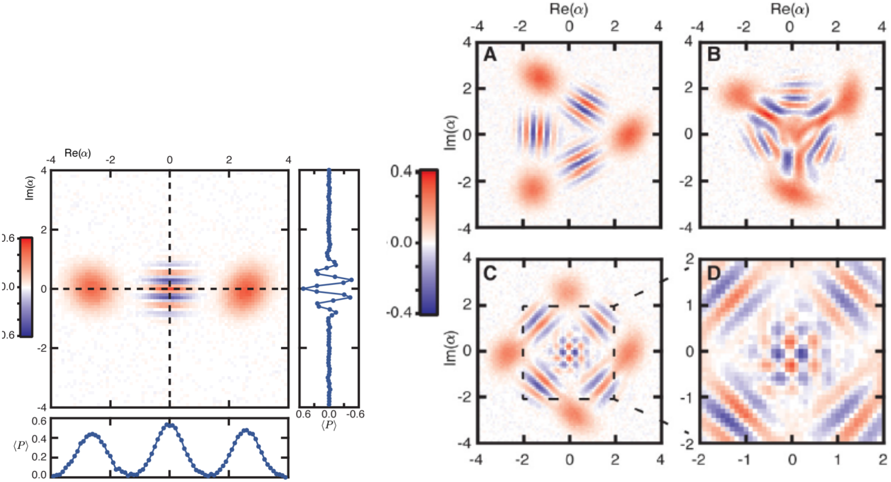


Wigner function W^ρ for different values of the density operator ρ

$$W^\rho : \mathbb{C} \ni \xi \to \frac{2}{\pi}\, \mathrm{Tr}\left(\left(\boldsymbol{D}_\xi e^{i\pi \boldsymbol{N}} \boldsymbol{D}_\xi^\dagger \right) \rho \right) \in [-2/\pi, 2/\pi]$$



Experimental Wigner functions of 2, 3 and 4-leg Schrödinger cat-states[3]



[3]Vlastakis, B.; Kirchmair, G.; Leghtas, Z.; Nigg, S. E.; Frunzio, L.; Girvin, S. M.; Mirrahimi, M.; Devoret, M. H., Schoelkopf, R. J. "Deterministically Encoding Quantum Information Using 100-Photon Schrödinger Cat States". Science, 2013, -

Master equation for a damped and driven ($\alpha \in \mathbb{R}$) harmonic oscillator [4]

$$\frac{d}{dt}\rho = \boldsymbol{L}\rho\boldsymbol{L}^\dagger - \frac{1}{2}\left(\boldsymbol{L}^\dagger\boldsymbol{L}\rho + \rho\boldsymbol{L}^\dagger\boldsymbol{L}\right) \quad \text{with} \quad \boldsymbol{L} = \boldsymbol{a} - \alpha\boldsymbol{I}$$

ρ can be represented by its Wigner function W^ρ defined by

$$\mathbb{C} \ni \xi = x + ip \mapsto W^\rho(\xi) = \frac{2}{\pi}\operatorname{Tr}\left(\left(e^{\xi\boldsymbol{a}^\dagger - \xi^*\boldsymbol{a}}\, e^{i\pi\boldsymbol{N}}\, e^{-\xi\boldsymbol{a}^\dagger + \xi^*\boldsymbol{a}}\right)\rho\right)$$

With the correspondences

$$\frac{\partial}{\partial\xi} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial p}\right), \quad \frac{\partial}{\partial\xi^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial p}\right)$$

$$W^{\rho\boldsymbol{a}} = \left(\xi - \frac{1}{2}\frac{\partial}{\partial\xi^*}\right)W^\rho, \quad W^{\boldsymbol{a}\rho} = \left(\xi + \frac{1}{2}\frac{\partial}{\partial\xi^*}\right)W^\rho$$

$$W^{\rho\boldsymbol{a}^\dagger} = \left(\xi^* + \frac{1}{2}\frac{\partial}{\partial\xi}\right)W^\rho, \quad W^{\boldsymbol{a}^\dagger\rho} = \left(\xi^* - \frac{1}{2}\frac{\partial}{\partial\xi}\right)W^\rho$$

