Codes correcteurs quantiques et feedback

GRETSI 2022

XXVIIIème Colloque Francophone de Traitement du Signal et des Images

Nancy, 05 – 09 Septembre 2022

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Quantic research team

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First and second "quantum revolution" [1]

- Current technologies based on the first quantum revolution with transistors and lasers: manipulation and control of a large number of identical objects described by quantum statistics.
- Emerging quantum technologies based on the second quantum revolution: manipulation and control of an individual object whose temporal evolution follows the Schrodinger differential equation. .

[1] Dowling, J. & Milburn, G.: *Quantum technology: the second quantum revolution*. Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 2003, 361, 1655-1674.

Serge Haroche

David J. Wineland

" This year's Nobel Prize in Physics honours the experimental inventions and discoveries that have allowed the **measurement and control of individual quantum systems***. They belong to two separate but related technologies: ions in a harmonic trap and photons in a cavity"*

From the Scientific Background on the Nobel Prize in Physics 2012 compiled by the Class for Physics of the Royal Swedish Academy of Sciences, 9 October 2012.

Entanglement and coherence, essential but fragile quantum resources for

- communications and cryptography: random generator, distribution of encryption keys via a quantum channel of transmission by BB84 protocol [2] . . . .
- computation and simulation: factorization of large RSA numbers and discrete logarithm (Digital Signature Standard) by polynomial algorithms; combinatorial optimization and machine learning [3]. . .
- metrology: clock, inertial sensor, gravimetry [4]. . .

Major difficulty: how to design machines which exploit quantum properties on a large scale, and efficiently protect them from external perturbations and noises (decoherence), which tend to suppress the quantum advantage?

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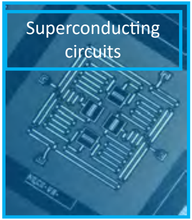
[2] https://www.idquantique.com/quantum-safe-security/overview/
[3] D-Wave, Rigetti, Google, IBM, Amazon WS, Pasqal, Alice&Bob, . . .
[4] Onera, Thales, https://www.muquans.com/

Investigated technologies for quantum computation [5]



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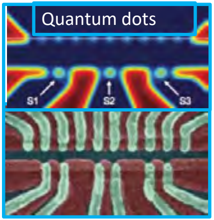

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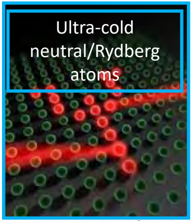

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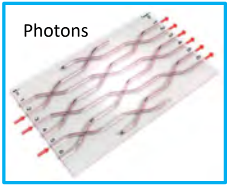

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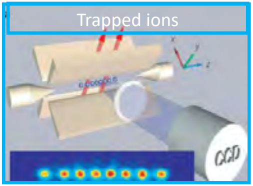

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Requirements:

- scalable modular architecture;
- control software from the very beginning.

[5] Courtesy of Walter Riess, IBM Research - Zurich.

Quantum Error Correction (QEC) is based on an elementary discrete-time feedback loop: a static-output feedback neglecting the finite bandwidth of the measurement and actuation processes.

- Current experiments: $\frac{1}{100}$ to $\frac{1}{1000}$ are typical error probabilities during elementary gates (manipulations) involving few physical qubits.
- High-order error-correcting codes with an important overhead; **more than 1000 physical qubits to encode a controllable logical qubit**[6].
- Today, no such controllable logical qubit has been built.
- Key issue: reduction by several magnitude orders such error rates, far below the threshold required by actual QEC, to build a controllable logical qubit encoded in a reasonable number of physical qubits and protected by QEC.

Control engineering can play a crucial role to built a controllable logical qubit protected by much **more elaborated feedback schemes increasing precision and stability.**

[6] A.G. Fowler, M. Mariantoni, J.M. Martinis, A.N. Cleland (2012): Surface codes: Towards practical large-scale quantum computation. Phys. Rev. A,86(3):032324.

Quantum Error Correction (QEC) from scratch
- Classical error correction
- QEC: the 9-qubit Shor code

Continuous-time dynamics of open quantum system
- Stochastic Master Equation (SME)
- Key characteristics of SME

Feedback schemes
- Measurement-based feedback and classical controller
- Coherent feedback and quantum controller

Storing a logical qubit in a high-quality harmonic oscillator
- Quantum harmonic oscillator
- Cat-qubit: autonomous correction of bit-flip
- GKP grid-state: robustness versus bit-flip and phase-flip

Conclusion

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• Single bit error model: the bit $b \in \{0,1\}$ flips with probability $p < 1/2$ during Δt (for usual DRAM: $p/\Delta t \leq 10^{-14}\ \mathrm{s}^{-1}$).
• Multi-bit error model: each bit $b_k \in \{0,1\}$ flips with probability $p < 1/2$ during Δt; no correlation between the bit flips.
•Use redundancy to construct with several physical bits b_k of flip probability p, a logical bit b_L with a flip probability $p_L < p$.
• The simplest solution, the 3-bit code (sampling time Δt):

$t = 0$: $b_L = [bbb]$ with $b \in \{0,1\}$

$t = \Delta t$: measure the three physical bits of $b_L = [b_1 b_2 b_3]$ (instantaneous) :

1. if all 3 bits coincide, nothing to do.
2. if one bit differs from the two other ones, flip this bit (instantaneous);

• Since the flip probability laws of the physical bits are independent, the probability that the logical bit b_L (protected with the above error correction code) flips during Δt is $\mathrm{p_L} = 3\mathrm{p}^2 - 2\mathrm{p}^3 < p$ since $p < 1/2$.

1. Schrödinger ($\hbar = 1$): wave funct. $|\psi\rangle \in \mathcal{H}$, density op. $\rho \sim |\psi\rangle\langle\psi|$

$$\frac{d}{dt}|\psi\rangle = -i\mathsf{H}|\psi\rangle, \quad \mathsf{H} = \mathsf{H}_0 + u\mathsf{H}_1 = \mathsf{H}^\dagger, \quad \frac{d}{dt}\rho = -i[\mathsf{H}, \rho].$$

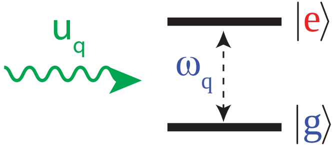
2. Origin of dissipation: collapse of the wave packet induced by the measurement of $\mathsf{O} = \mathsf{O}^\dagger$ with spectral decomp. $\sum_y \lambda_y \mathsf{P}_y$:
   - measurement outcome y with proba. $\mathbb{P}_y = \langle\psi|\mathsf{P}_y|\psi\rangle = \text{Tr}(\rho \mathsf{P}_y)$ depending on $|\psi\rangle$, ρ just before the measurement
   - measurement back-action if outcome y:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\mathsf{P}_y|\psi\rangle}{\sqrt{\langle\psi|\mathsf{P}_y|\psi\rangle}}, \quad \rho \mapsto \rho_+ = \frac{\mathsf{P}_y \rho \mathsf{P}_y}{\text{Tr}(\rho \mathsf{P}_y)}$$

3. Tensor product for the description of composite systems (S, C):
   - Hilbert space $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_c$
   - Hamiltonian $\mathsf{H} = \mathsf{H}_s \otimes \mathsf{I}_c + \mathsf{H}_{sc} + \mathsf{I}_s \otimes \mathsf{H}_c$
   - observable on sub-system C only: $\mathsf{O} = \mathsf{I}_s \otimes \mathsf{O}_c$.

[7] S. Haroche and J.M. Raimond (2006). *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts.

- Hilbert space with $|0\rangle \triangleq |e\rangle$ and $|1\rangle \triangleq |g\rangle$:
  $\mathcal{H}_M = \mathbb{C}^2 = \left\{ c_g|g\rangle + c_e|e\rangle,\ c_g, c_e \in \mathbb{C} \right\}$.
- Quantum state space:
  $\mathcal{D} = \{\rho \in \mathcal{L}(\mathcal{H}_M), \rho^\dagger = \rho,\ \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}$.
- Operators and commutations:
  $\sigma_- = |g\rangle\langle e|,\ \sigma_+ = \sigma_-{}^\dagger = |e\rangle\langle g|$
  $X \equiv \sigma_x = \sigma_- + \sigma_+ = |g\rangle\langle e| + |e\rangle\langle g|;$
  $Y \equiv \sigma_y = i\sigma_- - i\sigma_+ = i|g\rangle\langle e| - i|e\rangle\langle g|;$
  $Z \equiv \sigma_z = \sigma_+\sigma_- - \sigma_-\sigma_+ = |e\rangle\langle e| - |g\rangle\langle g|;$
  $\sigma_x{}^2 = I,\ \sigma_x\sigma_y = i\sigma_z,\ [\sigma_x, \sigma_y] = 2i\sigma_z,\ \ldots$



- Hamiltonian: $H_M = \omega_q\sigma_z/2 + u_q\sigma_x$.
- Bloch sphere representation:
  $\mathcal{D} = \left\{ \frac{1}{2}\left(I + x\sigma_x + y\sigma_y + z\sigma_z\right) \;\middle|\; (x, y, z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 \leq 1 \right\}$

[8] See S. M. Barnett, P.M. Radmore (2003): Methods in Theoretical Quantum Optics. Oxford University Press.

