Stabilization of discrete-time quantum systems subject to non-demolition measurements with imperfections and delays

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Delayed Complex Systems
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Outline

Feedback and quantum systems

Measurement-based feedback of photons

- The LKB experiment
- The ideal Markov model
- Open-loop convergence
- Closed-loop experimental data

Observer/controller design for the photon box

- A strict control Lyapunov control
- Quantum filter, separation principle and delay

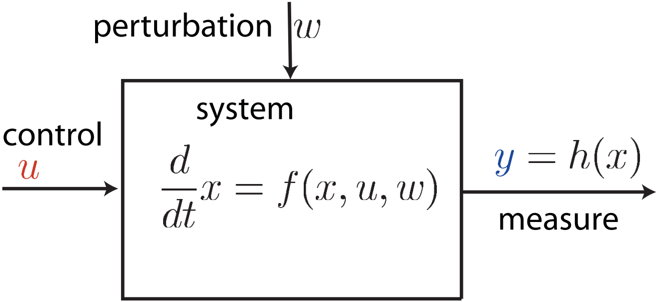
Conclusion: measurement-based versus coherent feedbacks

Generalization to any discrete-time QND systems [1]

- Controlled QND Markov chains
- Open-loop convergence
- Feedback, delay and closed-loop convergence
- Imperfect measurements

[1] H. Amini et al. Preprint arxiv:1201.1387, 2012.

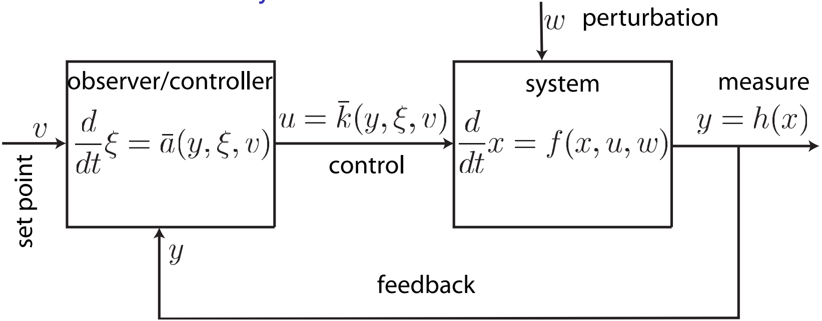
Model of classical systems



For the harmonic oscillator of pulsation ω with measured position y, controlled by the force u and subject to an additional unknown force w.

$$x = (x_1, x_2) \in \mathbb{R}^2, \qquad y = x_1$$
$$\tfrac{d}{dt}x_1 = x_2, \quad \tfrac{d}{dt}x_2 = -\omega^2 x_1 + u + w$$

Feedback for classical systems



Proportional Integral Derivative (PID) for $\frac{d^2}{dt^2}y = -\omega^2 y + u + w$ with the set point $v = y_{sp}$

$$u = -K_p(y - y_{sp}) - K_d \frac{d}{dt}(y - y_{sp}) - K_{int} \int (y - y_{sp})$$

with the positive gains (K_p, K_d, K_{int}) tuned as follows $(0 < \Omega_0 \sim \omega$, $0 < \xi \sim 1$, $0 < \epsilon \ll 1$:

$$K_p = \Omega_0^2, \quad K_d = 2\xi\Omega_0, \quad , K_{int} = \epsilon\Omega_0^3.$$

Feedback for the quantum system $\mathcal{S}$

Key issue: back-action due to the measurement process.

Measurement-based feedback: measurement back-action on $\mathcal{S}$ is stochastic (collapse of the wave-packet); controller is classical; the control input u is a classical variable appearing in some controlled Schrödinger equation; u depends on the past measures.

Coherent feedback: the system $\mathcal{S}$ is coupled to another quantum system (the controller); the composite system, $\mathcal{S} \otimes$ controller, is an open-quantum system relaxing to some target (separable) state (related to reservoir engineering).

This talk is devoted to the first experimental realization of a **measurement-based state feedback**. It has been done at Laboratoire Kastler Brossel of Ecole Normale Supérieure by the Cavity Quantum ElectroDynamics (CQED) group of Serge Haroche.[2]

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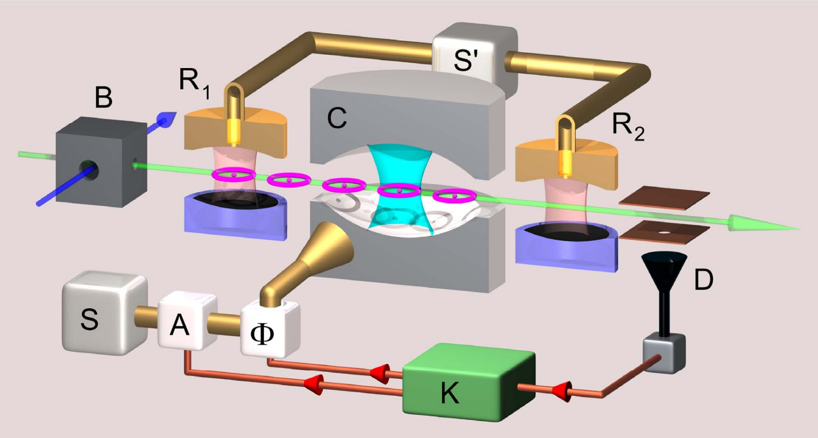
[2] C. Sayrin et al.: Real-time quantum feedback prepares and stabilizes photon number states. Nature, 477:73–77, 2011.

