Euler-Lagrange models with complex currents of three-phase electrical machines and observability issues[1]

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Outline

PM machines: usual models

In the (α, β) frame the dynamic equations read[2]:

$$\begin{cases} \frac{d}{dt}\left(J\dot{\theta}\right) = n_p \Im\left(\left(\bar{\phi} e^{\jmath n_p \theta}\right)^* \imath_s\right) - \tau_L \\ \frac{d}{dt}\left(\lambda \imath_s + \bar{\phi} e^{\jmath n_p \theta}\right) = u_s - R_s \imath_s \end{cases}$$

where

- * stands for complex-conjugation, $\jmath = \sqrt{-1}$ and n_p is the number of pairs of poles.
- θ is the rotor mechanical angle, J and τ_L are the inertia and load torque, respectively.
- $\imath_s = \imath_{s\alpha} + \jmath \imath_{s\beta}$ (resp $u_s = u_{s\alpha} + \jmath u_{s\beta}$) is the stator current (resp. voltage): complex quantities.
- $\lambda = (L_d + L_q)/2$ with inductances $L_d = L_q > 0$ (no saliency here).
- The stator flux is $\phi_s = \lambda \imath_s + \bar{\phi} e^{\jmath n_p \theta}$ with the constant $\bar{\phi} > 0$ representing to the rotor flux due to permanent magnets.

[2] See, e.g., J. Chiasson: Modeling and High Performance Control of Electric Machines, Wiley-IEEE Press, 2005.

PM machines: Euler-Lagrange setting[3]

Lagrangian: sum of kinetic and magnetic Lagrangian $\mathcal{L}_c + \mathcal{L}_m$:

$$\mathcal{L}_c = \frac{J}{2}\dot{\theta}^2, \quad \mathcal{L}_m = \frac{\lambda}{2}\left|\imath_s + \bar{\imath} e^{\jmath n_p \theta}\right|^2$$

where $\bar{\imath} = \bar{\phi}/\lambda > 0$ is the permanent magnetizing current.

Euler-Lagrange setting: with additional variable $q_s \in \mathbb{C}$ defined by $\frac{d}{dt} q_s = \imath_s$, take the Lagrangian $\mathcal{L} = \mathcal{L}_c + \mathcal{L}_m$ as a real function of $q = (\theta, q_{s\alpha}, q_{s\beta})$ and $\dot{q} = (\dot{\theta}, \imath_{s\alpha}, \imath_{s\beta})$:

$$\mathcal{L}(q, \dot{q}) = \frac{J}{2}\dot{\theta}^2 + \frac{\lambda}{2}\left((\imath_{s\alpha} + \bar{\imath}\cos n_p\theta)^2 + (\imath_{s\beta} + \bar{\imath}\sin n_p\theta)^2\right)$$

Then the dynamics (3 real ODE) read:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = -\tau_L$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_{s\alpha}}\right) - \frac{\partial \mathcal{L}}{\partial q_{s\alpha}} = u_{s\alpha} - R_s \imath_{s\alpha}, \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_{s\beta}}\right) - \frac{\partial \mathcal{L}}{\partial q_{s\beta}} = u_{s\beta} - R_s \imath_{s\beta}$$

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[3] See, e.g., R. Ortega, A. Loria, P. J. Nicklasson, and H. Sira-Ramirez. *Passivity–Based Control of Euler–Lagrange Systems*. Communications and Control Engineering. Springer-Verlag, Berlin, 1998.

Euler-Lagrange equation with complex variables[4]

Two generalized coordinates q_1 and q_2 correspond to a point $q = q_1 + \jmath q_2$ in the complex plane ($\jmath = \sqrt{-1}$). The Lagrangian $\mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2)$ is a real function and the Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) - \frac{\partial \mathcal{L}}{\partial q_1} = 0, \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right) - \frac{\partial \mathcal{L}}{\partial q_2} = 0.$$

