Invariant Observers for Mechanical systems

Pierre Rouchon
Ecole des Mines de Paris
Centre Automatique et Systèmes
pierre.rouchon@ensmp.fr

Dresden, March 2004

Work with Nasradine Aghannan

Outline :

Lagrangian dynamics $\mathcal{L} = \frac{1}{2} g_{ij}(q)\dot{q}^i\dot{q}^j - U(q)$ with position measures $y = q$. Asymptotic estimation of $\dot{q} = v$, independent of the coordinates chosen on the configuration space q.

1. The Euclidian case: $\ddot{q} = -\mathrm{grad}_q U$.

2. The non Euclidian case: $\nabla_{\dot{q}}\dot{q} = -\mathrm{grad}_q U$.

3. Observer convergence : contraction tools.

The Euclidian case

Lagrangian: $\mathcal{L} = \frac{1}{2}\dot{q}^2 - U(q)$ where q^i are Euclidian coordinates:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right) = \frac{\partial \mathcal{L}}{\partial q^i}, \quad \text{i.e.} \quad \ddot{q}^{\,i} = -\frac{\partial U}{\partial q^i}.$$

Nonlinear observer via input injection:

$$\dot{\widehat{q}}^i = \widehat{v}^i - \alpha(\widehat{q}^i - q^i), \quad \dot{\widehat{v}}^i = -\frac{\partial U}{\partial q^i}(q) - \beta(\widehat{q}^i - q^i).$$

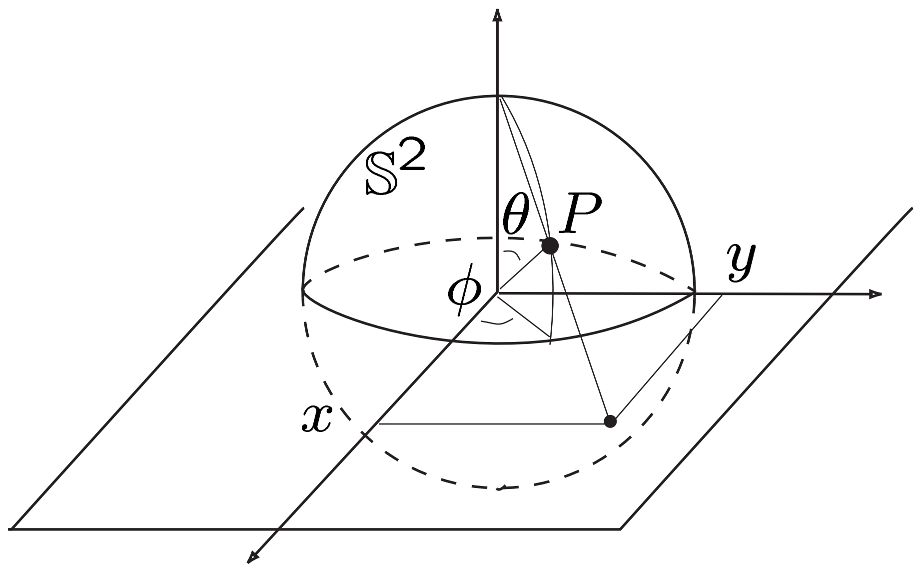
Error dynamics, $\widetilde{q}^i = \widehat{q}^i - q^i$, $\widetilde{v}^i = \widehat{v}^i - v^i$ (stable for $\alpha, \beta > 0$):

$$\dot{\widetilde{q}}^i = \widetilde{v}^i - \alpha \widetilde{q}^i, \quad \dot{\widetilde{v}}^i = -\beta \widetilde{q}^i.$$

What is going on when the q^i's are not Euclidian coordinates?
The same system but in another frame $q = \phi(Q)$:

$$\mathcal{L} = \frac{1}{2} g_{ij}(Q)\dot{Q}^i\dot{Q}^j - U(\phi(Q)) \quad \text{with} \quad (g_{ij}) = D\phi^T D\phi.$$

Configuration space and local coordinates.