The closed-loop CQED experiment

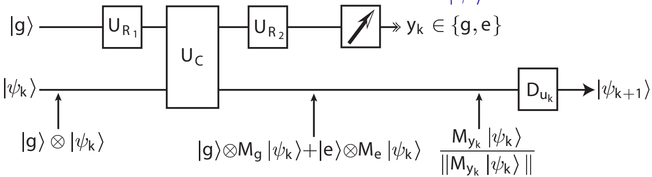
[3]



- Control input $u = Ae^{\imath\Phi}$; measure output $y \in \{g, e\}$.
- Sampling time 80 μs long enough for numerical computations.

[3] Courtesy of Igor Dotsenko

The ideal Markov chain for the wave function $|\psi\rangle$ [4]



Input u_k, state $|\psi_k\rangle = \sum_{n\geq 0} \psi_k^n |n\rangle$, output y_k:

$$\left|\psi_{k+1/2}\right\rangle = \frac{M_{y_k}|\psi_k\rangle}{\|M_{y_k}|\psi_k\rangle\|}, \quad |\psi_{k+1}\rangle = D_{u_k}\left|\psi_{k+1/2}\right\rangle \quad \text{with}$$

- $y_k = g$ (resp. e) with probability $\|M_g|\psi_k\rangle\|^2$ (resp. $\|M_e|\psi_k\rangle\|^2$);
- measurement Kraus operators $M_g = \cos\left(\frac{\phi_0 \mathbf{N} + \phi_R}{2}\right)$ and $M_e = \sin\left(\frac{\phi_0 \mathbf{N} + \phi_R}{2}\right)$: $M_g^\dagger M_g + M_e^\dagger M_e = \mathbf{1}$ with $\mathbf{N} = \mathbf{a}^\dagger \mathbf{a} = \text{diag}(0, 1, 2, \ldots)$ the photon number operator;
- displacement unitary operator ($u \in \mathbb{R}$): $D_u = e^{u\mathbf{a}^\dagger - u\mathbf{a}}$ with $\mathbf{a} = \text{upper diag}(\sqrt{1}, \sqrt{2}, \ldots)$ the photon annihilation operator.

[4] S. Haroche and J.M. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts, 2006.

The ideal Markov chain for the density operator $\rho = |\psi\rangle\langle\psi|$

Diagonal elements of ρ, $\rho^{nn} = \langle n|\rho|n\rangle = |\psi^n|^2$, form the photon number distribution.

$$\rho_{k+1} = \begin{cases} \dfrac{D_{u_k}\ M_g \rho_k M_g^\dagger D_{u_k}^\dagger}{\mathrm{Tr}\left(M_g \rho_k M_g^\dagger\right)} & y_k = g \text{ with probability } p_{g,k} = \mathrm{Tr}\left(M_g \rho_k M_g^\dagger\right) \\ \dfrac{D_{u_k}\ M_e \rho_k M_e^\dagger D_{u_k}^\dagger}{\mathrm{Tr}\left(M_e \rho_k M_e^\dagger\right)} & y_k = e \text{ with probability } p_{e,k} = \mathrm{Tr}\left(M_e \rho_k M_e^\dagger\right) \end{cases}$$

- Displacement unitary operator ($u \in \mathbb{R}$): $D_u = e^{u\mathbf{a}^\dagger - u\mathbf{a}}$ with $\mathbf{a} = \text{upper diag}(\sqrt{1}, \sqrt{2}, \ldots)$ the photon annihilation operator.
- Measurement Kraus operators $M_g = \cos\left(\frac{\phi_0 \mathbf{N} + \phi_R}{2}\right)$ and $M_e = \sin\left(\frac{\phi_0 \mathbf{N} + \phi_R}{2}\right)$: $M_g^\dagger M_g + M_e^\dagger M_e = \mathbf{1}$ with $\mathbf{N} = \mathbf{a}^\dagger \mathbf{a} = \text{diag}(0, 1, 2, \ldots)$ the photon number operator.