we get the following PDE for W^ρ :

$$\frac{\partial W^\rho}{\partial t} = \frac{1}{2}\left(\frac{\partial}{\partial x}\left((x-\alpha)W^\rho\right) + \frac{\partial}{\partial p}\left(pW^\rho\right) + \frac{1}{4}\frac{\partial^2}{\partial x^2}W^\rho + \frac{1}{4}\frac{\partial^2}{\partial p^2}W^\rho\right)$$

converging toward the Gaussian $W^{\rho_\infty}(x,p) = \boldsymbol{\frac{2}{\pi}e^{-2(x-\alpha)^2-2p^2}}$.

[4] See, e.g., S. Haroche and J.M. Raimond: Exploring the Quantum: Atoms, Cavities and Photons. Oxford University Press, 2006.

Difficulties to get the semi-groups $\left(e^{-t\mathfrak{A}}\right)_{t\geq 0}$ from unbounded generators $\mathfrak{A}$.

The minimal solution [5] of $\frac{d}{dt}\rho = -\mathfrak{A}(\rho)$ need not be trace-preserving.
We can see this on this example due to Davies [6]

$$\frac{d}{dt}\rho = -\mathfrak{A}(\rho) = \boldsymbol{L}\rho\boldsymbol{L}^\dagger - \frac{1}{2}\left(\boldsymbol{L}^\dagger\boldsymbol{L}\rho + \rho\boldsymbol{L}^\dagger\boldsymbol{L}\right) \quad \text{with} \quad \boldsymbol{L} = \left(\boldsymbol{a}^\dagger\right)^2$$

Formally with $\rho \geq 0$, $p_n = \langle n|\rho|n\rangle \geq 0$ and $\text{Tr}\left(\rho\right) = \sum_n p_n = 1$ we get

$$\begin{aligned}\frac{d}{dt}\text{Tr}\left(\rho\boldsymbol{N}\right) = \text{Tr}\left(\rho 2(\boldsymbol{N}+1)(\boldsymbol{N}+2)\right) &= \sum_{n\geq 0} p_n 2(n+1)(n+2) \\ &\geq 2\left(\sum_{n\geq 0} p_n n\right)^2 + 1 = 2\,\text{Tr}^2\left(\rho\boldsymbol{N}\right) + 1\end{aligned}$$

by convexity of $x \mapsto 2(x+1)(x+2)$ and $2(x+1)(x+2) \geq 2x^2 + 1$ for $x \geq 0$.
With $z = \text{Tr}\left(\boldsymbol{N}\rho\right)$, we have $\frac{d}{dt}z \geq 2z^2 + 1$ and thus for any initial condition $\rho_0 \geq 0$, $z_0 \geq 0$ and $z(t)$ reaches $+\infty$ in finite time. This implies that $\text{Tr}\left(\rho\right)$ is decreasing and that the above computations have to be re-considered.

[5] See, e.g., chapter 4 written by F. Fagnola and R. Rebolledo in the book edited by Attal, S.; Joye, A.; Pillet, C.-A. (Eds.) Open Quantum Systems III: Recent Developments Springer, Lecture notes in Mathematics 1882, 2006.

[6] E. Davies: Quantum dynamical semigroups and the neutron diffusion equation. Reports on Mathematical Physics, 1977, 11, 169-188

Quantum information processing with cat-qubits [8]

It is possible with quantum circuits to design an open quantum system governed by[7]

$$\frac{d}{dt}\rho = \boldsymbol{L}\rho\boldsymbol{L}^\dagger - \frac{1}{2}\left(\boldsymbol{L}^\dagger\boldsymbol{L}\rho + \rho\boldsymbol{L}^\dagger\boldsymbol{L}\right) + \epsilon\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \frac{\boldsymbol{N}\rho+\rho\boldsymbol{N}}{2}\right) \text{ with } \boldsymbol{L} = \boldsymbol{a}^2 - \alpha^2\boldsymbol{I}.$$

The supports of all solutions $\rho(t)$ converge to the decoherence free space spanned by the even and odd cat-state;

$$|C_\alpha^+\rangle \propto |\alpha\rangle + |\text{-}\alpha\rangle, \quad |C_\alpha^-\rangle \propto |\alpha\rangle - |\text{-}\alpha\rangle.$$

The corresponding PDE for W^ρ is of order 4 in x and p.