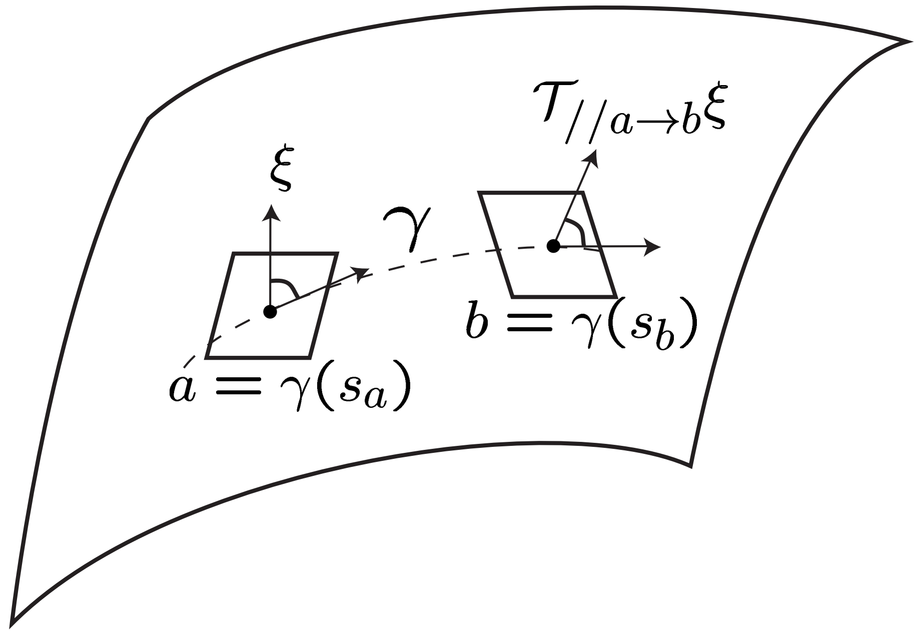


The goal is to have a design method that is independent of the coordinates chosen on the configuration space. This is not the case if we use classical design with observers of the form

$$\dot{\hat{x}} = f(\hat{x}) + k(h(\hat{x}) - y)$$

for nonlinear system $\dot{x} = f(x)$, $y = h(x)$.

Intrinsic interpretation of the dynamics



The positive definite matrix $(g_{ij}(q))$ define a scalar product on the tangent space at q to the configuration manifold (Riemannian manifold): we can measure distances and transport vectors along geodesics.

Intrinsic formulation

$$\dot{\widehat{q}}^i = \widehat{v}^i - \alpha(\widehat{q}^i - q^i), \quad \dot{\widehat{v}}^i = -\frac{\partial U}{\partial q^i}(q) - \beta(\widehat{q}^i - q^i).$$

The components $\widehat{q}^i - q^i$ are related to the gradient of the geodesic distance between the point q and $\widehat{q}$:

$$\widehat{q} - q = \text{grad}_{\widehat{q}} F(q, \widehat{q})$$

where F is the half square of the geodesic distance between q and $\widehat{q}$.

The injection term can be done via parallel transport: $\text{grad}_q U(q)$ is a tangent vector at q. To have a tangent vector at $\widehat{q}$, we take $\mathcal{T}_{//q \to \widehat{q}}(\text{grad}_q U(q))$.

It remains the term $\dot{\widehat{v}}^i$ that corresponds in fact the covariant derivative of $\widehat{v}$ along the curve followed by $\widehat{q}$: $\nabla_{\dot{\widehat{q}}} \widehat{v}$ when you gather $\dot{\widehat{v}}$ with "gyroscopic like terms".

Intrinsic formulation

$\dot{\widehat{q}} = \widehat{v} - \alpha \mathrm{grad}_{\widehat{q}} F(q,\widehat{q}), \quad \nabla_{\dot{\widehat{q}}} \widehat{v} = -\mathcal{T}_{//q\to\widehat{q}}(\mathrm{grad}_q U(q)) - \beta \mathrm{grad}_{\widehat{q}} F(q,\widehat{q}).$

For the metric g_{ij} we have in local coordinates:

$$\{\nabla_{\dot{\widehat{q}}} \widehat{v}\}^i = \dot{\widehat{v}}^i + \Gamma^i_{jk}(\widehat{q}) \widehat{v}^j \dot{\widehat{q}}^k, \quad \mathrm{grad}_q U(q) = g^{ij} \partial_{q^j} U, \quad \ldots.$$