Using the complex notation q

$$\tilde{\mathcal{L}}(q, q^*, \dot{q}, \dot{q}^*) \equiv \mathcal{L}\left(\frac{q+q^*}{2}, \frac{q-q^*}{2\jmath}, \frac{\dot{q}+\dot{q}^*}{2}, \frac{\dot{q}-\dot{q}^*}{2\jmath}\right).$$

Since $2\frac{\partial \tilde{\mathcal{L}}}{\partial q} = \frac{\partial \mathcal{L}}{\partial q_1} - \jmath \frac{\partial \mathcal{L}}{\partial q_2}$, $2\frac{\partial \tilde{\mathcal{L}}}{\partial q^*} = \frac{\partial \mathcal{L}}{\partial q_1} + \jmath \frac{\partial \mathcal{L}}{\partial q_2}$ we get

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1} + \jmath \frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right) = \frac{\partial \mathcal{L}}{\partial q_1} + \jmath \frac{\partial \mathcal{L}}{\partial q_2} \quad \text{that reads} \quad \frac{d}{dt}\left(2\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{q}^*}\right) - 2\frac{\partial \tilde{\mathcal{L}}}{\partial q^*} = 0.$$

[4] A usual method borrowed from quantum physics: see, e.g., C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg. *Photons and Atoms: Introduction to Quantum Electrodynamics*. Wiley, 1989.

PM machines: Lagrangian with complex stator currents

With $\frac{d}{dt}q_s = \imath_s$ (q_s complex cyclic variables) and the Lagrangian

$$\mathcal{L}(\theta, \dot\theta, \imath_s, \imath_s^*) = \frac{J}{2}\dot\theta^2 + \frac{\lambda}{2}\left(\imath_s + \bar\imath e^{\jmath n_p\theta}\right)\left(\imath_s^* + \bar\imath e^{-\jmath n_p\theta}\right)$$

the usual equations

$$\frac{d}{dt}\left(J\dot\theta\right) = n_p \Im\left(\left(\lambda\bar\imath e^{\jmath n_p\theta}\right)^* \imath_s\right) - \tau_L, \qquad \frac{d}{dt}\left(\lambda(\imath_s + \bar\imath e^{\jmath n_p\theta})\right) = u_s - R_s\imath_s$$

read

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta}\right) = \frac{\partial\mathcal{L}}{\partial\theta} - \tau_L, \quad 2\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\imath_s^*}\right) = u_s - R_s\imath_s$$

since $\frac{\partial\mathcal{L}}{\partial q_s^*} = 0$ and $\frac{\partial\mathcal{L}}{\partial\dot q_s^*} = \frac{\partial\mathcal{L}}{\partial\imath_s^*}$.

PM machines: structure of any dynamical models

More generally, the magnetic Lagrangian $\mathcal{L}_m$ is a real value function of θ, $\imath_s$ and $\imath_s^*$ that is $\frac{2\pi}{n_p}$ periodic versus θ. Thus any Lagrangian $\mathcal{L}_{\text{PM}}$ representing a 3-phases permanent magnet machine admits the following form

$$\mathcal{L}_{\text{PM}} = \frac{J}{2}\dot{\theta}^2 + \mathcal{L}_m(\theta, \imath_s, \imath_s^*)$$

Consequently, any model (with saliency, magnetic-saturation, space-harmonics, ...) of permanent magnet machines admits the following structure (J independent of θ here):

$$\frac{d}{dt}\left(J\dot{\theta}\right) = \frac{\partial \mathcal{L}_m}{\partial \theta} - \tau_L, \quad \frac{d}{dt}\left(2\frac{\partial \mathcal{L}_m}{\partial \imath_s^*}\right) = u_s - R_s \imath_s$$

with $\phi_s = 2\frac{\partial \mathcal{L}_m}{\partial \imath_s^*}$ as stator flux.