A similar system where $\boldsymbol{L} = \boldsymbol{a}^4 - \alpha^4\boldsymbol{I}$ could be very interesting for quantum information processing where the logical qubit is encoded in the planes spanned by even and odd cat-states:

$$\left\{|C_\alpha^+\rangle, |C_{i\alpha}^+\rangle\right\}, \quad \left\{|C_\alpha^-\rangle, |C_{i\alpha}^-\rangle\right\}..$$

The corresponding PDE for W^ρ is of order 8 in x and p.

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[7] Z. Leghtas et al.: Confining the state of light to a quantum manifold by engineered two-photon loss. Science, 2015, 347, 853-857.

[8] M. Mirrahimi et al: Dynamically protected cat-qubits: a new paradigm for universal quantum computation, New Journal of Physics, 2014, 16, 045014.

Outline

A bunch of spaces

- $\mathcal{H} = \left\{ |\psi\rangle = \sum_{n\in\mathbb{N}} \psi_n |n\rangle \;\middle|\; \psi_n \in \mathbb{C},\ \sum_{n\in\mathbb{N}} |\psi_n|^2 < +\infty \right\}$ separable Hilbert space.
- $\mathcal{H}^f = \left\{ |\psi\rangle = \sum_{n\in\mathbb{N}} \psi_n |n\rangle \;\middle|\; \exists \bar{n}, \forall n > \bar{n}, \psi_n = 0 \right\}$ dense in $\mathcal{H}$.
- $\mathcal{H}_k = \{ |\psi\rangle = \sum_{n\in\mathbb{N}} \psi_n |n\rangle \mid \sum_{n\in\mathbb{N}} n^k |\psi_n|^2 < +\infty \} \}$ dense in $\mathcal{H}$.
- $\mathcal{K}^1(\mathcal{H})$ the Banach space of Hermitian trace-class operators equipped with the trace norm: $\rho \in \mathcal{K}^1(\mathcal{H})$ compact Hermitian operator with spectral decomposition $\rho = \sum_{\mu\geq 1} \lambda_\mu |\psi_\mu\rangle\langle\psi_\mu|$ and such that $\sum_{\mu\geq 1} |\lambda_\mu| < +\infty$. The trace-norm is $\|\rho\|_{tr} = \operatorname{Tr}(|\rho|) = \sum_{\mu=1}^{\infty} |\lambda_\mu|$. We have $\rho = \rho^+ - \rho^-$ and $|\rho| = \rho^+ + \rho^-$ with $\rho^+ = \sum_{\mu\geq 1} \max(0, \lambda_\mu) |\psi_\mu\rangle\langle\psi_\mu|$ and $\rho^- = \sum_{\mu\geq 1} \max(0, -\lambda_\mu) |\psi_\mu\rangle\langle\psi_\mu|$.
- Quantum state-space: the convex set of density operators $\mathcal{D} = \left\{ \rho \in \mathcal{K}^1(\mathcal{H}) \;\middle|\; \sum_{\mu\geq 1} \lambda_\mu = 1, ;\ \lambda_\mu \geq 0 \text{ for all } \mu \geq 1 \right\}$.
- $\mathcal{K}^f(\mathcal{H}) = \left\{ \sum_{n,n'=1}^{\bar{n}} f_{n,n'} |n\rangle\langle n'| \;\middle|\; f_{n,n'} = f^*_{n',n},\ \bar{n} \in \mathbb{N} \right\}$ dense in $\mathcal{K}^1(\mathcal{H})$.
- For any $\rho \in \mathcal{K}^1(\mathcal{H})$ and any bounded operator $\boldsymbol{B}$ on $\mathcal{H}$ we have $\operatorname{Tr}(\boldsymbol{B}\rho) = \operatorname{Tr}(\rho\boldsymbol{B})$, $\quad \operatorname{Tr}(\boldsymbol{B}\rho) \leq \operatorname{Tr}(|\boldsymbol{B}\rho|) = \|\boldsymbol{B}\rho\|_{tr} \leq \|\boldsymbol{B}\| \operatorname{Tr}(|\rho|) = \|\boldsymbol{B}\| \|\rho\|_{tr}$