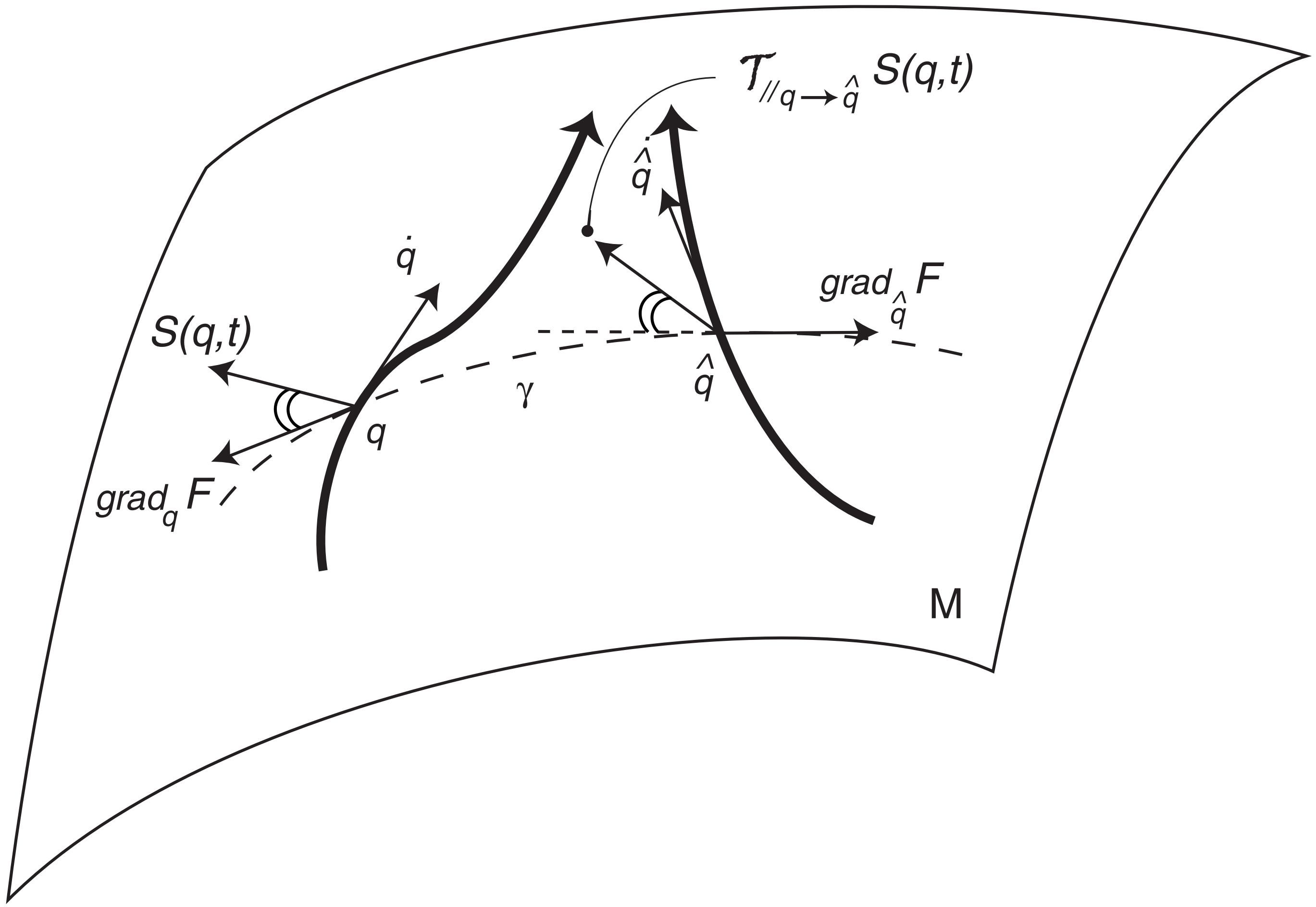
and the parallel transport along the geodesic joining q to $\widehat{q}$ is defined by solving a linear differential equation along this geodesic.

The Christoffel symbols Γ^i_{jk} are given by

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lk}}{\partial q^j} + \frac{\partial g_{jl}}{\partial q^k} - \frac{\partial g_{jk}}{\partial q^l} \right)$$

where g^{il} are the entries of $(g_{ij})^{-1}$.

Invariant observer representation



A first order approximation

The intrinsic formulation

$$\dot{\widehat{q}} = \widehat{v} - \alpha \text{grad}_{\widehat{q}} F(q, \widehat{q}), \quad \nabla_{\dot{\widehat{q}}} \widehat{v} = -\mathcal{T}_{//q \rightarrow \widehat{q}}(\text{grad}_q U(q)) - \beta \text{grad}_{\widehat{q}} F(q, \widehat{q}).$$

is not very useful in practice. But when $\widehat{q}$ is close to q we have the following explicit approximation

$$\begin{aligned} \dot{\widehat{q}}^i &= \widehat{v}^i - \alpha(\widehat{q}^i - q^i) \\ \dot{\widehat{v}}^i &= -\Gamma^i_{jk}(\widehat{q}) \widehat{v}^j \dot{\widehat{q}}^k - \partial_{q^i} U(q) - \Gamma^i_{jl}(q)(\partial_{q^j} U(q))(\widehat{q}^l - q^l) - \beta(\widehat{q}^i - q^i). \end{aligned}$$

We know that when the metric is Euclidian, i.e., when exist local coordinates such that $g_{ij} = \delta_{ij}$, such observer is asymptotically stable around any trajectories $(q, \dot{q})$.