Open-loop behavior ($u = 0$)

An experimental open-loop trajectory starting from coherent state $\rho_0 = |\psi_0\rangle\langle\psi_0|$ with $\bar{n} = 3$ photons: $|\psi_0\rangle = e^{-\bar{n}/2} \sum_{n \geq 0} \sqrt{\frac{\bar{n}^n}{n!}} |n\rangle$.

- A fast convergence towards $|n\rangle\langle n|$ for some n,
- followed by a slow relaxation towards vacuum $|0\rangle\langle 0|$: decoherence due to finite photon life time around 70 ms (not included into the ideal model).

Open-loop stability of $\rho_{k+1} = \frac{M_{y_k} \rho_k M_{y_k}^\dagger}{\text{Tr}\left(M_{y_k} \rho_k M_{y_k}^\dagger\right)}$ explaining this fast convergence when ϕ_0/π is irrational [5]

- for any n, $\rho_k^{nn} = \langle n| \rho_k |n\rangle$ is a martingale: $\mathbb{E}\left(\rho_{k+1}^{nn} \mid \rho_k\right) = \rho_k^{nn}$;
- almost all realizations starting from ρ_0 converge towards a photon number state $|n\rangle\langle n|$; the probability to converge towards $|n\rangle\langle n|$ is given by the initial population ρ_0^{nn}.

This convergence characterizes a Quantum Non Demolition (QND) measurement of photons (counting photons without destroying them).

[5] H. Amini et al., IEEE Trans. Automatic Control, in press, 2012.

Closed-loop experimental data

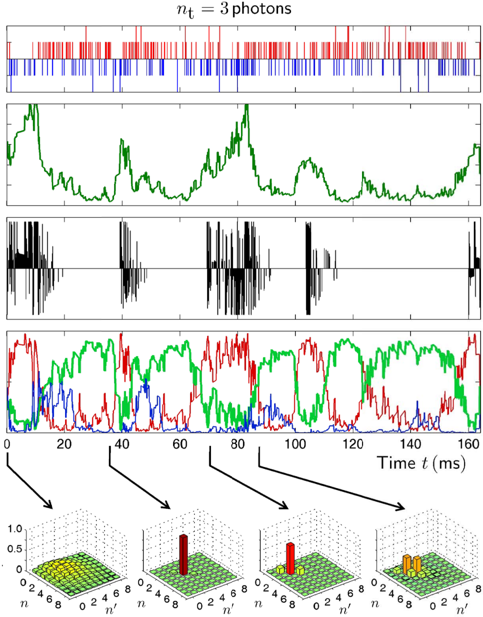
- Initial state coherent state with $\bar{n} = 3$ photons
- State estimation via a quantum filter of state ρ_k^{est}.
- Lyapunov state feedback $u_k = f(\rho_k)$ stabilizing towards $|\bar{n}\rangle\langle\bar{n}|$
- ρ_k is replaced by its estimate ρ_k^{est} in the feedback (quantum separation principle)

Sampling period 80 μs

Experience imperfections:

- detection efficiency 40%
- detection error rate 10%
- **delay: 4 steps**
- truncation to 9 photons
- finite photon life time
- atom occupancy 30%

Stabilization around 3-photon state



Fidelity as control Lyapunov function

In [6] we propose the following stabilizing state feedback law based on the fidelity towards the target state $|\bar{n}\rangle$,

$$u = f(\rho) =: \underset{\upsilon \in [-\bar{u}, \bar{u}]}{\text{Argmin}} \quad V(D_\upsilon \rho D_\upsilon^\dagger)$$

where $V(\rho) = 1 - F(|\bar{n}\rangle \langle \bar{n}|, \rho) = 1 - \rho^{\bar{n}\bar{n}}$ and $\bar{u} > 0$ is small. Two important issues.

- The state ρ is not directly measured; output delay is of 4 steps: it was solved by a quantum filter taking into account the delay.
- V is maximum and equal to 1 for any $\rho = |n\rangle \langle n|$ with $n \neq \bar{n}$: no distinction between $n = \bar{n} + 1$ (close to the target) and $\bar{n} + 1000$ (far from the target). This issue has been solved by changing the Lyapunov function V.

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[6] I. Dotsenko et al.: Quantum feedback by discrete quantum non-demolition measurements: towards on-demand generation of photon-number states. Physical Review A80:013805, 2009.