PM machines: saliency effects

With a positive magnetic Lagrangian of the form

$$\mathcal{L}_m = \frac{\lambda}{2}\left(\imath_s + \bar{\imath}e^{\jmath n_p\theta}\right)\left(\imath_s^* + \bar{\imath}e^{-\jmath n_p\theta}\right) - \frac{\mu}{4}\left(\left(\imath_s^* e^{\jmath n_p\theta}\right)^2 + \left(\imath_s e^{-\jmath n_p\theta}\right)^2\right)$$

where $\lambda = (L_d + L_q)/2$ and $\mu = (L_q - L_d)/2$ (inductances $L_d > 0$ and $L_q > 0$), we recover the usual model with saliency:

$$\begin{cases} \frac{d}{dt}\left(J\dot{\theta}\right) = n_p \Im\left((\lambda\imath_s^* + \lambda\bar{\imath}e^{-\jmath n_p\theta} - \mu\imath_s e^{-2\jmath n_p\theta})\imath_s\right) - \tau_L \\ \frac{d}{dt}\left(\lambda\imath_s + \lambda\bar{\imath}e^{\jmath n_p\theta} - \mu\imath_s^* e^{2\jmath n_p\theta}\right) = u_s - R_s\imath_s. \end{cases}$$

PM machines: magnetic-saturation and saliency effects

Inductances depend on the currents as, e.g.,

$$\lambda = \lambda(|\imath_s + \bar{\imath}e^{\jmath n_p \theta}|) = \lambda\left(\sqrt{(\imath_s + \bar{\imath}e^{\jmath n_p \theta})(\imath_s^* + \bar{\imath}e^{-\jmath n_p \theta})}\right)$$

where $\imath_s + \bar{\imath}e^{\jmath n_p \theta}$ stands for total magnetizing current. With magnetic Lagrangien

$$\mathcal{L}_m = \frac{\lambda\left(\left|\imath_s + \bar{\imath}e^{\jmath n_p \theta}\right|\right)}{2}\left|\imath_s + \bar{\imath}e^{\jmath n_p \theta}\right|^2 - \frac{\mu}{4}\left(\left(\imath_s^* e^{\jmath n_p \theta}\right)^2 + \left(\imath_s e^{-\jmath n_p \theta}\right)^2\right)$$

the dynamics read ($\Lambda = \lambda + \frac{\left|\imath_s + \bar{\imath}e^{\jmath n_p \theta}\right|}{2}\lambda'$):

$$\begin{cases} \frac{d}{dt}\left(J\dot{\theta}\right) = n_p \Im\left(\left(\Lambda\left(\imath_s^* + \bar{\imath}e^{-\jmath n_p \theta}\right) - \mu \imath_s e^{-2\jmath n_p \theta}\right)\imath_s\right) - \tau_L \\ \frac{d}{dt}\left(\Lambda\left(\imath_s + \bar{\imath}e^{\jmath n_p \theta}\right) - \mu \imath_s^* e^{2\jmath n_p \theta}\right) = u_s - R_s \imath_s \end{cases}$$

Similarly μ could also depend on $|\imath_s + \bar{\imath}e^{\jmath n_p \theta}|$.

Induction machines: usual models

Dynamics with complex stator and rotor currents:

$$\begin{cases} \frac{d}{dt}\left(J\dot{\theta}\right) = n_p \Im\left(L_m \imath_r^* e^{-\jmath n_p \theta} \imath_s\right) - \tau_L \\ \frac{d}{dt}\left(L_r \imath_r + L_m \imath_s e^{-\jmath n_p \theta}\right) = -R_r \imath_r \\ \frac{d}{dt}\left(L_s \imath_s + L_m \imath_r e^{\jmath n_p \theta}\right) = u_s - R_s \imath_s \end{cases}$$

where

- $\imath_r \in \mathbb{C}$ (resp. $\imath_s \in \mathbb{C}$) is the rotor (resp. stator) current; $u_s \in \mathbb{C}$ is the stator voltage
- $R_s > 0$ and $R_r > 0$ are stator and rotor resistances.
- $L_s > 0$, $L_r > 0$ and L_m are the inductances satisfying $L_s L_r > L_m^2$ for physical reasons (positive magnetic Lagrangien). They are constant here.
- the stator (resp. rotor) flux is $\phi_s = L_s \imath_s + L_m \imath_r e^{\jmath n_p \theta}$ (resp. $\phi_r = L_r \imath_r + L_m \imath_s e^{-\jmath n_p \theta}$).