- Local bit-flip errors: each physical qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ becomes $X|\psi\rangle = \alpha|1\rangle + \beta|0\rangle$ [9] with probability $p < 1/2$ during Δt. (for actual super-conducting qubit $p/\Delta t > 10^3\ s^{-1}$).
- $t = 0$: $|\psi_L\rangle = \alpha|0_L\rangle + \beta|1_L\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \equiv \mathbb{C}^8$ with $|0_L\rangle = |000\rangle$ and $|1_L\rangle = |111\rangle$.
- $t = \Delta t$: $|\psi_L\rangle$ becomes with

$$\text{1 flip:} \begin{cases} \alpha|100\rangle + \beta|011\rangle \\ \alpha|010\rangle + \beta|101\rangle \\ \alpha|001\rangle + \beta|110\rangle \end{cases} ;\ \text{2 flips:} \begin{cases} \alpha|110\rangle + \beta|001\rangle \\ \alpha|101\rangle + \beta|010\rangle \\ \alpha|011\rangle + \beta|100\rangle \end{cases} ;\ \text{3 flips: } \alpha|111\rangle + \beta|000\rangle.$$

- Key fact: 4 orthogonal planes $\mathcal{P}_c = \text{span}(|000\rangle, |111\rangle)$, $\mathcal{P}_1 = \text{span}(|100\rangle, |011\rangle)$, $\mathcal{P}_2 = \text{span}(|010\rangle, |101\rangle)$ and $\mathcal{P}_3 = \text{span}(|001\rangle, |110\rangle)$.
- Error syndromes: 3 commuting observables $S_1 = I \otimes Z \otimes Z$, $S_2 = Z \otimes I \otimes Z$ and $S_3 = Z \otimes Z \otimes I$ with spectrum $\{-1, +1\}$ and outcomes $(s_1, s_2, s_3) \in \{-1, +1\}$.

$$\text{-1- } s_1 = s_2 = s_3:\ \mathcal{P}_c \ni |\psi_L\rangle = \begin{cases} \alpha|000\rangle + \beta|111\rangle \text{ 0 flip} \\ \beta|000\rangle + \alpha|111\rangle \text{ 3 flips} \end{cases} ;\ \text{no correction}$$

$$\text{-2- } s_1 \neq s_2 = s_3:\ \mathcal{P}_1 \ni |\psi_L\rangle = \begin{cases} \alpha|100\rangle + \beta|011\rangle \text{ 1 flip} \\ \beta|100\rangle + \alpha|011\rangle \text{ 2 flips} \end{cases} ;\ (X \otimes I \otimes I)|\psi_L\rangle \in \mathcal{P}_c.$$

$$\text{-3- } s_2 \neq s_3 = s_1:\ \mathcal{P}_2 \ni |\psi_L\rangle = \begin{cases} \alpha|010\rangle + \beta|101\rangle \text{ 1 flip} \\ \beta|010\rangle + \alpha|101\rangle \text{ 2 flips} \end{cases} ;\ (I \otimes X \otimes I)|\psi_L\rangle \in \mathcal{P}_c.$$

$$\text{-4- } s_3 \neq s_1 = s_2:\ \mathcal{P}_3 \ni |\psi_L\rangle = \begin{cases} \alpha|001\rangle + \beta|110\rangle \text{ 1 flip} \\ \beta|001\rangle + \alpha|110\rangle \text{ 2 flips} \end{cases} ;\ (I \otimes I \otimes X)|\psi_L\rangle \in \mathcal{P}_c.$$

[9] $X = |1\rangle\langle 0| + |0\rangle\langle 1|$ and $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$.

[10] M.A Nielsen, I.L. Chuang (2000): Quantum Computation and Quantum Information.Cambridge University Press.

- Local phase-flip error: each physical qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ becomes $Z|\psi\rangle = \alpha|0\rangle - \beta|1\rangle$ [11] with probability $p < 1/2$ during Δt.
- Since $X = HZH$ and $Z = HXH$ ($H^2 = I$), use the 3-qubit bit flip code in the frame defined by H:

$$|0\rangle \mapsto \frac{|0\rangle + |1\rangle}{\sqrt{2}} \triangleq |+\rangle, \quad |1\rangle \mapsto \frac{|0\rangle - |1\rangle}{\sqrt{2}} \triangleq |-\rangle, \quad X \mapsto HXH = Z = |+\rangle\langle +| + |-\rangle\langle -|.$$

- $t = +$: $|\psi_L\rangle = \alpha|+_L\rangle + \beta|-_L\rangle$ with $|+_L\rangle = |+++\rangle$ and $|-_L\rangle = |---\rangle$.
- $t = \Delta t$: $|\psi_L\rangle$ becomes with

$$\text{1 flip: } \begin{cases} \alpha|-++\rangle + \beta|+--\rangle \\ \alpha|+-+\rangle + \beta|-+-\rangle \\ \alpha|++-\rangle + \beta|--+\rangle \end{cases} ; \text{ 2 flips: } \begin{cases} \alpha|--+\rangle + \beta|++-\rangle \\ \alpha|-+-\rangle + \beta|+-+\rangle \\ \alpha|+--\rangle + \beta|-++\rangle \end{cases} ; \text{ 3 flips: } \alpha|---\rangle + \beta|+++\rangle.$$

- Key fact: 4 orthogonal planes $\mathcal{P}_c = \text{span}(|+++\rangle, |---\rangle)$, $\mathcal{P}_1 = \text{span}(|-++\rangle, |+--\rangle)$, $\mathcal{P}_2 = \text{span}(|+-+\rangle, |-+-\rangle)$ and $\mathcal{P}_3 = \text{span}(|++-\rangle, |--+\rangle)$.
- Error syndromes: 3 commuting observables $S_1 = I \otimes X \otimes X$, $S_2 = X \otimes I \otimes X$ and $S_3 = X \otimes X \otimes I$ with spectrum $\{-1, +1\}$ and outcomes $(s_1, s_2, s_3) \in \{-1, +1\}$.

-1- $s_1 = s_2 = s_3$: $\mathcal{P}_c \ni |\psi_L\rangle = \begin{cases} \alpha|+++\rangle + \beta|---\rangle \text{ 0 flip} \\ \beta|+++\rangle + \alpha|---\rangle \text{ 3 flips} \end{cases}$; no correction

-2- $s_1 \neq s_2 = s_3$: $\mathcal{P}_1 \ni |\psi_L\rangle = \begin{cases} \alpha|-++\rangle + \beta|+--\rangle \text{ 1 flip} \\ \beta|-++\rangle + \alpha|+--\rangle \text{ 2 flips} \end{cases}$; $(Z \otimes I \otimes I)|\psi_L\rangle \in \mathcal{P}_c$.

-3- $s_2 \neq s_3 = s_1$: $\mathcal{P}_2 \ni |\psi_L\rangle = \begin{cases} \alpha|+-+\rangle + \beta|-+-\rangle \text{ 1 flip} \\ \beta|+-+\rangle + \alpha|-+-\rangle \text{ 2 flips} \end{cases}$; $(I \otimes Z \otimes I)|\psi_L\rangle \in \mathcal{P}_c$.

-4- $s_3 \neq s_1 = s_2$: $\mathcal{P}_3 \ni |\psi_L\rangle = \begin{cases} \alpha|++-\rangle + \beta|--+\rangle \text{ 1 flip} \\ \beta|++-\rangle + \alpha|--+\rangle \text{ 2 flips} \end{cases}$; $(I \otimes I \otimes Z)|\psi_L\rangle \in \mathcal{P}_c$.

[11] $X = |1\rangle\langle 0| + |0\rangle\langle 1|$, $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ and $H = \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\langle 0| + \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\langle 1|$.

- Take the phase flip code $|+++\rangle$ and $|---\rangle$. Replace each $|+\rangle$ (resp. $|-\rangle$) by $\frac{|000\rangle+|111\rangle}{\sqrt{2}}$ (resp. $\frac{|000\rangle-|111\rangle}{\sqrt{2}}$).

New logical qubit $|\psi_L\rangle = \alpha|0_L\rangle + \beta|1_L\rangle \in \mathbb{C}^{2^9} \equiv \mathbb{C}^{512}$:

$$|0_L\rangle = \frac{\big(|000\rangle+|111\rangle\big)\big(|000\rangle+|111\rangle\big)\big(|000\rangle+|111\rangle\big)}{2\sqrt{2}}, \quad |1_L\rangle = \frac{\big(|000\rangle-|111\rangle\big)\big(|000\rangle-|111\rangle\big)\big(|000\rangle-|111\rangle\big)}{2\sqrt{2}}$$

- Local errors: each of the 9 physical qubits can have a bit-flip X, a phase flip Z or a bit flip followed by a phase flip $ZX = iY$ [12] with probability p during Δt.
- Denote by X_k (resp. Y_k and Z_k), the local operator X (resp. Y and Z) acting on physical qubit no $k \in \{1, \dots, 9\}$. Denote by $\mathcal{P}_c = \text{span}(|0_L\rangle, |1_L\rangle)$ the code space.

