Models and feedback stabilization of open quantum systems

International Congress of Mathematicians (ICM2014)
August 13 - 21, Seoul , Korea.

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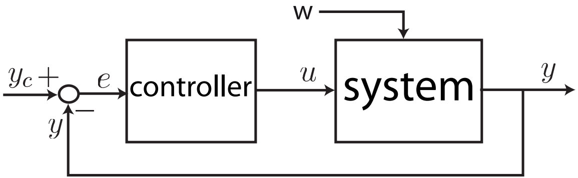
Quantum feedback: the back-action of the measurement.

A typical stabilizing feedback-loop for a classical system



Two kinds of stabilizing feedbacks for quantum systems

1. Measurement-based feedback: **controller is classical**; measurement back-action on the system $\mathcal{S}$ is stochastic (collapse of the wave-packet); the measured output y is a classical signal; the control input u is a classical variable appearing in some controlled Schrödinger equation; $u(t)$ depends on the past measurements $y(\tau)$, $\tau \leq t$.
2. Coherent/autonomous feedback and reservoir engineering: the system $\mathcal{S}$ is coupled to **the controller, another quantum system**; the composite system, $\mathcal{H}_S \otimes \mathcal{H}_{controller}$, is an open-quantum system relaxing to some target (separable) state.

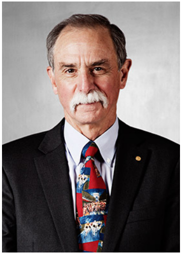
Controlling quantum degrees of freedom

Several applications:

- Nuclear Magnetic Resonance (NMR) applications;
- Quantum chemical synthesis;
- High resolution measurement devices (e.g. atomic/optic clocks);
- **Quantum information processing:** quantum computation and quantum communication.

Physics Nobel prize 2012:

Serge Haroche



David J. Wineland

Nobel prize: ground-breaking experimental methods that enable measuring and manipulation of individual quantum systems.

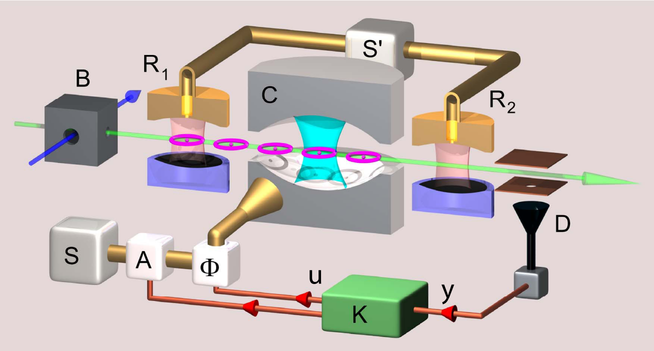
The LKB photon box

- First experimental realization of a quantum-state feedback (2011)
- Why density operator ρ instead of wave function $|\psi\rangle$
- Stabilization of "Schrödinger cats" by reservoir engineering

Model structure of open quantum systems

Conclusion: some open issues

The photon box of the Laboratoire Kastler-Brossel (LKB):
group of S.Haroche (Nobel Prize 2012), J.M.Raimond and M. Brune.[1]



Stabilization of a quantum state with exactly $n = 0, 1, 2, 3, \ldots$ photon(s).
Experiment: C. Sayrin et. al., Nature 477, 73-77, September 2011.
Theory: I. Dotsenko et al., Physical Review A, 80: 013805-013813, 2009.
R. Somaraju et al., Rev. Math. Phys., 25, 1350001, 2013.
H. Amini et. al., Automatica, 49 (9): 2683-2692, 2013.

[1] Courtesy of Igor Dotsenko. **Sampling period 80 μs.**

Three quantum features emphasized by the LKB photon box[2]

1. Schrödinger: wave funct. $|\psi\rangle \in \mathcal{H}$ or density op. $\rho \sim |\psi\rangle\langle\psi|$

$$\frac{d}{dt}|\psi\rangle = -\frac{i}{\hbar}\boldsymbol{H}|\psi\rangle, \quad \frac{d}{dt}\rho = -\frac{i}{\hbar}[\boldsymbol{H}, \rho], \quad \boldsymbol{H} = \boldsymbol{H}_0 + u\boldsymbol{H}_1$$

2. Origin of dissipation: collapse of the wave packet induced by the measurement of observable $\boldsymbol{O}$ with spectral decomp. $\sum_\mu \lambda_\mu \boldsymbol{P}_\mu$:
   - measurement outcome μ with proba. $\mathbb{P}_\mu = \langle\psi|\boldsymbol{P}_\mu|\psi\rangle = \text{Tr}\left(\rho \boldsymbol{P}_\mu\right)$ depending on $|\psi\rangle$, ρ just before the measurement
   - measurement back-action if outcome $\mu = y$:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\boldsymbol{P}_y|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{P}_y|\psi\rangle}}, \quad \rho \mapsto \rho_+ = \frac{\boldsymbol{P}_y \rho \boldsymbol{P}_y}{\text{Tr}\left(\rho \boldsymbol{P}_y\right)}$$

3. Tensor product for the description of composite systems (S, M):
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $\boldsymbol{H} = \boldsymbol{H}_S \otimes \boldsymbol{I}_M + \boldsymbol{H}_{int} + \boldsymbol{I}_S \otimes \boldsymbol{H}_M$
   - observable on sub-system M only: $\boldsymbol{O} = \boldsymbol{I}_S \otimes \boldsymbol{O}_M$.

[2] S. Haroche and J.M. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts, 2006.

- System S corresponds to a quantized harmonic oscillator:

$$\mathcal{H}_S = \left\{ \sum_{n=0}^{\infty} \psi_n |n\rangle \;\middle|\; (\psi_n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

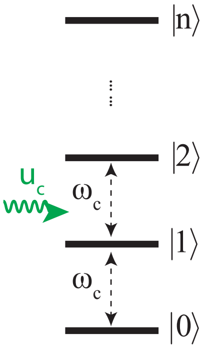
  where $|n\rangle$ represents the Fock state associated to exactly n photons inside the cavity
- Meter M is a qu-bit, a 2-level system (idem 1/2 spin system) : $\mathcal{H}_M = \mathbb{C}^2$, each atom admits two energy levels and is described by a wave function $c_g|g\rangle + c_e|e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$; atoms leaving B are all in state $|g\rangle$
- **State of the full system $|\Psi\rangle \in \mathcal{H}_S \otimes \mathcal{H}_M$:**

$$|\Psi\rangle = \sum_{n=0}^{+\infty} \Psi_{ng}|n\rangle \otimes |g\rangle + \Psi_{ne}|n\rangle \otimes |e\rangle, \qquad \Psi_{ne}, \Psi_{ng} \in \mathbb{C}.$$

  Ortho-normal basis: $(|n\rangle \otimes |g\rangle, |n\rangle \otimes |e\rangle)_{n\in\mathbb{N}}$.