Lyapunov-based feedback (goal photon number $\bar{n}$) [7]

$V(\rho) = \sum_n \left(-\epsilon \langle n | \rho | n \rangle^2 + \sigma_n \langle n | \rho | n \rangle\right)$ is a strict control Lyapunov function with $\epsilon > 0$ small enough,

$$\sigma_n = \begin{cases} \frac{1}{4} + \sum_{\nu=1}^{\bar{n}} \frac{1}{\nu} - \frac{1}{\nu^2}, & \text{if } n = 0; \\ \sum_{\nu=n+1}^{\bar{n}} \frac{1}{\nu} - \frac{1}{\nu^2}, & \text{if } n \in [1, \bar{n}-1]; \\ 0, & \text{if } n = \bar{n}; \\ \sum_{\nu=\bar{n}+1}^{n} \frac{1}{\nu} + \frac{1}{\nu^2}, & \text{if } n \in [\bar{n}+1, +\infty[, \end{cases}$$

and the feedback $u = f(\rho) =: \underset{\upsilon \in [-\bar{u}, \bar{u}]}{\text{Argmin}} \quad V_\epsilon\left(D_\upsilon \rho D_\upsilon^\dagger\right)$ ($\bar{u} > 0$ small).

In closed-loop, $V(\rho)$ becomes a strict super-martingale:

$$\mathbb{E}\left(V(\rho_{k+1} \mid \rho_k\right) = V(\rho_k) - Q(\rho_k)$$

with $Q(\rho)$ continuous, positive and vanishing only when $\rho = |\bar{n}\rangle\langle\bar{n}|$.
This feedback law yields

- global stabilization for any finite dimensional approximation consisting in truncation to $n^{\max} < +\infty$ photons.
- global approximate stabilization for $n^{\max} = +\infty$.

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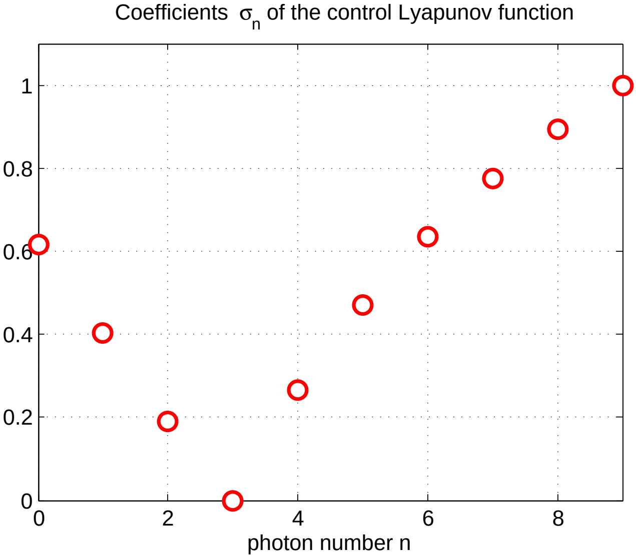
[7] H. Amini et al.: CDC-2011.

The control Lyapunov function used for the photon box $n^{max} = 9$.



$$V(\rho) = \sum_{n=0}^{9} \left(-\epsilon \langle n | \rho | n \rangle^2 + \sigma_n \langle n | \rho | n \rangle \right)$$

Global approximate stabilization ($n^{\max} = +\infty$) [8]

- The feedback $u = \underset{\upsilon \in [-\bar{u}, \bar{u}]}{\text{Argmin}} \quad V_\epsilon\left(D_\upsilon \rho D_\upsilon^\dagger\right)$ ensures a strict closed-loop Lyapunov function

$$V_\epsilon(\rho) = \sum_{n \geq 0} \left(-\epsilon \langle n | \rho | n \rangle^2 + \sigma_n \langle n | \rho | n \rangle \right)$$

with $\sigma_n \sim \log n$, for n large (high photon-number cut-off).

- For any $\eta > 0$ and $C > 0$, exist $\epsilon > 0$ and $\bar{u} > 0$ (small), such that, for any initial value ρ_0 with $V_\epsilon(\rho_0) \leq C$, $\rho_k^{\bar{n}\bar{n}}$ converges almost surely towards a number inside $[1 - \eta, 1]$. With $\text{Tr}(\rho_k) = 1$, and $\rho_k = \rho_k^\dagger \geq 0$, this means, that almost surely, for k large enough, ρ_k is close (weak-* topology) to the goal Fock state $\bar{\rho} = |\bar{n}\rangle \langle \bar{n}|$.

[8] R. Somaraju, M. Mirrahimi, P.R.: CDC 2011
http://arxiv.org/abs/1103.1724

Design of the strict control Lyapunov function[9]

Exploit open-loop stability: for each n, $\langle n | \rho | n \rangle$ is a martingale; $V(\rho) = -\frac{1}{2} \sum_n \langle n | \rho | n \rangle^2$ is a super-martingale with

$$\mathbb{E}\left(V(\rho_{k+1}) \;/\; \rho_k \right) = V(\rho_k) - Q(\rho_k)$$

where $Q(\rho) \geq 0$ and $Q(\rho) = 0$ iff, ρ is a Fock state.

For closing the loop take σ_n such that

$$u \mapsto \sum_n \sigma_n \left\langle n \left| D_u \rho D_u^\dagger \right| n \right\rangle$$

1. is strongly convex for $\rho = |\bar{n}\rangle \langle \bar{n}|$
2. is strongly concave for $\rho = |n\rangle \langle n|$, $n \neq \bar{n}$.