Summarize of the Euclidian case ($\dim q = 1$ and $U = 0$)

$\mathcal{L} = \frac{1}{2}\dot{q}^2$	$\mathcal{L} = \frac{1}{2}g(q)\dot{q}^2$
$\begin{cases} \dot{q} = v \\ \dot{v} = 0 \end{cases}$	$\begin{cases} \dot{q} = v \\ \nabla_{\dot{q}} v = 0 \end{cases}$
$\begin{cases} \dot{\widehat{q}} = \widehat{v} - \alpha(\widehat{q} - q) \\ \dot{\widehat{v}} = 0 - \beta(\widehat{q} - q) \end{cases}$	$\begin{cases} \dot{\widehat{q}} = \widehat{v} - \alpha \mathrm{grad}_{\widehat{q}} F(q, \widehat{q}) \\ \nabla_{\dot{\widehat{q}}} \widehat{v} = 0 - \beta \mathrm{grad}_{\widehat{q}} F(q, \widehat{q}) \end{cases}$
$F(q, \widehat{q}) = \frac{1}{2}(\widehat{q} - q)^2$	$F(q, \widehat{q}) = \frac{1}{2} d_G(q, \widehat{q})^2$
$\mathrm{grad}_{\widehat{q}} F(q, \widehat{q}) = \widehat{q} - q$	

An example

q-coordinate

$$\begin{cases} \dot{\widehat{q}} = \widehat{v} - \alpha(\widehat{q} - q) \\ \dot{\widehat{v}} = 0 - \beta(\widehat{q} - q) \end{cases}$$

$$\begin{cases} \dot{\widehat{q}} = \widehat{v} - \alpha \mathsf{grad}_{\widehat{q}} F(q, \widehat{q}) \\ \nabla_{\dot{\widehat{q}}} \widehat{v} = 0 - \beta \mathsf{grad}_{\widehat{q}} F(q, \widehat{q}) \end{cases}$$

r-coordinate : $r = \exp(q)$

$$\mathcal{L}(r, \dot{r}) = \frac{1}{2}\frac{\dot{r}^2}{r^2}$$

$$\begin{cases} \dot{r} = w \\ \dot{w} = \dfrac{w^2}{r} \end{cases}$$

$$\begin{cases} \dot{\widehat{r}} = \widehat{w} - \alpha\ \widehat{r}(\ln \widehat{r} - \ln r) \\ \dot{\widehat{w}} = \dfrac{\widehat{w}\dot{\widehat{r}}}{\widehat{r}} - \beta\ \widehat{r}(\ln \widehat{r} - \ln r) \end{cases}$$

Convergence ?

When the metric (g_{ij}) is Euclidian (flat space), we know that

$$\dot{\widehat{q}} = \widehat{v} - \alpha \text{grad}_{\widehat{q}} F(q, \widehat{q}), \quad \nabla_{\dot{\widehat{q}}} \widehat{v} = -\mathcal{T}_{//q \to \widehat{q}}(\text{grad}_q U(q)) - \beta \text{grad}_{\widehat{q}} F(q, \widehat{q}).$$

is convergent as soon as the gains $\alpha, \beta > 0$. It is not the case when the metric is not flat, i.e., when the Riemann curvature tensor R (order 4) is not identically zero (Gauss theorem): for any tangent vector ξ, ζ at q, $R(\xi, \zeta)$ is a linear application on the tangent space at q. In local coordinates, we have

$$\{R(\xi, \zeta)\eta\}^i = R^i_{jkl} \xi^k \zeta^l \eta^j$$

where R^i_{jkl} are the components of the curvature tensor:

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial q^l} - \frac{\partial \Gamma^i_{jl}}{\partial q^k} + \Gamma^i_{pl} \Gamma^p_{jk} - \Gamma^i_{pk} \Gamma^p_{jl}.$$

Jacobi equation

Take a geodesic dynamics (no potential): $\nabla_{\dot q}\dot q = 0$. Denoted by ξ the first variation of geodesic (ξ corresponds to δq): it obeys the Jacobi equation

$$\frac{D^2\xi}{Dt^2} = -R(\dot q, \xi)\dot q$$

where the operator $D/Dt = \nabla_{\dot q}$ corresponds to the covariant derivation along $t \mapsto q(t)$. Moreover $\xi \mapsto R(\dot q, \xi)\dot q$ is a symmetric operator. Thus we can write formally

$$R(\dot q, \xi)\dot q = \mathrm{grad}_\xi W(\xi), \quad \text{with} \quad W(\xi) = \langle R(\dot q, \xi)\dot q, \xi\rangle /2.$$

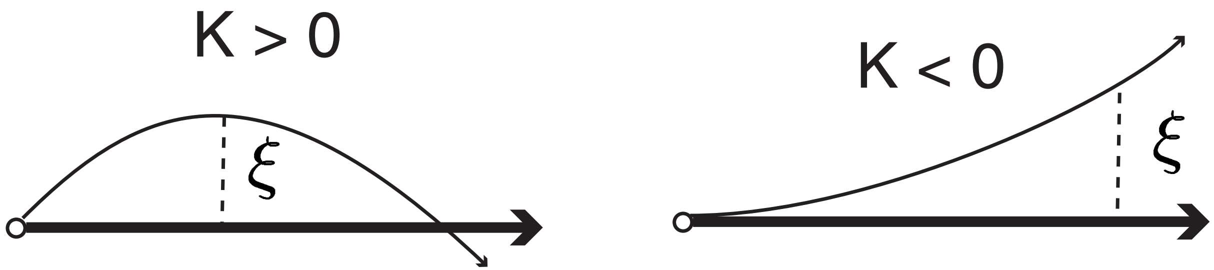
Formally the quadratic form W is positive (positive curvature) ξ oscillates and when it admits a negative part, ξ diverges exponentially (the geodesic flow is unstable).