Induction machines: Lagrangian with complex currents

The Lagrangian of the usual model is

$$\mathcal{L}_m = \frac{J}{2}\dot{\theta}^2 + \frac{L_m}{2}\left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|^2 + \frac{L_{fr}}{2}|\imath_r|^2 + \frac{L_{fs}}{2}|\imath_s|^2$$

where $L_s = L_m + L_{fs}$ and $L_r = L_m + L_{fr}$ with $L_m > 0$ and $0 < L_{fr}, L_{fs} \ll L_m$. More generally, a physically consistent model should be obtained with a Lagrangian of the form

$$\mathcal{L}_{\text{IM}} = \frac{J}{2}\dot{\theta}^2 + \mathcal{L}_m\left(\theta, \imath_r, \imath_r^*, \imath_s, \imath_s^*\right)$$

where $\mathcal{L}_m$ is the magnetic Lagrangien expressed with the rotor angle and currents. It is $\frac{2\pi}{n_p}$ periodic versus θ. Any physically admissible model reads (J independent of θ)

$$\frac{d}{dt}\left(J\dot{\theta}\right) = \frac{\partial \mathcal{L}_m}{\partial \theta} - \tau_L, \quad \frac{d}{dt}\phi_r = -R_r \imath_r, \quad \frac{d}{dt}\phi_s = u_s - R_s \imath_s,$$

where the rotor and stator fluxes are given by

$$\phi_r = 2\frac{\partial \mathcal{L}_m}{\partial \imath_r^*}, \qquad \phi_s = 2\frac{\partial \mathcal{L}_m}{\partial \imath_s^*}.$$

Induction machines: magnetic-saturation

With positive inductances of the form

$$L_m = L_m\left(\left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|\right), \; L_s = L_m + L_{fs}, \; L_r = L_m + L_{fr}$$

the magnetic Lagrangien remains positive

$$\mathcal{L}_m = \frac{L_m\left(\left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|\right)}{2}\left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|^2 + \frac{L_{fr}}{2}\imath_r \imath_r^* + \frac{L_{fs}}{2}\imath_s \imath_s^*$$

and the saturation model reads

$$\begin{cases} \frac{d}{dt}\left(J\dot{\theta}\right) = n_p \Im\left(\Lambda_m \imath_r^* e^{-\jmath n_p \theta}\imath_s\right) - \tau_L \\ \frac{d}{dt}\left(\Lambda_m\left(\imath_r + \imath_s e^{-\jmath n_p \theta}\right) + L_{fr}\imath_r\right) = -R_r \imath_r \\ \frac{d}{dt}\left(\Lambda_m\left(\imath_s + \imath_r e^{\jmath n_p \theta}\right) + L_{fs}\imath_s\right) = u_s - R_s \imath_s \end{cases}$$

with $\Lambda_m = L_m + \frac{\left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|}{2} L'_m$ function of $\left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|$.

Induction machines: space-harmonics and magnetic-saturation.

Add contribution of space harmonics to magnetic Lagrangien:

$$\frac{L_m\left(\left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|\right)}{2}\left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|^2 + \frac{L_{fr}}{2}\imath_r \imath_r^* + \frac{L_{fs}}{2}\imath_s \imath_s^* + \frac{L_\nu}{2}\left(\imath_s \imath_r^* e^{-\jmath \sigma_\nu \nu n_p \theta} + \imath_s^* \imath_r e^{\jmath \sigma_\nu \nu n_p \theta}\right)$$

with $L_\nu > 0$ a small parameter ($|L_\nu| \ll L_m$) and $\sigma_\nu = \pm 1$ depending on arithmetic conditions[5]. The dynamical model is changed as follows:

$$\frac{d}{dt}\left(J\dot{\theta}\right) = n_p \Im\left(\left(\Lambda_m e^{-\jmath n_p \theta} + L_\nu \sigma_\nu \nu e^{-\jmath \sigma_\nu \nu n_p \theta}\right)\imath_r^* \imath_s\right) - \tau_L$$

$$\frac{d}{dt}\left(\Lambda_m\left(\imath_r + \imath_s e^{-\jmath n_p \theta}\right) + L_{fr}\imath_r + L_\nu \imath_s e^{-\jmath \sigma_\nu \nu n_p \theta}\right) = -R_r \imath_r$$

$$\frac{d}{dt}\left(\Lambda_m\left(\imath_s + \imath_r e^{\jmath n_p \theta}\right) + L_{fs}\imath_s + L_\nu \imath_r e^{\jmath \sigma_\nu \nu n_p \theta}\right) = u_s - R_s \imath_s$$