This is achieved by inverting the Laplacian matrix associated to the control Hamiltionan $H = \imath(\mathbf{a} - \mathbf{a}^\dagger)$. Remember that $D_u = e^{-\imath u H}$.

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[9] H. Amini et al., CDC 2011, http://arxiv.org/abs/1103.1365

Estimation of ρ_k from the past measures $y_{\nu\leq k}$ via a quantum filter

$$
\begin{cases}
\rho_{k+1} = \dfrac{D_{u_k}\; M_{y_k}\rho_k M_{y_k}^\dagger D_{u_k}^\dagger}{\mathrm{Tr}\left(M_{y_k}\rho_k M_{y_k}^\dagger\right)} \\[2ex]
\rho_{k+1}^{\mathrm{est}} = \dfrac{D_{u_k}\; M_{y_k}\rho_k^{\mathrm{est}} M_{y_k}^\dagger D_{u_k}^\dagger}{\mathrm{Tr}\left(M_{y_k}\rho_k^{\mathrm{est}} M_{y_k}^\dagger\right)}
\end{cases}
$$

- Assume we know ρ_k and u_k. Outcome of measure no k, y_k, defines the jump operator M_{y_k} and we can compute ρ_{k+1}.
- Quantum filter and real-time estimation: initialize the estimation ρ^{est} to some initial value ρ_0^{est} and update at step k with measured jumps y_k and the known controls u_k.
- Quantum separation principle for stabilization towards a pure state[10]: assume that the feedback $u = f(\rho)$ ensures global asymptotic convergence towards a pure state; then, if $\ker(\rho_0^{\mathrm{est}}) \subset \ker(\rho_0)$, the feedback $u_k = f(\rho_k^{\mathrm{est}})$ ensures also global asymptotic convergence towards the same pure state.

[10] Bouten, van Handel, 2008.

A modified quantum filter with a measure delayed by one step

Without delay the stabilizing feedback reads

$$u_k = \text{Argmin} \quad V\left(D_v \frac{M_{y_k} \rho_k^{\text{est}} M_{y_k}^\dagger}{\text{Tr}\left(M_{y_k} \rho_k^{\text{est}} M_{y_k}^\dagger\right)} D_v^\dagger\right)$$

With delay, we have only access to y_{k-1} and the stabilizing feedback uses the Kraus map $\mathbb{K}(\rho) = M_g \rho M_g^\dagger + M_e \rho M_e^\dagger$:

$$u_k = \text{Argmin} \quad V\left(D_v \mathbb{K}(\rho_k^{\text{est}}) D_v^\dagger\right)$$

This is the same feedback law but with another state estimation at step k: $\mathbb{K}(\rho_k^{\text{est}})$ instead of $\frac{M_{y_k} \rho_k^{\text{est}} M_{y_k}^\dagger}{\text{Tr}\left(M_{y_k} \rho_k^{\text{est}} M_{y_k}^\dagger\right)}$.

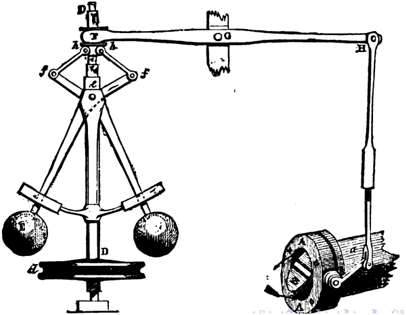
- **System theoretical interpretation:** $\mathbb{K}(\rho_k^{\text{est}})$ stands for the the prediction of cavity state at step k. This prediction is in average (expectation value) since $y_k \in \{g, e\}$ can take two values.
- **Quantum physics interpretation:** $\mathbb{K}(\rho_k^{\text{est}})$ corresponds to tracing over the atom that has already interacted with the cavity (entangled with cavity state) but that has not been measured at step k.

A delay of two steps involves two iterations of such Kraus maps, ...

Conclusion: measurement-based versus coherent feedback.

- Classical state-feedback stabilization: continuous time systems with QND measurement (possible extension of M. Mirrahimi and R. van Handel, SIAM JOC, 2007), filtering stability (Belavkin seminal contributions, see also van Handel, ...).
- Stabilization by coherent feedback: similarly to the Watt regulator where a mechanical system is controlled by another one, the controller is a quantum system coupled to the original one (Mabuchi, Nurdin, Gough, James, Petersen, ...); related to "quantum circuit" theory (see last chapters of Gardiner-Zoller book and the courses of Michel Devoret at Collège de France);
- Coherent feedback is closely related to reservoir engineering: exploit and design the measurement process (here operators M_μ) and its intrinsic back-action to ensure convergence of the ensemble-average dynamics towards a unique pure state (Ticozzi, Viola, . . .)