Jacobi equation (end)

Formally, the Jacobi equation

$$\frac{D^2\xi}{Dt^2} = -\text{grad}_{\xi} W(\xi)$$

is stable when the quadratic form, the potential W is positive (positive sectional curvature K) and ξ oscillates. When the potential W admits a negative part, ξ diverges exponentially (the geodesic flow is unstable).



The convergent observer in the non Euclidian case

The locally convergent observer of the mechanical system

$$\dot{q} = v$$
$$\nabla_{\dot{q}} v = S(q,t)$$

is then

$$\dot{\widehat{q}} = \widehat{v} - \alpha \ \mathsf{grad}_{\widehat{q}} F(\widehat{q}, q)$$
$$\nabla_{\dot{\widehat{q}}} \widehat{v} = \mathcal{T}_{//q \to \widehat{q}} S(q,t) - \beta \ \mathsf{grad}_{\widehat{q}} F(\widehat{q}, q) + R(\widehat{v}, \mathsf{grad}_{\widehat{q}} F(\widehat{q}, q))\widehat{v}$$

where we have added a curvature term to compensate the effect of a non Euclidian metric.

The approximate observer in the non Euclidian case

Since

$$\{\text{grad}_{\widehat{q}} F\}^i = \widehat{q}^i - q^i + O(\|\widehat{q} - q\|^2)$$
$$\{\mathcal{T}_{//q \to \widehat{q}} w\}^i = w^i - \Gamma^i_{jl}(q) w^j (\widehat{q}^l - q^l) + O(\|\widehat{q} - q\|^2)$$

we have the following approximate observer

$$\begin{aligned}
\dot{\widehat{q}}^i &= \widehat{v}^i - \alpha(\widehat{q}^i - q^i) \\
\dot{\widehat{v}}^i &= -\Gamma^i_{jk}(\widehat{q}) \widehat{v}^j \dot{\widehat{q}}^k + S^i(q,t) - \Gamma^i_{jl}(q) S^j(q,t)(\widehat{q}^l - q^l) - \beta(\widehat{q}^i - q^i) \\
&\quad + R^i_{jkl}(q) \widehat{v}^k (\widehat{q}^l - q^l) \widehat{v}^j .
\end{aligned}$$

of the mechanical system

$$\begin{aligned}
\dot{q}^i &= v^i \\
\dot{v}^i &= -\Gamma^i_{jk}(q) v^j v^k + S^i(q,t)
\end{aligned}$$

First order variation.

The linearized dynamics around $\widehat{q}$: as for the Jacobi equation, use $D/Dt = \nabla_{\dot{\widehat{q}}}$ instead of d/dt. Denote by $\xi = \delta\widehat{q}$ and ζ the covariant variation of the estimated velocity. Then tedious computations in local coordinates gives, when written in intrinsic manner:

$$\begin{aligned}
\nabla_{\dot{\widehat{q}}}\xi &= \zeta - \alpha\nabla_{\xi}\mathsf{grad}_{\widehat{q}}F(\widehat{q},q) \\
\nabla_{\dot{\widehat{q}}}\zeta &= -R(\dot{\widehat{q}},\xi)\widehat{v} + \nabla_{\xi}\left(\mathcal{T}_{//q\to\widehat{q}}S(q,t)\right) - \beta\nabla_{\xi}\mathsf{grad}_{\widehat{q}}F(\widehat{q},q) \\
&\quad (\nabla_{\xi}R)(\widehat{v},\mathsf{grad}_{\widehat{q}}F(\widehat{q},q))\widehat{v} + 2R(\zeta,\mathsf{grad}_{\widehat{q}}F(\widehat{q},q))\widehat{v} \\
&\quad + R(\widehat{v},\nabla_{\xi}\mathsf{grad}_{\widehat{q}}F(\widehat{q},q))\widehat{v}
\end{aligned}$$

which gives when $\widehat{q} = q$

$$\frac{D\xi}{Dt} = \zeta - \alpha\ \xi, \qquad \frac{D\zeta}{Dt} = -\beta\xi$$

Convergence analysis: local contraction for a good metric on the tangent bundle.