Adapted Banach space for $\frac{d}{dt}\rho = \boldsymbol{L}\rho\boldsymbol{L}^\dagger - \frac{1}{2}(\boldsymbol{L}^\dagger\boldsymbol{L}\rho + \rho\boldsymbol{L}^\dagger\boldsymbol{L})$ with $\boldsymbol{L} = \boldsymbol{a}^k - \alpha^k\boldsymbol{I}$.

- The operator $\boldsymbol{L}^\dagger\boldsymbol{L}$ with domain $\mathcal{H}_{2k}$ admits a spectral decomposition $\boldsymbol{L}^\dagger\boldsymbol{L} = \sum_{\mu=1}^{\infty} d_\mu |g_\mu\rangle\langle g_\mu|$ where $\left(|g_\mu\rangle\right)_{\mu\geq 1}$ is an Hilbert basis of $\mathcal{H}$ and $d_\mu \geq 0$.
  Proof: $(\boldsymbol{I} + \boldsymbol{L}^\dagger\boldsymbol{L})^{-1}$ is a compact Hermitian operator.
- $\mathcal{K}_L(\mathcal{H}) \triangleq \left\{\rho \in \mathcal{K}^1(\mathcal{H}) \;\middle|\; \operatorname{Tr}\left(\left|\sqrt{\boldsymbol{I}+\boldsymbol{L}^\dagger\boldsymbol{L}}\,\rho\,\sqrt{\boldsymbol{I}+\boldsymbol{L}^\dagger\boldsymbol{L}}\right|\right) < +\infty\right\}$ equipped with the norm $\|\rho\|_L = \operatorname{Tr}\left(\left|\sqrt{\boldsymbol{I}+\boldsymbol{L}^\dagger\boldsymbol{L}}\,\rho\,\sqrt{\boldsymbol{I}+\boldsymbol{L}^\dagger\boldsymbol{L}}\right|\right)$ is a Banach space.
  Moreover $\rho \in \mathcal{K}_L(\mathcal{H})$ implies $\boldsymbol{L}\rho\boldsymbol{L}^\dagger \in \mathcal{K}^1(\mathcal{H})$.
- We have $[\boldsymbol{L}, \boldsymbol{L}^\dagger] = \boldsymbol{a}^k(\boldsymbol{a}^\dagger)^k - (\boldsymbol{a}^\dagger)^k\boldsymbol{a}^k = \boldsymbol{M}$ with

  $$\boldsymbol{M} = (\boldsymbol{N}+\boldsymbol{I})(\boldsymbol{N}+2\boldsymbol{I})\ldots(\boldsymbol{N}+k\boldsymbol{I}) \;-\; \boldsymbol{N}(\boldsymbol{N}-\boldsymbol{I})^+\ldots(\boldsymbol{N}-(k-1)\boldsymbol{I})^+ \geq k!\boldsymbol{I}.$$
- $\operatorname{Tr}\left(\boldsymbol{L}\rho\boldsymbol{L}^\dagger\right)$ satisfies $\frac{d}{dt}\operatorname{Tr}\left(\boldsymbol{L}\rho\boldsymbol{L}^\dagger\right) = -\operatorname{Tr}\left(\boldsymbol{L}\rho\boldsymbol{L}^\dagger\boldsymbol{M}\right) \leq -k!\operatorname{Tr}\left(\boldsymbol{L}\rho\boldsymbol{L}^\dagger\right)$.