Watt regulator: a classical analogue of quantum coherent feedback. [11]



The first variations of speed $\delta\omega$ and governor angle $\delta\theta$ obey to

$$\frac{d}{dt}\delta\omega = -a\delta\theta$$
$$\frac{d^2}{dt^2}\delta\theta = -\Lambda\frac{d}{dt}\delta\theta - \Omega^2(\delta\theta - b\delta\omega)$$

with (a, b, Λ, Ω) positive parameters.

Third order system

$$\frac{d^3}{dt^3}\delta\omega = -\Lambda\frac{d^2}{dt^2}\delta\omega - \Omega^2\frac{d}{dt}\delta\omega - ab\Omega^2\delta\omega = 0$$

Characteristic polynomial $P(s) = s^3 + \Lambda s^2 + \Omega^2 s + ab\Omega^2$ with roots having negative real parts iff $\Lambda > ab$: governor damping must be strong enough to ensure asymptotic stability of the closed-loop system.

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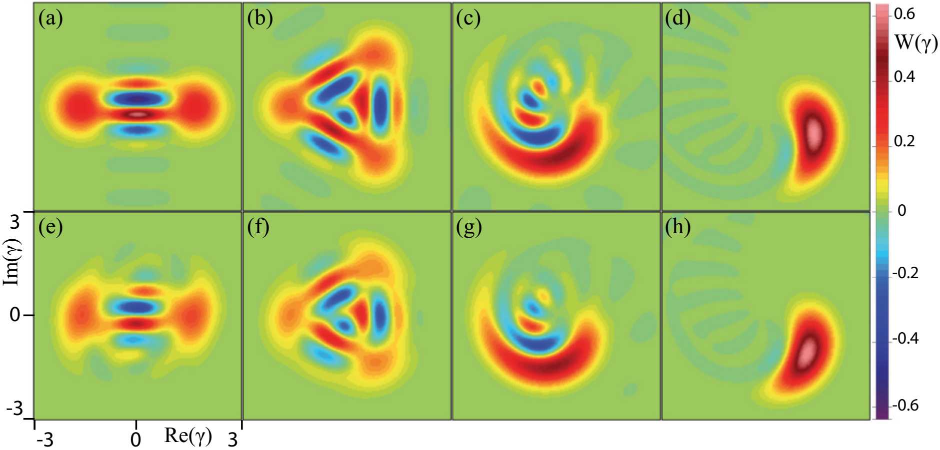
[11] J.C. Maxwell: On governors. Proc. of the Royal Society, No.100, 1868.

Reservoir engineering stabilizing Schrödinger cats for the photon box [12]

Wigner functions of the various states that can be produced by such reservoir based on composite dispersive/resonant atom/cavity interaction.



[12] Sarlette et al: PRL 107:010402,2011 and PRA to appear in 2012.

Control of a QND Markov chain with delay τ

$$\rho_{k+1} = \mathbb{M}_{\mu_k}^{u_{k-\tau}}(\rho_k) =: \frac{M_{\mu_k}^{u_{k-\tau}} \rho_k M_{\mu_k}^{u_{k-\tau}\dagger}}{\text{Tr}\left(M_{\mu_k}^{u_{k-\tau}} \rho_k M_{\mu_k}^{u_{k-\tau}\dagger}\right)}$$

- To each measurement outcome μ is attached the Kraus operator $M_\mu^u \in \mathbb{C}^{d\times d}$ depending on μ and also on a scalar control input $u \in \mathbb{R}$. For each u, $\sum_{\mu=1}^m M_\mu^{u\dagger} M_\mu^u = I$, and we have the Kraus map $\mathbb{K}^u(\rho) = \sum_{\mu=1}^m M_\mu^u \rho M_\mu^{u\dagger}$
- μ_k is a random variable taking values μ in $\{1, \cdots, m\}$ with probability $p_{\mu,\rho_k}^{u_{k-\tau}} = \text{Tr}\left(M_\mu^{u_{k-\tau}} \rho_k M_\mu^{u_{k-\tau}\dagger}\right)$.
- For $u = 0$, the measurement operators M_μ^0 are diagonal in the same orthonormal basis $\{\, |n\rangle \mid n \in \{1, \cdots, d\}\}$, therefore $M_\mu^0 = \sum_{n=1}^d c_{\mu,n} |n\rangle \langle n|$ with $c_{\mu,n} \in \mathbb{C}$.
- For all $n_1 \neq n_2$ in $\{1, \cdots, d\}$, there exists $\mu \in \{1, \cdots, m\}$ such that $|c_{\mu,n_1}|^2 \neq |c_{\mu,n_2}|^2$.