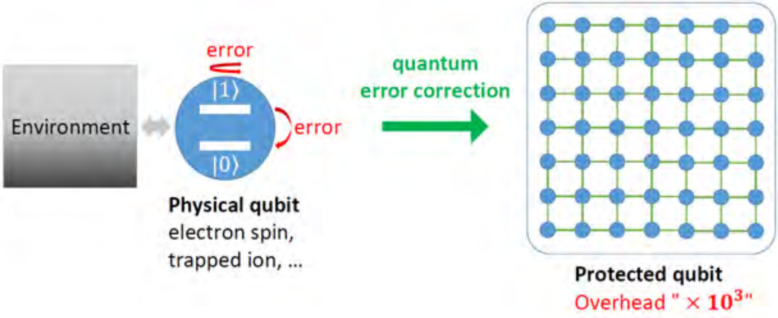
One get a family of the $1 + 3 \times 9 = 28$ **orthogonal planes:**

$$\mathcal{P}_c, \quad \big(X_k\mathcal{P}_c\big)_{k=1,\dots,9}, \quad \big(Y_k\mathcal{P}_c\big)_{k=1,\dots,9}, \quad \big(Z_k\mathcal{P}_c\big)_{k=1,\dots,9}.$$

- One can always construct error syndromes to obtain, when there is only one error among the 9 qubits during Δt, the number k of the qubit and the error type it has undergone (X, Y or Z). These 28 planes are then eigen-planes by the syndromes.
- If the physical qubit k is subject to **any kind of local errors** associated to arbitrary operator $M_k = gI + aX_k + bY_k + cZ_k$ $(g, a, b, c \in \mathbb{C})$, $|\psi_L\rangle \mapsto \frac{M_k|\psi_L\rangle}{\sqrt{\langle\psi_L|M_k^\dagger M_k|\psi_L\rangle}}$, the syndrome measurements will project the corrupted logical qubit on one of the 4 planes $\mathcal{P}_c$, $X_k\mathcal{P}_c$, $Y_k\mathcal{P}_c$ or $Z_k\mathcal{P}_c$. It is then simple by using either I, X_k, Y_k or Z_k, to recover up to a global phase the original logical qubit $|\psi_L\rangle$.

[12] $X = |1\rangle\langle 0| + |0\rangle\langle 1|$, $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ and $Y = i|1\rangle|0\rangle - i|0\rangle|1\rangle$.

QEC: 2D redundancy to correct bit-flip and phase-flip errors



- For a logical qubit relying on n physical qubits, the dimension of the Hilbert has to be larger than $2(1+3n)$ to recover a single but arbitrary qubit error: $2^n \geq 2(1+3n)$ imposing $n \geq 5$ ($\mathcal{H} = \mathbb{C}^{2^5} = \mathbb{C}^{32}$)

- Efficient constructions of quantum error-correcting codes: stabilizer codes, surface codes where the physical qubits are located on a 2D-lattice, topological codes, ...

- Fault tolerant computations: computing on encoded quantum states; fault-tolerant operations to avoid propagations of errors during encoding, gates and measurement; concatenation and threshold theorem, ...

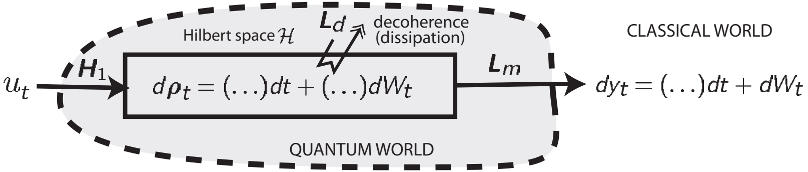
- Actual experiments: 10^{-3} is the typical error probability during elementary gates involving few physical qubits.

- High-order error-correcting codes with an important overhead; **more than 1000 physical qubits to encode a logical qubit** [13] $\mathcal{H} \sim \mathbb{C}^{2^{1000}}$.

[13] A.G. Fowler, M. Mariantoni, J.M. Martinis, A.N. Cleland (2012): Surface codes: Towards practical large-scale quantum computation. Phys. Rev. A,86(3):032324.

Outline

Classical I/O dynamics based on Stochastic Master Equation (SME) [14]



Continuous-time models: stochastic differential systems (Itō formulation)
density operator ρ ($\rho^\dagger = \rho$, $\rho \geq 0$, $\text{Tr}(\rho) = 1$) as state ($\hbar \equiv 1$ here):

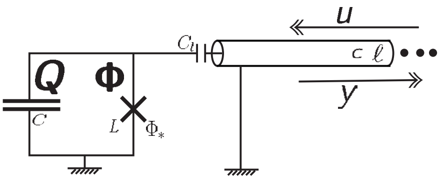
$$d\rho_t = \Big(-i[H_0 + u_t H_1, \rho_t] + \sum_{\nu=d,m} L_\nu \rho_t L_\nu^\dagger - \frac{1}{2}(L_\nu^\dagger L_\nu \rho_t + \rho_t L_\nu^\dagger L_\nu) \Big) dt$$
$$+ \sqrt{\eta_m} \Big(L_m \rho_t + \rho_t L_m^\dagger - \text{Tr}\left((L_m + L_m^\dagger)\rho_t \right) \rho_t \Big) dW_t$$

driven by the Wiener process W_t, with measurement y_t,

$$dy_t = \sqrt{\eta_m}\, \text{Tr}\left((L_m + L_m^\dagger)\, \rho_t \right) dt + dW_t \quad \text{detection efficiencies } \eta_m \in [0, 1].$$

Measurement backaction: $d\rho$ **and** dy **share the same noises** dW. Very different from the usual Kalman I/O state-space description.

[14] A. Barchielli, M. Gregoratti (2009): Quantum Trajectories and Measurements in Continuous Time: the Diffusive Case. Springer Verlag.



Classical model ($\frac{C_l}{C+C_l} = \epsilon \ll 1$):

$$\frac{d}{dt}\Phi = \frac{1}{C}Q + 2\epsilon u - \epsilon^2\sqrt{\frac{\ell}{c}}\frac{\Phi_*}{L}\sin\left(\frac{1}{\Phi_*}\Phi\right)$$
$$\frac{d}{dt}Q = -\frac{\Phi_*}{L}\sin\left(\frac{1}{\Phi_*}\Phi\right)$$

with $y = u - \epsilon\sqrt{\frac{\ell}{c}}\frac{\Phi_*}{L}\sin\left(\frac{1}{\Phi_*}\Phi\right)$.

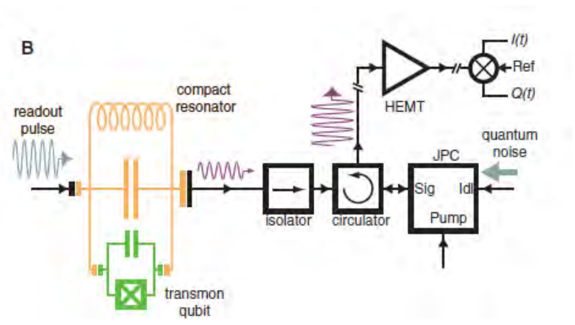
$H_s(\Phi, Q) = \frac{1}{2C}Q^2 - \frac{\Phi_*^2}{L}\cos\left(\frac{1}{\Phi_*}\Phi\right)$ with nonlinearity ($\Phi_* < (L/C)^{1/4}$):

- anharmonic spectrum: frequency transition between the ground and first excited states larger than frequency transition between first and second excited states, . . .
- qubit model based on restriction to these two slowest energy levels, $|g\rangle$ and $|e\rangle$, with pulsation $\omega_q \sim 1/\sqrt{LC}$.

Two weak coupling regimes of the transmon qubit[15]:

- resonant, in/out wave pulsation ω_q;
- off-resonant, in/out wave pulsation $\omega_q + \Delta$ with $|\Delta| \ll \omega_q$.

[15] J. Koch et al. (2007): Charge-insensitive qubit design derived from the Cooper pair box. Phys. Rev. A, 76:042319.



Superconducting qubit dispersively coupled to a cavity traversed by a microwave signal (input/output theory). The back-action on the qubit state of a single measurement of one output field quadrature y is described by a simple SME for the qubit density operator ρ, 2×2 Hermitian ≥ 0 matrix.

$$d\rho_t = \Big(-\tfrac{i}{2}[\omega_q Z, \rho_t] + \gamma(Z\rho Z - \rho_t)\Big) dt + \sqrt{\eta\gamma}\Big(Z\rho_t + \rho_t Z - 2\,\mathrm{Tr}\,(Z\rho_t)\,\rho_t \Big) dW_t$$

with y_t given by $dy_t = 2\sqrt{\eta\gamma}\,\mathrm{Tr}\,(Z\rho_t)\,dt + dW_t$ where $\gamma \geq 0$ is related to the measurement strength and $\eta \in [0,1]$ is the detection efficiency.

[16] M. Hatridge et al. (2013): Quantum Back-Action of an Individual Variable-Strength Measurement. Science, 339, 178-181.

With a single imperfect measurement $dy_t = \sqrt{\eta}\ \mathrm{Tr}\left((L+L^\dagger)\,\rho_t\right)\, dt + dW_t$ and detection efficiency $\eta \in [0,1]$, the quantum state ρ_t obeys to

$$d\rho_t = \left(-i[H_0 + u_t H_1, \rho_t] + L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L \rho_t + \rho_t L^\dagger L)\right) dt \\ + \sqrt{\eta}\left(L\rho_t + \rho_t L^\dagger - \ \mathrm{Tr}\left((L+L^\dagger)\rho_t\right)\rho_t\right) dW_t$$

driven by the Wiener process dW_t

With Itō rules, it can be written as the following "discrete-time" Markov model

$$\rho_{t+dt} \triangleq \rho_t + d\rho_t = \frac{M_{u_t,dy_t}\rho_t M^\dagger_{u_t,dy_t} + (1-\eta)L\rho_t L^\dagger dt}{\mathrm{Tr}\left(M_{u_t,dy_t}\rho_t M^\dagger_{u_t,dy_t} + (1-\eta)L\rho_t L^\dagger dt\right)}$$

with $M_{u_t,dy_t} = I - \left(i(H_0 + u_t H_1) + \frac{1}{2}\left(L^\dagger L\right)\right) dt + \sqrt{\eta} L dy_t$.