Well posedness in $\mathcal{K}_L(\mathcal{H})$ based on Hill-Yosida thm for Banach space [10]

Consider the Cauchy problem $\frac{d}{dt}\rho = -\mathfrak{A}(\rho)$ associated to the super-operator

$$\mathcal{K}_L(\mathcal{H}) \supset D_{\mathfrak{A}} \ni \rho \mapsto \mathfrak{A}(\rho) = (\boldsymbol{L}^\dagger \boldsymbol{L}\rho + \rho \boldsymbol{L}^\dagger \boldsymbol{L})/2 - \boldsymbol{L}\rho\boldsymbol{L}^\dagger \in \mathcal{K}_L(\mathcal{H})$$

with $\boldsymbol{L} = \boldsymbol{a}^k - \alpha^k$. For any integer $k > 0$, any real $\alpha > 0$ and any ρ_0 in the domain of $\mathfrak{A}$,

1. there exists a unique C^1 function $[0, +\infty[\ni t \mapsto \rho(t) \in \mathcal{K}_L(\mathcal{H})$, such that $\rho(t)$ belongs to the domain of $\mathfrak{A}$ for all $t \geq 0$ and solves the initial value problem with $\rho(0) = \rho_0$
2. $\forall t \geq 0$, $\text{Tr}\,(\rho(t)) = \text{Tr}\,(\rho_0)$, $\|\rho(t)\|_L \leq \|\rho_0\|_L$ and $\|\mathfrak{A}(\rho(t))\|_L \leq \|\mathfrak{A}(\rho_0)\|_L$.
3. If ρ_0 is non-negative then $\rho(t)$ remains also non negative.

Proof: for any $\lambda > 0$ and $f \in \mathcal{K}_L(\mathcal{H})$, exits $\rho \in \mathcal{K}_L(\mathcal{H})$ such that $\rho + \lambda\mathfrak{A}(\rho) = f$ and $\|\rho\|_L \leq \|f\|_L$. We prove that $(\boldsymbol{I} + \lambda\mathfrak{A})^{-1}$ is a completely positive map, i.e. a quantum (Kraus) map, from $\mathcal{K}_L(\mathcal{H})$ to $D_{\mathfrak{A}}$. We combine arguments due to E. Davies[9] with the fact that $[\boldsymbol{L}, \boldsymbol{L}^\dagger] > 0$. See the forthcoming special issue of COCV or the preprint `http://arxiv.org/abs/1511.03898`.

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[9] E. Davies: Quantum dynamical semigroups and the neutron diffusion equation. Reports on Mathematical Physics, 1977, 11, 169-188

[10] H. Brezis: Analyse fonctionnelle. Masson, Paris, 1987.

- The set of steady-states characterized by $\mathfrak{A}(\rho) = 0$ corresponds to Hermitian operators $\bar{\rho}$ with range included in the k-dimensional complex vector space, the decoherence-free sub-space,

$$\mathcal{H}_{\alpha,k} = \text{span}\left\{ |\alpha_m\rangle \;\middle|\; \alpha_m = \alpha\, e^{2i\pi m/k} \;,\quad m = 1, 2, ..., k \right\}.$$

- Consider the unique trajectory $[0, +\infty[\ni t \mapsto \rho(t) \in \mathcal{K}_L(\mathcal{H})$ solution of $\frac{d}{dt}\rho = -\mathfrak{A}(\rho)$ with initial condition $\rho(0) = \rho_0$ non-negative, of trace one and in the domain of $\mathfrak{A}$. Then there exists $\bar{\rho}_{\rho_0} \in \mathcal{K}_L(\mathcal{H})$ nonnegative and of trace one, with support in $\mathcal{H}_{\alpha,k}$ such that ρ converges to $\bar{\rho}$ in $\mathcal{K}_L(\mathcal{H})$. Moreover, we have exponential convergence towards $\mathcal{H}_{\alpha,k}$ in the sense:

$$\text{Tr}\left(\left| \boldsymbol{L}(\rho(t) - \bar{\rho}_{\rho_0}) \boldsymbol{L}^\dagger \right| \right) \leq \text{Tr}\left(\boldsymbol{L} |\rho_0 - \bar{\rho}_{\rho_0}| \boldsymbol{L}^\dagger \right) \; e^{-k!\,t} \,.$$

Proof: the Lyapunov function $V(\rho) = \text{Tr}\left(\boldsymbol{L}\rho\boldsymbol{L}^\dagger\right)$ and $\frac{d}{dt}V \leq -k!\,V$.