Open-loop convergence $\rho_{k+1} = \mathbb{M}^0_{\mu_k}(\rho_k)$

For any initial condition ρ_0,

- with probability one, ρ_k converges to one of the d states $|n\rangle\langle n|$ with $n \in \{1, \cdots, d\}$.
- the probability of convergence towards the state $|n\rangle\langle n|$ depends only on ρ_0 and is given by $\langle n|\rho_0|n\rangle$.

Proof based on

- the martingales $\langle n|\rho|n\rangle$
- the super-martingale $V(\rho) := -\sum_n \frac{\left(\langle n|\rho|n\rangle\right)^2}{2}$ satisfying

$$\mathbb{E}\left(V(\rho_{k+1})|\rho_k\right) - V(\rho_k) = -Q(\rho_k) \leq 0$$

  with $Q(\rho) = \frac{1}{4}\sum_{n,\mu,\nu} p^0_{\mu,\rho} p^0_{\nu,\rho} \left(\frac{|c_{\mu,n}|^2\langle n|\rho|n\rangle}{p^0_{\mu,\rho}} - \frac{|c_{\nu,n}|^2\langle n|\rho|n\rangle}{p^0_{\nu,\rho}}\right)^2$.

- $Q(\rho) = 0$ iff exists $n \in \{1, \ldots, d\}$ such that $\rho = |n\rangle\langle n|$.

Feedback stabilization of $\rho_{k+1} = \mathbb{M}_{\mu_k}^{u_{k-\tau}}(\rho_k)$ towards $|\bar{n}\rangle\langle\bar{n}|$

- $V_0(\rho) = \sum_{n=1}^{d} \sigma_n \langle n| \rho |n\rangle$ with $\sigma_n \geq 0$ chosen such that $\sigma_{\bar{n}} = 0$ and for any $n \neq \bar{n}$, the second-order u-derivative of $V_0(\mathbb{K}^u(|n\rangle\langle n|))$ at $u = 0$ is strictly negative ($\mathbb{K}^u$ is the Kraus map): set of linear equations in σ_n solved by inverting an irreducible *M*-matrix (Perron-Frobenius theorem).
- The function ($\epsilon > 0$ small enough): $V_\epsilon(\rho) = V_0(\rho) - \frac{\epsilon}{2}\sum_{n=1}^{d}(\langle n| \rho |n\rangle)^2$ still admits a unique global minimum at $|\bar{n}\rangle\langle\bar{n}|$; for u close to 0, $u \mapsto V_\epsilon(\mathbb{K}^u(|n\rangle\langle n|))$ is strongly concave for any $n \neq \bar{n}$ and strongly convexe for $n = \bar{n}$.
- The delay of τ steps: stabilize the state $\chi = (\rho, \beta_1, \cdots, \beta_\tau)$ (β_l control input u delayed l steps) towards $\bar{\chi} = (|\bar{n}\rangle\langle\bar{n}|, 0, \ldots, 0)$ using the control-Lyapunov function
$$W_\epsilon(\chi) = V_\epsilon(\mathbb{K}^{\beta_1}(\mathbb{K}^{\beta_2}(\ldots\ldots\mathbb{K}^{\beta_\tau}(\rho)\ldots))).$$

For $\bar{u}$ and ϵ small enough, the feedback

$$u_k = f(\chi_k) =: \underset{\xi\in[-\bar{u},\bar{u}]}{\operatorname{argmin}}\left(\mathbb{E}\left(W_\epsilon(\chi_{k+1})|\chi_k, u_k = \xi\right)\right)$$

ensures global stabilization towards $\bar{\chi}$.

Quantum separation principle

- Estimate the hidden state ρ by ρ^{est} satisfying

$$\rho^{\text{est}}_{k+1} = \mathbb{M}^{u_{k-\tau}}_{\mu_k}(\rho^{\text{est}}_k)$$

where ρ obeys to $\rho_{k+1} = \mathbb{M}^{u_{k-\tau}}_{\mu_k}(\rho_k)$ with the stabilizing feedback $u_k = f(\rho^{\text{est}}_k, u_{k-1}, \ldots, u_{k-\tau})$ computed using ρ^{est} instead of ρ.

- If $\ker(\rho^{\text{est}}_0) \subset \ker(\rho_0)$, ρ_k and ρ^{est}_k converge almost surely towards the target state $|\bar{n}\rangle\langle\bar{n}|$.
  Proof based on[13]:
  - $\langle\bar{n}|\rho_k|\bar{n}\rangle \in [0,1]$,
  - linearity of $\mathbb{E}\left(\langle\bar{n}|\rho_k|\bar{n}\rangle \,|\rho_0, \rho^{\text{est}}_0\right)$ versus ρ_0,
  - decomposition $\rho^{\text{est}}_0 = \gamma\rho_0 + (1-\gamma)\rho^c_0$ with $\gamma \in]0,1[$.

[13] Bouten, van Handel, 2008.