S: quantum harmonic oscillator (spring system)

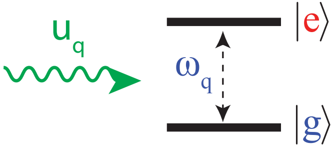
- Hilbert space:
  $\mathcal{H}_S = \left\{ \sum_{n\geq 0} \psi_n |n\rangle, \ (\psi_n)_{n\geq 0} \in l^2(\mathbb{C}) \right\} \equiv L^2(\mathbb{R}, \mathbb{C})$
- Quantum state space:
  $\mathcal{D} = \{\rho \in \mathcal{L}(\mathcal{H}_S), \rho^\dagger = \rho, \ \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}$.
- Operators and commutations:
  $\boldsymbol{a}|n\rangle = \sqrt{n}\ |\boldsymbol{n\text{-}1}\rangle$, $\boldsymbol{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$;
  $\boldsymbol{N} = \boldsymbol{a}^\dagger\boldsymbol{a}$, $\boldsymbol{N}|n\rangle = n|n\rangle$;
  $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = \boldsymbol{I}$, $\boldsymbol{a}f(\boldsymbol{N}) = f(\boldsymbol{N}+\boldsymbol{I})\boldsymbol{a}$;
  $\boldsymbol{D}_\alpha = e^{\alpha\boldsymbol{a}^\dagger - \alpha^\dagger\boldsymbol{a}}$.
  $\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$, $[\boldsymbol{X}, \boldsymbol{P}] = \imath\boldsymbol{I}/2$.
- Hamiltonian: $\boldsymbol{H}_S/\hbar = \omega_c\boldsymbol{a}^\dagger\boldsymbol{a} + \boldsymbol{u_c}(\boldsymbol{a}+\boldsymbol{a}^\dagger)$.
  (associated classical dynamics:
  $\frac{dx}{dt} = \omega_c p, \ \frac{dp}{dt} = -\omega_c x - \sqrt{2}u_c)$.



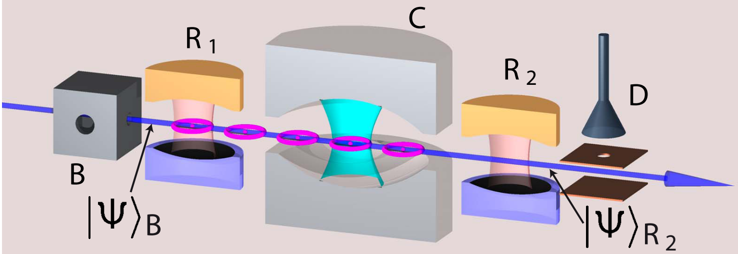
- Classical pure state $\equiv$ coherent state $|\alpha\rangle$
  $\alpha \in \mathbb{C}: \ |\alpha\rangle = \sum_{n\geq 0}\left(e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\right)|n\rangle$; $|\alpha\rangle \equiv \frac{1}{\pi^{1/4}}e^{\imath\sqrt{2}x\Im\alpha}e^{-\frac{(x-\sqrt{2}\Re\alpha)^2}{2}}$
  $\boldsymbol{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\boldsymbol{D}_\alpha|0\rangle = |\alpha\rangle$.

M: 2-level system, i.e. a qubit (half-spin system)

- Hilbert space:
  $\mathcal{H}_M = \mathbb{C}^2 = \left\{ c_g|g\rangle + c_e|e\rangle,\ c_g, c_e \in \mathbb{C} \right\}.$
- Quantum state space:
  $\mathcal{D} = \{\rho \in \mathcal{L}(\mathcal{H}_M), \rho^\dagger = \rho,\ \mathrm{Tr}(\rho) = 1, \rho \geq 0\}$.
- Operators and commutations:
  $\boldsymbol{\sigma_-} = |g\rangle\langle e|$, $\boldsymbol{\sigma_+} = \boldsymbol{\sigma_-}^\dagger = |e\rangle\langle g|$
  $\boldsymbol{\sigma_x} = \boldsymbol{\sigma_-} + \boldsymbol{\sigma_+} = |g\rangle\langle e| + |e\rangle\langle g|$;
  $\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_-} - i\boldsymbol{\sigma_+} = i|g\rangle\langle e| - i|e\rangle\langle g|$;
  $\boldsymbol{\sigma_z} = \boldsymbol{\sigma_+}\boldsymbol{\sigma_-} - \boldsymbol{\sigma_-}\boldsymbol{\sigma_+} = |e\rangle\langle e| - |g\rangle\langle g|$;
  $\boldsymbol{\sigma_x}^2 = \boldsymbol{I}$, $\boldsymbol{\sigma_x}\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_z}$, $[\boldsymbol{\sigma_x}, \boldsymbol{\sigma_y}] = 2i\boldsymbol{\sigma_z}$, ...



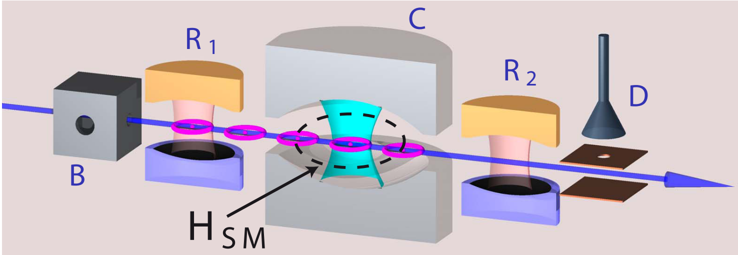
- Hamiltonian: $\boldsymbol{H}_M/\hbar = \omega_q \boldsymbol{\sigma_z}/2 + \boldsymbol{u_q}\boldsymbol{\sigma_x}$.
- Bloch sphere representation:
  $\mathcal{D} = \left\{ \frac{1}{2}\left(\boldsymbol{I} + x\boldsymbol{\sigma_x} + y\boldsymbol{\sigma_y} + z\boldsymbol{\sigma_z}\right) \mid (x, y, z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 \leq 1 \right\}$