[17] PR (2014): Models and Feedback Stabilization of Open Quantum Systems. Proc. of Int. Congress of Mathematicians, vol. IV, pp 921–946, Seoul. (`http://arxiv.org/abs/1407.7810`).

Measured output map $dy_t = \sqrt{\eta}\ \mathrm{Tr}\left((\mathrm{L} + \mathrm{L}^\dagger)\,\rho_t\right)\ dt + dW_t$ and measurement backaction described by

$$\rho_{t+dt} \triangleq \rho_t + d\rho_t = \frac{\mathrm{M}_{u_t,dy_t}\rho_t \mathrm{M}_{u_t,dy_t}^\dagger + (1-\eta)\mathrm{L}\rho_t\mathrm{L}^\dagger dt}{\mathrm{Tr}\left(\mathrm{M}_{u_t,dy_t}\rho_t \mathrm{M}_{u_t,dy_t}^\dagger + (1-\eta)\mathrm{L}\rho_t\mathrm{L}^\dagger dt\right)}$$

- **if ρ_0 density operator, then, for all $t > 0$, ρ_t remains a density operator**
  The dynamics preserve the cone of non-negative Hermitian operators.
- **Positivity and trace preserving numerical scheme for quantum Monte-Carlo simulations.**
- When $\eta = 1$, $\mathrm{rank}(\rho_t) \leq \mathrm{rank}(\rho_0)$ for all $t \geq 0$. In particular if ρ_0 is a rank one projector, then ρ_t remains a rank one projector (pure state).

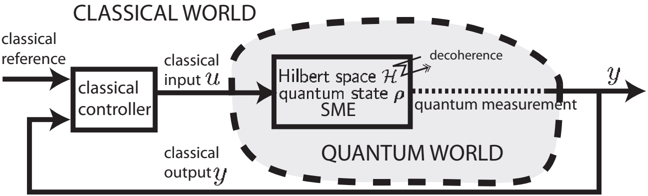
$$d\rho_t = \left(-i[H_0 + uH_1, \rho_t] + L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L\rho_t + \rho_t L^\dagger L)\right) dt$$
$$+ \sqrt{\eta}\left(L\rho_t + \rho_t L^\dagger - \operatorname{Tr}\left((L + L^\dagger)\rho_t\right) \rho_t\right) dW_t$$

with measured output map $dy_t = \sqrt{\eta}\, \operatorname{Tr}\left((L + L^\dagger)\, \rho_t\right)\, dt + dW_t$

- Invariance of the SME structure under unitary transformations.
  A time-varying change of frame $\widetilde{\rho} = U_t^\dagger \rho U_t$ with U_t unitary.
  The new density operator $\widetilde{\rho}$ obeys to a similar SME where
  $\widetilde{H}_0 + u\widetilde{H}_1 = U_t^\dagger (H_0 + uH_0) U_t + iU_t^\dagger \left(\frac{d}{dt} U_t\right)$ and $\widetilde{L} = U_t^\dagger L U_t$.
- Ensemble average. "L^1-contraction" of Lindblad dynamics
  $$\frac{d}{dt}\rho = -i[H_0 + uH_1, \rho_t] + L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L\rho_t + \rho_t L^\dagger L)$$
  generating a contraction semi-group for many distances (nuclear distance[18], Hilbert metric on the cone of non negative operators[19]).
- If the non-negative Hermitian operator A satisfies the operator inequality
  $$i[H_0 + uH_1, A] + L^\dagger A L - \frac{1}{2}(L^\dagger L A + A L^\dagger L) \leq 0$$
  then $V(\rho) = \operatorname{Tr}(A\rho)$ is a super-martingale (Lyapunov function).

[18] D.Petz (1996). Monotone metrics on matrix spaces. Linear Algebra and its Applications, 244, 81-96.

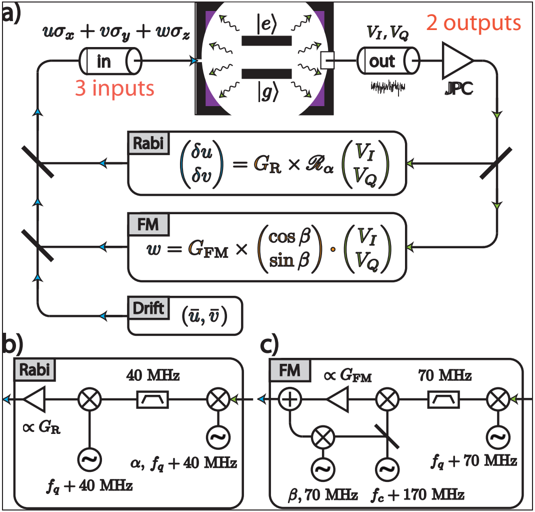
[19] R. Sepulchre, A. Sarlette, PR (2010). Consensus in non-commutative spaces. IEEE-CDC.



- P-controller (Markovian feedback[20]) for $u_t \, dt = k \, dy_t$, the ensemble average closed-loop dynamics of ρ remains governed by a linear Lindblad master equation.
- PID controller: no Lindblad master equation in closed-loop for dynamics output feedback
- Nonlinear hidden-state stochastic systems: Lyapunov state-feedback[21]; many open issues on convergence rates, delays, robustness, ...
- **Short sampling times limit feedback complexity**

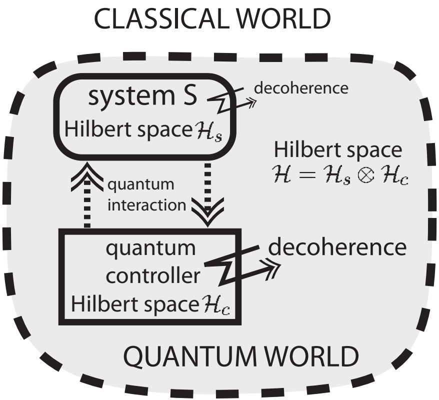
[20] H. Wiseman, G. Milburn (2009). Quantum Measurement and Control. Cambridge University Press.

[21] See e.g.: C. Ahn et. al (2002): Continuous quantum error correction via quantum feedback control. Phys. Rev. A 65;
M. Mirrahimi, R. Handel (2007): Stabilizing feedback controls for quantum systems. SIAM Journal on Control and Optimization, 46(2), 445-467;
G. Cardona, A. Sarlette, PR (2019): Continuous-time quantum error correction with noise-assisted quantum feedback. IFAC Mechatronics & Nolcos Conf.



[22] P. Campagne-Ibarcq, . . . , PR, B. Huard (2016): Using Spontaneous Emission of a Qubit as a Resource for Feedback Control. Phys. Rev. Lett. 117(6).

Quantum analogue of Watt speed governor: a **dissipative** mechanical system controls another mechanical system [23]



Optical pumping (Kastler 1950), coherent population trapping (Arimondo 1996)

Dissipation engineering, autonomous feedback: (Zoller, Cirac, Wolf, Verstraete, Devoret, Schoelkopf, Siddiqi, Martinis, Raimond, Brune,. . . , Lloyd, Viola, Ticozzi, Leghtas, Mirrahimi, Sarlette, PR, ...)

(S,L,H) theory and linear quantum systems: quantum feedback networks based on stochastic Schrödinger equation, Heisenberg picture (Gardiner, Yurke, Mabuchi, Genoni, Serafini, Milburn, Wiseman, Doherty, . . . , Gough, James, Petersen, Nurdin, Yamamoto, Zhang, Dong, . . .)

Stability analysis: Kraus maps and Lindblad propagators are always contractions (non commutative diffusion and consensus).

[23] J.C. Maxwell (1868): On governors. Proc. of the Royal Society, No.100.

MINES PARIS | PSL

The closed-loop Lindblad master equation on $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_c$:

$$\frac{d}{dt}\rho = -i\Big[\mathsf{H}_s \otimes \mathsf{I}_c + \mathsf{I}_s \otimes \mathsf{H}_c + \mathsf{H}_{sc}\ ,\ \rho\Big] + \sum_{\nu} \mathbb{D}_{\mathsf{L}_{s,\nu}\otimes \mathsf{I}_c}(\rho) + \sum_{\nu'} \mathbb{D}_{\mathsf{I}_s \otimes \mathsf{L}_{c,\nu'}}(\rho)$$

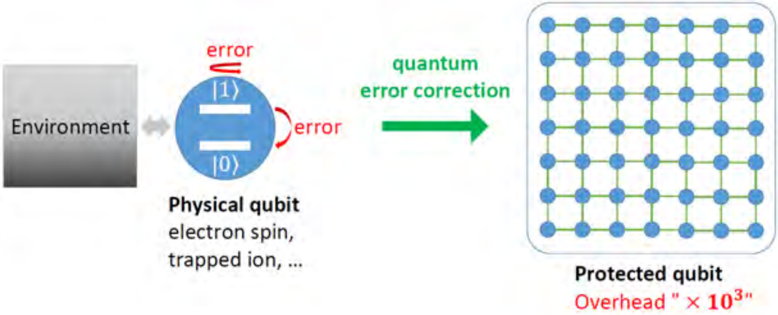
with $\mathbb{D}_{\mathsf{L}}(\rho) = \mathsf{L}\rho\mathsf{L}^\dagger - \frac{1}{2}\left(\mathsf{L}^\dagger\mathsf{L}\rho + \rho\mathsf{L}^\dagger\mathsf{L}\right)$ and operators made of tensor products.