[5] See H.R. Fudeh and C.M. Ong: Modeling and analysis of induction machines containing space harmonics. Part-I: modeling and transformation. *IEEE Transactions on Power Apparatus and Systems*, 102:2608–2615, 1983.

Sensorless control: a load torque τ_L constant but unknown, control inputs u_s and measured outputs $\imath_s$ [6]
Physical models including saliency and magnetic saturation associated to Lagrangian $\mathcal{L}_{\text{PM}} = \frac{J}{2}\dot{\theta}^2 + \mathcal{L}_m(\theta, \imath_s, \imath_s^*)$,

$$\frac{d}{dt}\left(J\dot{\theta}\right) = \frac{\partial \mathcal{L}_m}{\partial \theta} - \tau_L, \quad \frac{d}{dt}\left(2\frac{\partial \mathcal{L}_m}{\partial \imath_s^*}\right) = u_s - R_s \imath_s$$

can be always written in state-space form

$$\frac{d}{dt}X = f(X, U), \qquad Y = h(X)$$

where $X = (\tau_L, \theta, \dot{\theta}, \Re(\imath_s), \Im(\imath_s))$ with $U = (\Re(u_s), \Im(u_s))$, $Y = (\Re(\imath_s), \Im(\imath_s))$ and $\frac{d}{dt}\tau_L = 0$.

[6] For a nice exposure see J. Holtz: Sensorless control of induction motor drives. *Proc. of the IEEE*, 90(8):1359–1394, 2002.

Sensorless control around zero stator frequency

A stationary regime at zero stator frequency corresponds then to a steady state $(\bar{X}, \bar{U}, \bar{Y})$ satisfying $f(\bar{X}, \bar{U}) = 0$, $\bar{Y} = h(\bar{X})$. For a PM machines we get

$$\frac{\partial \mathcal{L}_m}{\partial \theta}(\theta, \imath_s, \imath_s^*) - \tau_L = 0, \quad \imath_s = \bar{\imath_s}$$

to recover $(\theta, \imath_s, \tau_L)$ from the stationary values $\bar{u}_s$ and $\bar{\imath_s}$. This implies severe observability difficulties:

- to any constant input and output $\bar{u}_s$ and $\bar{\imath}_s$ satisfying $\bar{u}_s = R_s \bar{\imath}_s$ correspond a one dimensional family of steady states parameterized by the scalar variable ξ with $\tau_L = \frac{\partial \mathcal{L}_m}{\partial \theta}(\xi, \bar{\imath}_s, \bar{\imath}_s^*)$, $\theta = \xi$, $\quad \imath_s = \bar{\imath}_s$.
- the linear tangent systems around such steady-states are not observable;

The situation is similar for induction machines: including space-harmonic and magnetic-saturation does not canceled such lack of observability.

Concluding remarks

- Extensions to network of machines and generators connected via long lines can also be developed with similar variational principles and Euler-Lagrange equations with complex currents and voltages (ODE or PDE).
- Observability issues at zero stator frequency: a strong motivation for theoretical works on the following specific stabilization problem involving an unknown constant parameter p: take $\frac{d}{dt}x = f(x, u, p)$, $y = h(x)$ a nonlinear system where $\{(x, p) \mid f(x, \bar{u}, p) = 0, h(x) = \bar{y}\}$ is a smooth curve; take any $(\bar{x}, \bar{p})$ on this equilibrium curve; under which conditions is it possible to construct (without knowing p) a (dynamic) output feedback stabilizing x around $\bar{x}$ in a robust way.