- When atom comes out B, $|\Psi\rangle_B$ of the full system is separable $|\Psi\rangle_B = |\psi\rangle \otimes |g\rangle$.
- Just before the measurement in D, the state is in general entangled (not separable):

$$|\Psi\rangle_{R_2} = \boldsymbol{U}_{SM}\big(|\psi\rangle \otimes |g\rangle\big) = \big(\boldsymbol{M}_g|\psi\rangle\big) \otimes |g\rangle + \big(\boldsymbol{M}_e|\psi\rangle\big) \otimes |e\rangle$$

where $\boldsymbol{U}_{SM}$ is a unitary transformation (Schrödinger propagator) defining the linear measurement operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ on $\mathcal{H}_S$. Since $\boldsymbol{U}_{SM}$ is unitary, $\boldsymbol{M}_g^\dagger \boldsymbol{M}_g + \boldsymbol{M}_e^\dagger \boldsymbol{M}_e = \boldsymbol{I}$.



The unitary propagator $\boldsymbol{U}_{SM}$ is derived from Jaynes-Cummings Hamiltonian $\boldsymbol{H}_{SM}$ in the interaction frame.
Two kind of qubit/cavity Halmitonians:
resonant, $\boldsymbol{H}_{SM}/\hbar = i\big(\Omega(vt)/2\big)\ \big(\boldsymbol{a}^\dagger \otimes \boldsymbol{\sigma}_{\text{-}} - \boldsymbol{a} \otimes \boldsymbol{\sigma}_{\text{+}}\big)$,
dispersive, $\boldsymbol{H}_{SM}/\hbar = \big(\Omega^2(vt)/(2\delta)\big)\ \boldsymbol{N} \otimes \boldsymbol{\sigma_z}$,
where $\Omega(x) = \Omega_0 e^{-\frac{x^2}{w^2}}$, $x = vt$ with v atom velocity, Ω_0 vacuum Rabi pulsation, w radial mode-width and where $\delta = \omega_q - \omega_c$ is the detuning between qubit pulsation ω_q and cavity pulsation ω_c ($|\delta| \ll \Omega_0$).

Just before D, the field/atom state is **entangled**:

$$\boldsymbol{M}_g|\psi\rangle \otimes |g\rangle + \boldsymbol{M}_e|\psi\rangle \otimes |e\rangle$$

Denote by $\mu \in \{g, e\}$ the measurement outcome in detector D: with probability $\mathbb{P}_\mu = \langle\psi|\boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu|\psi\rangle$ we get μ. Just after the measurement outcome $\mu = y$, the state becomes separable:

$$|\Psi\rangle_D = \frac{1}{\sqrt{\mathbb{P}_y}} \left(\boldsymbol{M}_y|\psi\rangle\right) \otimes |y\rangle = \left(\frac{\boldsymbol{M}_y}{\sqrt{\langle\psi|\boldsymbol{M}_y^\dagger \boldsymbol{M}_y|\psi\rangle}}|\psi\rangle\right) \otimes |y\rangle.$$

Markov process (density matrix formulation $\rho \sim |\psi\rangle\langle\psi|$)

$$\rho_+ = \begin{cases} \frac{\boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger\right)}, & \text{with probability } \mathbb{P}_g = \mathrm{Tr}\left(\boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger\right); \\ \frac{\boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger\right)}, & \text{with probability } \mathbb{P}_e = \mathrm{Tr}\left(\boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger\right). \end{cases}$$

Kraus map: $\mathbb{E}\left(\rho_+/\rho\right) = \boldsymbol{K}(\rho) = \boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger.$

A controlled Markov process (input u, hidden state ρ, output y)

Input u: classical amplitude of a coherent micro-wave pulse.
State ρ: the density operator of the photon(s) trapped in the cavity.
Output y: quantum projective measurement of the probe atom.
The ideal model reads

$$\rho_{k+1} = \begin{cases} \dfrac{\boldsymbol{D}_{u_k} \boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger \boldsymbol{D}_{u_k}^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger\right)} & y_k = g \text{ with probability } \mathbb{P}_{g,k} = \mathrm{Tr}\left(\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger\right) \\ \dfrac{\boldsymbol{D}_{u_k} \boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger \boldsymbol{D}_{u_k}^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger\right)} & y_k = e \text{ with probability } \mathbb{P}_{e,k} = \mathrm{Tr}\left(\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger\right) \end{cases}$$

- Displacement unitary operator ($u \in \mathbb{R}$): $\boldsymbol{D}_u = e^{u\boldsymbol{a}^\dagger - u\boldsymbol{a}}$ with $\boldsymbol{a} = \text{upper diag}(\sqrt{1}, \sqrt{2}, \ldots)$ the photon annihilation operator.
- Measurement Kraus operators in the linear dispersive case $\boldsymbol{M}_g = \cos\left(\frac{\phi_0 \boldsymbol{N} + \phi_R}{2}\right)$ and $\boldsymbol{M}_e = \sin\left(\frac{\phi_0 \boldsymbol{N} + \phi_R}{2}\right)$: $\boldsymbol{M}_g^\dagger \boldsymbol{M}_g + \boldsymbol{M}_e^\dagger \boldsymbol{M}_e = \boldsymbol{I}$ with $\boldsymbol{N} = \boldsymbol{a}^\dagger \boldsymbol{a} = \text{diag}(0, 1, 2, \ldots)$ the photon number operator.

$$\rho_{k+1} = \begin{cases} \dfrac{\cos\left(\frac{\phi_0 \boldsymbol{N}+\phi_R}{2}\right)\boldsymbol{\rho_k}\cos\left(\frac{\phi_0 \boldsymbol{N}+\phi_R}{2}\right)}{\mathrm{Tr}\left(\cos^2\left(\frac{\phi_0 \boldsymbol{N}+\phi_R}{2}\right)\boldsymbol{\rho_k}\right)} & \text{with prob. } \mathrm{Tr}\left(\cos^2\left(\frac{\phi_0 \boldsymbol{N}+\phi_R}{2}\right)\rho_k\right) \\ \dfrac{\sin\left(\frac{\phi_0 \boldsymbol{N}+\phi_R}{2}\right)\boldsymbol{\rho_k}\sin\left(\frac{\phi_0 \boldsymbol{N}+\phi_R}{2}\right)}{\mathrm{Tr}\left(\sin^2\left(\frac{\phi_0 \boldsymbol{N}+\phi_R}{2}\right)\boldsymbol{\rho_k}\right)} & \text{with prob. } \mathrm{Tr}\left(\sin^2\left(\frac{\phi_0 \boldsymbol{N}+\phi_R}{2}\right)\rho_k\right) \end{cases}$$