• Consider a convex subset $\overline{\mathcal{D}}_s$ of steady-states for original system S: each density operator $\overline{\rho}_s$ on $\mathcal{H}_s$ belonging to $\overline{\mathcal{D}}_s$ satisfy $i[\mathsf{H}_s, \overline{\rho}_s] = \sum_\nu \mathbb{D}_{\mathsf{L}_{s,\nu}}(\overline{\rho}_s)$.

• Designing a **realistic** quantum controller C $(\mathsf{H}_c, \mathsf{L}_{c,\nu'})$ and coupling Hamiltonian H_{sc} stabilizing $\overline{\mathcal{D}}_s$ is non trivial. **Realistic** means in particular relying on physical time-scales and constraints:

- Fastest time-scales attached to H_s and H_c (Bohr frequencies) and averaging approximations: $\|\mathsf{H}_s\|, \|\mathsf{H}_c\| \gg \|\mathsf{H}_{sc}\|$,
- High-quality oscillations: $\|\mathsf{H}_s\| \gg \|\mathsf{L}_{s,\nu}^\dagger \mathsf{L}_{s,\nu}\|$ and $\|\mathsf{H}_c\| \gg \|\mathsf{L}_{c,\nu'}^\dagger \mathsf{L}_{c,\nu'}\|$.
- Decoherence rates of S much slower than those of C: $\|\mathsf{L}_{s,\nu}^\dagger \mathsf{L}_{s,\nu}\| \ll \|\mathsf{L}_{c,\nu'}^\dagger \mathsf{L}_{c,\nu'}\|$: model reduction by quasi-static approximations (adiabatic elimination, singular perturbations).

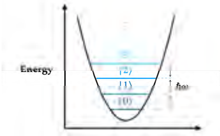
QEC: 2D redundancy to correct bit-flip and phase-flip errors



APPROACH: HARDWARE SHORTCUTS TO QEC

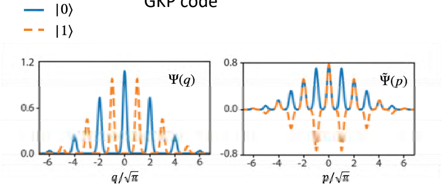
Idea 1: Hardware-efficient delocalization: bosonic codes

- Infinite dimensional Hilbert space of a single quantum harmonic oscillator to encode information non-locally.

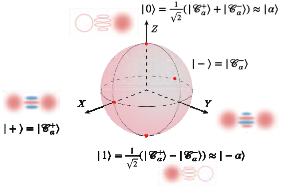


$$\mathscr{H} = \text{span}\{\,|n\rangle, n \in \mathbb{N}\}$$

GKP code



Cat code (Quantic & Yale)



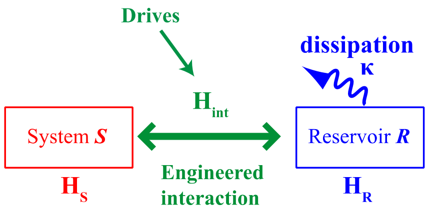
D. Gottesman, A. Kitaev, J. Preskill, Phys. Rev. A 64, 2001.

Z. Leghtas, Phys. Rev. L 111, 2013.

APPROACH: HARDWARE SHORTCUTS TO QEC

Idea 2: Autonomous error correction through control by dissipativity

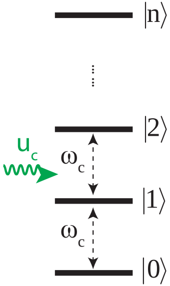
- Engineer nonlinear dissipative mechanisms that stabilize the manifold of quantum states where the information is encoded.



Mirrahimi et al., New Journal of Physics, 2014.

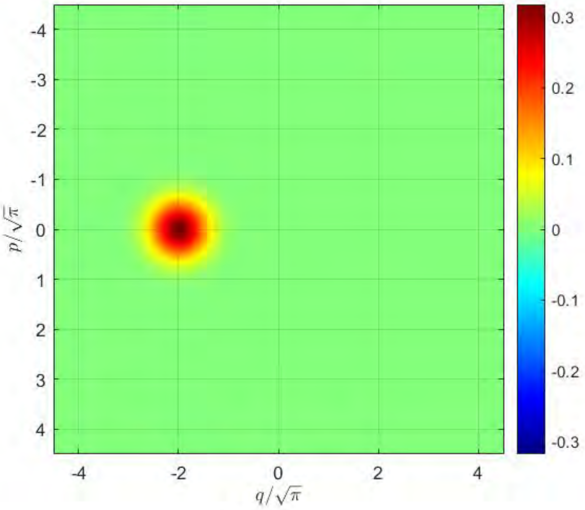
Quantum harmonic oscillator (spring system) [8]

- Hilbert space:
  $\mathcal{H}_S = \left\{ \sum_{n\geq 0} \psi_n |n\rangle,\ (\psi_n)_{n\geq 0} \in l^2(\mathbb{C}) \right\} \equiv L^2(\mathbb{R}, \mathbb{C})$
- Quantum state space:
  $\mathcal{D} = \{\rho \in \mathcal{L}(\mathcal{H}_S), \rho^\dagger = \rho,\ \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}$.
- Operators and commutations:
  $\mathsf{a}|n\rangle = \sqrt{n}\ |n\text{-}1\rangle$, $\mathsf{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$;
  $\mathsf{N} = \mathsf{a}^\dagger \mathsf{a}$, $\mathsf{N}|n\rangle = n|n\rangle$;
  $[\mathsf{a}, \mathsf{a}^\dagger] = \mathsf{I}$, $\mathsf{a} f(\mathsf{N}) = f(\mathsf{N}+\mathsf{I})\mathsf{a}$;
  $\mathsf{D}_\alpha = e^{\alpha \mathsf{a}^\dagger - \alpha^\dagger \mathsf{a}}$.
  $\mathsf{a} = \frac{\mathsf{Q}+i\mathsf{P}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(q + \frac{\partial}{\partial q}\right)$, $[\mathsf{Q}, \mathsf{P}] = \imath \mathsf{I}$.
- Hamiltonian: $\mathsf{H}_S = \omega_c \mathsf{a}^\dagger \mathsf{a} + u_c(\mathsf{a} + \mathsf{a}^\dagger)$.
  (associated classical dynamics: $\frac{dq}{dt} = \omega_c p$, $\frac{dp}{dt} = -\omega_c q - u_c$).



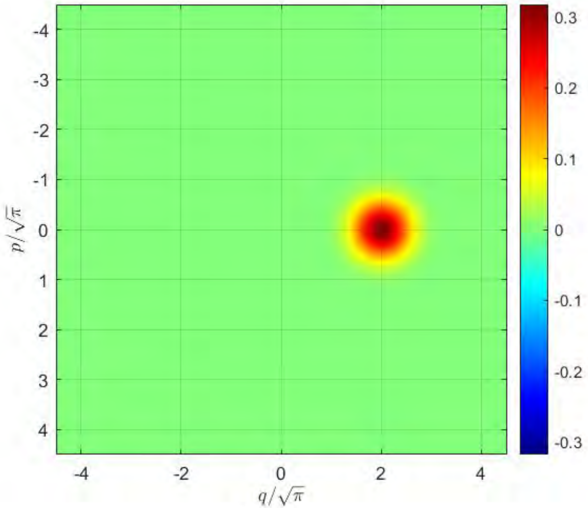
- Classical pure state $\equiv$ coherent state $|\alpha\rangle$
  $$\alpha \in \mathbb{C}:\ |\alpha\rangle = \sum_{n\geq 0}\left(e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\right)|n\rangle;\ |\alpha\rangle \equiv \frac{1}{\pi^{1/4}} e^{\imath\sqrt{2}q\Im\alpha} e^{-\frac{(q-\sqrt{2}\Re\alpha)^2}{2}}$$
  $\mathsf{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\mathsf{D}_\alpha|0\rangle = |\alpha\rangle$.
- Wigner function of a density operator ρ:
  $$\mathbb{C} \ni \alpha = \frac{q+ip}{\sqrt{2}} \mapsto W^\rho(q,p) = \mathrm{Tr}\left(e^{i\pi\mathsf{N}}\ \mathsf{D}_\alpha\ \rho\ \mathsf{D}_{-\alpha}\right)$$

Wigner function[24] of $|0_L\rangle \triangleq |-\sqrt{2\pi}\rangle$



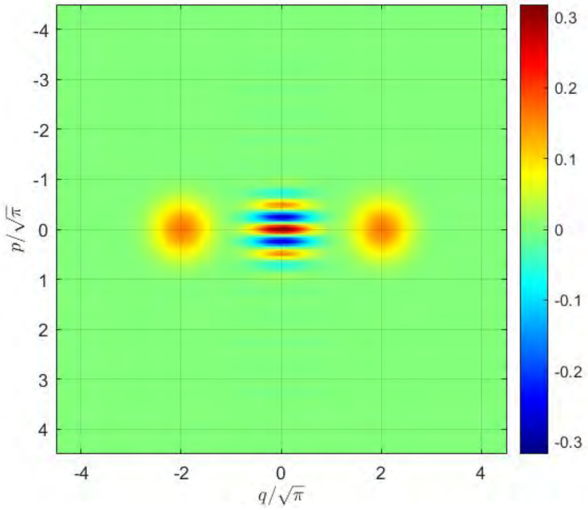
[24] For $\psi \in L^2(\mathbb{R}, \mathbb{C})$: $W^{|\psi\rangle\langle\psi|}(q,p) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \psi^*(q - \frac{u}{2}) \psi(q + \frac{u}{2}) e^{-2ipu} du$.