$$\frac{D\xi}{Dt} = \zeta - \alpha\ \xi, \qquad \frac{D\zeta}{Dt} = -\beta\xi$$

For $\alpha, \beta > 0$, $A = \begin{pmatrix} -\alpha & 1 \\ -\beta & 0 \end{pmatrix}$ is Hurwitz. There exists a positive definite quadratic form $Q = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ such that $A^t Q + QA = -I$. Equipped the tangent bundle with the following metric

$$\frac{a}{2}\langle \xi, \xi \rangle + c \langle \xi, \zeta \rangle + \frac{b}{2} \langle \zeta, \zeta \rangle .$$

Convergence analysis: the metric on the tangent bundle.

$$\frac{a}{2}\langle \xi, \xi \rangle + c\,\langle \xi, \zeta \rangle + \frac{b}{2}\langle \zeta, \zeta \rangle\,.$$

In local coordinates (q^i, v^i), the length of the small vector $(\delta q^i, \delta v^i)$ tangent to (q, v) is

$$V\Big(\delta q\ , (\delta v^i + \Gamma^i_{kl}(q) v^k \delta q^l)_{i=1\dots n}\Big) = \frac{a}{2}\ g_{ij}\ \delta q^i\ \delta q^j$$
$$+ c\ g_{ij}\ (\delta v^i + \Gamma^i_{kl}(q) v^k \delta q^l)\ \delta q^j$$
$$+ \frac{b}{2}\ g_{ij}\ (\delta v^i + \Gamma^i_{kl}(q) v^k \delta q^l)\ (\delta v^j + \Gamma^j_{kl}(q) v^k \delta q^l)$$

This defines a Riemannian structure on the tangent bundle. In the local coordinates (q^i, v^i), the metric is a $2n \times 2n$ matrix with entries function of q and v.

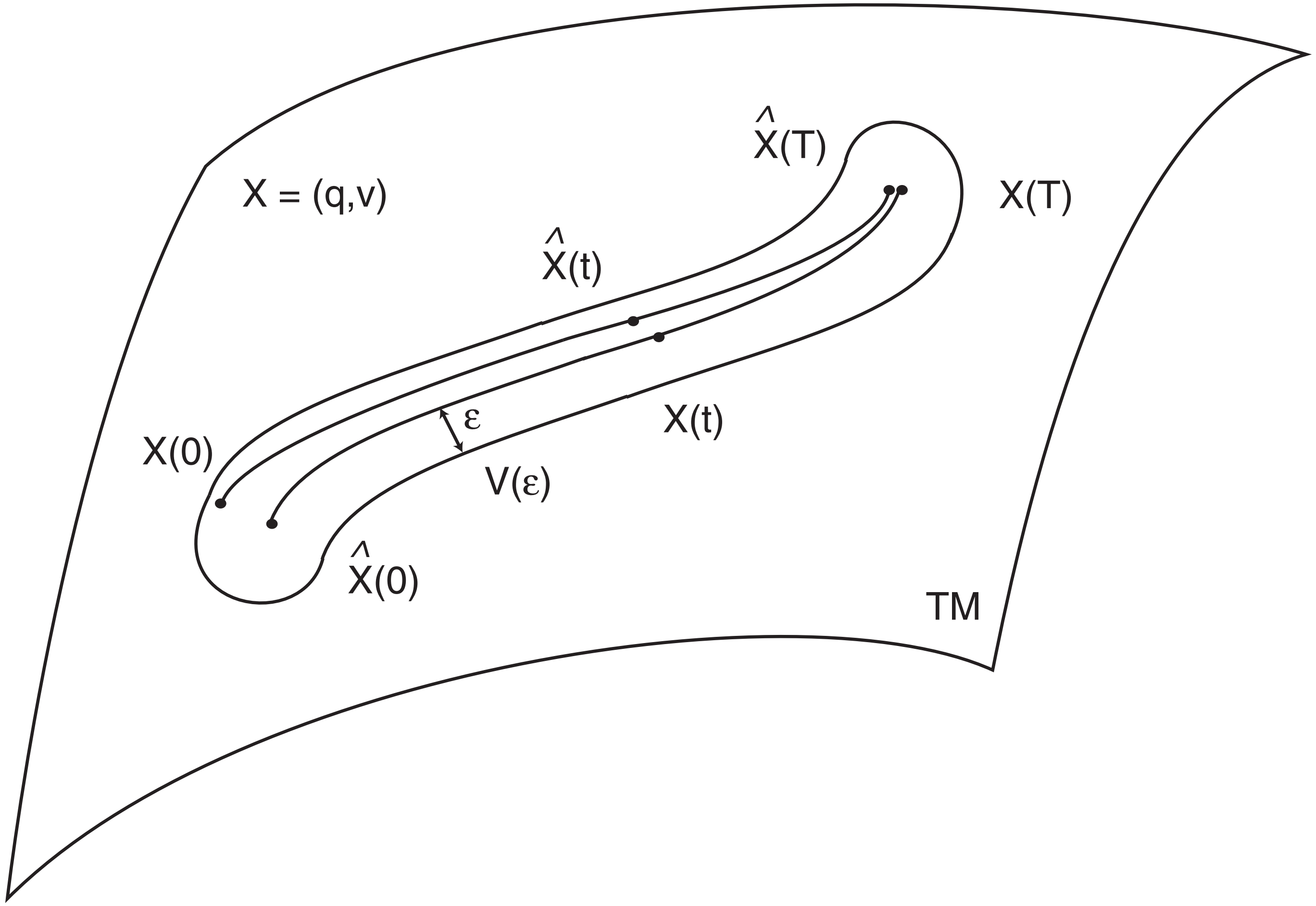
Convergence analysis: local contraction around q.