Steady state: any Fock state $\rho = |\bar{n}\rangle\langle\bar{n}|$ ($\bar{n} \in \mathbb{N}$) is a steady-state (no other steady state when (ϕ_R, ϕ_0, π) are $\mathbb{Q}$-independent)

Martingales: for any real function g, $V_g(\rho) = \mathrm{Tr}\left(g(\boldsymbol{N})\rho\right)$ is a martingale:

$$\mathbb{E}\left(V_g(\rho_{k+1}) \ / \ \rho_k\right) = V_g(\rho_k).$$

Convergence to a Fock state when (ϕ_R, ϕ_0, π) are $\mathbb{Q}$-independent:

$V(\rho) = -\frac{1}{2}\sum_n \langle n|\rho|n\rangle^2$ is a super-martingale with

$$\mathbb{E}\left(V(\rho_{k+1}) \ / \ \rho_k\right) = V(\rho_k) - Q(\rho_k)$$

where $Q(\rho) \geq 0$ and $Q(\rho) = 0$ iff, ρ is a Fock state.

For a realization starting from ρ_0, the probability to converge towards the Fock state $|\bar{n}\rangle\langle\bar{n}|$ is equal to $\mathrm{Tr}\left(|\bar{n}\rangle\langle\bar{n}|\rho_0\right) = \langle\bar{n}|\rho_0|\bar{n}\rangle$.

Structure of the stabilizing quantum-state feedback scheme

With a sampling time of 80 μs, the controller is classical

- Goal: stabilization of the steady-state $|\bar{n}\rangle\langle\bar{n}|$ (controller set-point).
- At each time step k:
  1. read y_k the measurement outcome for probe atom k.
  2. update the quantum state estimation $\boldsymbol{\rho_{k-1}}$ to $\boldsymbol{\rho_k}$ from y_k
  3. compute u_k as a function of $\boldsymbol{\rho_k}$ (state feedback).
  4. apply the micro-wave pulse of amplitude u_k.

Observer/controller exploiting the **quantum separation principle**[3]:

1. real-time state estimation based on asymptotic observer: here quantum filtering techniques;
2. state feedback stabilization towards a stationary regime: here control Lyapunov techniques constructed with open-loop martingales $\text{Tr}\,(g(\boldsymbol{N})\rho)$ and inversion of a Laplacian matrix.

[3] L. Bouten and R. van Handel: On the separation principle of quantum control. In *Quantum Stochastics and Information: Statistics, Filtering and Control*, V. P Belavkin and M. I. Guta (Eds.) World Scientific, 2008.

Experimental closed-loop data

Stabilization around 3-photon state

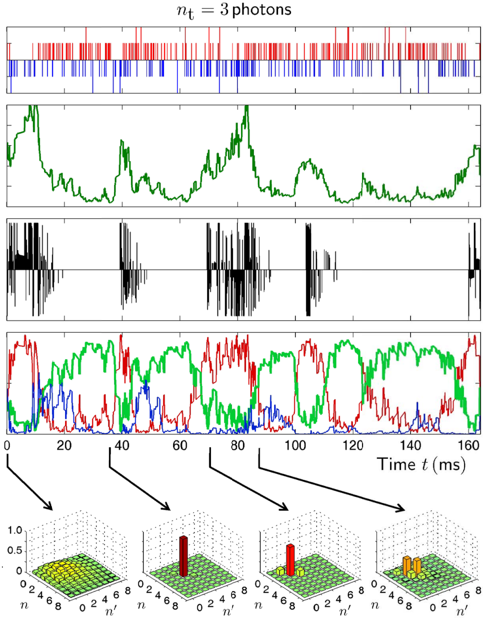
C. Sayrin et. al., Nature 477, 73-77, Sept. 2011.

Decoherence due to finite photon life time around 70 ms)

Detection efficiency 40%
Detection error rate 10%
Delay 4 sampling periods

The quantum filter takes into account cavity decoherence, measure imperfections and delays (Bayes law).

Truncation to 9 photons



The LKB photon box

First experimental realization of a quantum-state feedback (2011)

Why density operator ρ instead of wave function $|\psi\rangle$

Stabilization of "Schrödinger cats" by reservoir engineering

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Conclusion: some open issues

- With pure state $\rho = |\psi\rangle\langle\psi|$, we have
$$\rho_+ = |\psi_+\rangle\langle\psi_+| = \frac{1}{\text{Tr}\left(\boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger\right)} \boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger$$
when the atom collapses in $\mu = g, e$ with proba. $\text{Tr}\left(\boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger\right)$.
- Detection error rates: $\mathbb{P}(y = e/\mu = g) = \eta_g \in [0, 1]$ the probability of erroneous assignation to e when the atom collapses in g; $\mathbb{P}(y = g/\mu = e) = \eta_e \in [0, 1]$ (given by the contrast of the Ramsey fringes).

Bayes law: expectation ρ_+ of $|\psi_+\rangle\langle\psi_+|$ knowing ρ and the imperfect detection y.

$$\rho_+ = \begin{cases} \frac{(1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + \eta_e \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger}{\text{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + \eta_e \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)} \text{ if } y = g, \text{ prob. } \text{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + \eta_e \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right); \\ \frac{\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + (1-\eta_e) \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger}{\text{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + (1-\eta_e) \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)} \text{ if } y = e, \text{ prob. } \text{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + (1-\eta_e) \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right). \end{cases}$$

ρ_+ does not remain pure: the quantum state ρ_+ becomes a mixed state; $|\psi_+\rangle$ becomes physically irrelevant (not numerically).