There exist k^2 linearly independent Hermitian bounded operators $\boldsymbol{Q}_{m,m'}$, $m, m' = 1, 2, ..., k$, which are invariant under $\frac{d}{dt}\rho = -\mathfrak{A}(\rho)$, i.e. for which

$$\operatorname{Tr}\left(\boldsymbol{Q}_{m,m'}\ \rho_t\right) = \operatorname{Tr}\left(\boldsymbol{Q}_{m,m'}\ \rho_0\right)$$

for any trajectory $[0, +\infty) \ni t \mapsto \rho_t \in \mathcal{K}_L(\mathcal{H})$.
Moreover, the linear space of invariant Hermitian operators spanned by $\{\boldsymbol{Q}_{m,m'}\}_{m,m'=1...k}$ contains in particular the k operators

$$\boldsymbol{Q}_m^{\cos} = \sum_{n\in\mathbb{N}} \cos\left(\frac{2\pi mn}{k}\right) |n\rangle\langle n| \qquad \text{for} \quad m = 0, 1, ..., \lceil \tfrac{k-1}{2} \rceil \,;$$

$$\boldsymbol{Q}_m^{\sin} = \sum_{n\in\mathbb{N}} \sin\left(\frac{2\pi mn}{k}\right) |n\rangle\langle n| \qquad \text{for} \quad m = 1, ..., \lfloor \tfrac{k-1}{2} \rfloor \,.$$

Proof: use the fact that $\rho \mapsto \lim_{t\mapsto+\infty} e^{-t\mathfrak{A}}(\rho)$ is a complete positive map, i.e. a quantum channel and the fact that the dual of $\mathcal{K}^1(\mathcal{H})$ is the set of bounded operators.

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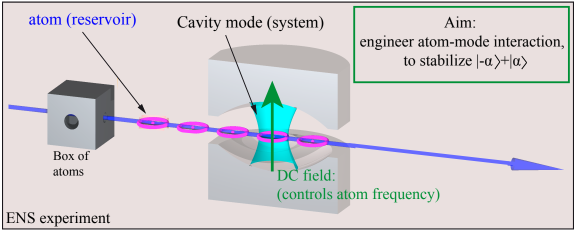
Reservoir with the cavity deoherence ($1/\kappa$ photon life-time)[11]



$$\frac{d}{dt}\rho = \overbrace{(\boldsymbol{a}-\alpha)\rho(\boldsymbol{a}-\alpha)^\dagger - \frac{1}{2}\left((\boldsymbol{a}-\alpha)^\dagger(\boldsymbol{a}-\alpha)\rho + \rho(\boldsymbol{a}-\alpha)^\dagger(\boldsymbol{a}-\alpha)\right)}^{\text{reservoir relaxation}}$$

$$+ \underbrace{\kappa\left(\boldsymbol{a}e^{i\pi \boldsymbol{N}}\rho e^{-i\pi \boldsymbol{N}}\boldsymbol{a}^\dagger - \frac{1}{2}(\boldsymbol{a}^\dagger\boldsymbol{a}\rho + \rho\boldsymbol{a}^\dagger\boldsymbol{a})\right)}_{\text{decoherence}}.$$

[11] A. Sarlette, ; Brune, M.; Raimond, J.M.; P.R. "Stabilization of nonclassical states of the radiation field in a cavity by reservoir engineering", Phys. Rev. Lett., 2011, 107, 010402.
A. Sarlette ; Leghtas, Z.; Brune, M.; Raimond, J.; P.R. " Stabilization of nonclassical states of one and two-mode radiation fields by reservoir engineering." Phys. Rev. A, 2012, 86, 012114

Robustness of the reservoir stabilizing the two-leg cat.