Wigner function[25] of $|1_L\rangle \triangleq |\sqrt{2\pi}\rangle$



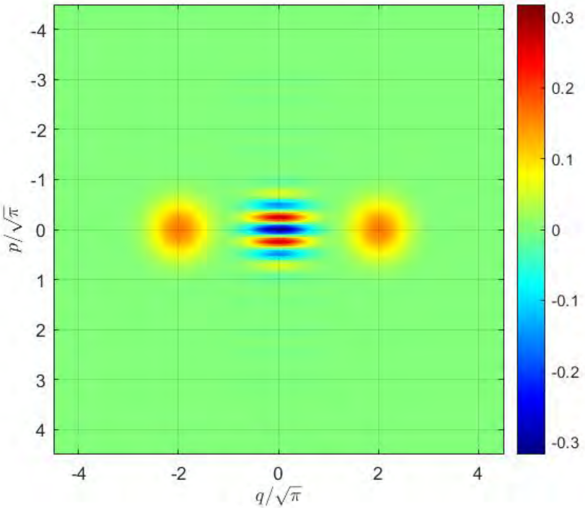
[25] For $\psi \in L^2(\mathbb{R}, \mathbb{C})$: $W^{|\psi\rangle\langle\psi|}(q,p) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \psi^*(q - \frac{u}{2})\psi(q + \frac{u}{2})e^{-2ipu}du$.

Wigner function[26] of $|+_L\rangle = \frac{|-\sqrt{2\pi}\rangle + |\sqrt{2\pi}\rangle}{\sqrt{2}}$ ("Schrödinger phase cat")



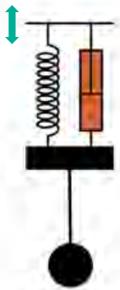
[26] For $\psi \in L^2(\mathbb{R}, \mathbb{C})$: $W^{|\psi\rangle\langle\psi|}(q,p) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \psi^*(q - \frac{u}{2})\psi(q + \frac{u}{2})e^{-2ipu}du$.

Wigner function[27] of $|-_L\rangle = \frac{|-\sqrt{2\pi}\rangle - |\sqrt{2\pi}\rangle}{\sqrt{2}}$ ("Schrödinger phase cat")



[27] For $\psi \in L^2(\mathbb{R}, \mathbb{C})$: $W^{|\psi\rangle\langle\psi|}(q,p) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \psi^*(q - \frac{u}{2})\psi(q + \frac{u}{2})e^{-2ipu}du$.

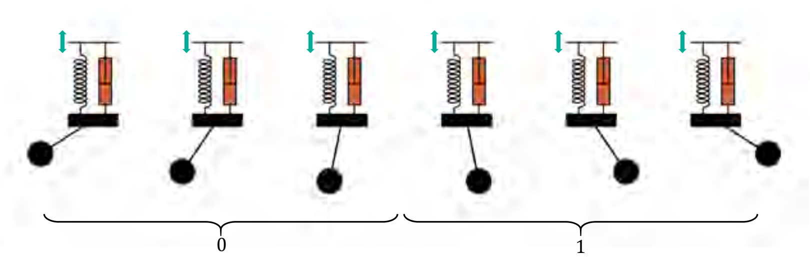
MAIN IDEA IN A CLASSICAL PICTURE



Driven damped oscillator
coupled to a pendulum.

Courtesy of Raphaël Lescanne

A BI-STABLE SYSTEM

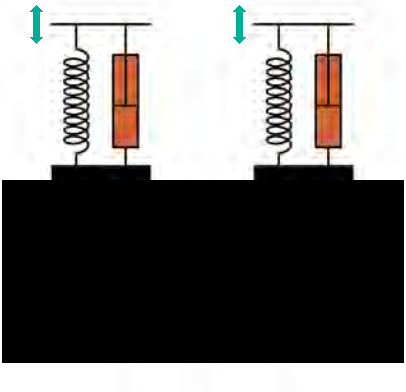


There are **2 steady states** in which we can encode information

Courtesy of Raphaël Lescanne

MAIN IDEA IN A CLASSICAL PICTURE

Stabilization regardless of the state



[redacted]

Neither the **drive** nor the **dissipation** can **distinguish** between 0 and 1

Important to preserve quantum coherence

Courtesy of Raphaël Lescanne

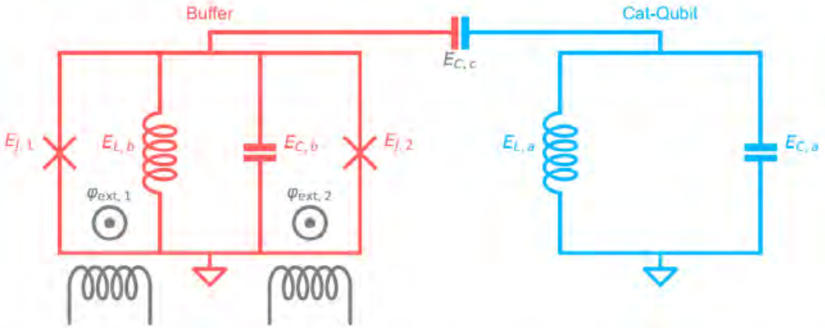


Figure S3. Equivalent circuit diagram. The cat-qubit (blue), a linear resonator, is capacitively coupled to the buffer (red). One recovers the circuit of Fig. 2 by replacing the buffer inductance with a 5-junction array and by setting $\varphi_\Sigma = (\varphi_{ext,1} + \varphi_{ext,2})/2$ and $\varphi_\Delta = (\varphi_{ext,1} - \varphi_{ext,2})/2$. Not shown here: the buffer is capacitively coupled to a transmission line, the cat-qubit resonator is coupled to a transmon qubit

[28] R. Lescanne, ..., M. Mirrahimi, M. and Z. Leghtas: Exponential suppression of bit-flips in a qubit encoded in an oscillator. 2020, Nat. Phys. , Vol. 16, p. 509-513.

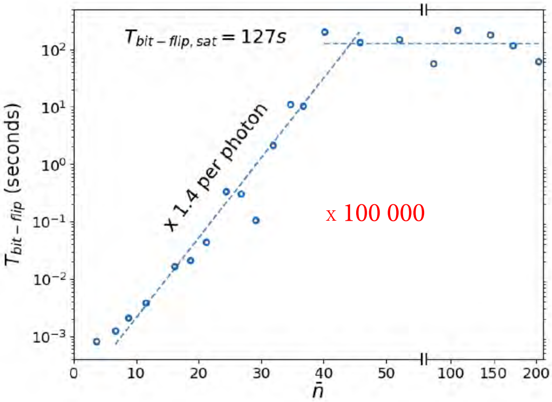


FIG. 4. Exponential suppression of bit flips. The bit-flip time (y-axis, log-scale) is measured (open circles) as a function of cat-size (x-axis). The bit-flip time increases exponentially, multiplying by 1.4 per photon (solid line) before saturating at approximately 127 s (horizontal dashed line).

Classical Hamiltonian of **two harmonic oscillators** of pulsations $\omega_a \neq \omega_b$

$$H(q_a, p_a, q_b, p_b, t) = \tfrac{\omega_a}{2}(q_a^2 + p_a^2) + \tfrac{\omega_b}{2}(q_b^2 + p_b^2) + 2g\cos\left(\sqrt{2}\phi_a q_a + \sqrt{2}\phi_b q_b + (2\omega_a - \omega_b)t\right)$$

including **oscillatory non-linear coupling** ($|g| \ll \omega_a, \omega_b$) and parameters $1 \gg \phi_a\phi_b > 0$. Dynamical Hamilton equations

$$\frac{d}{dt}q_a = \omega_a p_a, \quad \frac{d}{dt}p_a = -\omega_a q_a + 2ig\sqrt{2}\phi_a \sin\left(\sqrt{2}\phi_a q_a + \sqrt{2}\phi_b q_b + (2\omega_a - \omega_b)t\right)$$

$$\frac{d}{dt}q_b = \omega_b p_b, \quad \frac{d}{dt}p_b = -\omega_b q_b - \kappa_b p_b + 2ig\sqrt{2}\phi_b \sin\left(\sqrt{2}\phi_a q_a + \sqrt{2}\phi_b q_b + (2\omega_a - \omega_b)t\right) + v\cos\omega_b t + w\sin\omega_b t$$

including **weak damping rate** $0 < \kappa_b \ll \omega_b$ and resonant drive $|v|, |w| \ll \omega_b$. With complex variable $z_a = (q_a + ip_a)/\sqrt{2}$ and $z_b = (q_b + ip_b)/\sqrt{2}$ one gets

$$\frac{d}{dt}z_a = -i\omega_a z_a + 2ig\phi_a \sin\left(\phi_a(z_a + z_a^*) + \phi_b(z_b + z_b^*) + (2\omega_a - \omega_b)t\right)$$

$$\frac{d}{dt}z_b = -i\omega_b z_b - \tfrac{\kappa_b}{2}(z_b - z_b^*) + 2ig\phi_b \sin\left(\phi_a(z_a + z_a^*) + \phi_b(z_b + z_b^*) + (2\omega_a - \omega_b)t\right) + ue^{-i\omega_b t} - u^* e^{i\omega_b t}$$

with $(w + iv)/2\sqrt{2} = u \in \mathbb{C}$.