We get

$$\rho_+ = \begin{cases} \frac{(1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+\eta_e\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger}{\text{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+\eta_e\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)}, & \text{with prob. } \text{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+\eta_e\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right); \\ \frac{\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+(1-\eta_e)\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger}{\text{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+(1-\eta_e)\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)} & \text{with prob. } \text{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+(1-\eta_e)\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right). \end{cases}$$

Key point:

$$\text{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+\eta_e\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right) \text{ and } \text{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+(1-\eta_e)\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)$$

are the probabilities to detect $y = g$ and e, knowing ρ.
Generalization by merging a Kraus map $\boldsymbol{K}(\rho) = \sum_\mu \boldsymbol{M}_\mu\rho\boldsymbol{M}_\mu^\dagger$ where $\sum_\mu \boldsymbol{M}_\mu^\dagger\boldsymbol{M}_\mu = \boldsymbol{I}$ with a left stochastic matrix $(\eta_{\mu',\mu})$:

$$\rho_+ = \frac{\sum_\mu \eta_{y,\mu}\boldsymbol{M}_\mu\rho\boldsymbol{M}_\mu^\dagger}{\text{Tr}\left(\sum_\mu \eta_{y,\mu}\boldsymbol{M}_\mu\rho\boldsymbol{M}_\mu^\dagger\right)} \quad \text{when we detect } y = \mu'.$$

The probability to detect $y = \mu'$ knowing ρ is $\text{Tr}\left(\sum_\mu \eta_{\mu',\mu}\boldsymbol{M}_\mu\rho\boldsymbol{M}_\mu^\dagger\right)$.

Photon-box quantum filter: 6×21 left stochastic matrix $(\eta_{\mu',\mu})$

$$\rho_{k+1} = \frac{1}{\mathrm{Tr}\left(\sum_\mu \eta_{y_k,\mu} M_\mu \rho_k M_\mu^\dagger\right)} \left(\sum_\mu \eta_{y_k,\mu} M_\mu \rho_k M_\mu^\dagger\right) \text{ where}$$

- we have a total of $m = 3 \times 7 = 21$ Kraus operators M_μ. The "jumps" are labeled by $\mu = (\mu^a, \mu^c)$ with $\mu^a \in \{no, g, e, gg, ge, eg, ee\}$ labeling atom related jumps and $\mu^c \in \{o, +, -\}$ cavity decoherence jumps.
- we have only $m' = 6$ real detection possibilities $y = \mu' \in \{no, g, e, gg, ge, ee\}$ corresponding respectively to no detection, a single detection in g, a single detection in e, a double detection both in g, a double detection one in g and the other in e, and a double detection both in e.

$\mu' \setminus \mu$	(no, μ^c)	(g, μ^c)	(e, μ^c)	(gg, μ^c)	(ee, μ^c)	(ge, μ^c) (eg, μ^c)
no	1	$1-\epsilon_d$	$1-\epsilon_d$	$(1-\epsilon_d)^2$	$(1-\epsilon_d)^2$	$(1-\epsilon_d)^2$
g	0	$\epsilon_d(1-\eta_g)$	$\epsilon_d\eta_e$	$2\epsilon_d(1-\epsilon_d)(1-\eta_g)$	$2\epsilon_d(1-\epsilon_d)\eta_e$	$\epsilon_d(1-\epsilon_d)(1-\eta_g+\eta_e)$
e	0	$\epsilon_d\eta_g$	$\epsilon_d(1-\eta_e)$	$2\epsilon_d(1-\epsilon_d)\eta_g$	$2\epsilon_d(1-\epsilon_d)(1-\eta_e)$	$\epsilon_d(1-\epsilon_d)(1-\eta_e+\eta_g)$
gg	0	0	0	$\epsilon_d^2(1-\eta_g)^2$	$\epsilon_d^2\eta_e^2$	$\epsilon_d^2\eta_e(1-\eta_g)$
ge	0	0	0	$2\epsilon_d^2\eta_g(1-\eta_g)$	$2\epsilon_d^2\eta_e(1-\eta_e)$	$\epsilon_d^2((1-\eta_g)(1-\eta_e)+\eta_g\eta_e)$
ee	0	0	0	$\epsilon_d^2\eta_g^2$	$\epsilon_d^2(1-\eta_e)^2$	$\epsilon_d^2\eta_g(1-\eta_e)$

Outline

The LKB photon box

First experimental realization of a quantum-state feedback (2011)

Why density operator ρ instead of wave function $|\psi\rangle$

Stabilization of "Schrödinger cats" by reservoir engineering

Model structure of open quantum systems

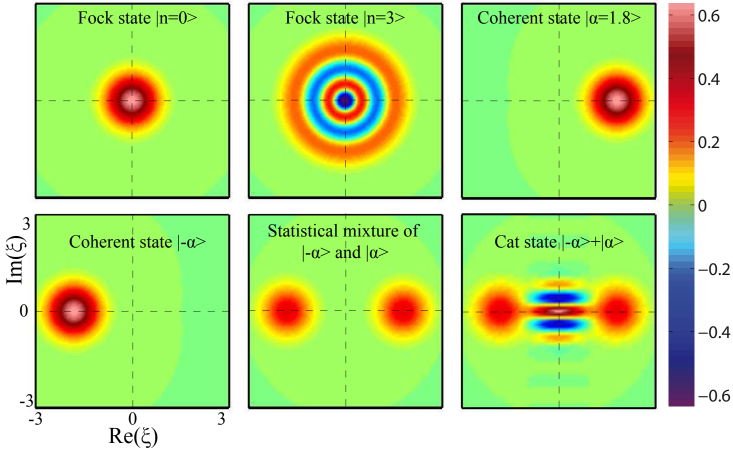
Conclusion: some open issues

Wigner functions of some quantum states for an harmonic oscillator

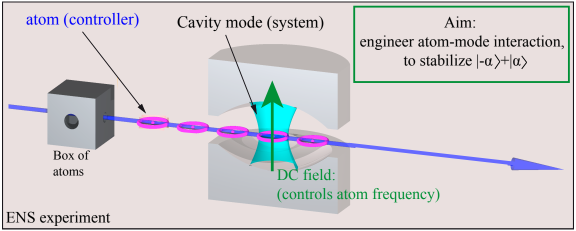
Classical state of amplitude $\alpha \in \mathbb{C}$: $|\alpha\rangle = \sum_{n \geq 0} \left(e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \right) |n\rangle$;

Phase-cat states: $\mathcal{N}\big(|\alpha\rangle + |-\alpha\rangle\big)$.