Set $X = (q, v)$. Denote by $G(X)$ the matrix defining the metric and by $\dot{\hat{X}} = \Upsilon(X, \hat{X})$ the observer.

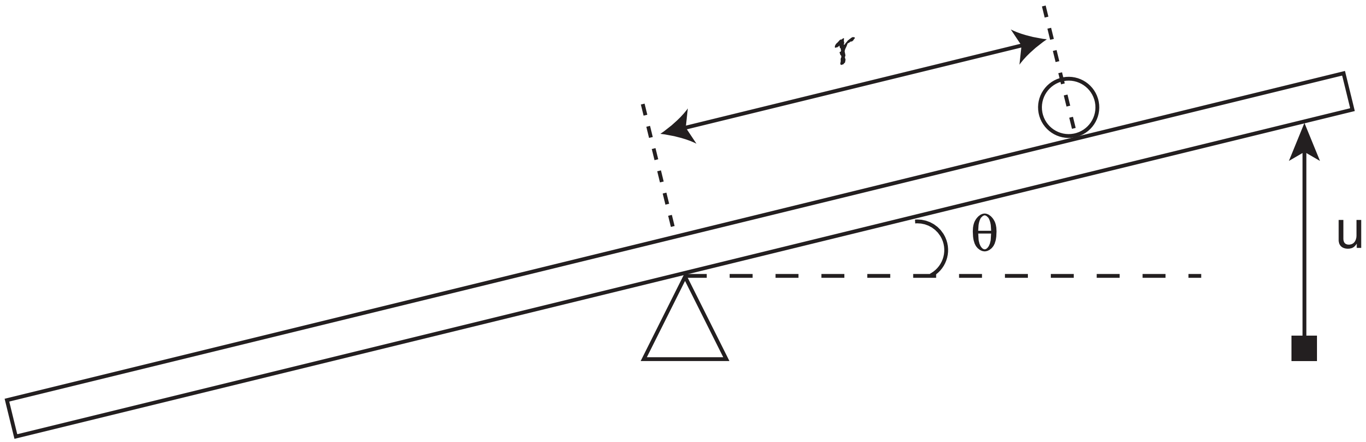
By construction $\dot{X} = \Upsilon(X, X)$ corresponds to the true dynamics. The above developments prove in fact that, for $\hat{X} = X$, we have the following matrix inequality

$$\frac{\partial G}{\partial X}\Big|_{\hat{X}} \Upsilon(X, \hat{X}) + \left(\frac{\partial \Upsilon}{\partial \hat{X}}\Big|_{(X,\hat{X})}\right)^T G(\hat{X}) + G(\hat{X}) \left(\frac{\partial \Upsilon}{\partial \hat{X}}\Big|_{(X,\hat{X})}\right) \le -\lambda G(\hat{X}).$$

proving that the observer dynamics is a contraction when the estimated position $\widehat{q}$ is close to the real one q.


X = (q,v)
$\hat{X}$(T)
X(T)
$\hat{X}$(t)
X(0)
ε
X(t)
V(ε)
$\hat{X}$(0)
TM

The Ball and Beam system: equations



$$\dot{r} = v_r$$
$$\dot{\theta} = v_\theta$$
$$\dot{v}_r = r{v_\theta}^2 - \sin\theta$$
$$\dot{v}_\theta = \frac{-2r}{1+r^2} v_r v_\theta - \frac{r}{1+r^2}\cos\theta + \frac{u}{1+r^2}$$

The approximated invariant observer

$$\dot{\hat{r}} = \hat{v}_r - \alpha(\hat{r} - r)$$
$$\dot{\theta} = \hat{v}_\theta - \alpha(\hat{\theta} - \theta)$$

$$\dot{\hat{v}}_r = \hat{r}\dot{\hat{\theta}}\hat{v}_\theta - \left(\sin\theta + \hat{r}(\hat{r} - r)\frac{r\cos\theta - u}{1 + r^2}\right)$$
$$- \beta(\hat{r} - r) + \left(\frac{1}{1 + \hat{r}^2}\hat{v}_r\hat{v}_\theta(\hat{\theta} - \theta) + \frac{-1}{1 + \hat{r}^2}\hat{v}_\theta^2(\hat{r} - r)\right)$$