$$\frac{d}{dt} z_a = -i\omega_a z_a + 2ig\phi_a \sin\Big(\phi_a(z_a + z_a^*) + \phi_b(z_b + z_b^*) + (2\omega_a - \omega_b)t\Big)$$

$$\frac{d}{dt} z_b = -i\omega_b z_b - \tfrac{\kappa_b}{2}(z_b - z_b^*) + 2ig\phi_b \sin\Big(\phi_a(z_a + z_a^*) + \phi_b(z_b + z_b^*) + (2\omega_a - \omega_b)t\Big) + ue^{-i\omega_b t} - u^* e^{i\omega_b t}$$

The **time-varying change of variables** $z_a = \bar{z}_a e^{-i\omega_a t}$ and $z_b = \bar{z}_b e^{-i\omega_b t}$ yields to

$$\frac{d}{dt} \bar{z}_a = 2ig\phi_a e^{i\omega_a t} \sin\Big(\phi_a(\bar{z}_a e^{-i\omega_a t} + \bar{z}_a^* e^{+i\omega_a t}) + \phi_b(\bar{z}_b e^{-i\omega_b t} + \bar{z}_b^* e^{+i\omega_b t}) + (2\omega_a - \omega_b)t\Big)$$

$$\frac{d}{dt} \bar{z}_b = -\tfrac{\kappa_b}{2}(\bar{z}_b - \bar{z}_b^* e^{2i\omega_b t}) + u - u^* e^{2i\omega_b t} + 2ig\phi_b e^{i\omega_b t} \sin\Big(\phi_a(\bar{z}_a e^{-i\omega_a t} + \bar{z}_a^* e^{+i\omega_a t}) + \phi_b(\bar{z}_b e^{-i\omega_b t} + \bar{z}_b^* e^{+i\omega_b t}) + (2\omega_a - \omega_b)t\Big).$$

First order **averaging** based on asymptotic expansion up-to order 3 versus $\phi_a, \phi_b \ll 1$ (weak non-linearity) gives with $g_2 = \frac{g\phi_a^2\phi_b}{2}$

$$\frac{d}{dt} \bar{z}_a = 2g_2 \bar{z}_a^* \bar{z}_b, \quad \frac{d}{dt} \bar{z}_b = u - g_2 \bar{z}_a^2 - \tfrac{\kappa_b}{2} \bar{z}_b.$$

2 stable steady-states $\bar{z}_a, \bar{z}_b) = (\pm\alpha, 0)$ **with** $\alpha^2 = u/g_2$, an unstable one $(0, 2u/\kappa_b)$. When $\kappa_b \gg |g_2|$, $\bar{z}_b$ relaxes rapidly to $u - g_2 \bar{z}_a^2$ (**singular perturbations**). The slow evolution of $\bar{z}_a$ obeys to

$$\frac{d}{dt} \bar{z}_a = -\frac{4g_2^2}{\kappa_b} \bar{z}_a^* (\bar{z}_a^2 - \alpha^2)$$

Quantum Hamiltonian: two commuting annihilation operators $\mathrm{a} = (q_a + \frac{\partial}{\partial p_a})/\sqrt{2}$ and $\mathrm{b} = (q_b + \frac{\partial}{\partial p_b})/\sqrt{2}$ with $[\mathrm{a}, \mathrm{a}^\dagger] = \mathrm{I}$, $[\mathrm{b}, \mathrm{b}^\dagger] = \mathrm{I}$

$$\mathrm{H}_1(t) = \omega_a \mathrm{a}^\dagger \mathrm{a} + \omega_b \mathrm{b}^\dagger \mathrm{b} + g e^{i(2\omega_a - \omega_b)t} \exp\Big(i\phi_a(\mathrm{a} + \mathrm{a}^\dagger) + i\phi_b(\mathrm{b} + \mathrm{b}^\dagger)\Big) \\ + g e^{-i(2\omega_a - \omega_b)t} \exp\Big(-i\phi_a(\mathrm{a} + \mathrm{a}^\dagger) + i\phi_b(\mathrm{b} + \mathrm{b}^\dagger)\Big).$$

Change of frame for $\frac{d}{dt}\rho_1 = -i[\mathrm{H}_1(t), \rho_1]$: new density operator

$$\rho_2 = \exp\Big(i\omega_a t \mathrm{a}^\dagger \mathrm{a} + i\omega_b t \mathrm{b}^\dagger \mathrm{b}\Big)\, \rho_1 \exp\Big(-i\omega_a t \mathrm{a}^\dagger \mathrm{a} - i\omega_b t \mathrm{b}^\dagger \mathrm{b}\Big)$$

is governed by $\frac{d}{dt}\rho_2 = -i[\mathrm{H}_2(t), \rho_2]$ with

$$\mathrm{H}_2(t) = g e^{i(2\omega_a - \omega_b)t} \exp\Big(i\phi_a(e^{-i\omega_a t}\mathrm{a} + e^{i\omega_a t}\mathrm{a}^\dagger) + i\phi_b(e^{-i\omega_b t}\mathrm{b} + e^{i\omega_b t}\mathrm{b}^\dagger)\Big) + h.c.$$

Expansion up-to order 3 versus $\phi_a, \phi_b \ll 1$:

$$\mathrm{H}_2(t) = g e^{i(2\omega_a - \omega_b)t}\Big(\mathrm{I} + i\phi_a(e^{-i\omega_a t}\mathbf{a} + e^{i\omega_a t}\mathbf{a}^\dagger) - \frac{\phi_a^2}{2}(e^{-i\omega_a t}\mathbf{a} + e^{i\omega_a t}\mathbf{a}^\dagger)^2 - \frac{i\phi_a^3}{3}(e^{-i\omega_a t}\mathbf{a} + e^{i\omega_a t}\mathbf{a}^\dagger)^3\Big) \ldots \\ \Big(\mathrm{I} + i\phi_b(e^{-i\omega_b t}\mathbf{b} + e^{i\omega_b t}\mathbf{b}^\dagger) - \frac{\phi_b^2}{2}(e^{-i\omega_b t}\mathbf{b} + e^{i\omega_b t}\mathbf{b}^\dagger)^2 - \frac{i\phi_b^3}{6}(e^{-i\omega_b t}\mathbf{b} + e^{i\omega_b t}\mathbf{b}^\dagger)^3\Big) + h.c.$$

$$\mathrm{H}_2(t) = g e^{i(2\omega_a - \omega_b)t} \dots$$
$$\Big(\mathrm{I} + i\phi_a\big(e^{-i\omega_a t}\mathrm{a} + e^{i\omega_a t}\mathrm{a}^\dagger\big) - \tfrac{\phi_a^2}{2}\big(e^{-i\omega_a t}\mathrm{a} + e^{i\omega_a t}\mathrm{a}^\dagger\big)^2 - \tfrac{i\phi_a^3}{3}\big(e^{-i\omega_a t}\mathrm{a} + e^{i\omega_a t}\mathrm{a}^\dagger\big)^3\Big)\dots$$
$$\Big(\mathrm{I} + i\phi_b\big(e^{-i\omega_b t}\mathrm{b} + e^{i\omega_b t}\mathrm{b}^\dagger\big) - \tfrac{\phi_b^2}{2}\big(e^{-i\omega_b t}\mathrm{b} + e^{i\omega_b t}\mathrm{b}^\dagger\big)^2 - \tfrac{i\phi_b^3}{6}\big(e^{-i\omega_b t}\mathrm{b} + e^{i\omega_b t}\mathrm{b}^\dagger\big)^3\Big)$$
$$+\, h.c.$$

Only two secular terms (i.e. non-oscillatory): $-ig_2\mathrm{a}^2\mathrm{b}^\dagger$ and its hermitian conjugate $ig_2\left(\mathrm{a}^\dagger\right)^2\mathrm{b}$ where $g_2 = g\phi_a^2\phi_b/2$. Justify the following approximate time-invariant Hamiltonian for H_2 (**rotating wave approximation**): :

$$\mathrm{H}_2(t) \approx -ig_2\mathrm{a}^2\mathrm{b}^\dagger + ig_2\left(\mathrm{a}^\dagger\right)^2\mathrm{b}.$$

Finer approximations via high-order **averaging** techniques.

Cat-qubit stored in oscillator a, **controller based on a damped oscillator** b stabilizing against one decoherence channel (bit-fip):

$$\frac{d}{dt}\rho = -\left[g_2 \mathrm{a}^2\mathrm{b}^\dagger - g_2(\mathrm{a}^\dagger)^2\mathrm{b}\ ,\ \rho\right] + \left[u\mathrm{b}^\dagger - u^*\mathrm{b}\ ,\ \rho\right] + \kappa_b\Big(\mathrm{b}\rho\mathrm{b}^\dagger - (\mathrm{b}^\dagger\mathrm{b}\rho + \rho\mathrm{b}^\dagger\mathrm{b})/2\Big)$$

$$= -\Big[g_2(\mathrm{a}^2 - \alpha^2)\mathrm{b}^\dagger - g_2^*((\mathrm{a}^\dagger)^2 - (\alpha)^2)\mathrm{b}\ ,\ \rho\Big] + \kappa_b\Big(\mathrm{b}\rho\mathrm{b}^\dagger - (\mathrm{b}^\dagger\mathrm{b}\rho + \rho\mathrm{b}^\dagger\mathrm{b})/2\Big)$$

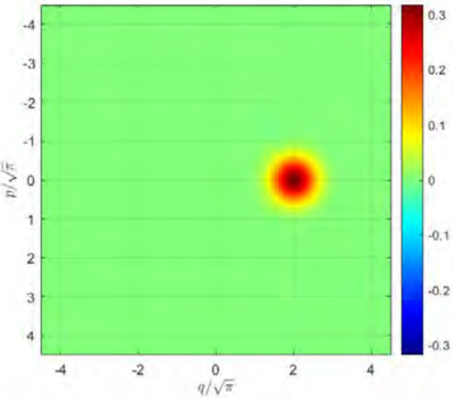
with $\alpha \in \mathbb{C}$ such that $\alpha^2 = u/g_2$, the drive amplitude $u \in \mathbb{C}$ applied to mode b and $1/\kappa_b > 0$ the life-time of photon in mode b.