Wigner function W^ρ associated ρ: $W^\rho : \mathbb{C} \ni \xi \to \frac{2}{\pi} \operatorname{Tr}\left(e^{i\pi \boldsymbol{N}} \boldsymbol{D}_{-\xi} \rho \boldsymbol{D}_{\xi}\right)$



Stabilizing "Schrödinger cats" $(|\alpha_\infty\rangle + i|\text{-}\alpha_\infty\rangle)/\sqrt{2}$. [4]



Jaynes-Cumming Hamiltionian

$$\boldsymbol{H}(t)/\hbar = \omega_c \boldsymbol{a}^\dagger \boldsymbol{a} \otimes \boldsymbol{I}_M + \boldsymbol{\omega_q(t)} \boldsymbol{I}_S \otimes \boldsymbol{\sigma_z}/2 + i\Omega(t)\left(\boldsymbol{a}^\dagger \otimes \boldsymbol{\sigma_-} - \boldsymbol{a} \otimes \boldsymbol{\sigma_+}\right)/2$$

with the open-loop control $t \mapsto \boldsymbol{\omega_q(t)}$ combining dispersive $\omega_q \neq \omega_c$ and resonant $\omega_q = \omega_c$ interactions.

Key issues: **convergence** of $\rho_{k+1} = \boldsymbol{K}(\rho_k) = \boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger$

[4] A. Sarlette et al: Stabilization of Nonclassical States of the Radiation Field in a Cavity by Reservoir Engineering. Physical Review Letters, Volume 107, Issue 1, 2011.

Convergence of $\boldsymbol{K}$ iterates towards $(|\alpha_\infty\rangle + i|\text{-}\alpha_\infty\rangle)/\sqrt{2}$

Iterations $\rho_{k+1} = \boldsymbol{K}(\rho_k) = \boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger$ in the Kerr frame $\boldsymbol{\rho = e^{-ih_N^{\text{Kerr}}} \rho^{\text{Kerr}} e^{ih_N^{\text{Kerr}}}}$ yields

$$\rho_{k+1}^{\text{Kerr}} = \boldsymbol{K}^{\text{Kerr}}(\rho_k^{\text{Kerr}}) = \boldsymbol{M}_g^{\text{Kerr}} \rho_k^{\text{Kerr}} (\boldsymbol{M}_g^{\text{Kerr}})^\dagger + \boldsymbol{M}_e^{\text{Kerr}} \rho_k^{\text{Kerr}} (\boldsymbol{M}_e^{\text{Kerr}})^\dagger.$$

with $\boldsymbol{M}_g^{\text{Kerr}} = \cos(\frac{u}{2}) \cos(\theta_{\boldsymbol{N}}/2) + \sin(\frac{u}{2}) \frac{\sin(\theta_{\boldsymbol{N}}/2)}{\sqrt{\boldsymbol{N}}} \boldsymbol{a}^\dagger$ and $\boldsymbol{M}_e^{\text{Kerr}} = \sin(\frac{u}{2}) \cos(\theta_{\boldsymbol{N}+1}/2) - \cos(\frac{u}{2}) \boldsymbol{a} \frac{\sin(\theta_{\boldsymbol{N}}/2)}{\sqrt{\boldsymbol{N}}}$.

Assume $|u| \leq \pi/2$, $\theta_0 = 0$, $\theta_n \in]0, \pi[$ for $n > 0$ and $\lim_{n \mapsto +\infty} \theta_n = \pi/2$, then (Zaki Leghtas, PhD thesis (2012))

- exists a unique common eigen-state $|\psi^{\text{Kerr}}\rangle$ of $\boldsymbol{M}_g^{\text{Kerr}}$ and $\boldsymbol{M}_e^{\text{Kerr}}$: $\rho_\infty^{\text{Kerr}} = |\psi^{\text{Kerr}}\rangle\langle\psi^{\text{Kerr}}|$ fixed point of $\boldsymbol{K}^{\text{Kerr}}$.
- if, moreover $n \mapsto \theta_n$ is increasing, $\lim_{k \mapsto +\infty} \rho_k^{\text{Kerr}} = \rho_\infty^{\text{Kerr}}$.

For well chosen experimental parameters, $\boldsymbol{\rho_\infty^{\text{Kerr}} \approx |\alpha_\infty\rangle\langle\alpha_\infty|}$ and $h_{\boldsymbol{N}}^{\text{Kerr}} \approx \pi \boldsymbol{N}^2/2$. Since $e^{-i\frac{\pi}{2}\boldsymbol{N}^2}|\alpha_\infty\rangle = \frac{e^{-i\pi/4}}{\sqrt{2}}(|\alpha_\infty\rangle + i|\text{-}\alpha_\infty\rangle)$:

$$\lim_{k \mapsto +\infty} \rho_k = \tfrac{1}{2}\Big(|\alpha_\infty\rangle + i|\text{-}\alpha_\infty\rangle\Big)\Big(\langle\alpha_\infty| + i\langle\text{-}\alpha_\infty|\Big)$$

$$\neq \tfrac{1}{2}|\alpha_\infty\rangle\langle\alpha_\infty| + \tfrac{1}{2}|\text{-}\alpha_\infty\rangle\langle\text{-}\alpha_\infty|.$$

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Discrete-time models of open quantum systems

Four features:

1. Bayes law: $\mathbb{P}(\mu'/\mu) = \mathbb{P}(\mu/\mu')\mathbb{P}(\mu') \;/\; \left(\sum_{\nu'} \mathbb{P}(\mu/\nu')\mathbb{P}(\nu')\right)$,
2. Schrödinger equations defining unitary transformations.
3. Partial collapse of the wave packet: irreversibility and dissipation are induced by the measurement of observables with degenerate spectra.
4. Tensor product for the description of composite systems.

$\Rightarrow$ **Discrete-time models:** Markov processes of state ρ, (density op.):

$\rho_{k+1} = \frac{\sum_{\mu=1}^{m} \eta_{\mu',\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger}{\mathrm{Tr}\left(\sum_{\mu=1}^{m} \eta_{\mu',\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)}$, with proba. $\mathbb{P}_{\mu'}(\rho_k) = \sum_{\mu=1}^{m} \eta_{\mu',\mu} \,\mathrm{Tr}\left(\boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)$

associated to Kraus maps (ensemble average, quantum channel)

$$\mathbb{E}\left(\rho_{k+1}|\rho_k\right) = \boldsymbol{K}(\rho_k) = \sum_\mu \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger \quad \text{with} \quad \sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \boldsymbol{I}$$

and left stochastic matrices (imperfections, decoherences) $(\eta_{\mu',\mu})$.