$$\dot{\hat{v}}_\theta = \frac{-\hat{r}}{1 + \hat{r}^2}(\dot{\hat{r}}\hat{v}_\theta + \hat{v}_r\dot{\hat{\theta}}) - \left(\frac{r\cos\theta}{1 + r^2} - \frac{\hat{r}}{1 + \hat{r}^2}\left((\hat{r} - r)\frac{r\cos\theta - u}{1 + r^2} + (\hat{\theta} - \theta)\sin\theta\right)\right)$$
$$- \beta(\hat{\theta} - \theta) + \left(\frac{1}{(1 + \hat{r}^2)^2}\hat{v}_r^2(\hat{\theta} - \theta) + \frac{-1}{(1 + \hat{r}^2)^2}\hat{v}_r\hat{v}_\theta(\hat{r} - r)\right)$$

Perfect incompressible fluid

The configuration space M is the Lie group of volume preserving diffeomorphisms on Ω, a bounded connected domain of $\mathbb{R}^3$ (J.J.Moreau, V. Arnol'd, ...).

$\mathcal{U} = T_{I_d}M$ is the Lie algebra of vector fields in Ω of zero divergence and tangent to the boundary $\partial\Omega$.

M is The scalar product on $\mathcal{U}$ is derived from the kinetic energy,

$$< \vec{v}, \vec{\xi} > = \iiint_\Omega \vec{v}(x) \cdot \vec{\xi}(x) dx$$

and is invariant through the right translations ($g \in M$):

$$R_g : \ h \in M \to h \circ g \in M$$

The covariant derivation is

$$\nabla_{\vec{v}}\vec{\xi} = \frac{\partial\vec{\xi}}{\partial t} + (\vec{v}\cdot\nabla)\vec{\xi} + \nabla\eta$$

with $\vec{v}(t,\centerdot)$ and $\vec{\xi}(t,\centerdot)$ in $\mathcal{U}$. The gradient field $\nabla\eta$ is completely defined by the fact that $\nabla_{\vec{v}}\vec{\xi}$ must belong to $\mathcal{U}$ (it is solution of a Laplace equation in Ω with Neuman conditions on $\partial\Omega$).

If $\vec{v}(t,\centerdot) \in \mathcal{U}$ is solution of the Euler equation, i.e., $\nabla_{\vec{v}}\vec{v} = 0$, the curve $t \longrightarrow \phi_t^{\vec{v}}$ is a geodesic on M where $\phi_t^{\vec{v}}$ is the flow of the vector field $\vec{v}$.

The large nabla "∇" is used for the covariant derivation on M and the small nabla "∇" for the gradient operator in the 3-D Euclidian space $\mathbb{R}^3$.

Here the role of q is played by the flow ϕ, the role of v by the vector field $\vec{v}$. The analogue of the first order approximation of the invariant observer reads:

$$\frac{\partial \widehat{\phi}}{\partial t}(t,x) = \widehat{\vec{v}}(t, \widehat{\phi}(t,x)) - \alpha \vec{e}(t, \widehat{\phi}(t,x))$$

$$\frac{\partial \widehat{\vec{v}}}{\partial t} = -\nabla \eta - \Big((\widehat{\vec{v}} - \alpha \vec{e}) \cdot \nabla\Big) \widehat{\vec{v}} - \beta \vec{e} + (\vec{e} \cdot \nabla) \nabla \widehat{p} - (\widehat{\vec{v}} \cdot \nabla) \nabla \widehat{\eta}$$

where

- $\vec{e} \in \mathcal{U}$ corresponds to the position errors $\widehat{q} - q$, i.e., $\vec{e}(t, \phi(t,x)) \approx \widehat{\phi}(t,x) - \phi(t,x)$. The gradient field $\nabla \eta$ ensures $\frac{\partial \widehat{\vec{v}}}{\partial t} \in \mathcal{U}$.

- the term $(\vec{e} \cdot \nabla)\nabla \widehat{p} - (\widehat{\vec{v}} \cdot \nabla)\nabla \widehat{\eta}$ corresponds to the curvature term $R(\widehat{v}, \widehat{q} - q)\widehat{v}$ (PR 1992); the gradient field $\nabla \widehat{p}$ is such that $\nabla \widehat{p} + (\widehat{\vec{v}} \cdot \nabla)\widehat{\vec{v}} \in \mathcal{U}$ and $\nabla \widehat{\eta}$ such that $\nabla \widehat{\eta} + (\widehat{\vec{v}} \cdot \nabla)\vec{e} \in \mathcal{U}$.

Conclusion

- Observer design locally convergent and independent of the coordinates used on the configuration space. Practically, the gain scheduling is automatically done via geometric object such a the Christoffel symbol and the curvature tensor

- Possible extension to other nonlinear system via an adapted notion of error.