Since $\boldsymbol{W^{e^{i\pi N}\rho e^{-i\pi N}}(\xi) = W^{\rho}(-\xi)}$ the master Lindblad equation

$$\frac{d}{dt}\rho = \overbrace{\Big(\boldsymbol{a}-\alpha)\rho(\boldsymbol{a}-\alpha)^\dagger - \tfrac{1}{2}\big((\boldsymbol{a}-\alpha)^\dagger(\boldsymbol{a}-\alpha)\rho + \rho(\boldsymbol{a}-\alpha)^\dagger(\boldsymbol{a}-\alpha)\big)\Big)}^{\text{reservoir relaxation}}$$
$$+ \underbrace{\kappa\Big(\boldsymbol{a}\boldsymbol{e^{i\pi N}\rho e^{-i\pi N}}\boldsymbol{a}^\dagger - \tfrac{1}{2}(\boldsymbol{a}^\dagger\boldsymbol{a}\rho + \rho\boldsymbol{a}^\dagger\boldsymbol{a})\Big)}_{\text{decoherence}}.$$

yields to the following non local diffusion PDE (quantum Fokker-Planck equation):

$$\left.\frac{\partial W^\rho}{\partial t}\right|_{(x,p)} = \tfrac{1+\kappa}{2}\left(\frac{\partial}{\partial x}\Big((x-\alpha)W^\rho\Big) + \frac{\partial}{\partial p}\Big(pW^\rho\Big) + \tfrac{1}{4}\Delta W^\rho\right)_{(x,p)}$$
$$+\kappa\Big((x^2+p^2+\tfrac{1}{2})\left(W^\rho|_{(-x,-p)} - W^\rho|_{(x,p)}\right) + \tfrac{1}{16}\left(\Delta W^\rho|_{(-x,-p)} - \Delta W^\rho|_{(x,p)}\right)\Big)$$
$$-\kappa\left(\tfrac{x}{2}\left(\left.\frac{\partial W^\rho}{\partial x}\right|_{(-x,-p)} + \left.\frac{\partial W^\rho}{\partial x}\right|_{(x,p)}\right) + \tfrac{p}{2}\left(\left.\frac{\partial W^\rho}{\partial p}\right|_{(-x,-p)} + \left.\frac{\partial W^\rho}{\partial p}\right|_{(x,p)}\right)\right)$$

Lindblad master equation

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho] + \sum_{\nu}\left(L_\nu \rho L_\nu^\dagger - (L_\nu^\dagger L_\nu \rho + \rho L_\nu^\dagger L_\nu)/2\right),$$

for composite systems made of qubit(s) (Pauli operator $\boldsymbol{\sigma_x}$,$\boldsymbol{\sigma_y}$ and $\boldsymbol{\sigma_z}$) and harmonic oscillator(s) (annihilation operator $\boldsymbol{a}$, number operator $\boldsymbol{N}$) with e.g. (Hamiltonian coupling)

$$H = \omega_c \boldsymbol{N} \otimes \boldsymbol{I}_q + \chi \boldsymbol{N}^2 \otimes \boldsymbol{I}_q + u_c(\boldsymbol{a} + \boldsymbol{a}^\dagger) \otimes \boldsymbol{I}_q + \frac{\omega_q}{2} \boldsymbol{I}_c \otimes \boldsymbol{\sigma_z} + u_q \boldsymbol{I}_c \otimes \boldsymbol{\sigma_x} + \boldsymbol{g(a + a^\dagger) \otimes \sigma_x}$$

and local decoherence

$$L_1 = \sqrt{\kappa(1+n_{th})}\boldsymbol{a} \otimes \boldsymbol{I}_q,\ L_2 = \sqrt{\kappa n_{th}}\boldsymbol{a}^\dagger \otimes \boldsymbol{I}_q,$$
$$L_3 = \sqrt{\frac{1}{T_1}}\boldsymbol{I}_c \otimes (\boldsymbol{\sigma_x} - i\boldsymbol{\sigma_y}),\ L_4 = \sqrt{\frac{1}{T_\phi}}\boldsymbol{I}_c \otimes \boldsymbol{\sigma_z}.$$

Conclusion pour Jean-Michel

Bon anniversaire **Jean-Michel** et un **grand merci** pour tout ce que tu as fait pour

1. la communauté du contrôle,
2. la théorie mathématique des systèmes,
3. et plus largement les mathématiques en France et dans le Monde,
4. ...