Discrete-time models: Markov chains

$$\rho_{k+1} = \frac{\sum_{\mu=1}^{m} \eta_{\mu',\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger}{\text{Tr}\left(\sum_{\mu=1}^{m} \eta_{\mu',\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)}, \text{ with proba. } \mathbb{P}_{\mu'}(\rho_k) = \sum_{\mu=1}^{m} \eta_{\mu',\mu} \text{ Tr}\left(\boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)$$

with ensemble averages corresponding to Kraus linear maps

$$\mathbb{E}\left(\rho_{k+1}|\rho_k\right) = \boldsymbol{K}(\rho_k) = \sum_\mu \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger \quad \text{with} \quad \sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \boldsymbol{I}$$

Continuous-time models: stochastic differential systems

$$d\rho_t = \left(-\tfrac{i}{\hbar}[\boldsymbol{H}, \rho_t] + \sum_\nu \boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)\right) dt + \sum_\nu \sqrt{\eta_\nu}\left(\boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger - \text{ Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\rho_t\right) \rho_t\right) dW_{\nu,t}$$

driven by Wiener processes $dW_{\nu,t}$, with measurements $y_{\nu,t}$, $dy_{\nu,t} = \sqrt{\eta_\nu} \text{ Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\, \rho_t\right) dt + dW_{\nu,t}$, detection efficiencies $\eta_\nu \in [0, 1]$ and Lindblad-Kossakowski master equations ($\eta_\nu \equiv 0$):

$$\frac{d}{dt}\rho = -\tfrac{i}{\hbar}[\boldsymbol{H}, \rho] + \sum_\nu \boldsymbol{L}_\nu \rho \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho + \rho \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)$$

With a single imperfect measurement
$dy_t = \sqrt{\eta}\,\text{Tr}\left(\left(L + L^\dagger\right)\rho_t\right)\,dt + dW_t$ and detection efficiency $\eta \in [0, 1]$,
the quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-\frac{i}{\hbar}[H, \rho_t] + L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L\rho_t + \rho_t L^\dagger L)\right)dt + \sqrt{\eta}\left(L\rho_t + \rho_t L^\dagger - \text{Tr}\left((L + L^\dagger)\rho_t\right)\rho_t\right)dW_t$$

driven by the Wiener process dW_t

With Itō rules, it can be written as the following "discrete-time" Markov model

$$\rho_{t+dt} = \frac{M_{dy_t}\rho_t M_{dy_t}^\dagger + (1-\eta)L\rho_t L^\dagger dt}{\text{Tr}\left(M_{dy_t}\rho_t M_{dy_t}^\dagger + (1-\eta)L\rho_t L^\dagger dt\right)}$$

with $M_{dy_t} = I + \left(-\frac{i}{\hbar}H - \frac{1}{2}\left(L^\dagger L\right)\right)dt + \sqrt{\eta}\,dy_t L$.

With Poisson process $\boldsymbol{N(t)}$, $\langle \boldsymbol{dN(t)} \rangle = \left(\overline{\theta} + \overline{\eta} \operatorname{Tr}\left(V\rho_t V^\dagger\right)\right) dt$, and detection imperfections modeled by $\overline{\theta} \geq 0$ and $\overline{\eta} \in [0, 1]$, the quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \Big(-i[H, \rho_t] + V\rho_t V^\dagger - \tfrac{1}{2}(V^\dagger V \rho_t + \rho_t V^\dagger V)\Big)\, dt$$
$$+ \left(\frac{\overline{\theta}\rho_t + \overline{\eta} V\rho_t V^\dagger}{\overline{\theta} + \overline{\eta} \operatorname{Tr}\left(V\rho_t V^\dagger\right)} - \rho_t\right)\left(\boldsymbol{dN(t)} - \left(\overline{\theta} + \overline{\eta} \operatorname{Tr}\left(V\rho_t V^\dagger\right)\right) dt\right)$$

For $\boldsymbol{dN(t) = 0}$ we have

$$\rho_{t+dt} = \frac{M_{\mathbf{0}}\rho_t M_{\mathbf{0}}^\dagger + (1-\overline{\eta}) V\rho_t V^\dagger dt}{\operatorname{Tr}\Big(M_{\mathbf{0}}\rho_t M_{\mathbf{0}}^\dagger + (1-\overline{\eta}) V\rho_t V^\dagger dt\Big)}$$

with $M_{\mathbf{0}} = I - \left(iH + \frac{1}{2}V^\dagger V\right) dt$.
For $\boldsymbol{N(t+dt) - N(t) = 1}$ we have a similar transition rule $\rho_{t+dt} = \frac{\bullet}{\operatorname{Tr}(\bullet)}$ where ρ_t is replaced by $\tilde{\rho}_t = \dfrac{\overline{\theta}\rho_t + \overline{\eta} V\rho_t V^\dagger}{\overline{\theta} + \overline{\eta} \operatorname{Tr}\left(V\rho_t V^\dagger\right)}$.

Continuous/discrete-time diffusive-jump SME

The quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-i[H,\rho_t] + L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L\rho_t + \rho_t L^\dagger L) + V\rho_t V^\dagger - \frac{1}{2}(V^\dagger V\rho_t + \rho_t V^\dagger V)\right) dt$$

$$+ \sqrt{\eta}\left(L\rho_t + \rho_t L^\dagger - \operatorname{Tr}\left((L+L^\dagger)\rho_t\right)\rho_t\right) \boldsymbol{dW_t}$$

$$+ \left(\frac{\overline{\theta}\rho_t + \overline{\eta} V\rho_t V^\dagger}{\overline{\theta} + \overline{\eta}\operatorname{Tr}(V\rho_t V^\dagger)} - \rho_t\right)\left(\boldsymbol{dN(t)} - \left(\overline{\theta} + \overline{\eta}\operatorname{Tr}\left(V\rho_t V^\dagger\right)\right) dt\right)$$

For $\boldsymbol{dN(t) = 0}$ we have

$$\rho_{t+dt} = \frac{M_{\boldsymbol{dy_t}}\rho_t M_{\boldsymbol{dy_t}}^\dagger + (1-\eta)L\rho_t L^\dagger dt + (1-\overline{\eta})V\rho_t V^\dagger dt}{\operatorname{Tr}\left(M_{\boldsymbol{dy_t}}\rho_t M_{\boldsymbol{dy_t}}^\dagger + (1-\eta)L\rho_t L^\dagger dt + (1-\overline{\eta})V\rho_t V^\dagger dt\right)}$$

with $M_{\boldsymbol{dy_t}} = I - \left(iH + \frac{1}{2}L^\dagger L + \frac{1}{2}V^\dagger V\right) dt + \sqrt{\eta}\,\boldsymbol{dy_t} L$.
For $\boldsymbol{N(t+dt) - N(t) = 1}$ we have a similar transition $\rho_{t+dt} = \frac{\bullet}{\operatorname{Tr}(\bullet)}$ where ρ_t is replaced by $\tilde{\rho}_t = \dfrac{\overline{\theta}\rho_t + \overline{\eta} V\rho_t V^\dagger}{\overline{\theta} + \overline{\eta}\operatorname{Tr}(V\rho_t V^\dagger)}$.