Any density operators $\bar{\rho} = \bar{\rho}_a \otimes |0\rangle\langle 0|_b$ **is a steady-state as soon as the support of $\bar{\rho}_a$ belongs to the two dimensional vector space spanned by the quasi-classical wave functions $|\alpha\rangle$ and $|\text{-}\alpha\rangle$** (range($\bar{\rho}_a$) $\subset$ span$\{|\alpha\rangle, |\text{-}\alpha\rangle\}$) (Schrödinger cat-qubit).

Usually $\kappa_b \gg |g_2|$, mode b relaxes rapidly to vaccuum $|0\rangle\langle 0|_b$, can be eliminated adiabatically (**singular perturbations**, second order corrections) to provides the slow evolution of mode a [29]

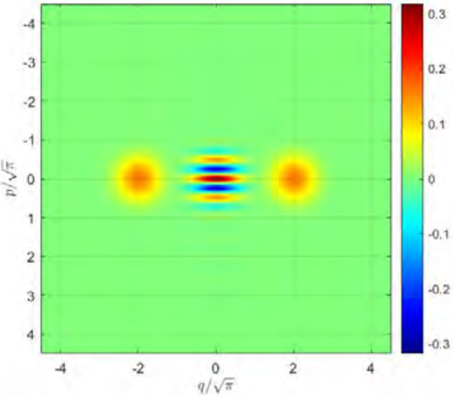
$$\frac{d}{dt}\rho_a = \frac{4|g_2|^2}{\kappa_b}\Big((\mathrm{a}^2-\alpha^2)\rho_a(\mathrm{a}^2-\alpha^2)^\dagger - ((\mathrm{a}^2-\alpha^2)^\dagger(\mathrm{a}^2-\alpha^2)\rho_a + \rho_a(\mathrm{a}^2-\alpha^2)^\dagger(\mathrm{a}^2-\alpha^2))/2\Big).$$

[29] For a mathematical proof of convergence analysis in an adapted Banach space, see : R. Azouit, A. Sarlette, PR: Well-posedness and convergence of the Lindblad master equation for a quantum harmonic oscillator with multi-photon drive and damping. 2016, ESAIM: COCV , Vol. 22, No. 4, p. 1353 -1369.



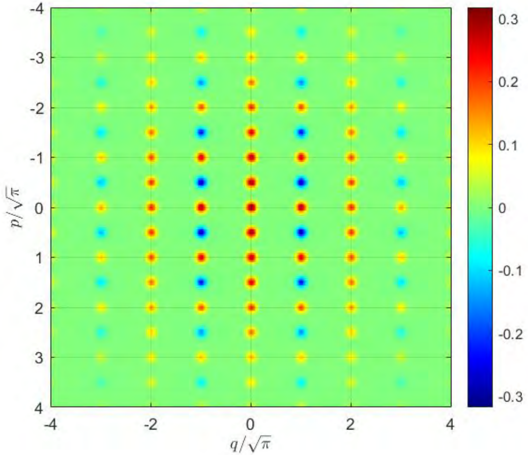
Bit-flip and phase-flip errors correspond to local diffusion on $W^{\rho}(q,p)$:
$|\pm\sqrt{2\pi}\rangle$ robust versus local diffusion

Wigner function of $(|\sqrt{2\pi}\rangle + |\text{-}\sqrt{2\pi}\rangle)/\sqrt{2}$



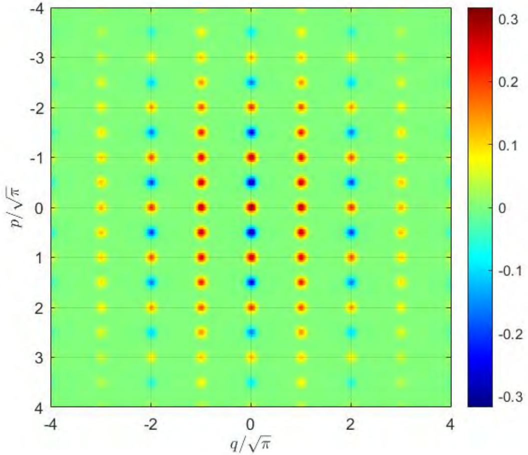
Bit-flip and phase-flip errors correspond to local diffusion on $W^\rho(q,p)$:
$(|\sqrt{2\pi}\rangle \pm |\text{-}\sqrt{2\pi}\rangle)/\sqrt{2}$ not robust versus diffusion

Wigner function of the GKP finite energy grid-state $|0_L\rangle$ [30]



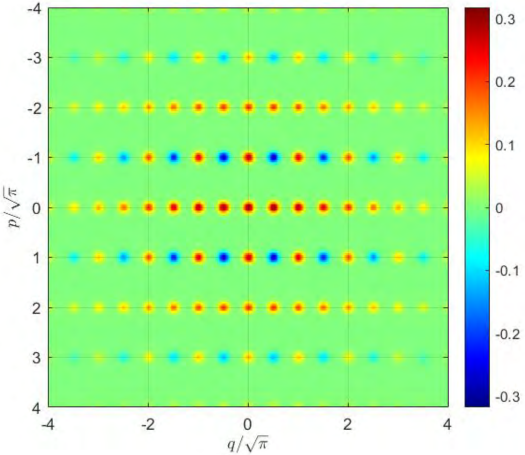
[30] $|0_L\rangle \equiv e^{-\epsilon q^2} \sum_k e^{-\frac{(q-2k\sqrt{\pi})^2}{\epsilon}}$ with $\epsilon = \frac{1}{30}$.

Wigner function of the GKP finite energy grid-state $|1_L\rangle$ [31]



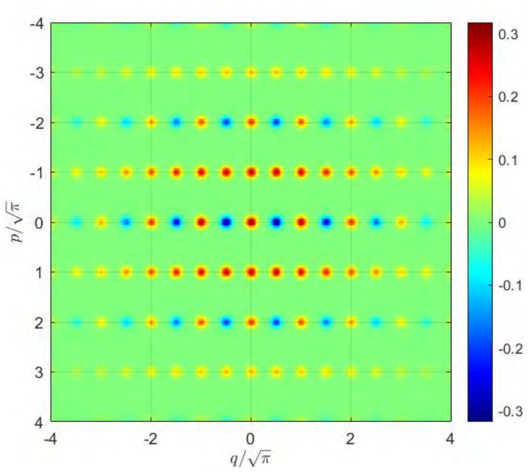
[31] $|1_L\rangle \equiv e^{-\epsilon q^2} \sum_k e^{-\frac{(q-(2k+1)\sqrt{\pi})^2}{\epsilon}}$ with $\epsilon = \frac{1}{30}$.

Wigner function of the GKPfinite energy grid-state $|+_L\rangle$ [32]

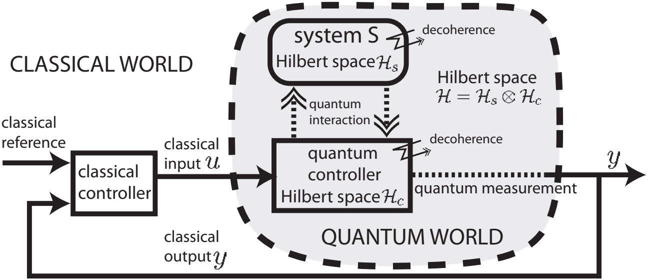


[32] $|+_L\rangle \equiv e^{-\epsilon q^2} \sum_k e^{\frac{(q-k\sqrt{\pi})^2}{\epsilon}} \equiv e^{-\epsilon p^2} \sum_k e^{\frac{(p-2k\sqrt{\pi})^2}{\epsilon}}$.

Wigner function of the GKP finite energy grid-state $|-_L\rangle$ [33]



[33] $|-_L\rangle \equiv e^{-\epsilon q^2} \sum_k (-1)^k e^{-\frac{(q-k\sqrt{\pi})^2}{\epsilon}} \equiv e^{-\epsilon p^2} \sum_k e^{-\frac{(p-(2k+1)\sqrt{\pi})^2}{\epsilon}}$.



To protect quantum information stored in system S (alternative to usual QEC):

- fast stabilization and protection mainly achieved by a **quantum controller** (coherent feedback stabilizing decoherence-free sub-spaces);
- slow decoherence and perturbations mainly tackled by a **classical controller** (measurement-based feedback "finishing the job")

Underlying mathematical methods for high-precision dynamical modeling and control based on stochastic master equations (SME):

- High-order averaging methods and geometric singular perturbations for coherent feedback.
- Stochastic control Lyapunov methods for exponential stabilization via measurement-based feedback.