Continuous/discrete-time general diffusive-jump SME

The quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \Big(-i[H,\rho_t] + \sum_\nu L_\nu \rho_t L_\nu^\dagger - \tfrac{1}{2}(L_\nu^\dagger L_\nu \rho_t + \rho_t L_\nu^\dagger L_\nu) + V_\mu \rho_t V_\mu^\dagger - \tfrac{1}{2}(V_\mu^\dagger V_\mu \rho_t + \rho_t V_\mu^\dagger V_\mu)\Big)\, dt$$

$$+ \sum_\nu \sqrt{\eta_\nu}\Big(L_\nu \rho_t + \rho_t L_\nu^\dagger - \operatorname{Tr}\big((L_\nu + L_\nu^\dagger)\rho_t\big)\,\rho_t\Big)\, dW_{\nu,t}$$

$$+ \sum_\mu \left(\frac{\overline{\theta}_\mu \rho_t + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} V_{\mu'} \rho_t V_{\mu'}^\dagger}{\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \operatorname{Tr}\left(V_{\mu'}\rho_t V_{\mu'}^\dagger\right)} - \rho_t\right)\left(dN_\mu(t) - \Big(\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \operatorname{Tr}\left(V_{\mu'}\rho_t V_{\mu'}^\dagger\right)\Big)\, dt\right)$$

where $\eta_\nu \in [0,1]$, $\overline{\theta}_\mu, \overline{\eta}_{\mu,\mu'} \geq 0$ with $\overline{\eta}_{\mu'} = \sum_\mu \overline{\eta}_{\mu,\mu'} \leq 1$ are parameters modelling measurements imperfections.
When $\forall \mu,\ dN_\mu(t) = 0$, we have

$$\rho_{t+dt} = \frac{M_{dy_t}\rho_t M_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) L_\nu \rho_t L_\nu^\dagger dt + \sum_\mu (1-\overline{\eta}_\mu) V_\mu \rho_t V_\mu^\dagger dt}{\operatorname{Tr}\left(M_{dy_t}\rho_t M_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) L_\nu \rho_t L_\nu^\dagger dt + \sum_\mu (1-\overline{\eta}_\mu) V_\mu \rho_t V_\mu^\dagger dt\right)}$$

with $M_{dy_t} = I - \left(iH + \frac{1}{2}\sum_\nu L_\nu^\dagger L_\nu + \frac{1}{2}\sum_\mu V_\mu^\dagger V_\mu\right) dt + \sum_\nu \sqrt{\eta_\nu}\, dy_{\nu t} L_\nu$ and where $dy_{\nu,t} = \sqrt{\eta_\nu} \operatorname{Tr}\left((L_\nu + L_\nu^\dagger)\,\rho_t\right)\, dt + dW_{\nu,t}$.

If, for some μ, $N_\mu(t+dt) - N_\mu(t) = 1$, we have a similar transition rule $\rho_{t+dt} = \frac{\bullet}{\operatorname{Tr}(\bullet)}$ but where ρ_t is replaced by $\tilde{\rho}_t = \dfrac{\overline{\theta}_\mu \rho_t + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} V_{\mu'}\rho_t V_{\mu'}^\dagger}{\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \operatorname{Tr}\left(V_{\mu'}\rho_t V_{\mu'}^\dagger\right)}$.

Useful for positiveness-preserving numerical schemes

- Few available convergence results in the low rank case:
  most of available results are for full rank density operators either for Kraus maps (quantum channels)
  $\rho_{k+1} = \boldsymbol{K}(\rho_k) = \sum_\mu \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger$, or for Lindblad-Kossakowski master equations :
  $\frac{d}{dt}\rho = -\frac{i}{\hbar}[\boldsymbol{H}, \rho] + \sum_\nu \boldsymbol{L}_\nu \rho \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho + \rho_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)$.
- Continuous-time models with quantum input signal ?
  Stochastic master equations driven by Wiener processes valid for classical (coherent) input signals of amplitude u (see, e.g., the (S, L, H)-theory of quantum networks, J. Gough and M. James, IEEE Trans. AC 2009); modelling issues for quantum input signals such as $|\boldsymbol{u}\rangle + |\text{-}\boldsymbol{u}\rangle$.
- The curse of dimensionality: composite quantum systems rely on tensor products whereas composite classical systems rely on Cartesian products . . . .

Acknowledgments

- **Former PhDs and PostDocs:** Hadis Amini, Hector Bessa Silveira, Zaki Leghtas, Ram Somaraju.
- **LKB Physicists:** Michel Brune, Igor Dotsenko, Sébastien Gleyzes, Serge Haroche, Jean-Michel Raimond, Bruno Peaudecerf, Clément Sayrin, Xingxing Zhou.
- **Quantic project (INRIA/ENS/MINES):** Benjamin Huard, François Mallet, Mazyar Mirrahimi, Alain Sarlette, Rémi Azouit, Landry Bretheau, Philippe Campagne, Joachim Cohen, Emmanuel Flurin, Ananda Roy, Pierre Six.
- **Mathematicians:** Karine Beauchard, Jean-Michel Coron, Thomas Chambrion, Bernard Bonnard, Ugo Boscain, Sylvain Ervedoza, Stéphane Gaubert, Andrea Grigoriu, Claude Le Bris, Yvon Maday, Vahagn Nersesyan, Clément Pellégrini, Paulo Sergio Pereira da Silva, Jean-Pierre Puel, Lionel Rosier, Julien Salomon, Rodolphe Sepulchre, Mario Sigalotti, Gabriel Turinici.
- **Mines Paris-Tech:** Brigitte d'Andrea-Novel, Silvère Bonnabel, François Chaplais, Florent Di Méglio, Jean Lévine, Philippe Martin, Nicolas Petit, Laurent Praly.
- **And also:** Lectures at Collège de France, ANR projects CQUID and EMAQS, UPS-COFECUB, ED-SMI, . . .