Imperfect measurements: the new "observable" state $\widehat{\rho}$

- The left stochastic matrix η: $\eta_{\mu',\mu} \in [0,1]$ is the probability of having the imperfect outcome $\mu' \in \{1,\ldots,m'\}$ knowing that the perfect one is $\mu \in \{1,\ldots,m\}$.
- $\widehat{\rho}_k = \mathbb{E}\left(\rho_k | \rho_0, \mu'_0, \ldots, \mu'_{k-1}, u_{-\tau}, \ldots, u_{k-\tau-1}\right)$ obeys to[14]

$$\widehat{\rho}_{k+1} = \mathbb{L}^{u_{k-\tau}}_{\mu'_k}(\widehat{\rho}_k), \quad \text{where}$$

  - $\mathbb{L}^u_{\mu'}(\widehat{\rho}) = \frac{\mathbf{L}^u_{\mu'}(\widehat{\rho})}{\mathrm{Tr}\left(\mathbf{L}^u_{\mu'}(\widehat{\rho})\right)}$ with $\mathbf{L}^u_{\mu'}(\widehat{\rho}) = \sum_{\mu=1}^m \eta_{\mu',\mu} M^u_\mu \widehat{\rho} M^{u\dagger}_\mu$;
  - μ'_k is a random variable taking values μ' in $\{1,\cdots,m'\}$ with probability $p^{u_{k-\tau}}_{\mu',\widehat{\rho}_k} = \mathrm{Tr}\left(\mathbf{L}^{u_{k-\tau}}_{\mu'}(\widehat{\rho}_k)\right)$.
  - $\mathbb{E}\left(\widehat{\rho}_{k+1} | \widehat{\rho}_k = \rho, u_{k-\tau} = u\right) = \mathbb{K}^u(\widehat{\rho})$
- Assumption: for all $n_1 \neq n_2$ in $\{1,\cdots,d\}$, there exists $\mu' \in \{1,\cdots,m'\}$, s.t. $\mathrm{Tr}\left(\mathbf{L}^0_{\mu'}(|n_1\rangle\langle n_1|)\right) \neq \mathrm{Tr}\left(\mathbf{L}^0_{\mu'}(|n_2\rangle\langle n_2|)\right)$.

Open-loop convergence of $\widehat{\rho}_k$ towards $|n\rangle\langle n|$ with prob. $\langle n|\widehat{\rho}_0|n\rangle$.

[14] R. Somaraju et al., ACC 2012 (http://arxiv.org/abs/1109.5344)

Feedback stabilization of $\widehat{\rho}_{k+1} = \mathbb{L}_{\mu_k}^{u_{k-\tau}}(\widehat{\rho}_k)$ towards $|\bar{n}\rangle\langle\bar{n}|$

- With the previous function $V_\epsilon(\rho) = V_0(\rho) - \frac{\epsilon}{2}\sum_{n=1}^{d}(\langle n|\rho|n\rangle)^2$ stabilize $\widehat{\chi} = (\widehat{\rho}, \beta_1, \cdots, \beta_\tau)$ towards $\bar{\chi} = (|\bar{n}\rangle\langle\bar{n}|, 0, \ldots, 0)$ using the control-Lyapunov function
$$W_\epsilon(\widehat{\chi}) = V_\epsilon(\mathbb{K}^{\beta_1}(\mathbb{K}^{\beta_2}(\ldots\ldots\mathbb{K}^{\beta_\tau}(\widehat{\rho})\ldots))).$$
- For $\bar{u}$ and ϵ small enough, the feedback
$$u_k = f(\widehat{\chi}_k) =: \underset{\xi\in[-\bar{u},\bar{u}]}{\operatorname{argmin}}\Big(\mathbb{E}\left(W_\epsilon(\widehat{\chi}_{k+1})|\widehat{\chi}_k, u_k = \xi\right)\Big)$$
ensures global stabilization of $\widehat{\chi}_k$ towards $\bar{\chi}$.
- Since $\widehat{\rho}_k = \mathbb{E}\left(\rho_k|\rho_0, \mu'_0, \ldots, \mu'_{k-1}, u_{-\tau}, \ldots, u_{k-\tau-1}\right)$ convergences towards the pure state $|\bar{n}\rangle\langle\bar{n}|$, ρ_k converges also towards the same pure state.

Quantum separation principle

- Estimate the hidden state $\widehat{\rho}$ by $\widehat{\rho}^{\text{est}}$ satisfying

$$\widehat{\rho}^{\text{est}}_{k+1} = \mathbb{L}^{u_{k-\tau}}_{\mu'_k}(\widehat{\rho}^{\text{est}}_k)$$

  where

  - ρ_k obeys to

    $$\rho_{k+1} = \mathbb{M}^{u_{k-\tau}}_{\mu_k}(\rho_k)$$

    with the stabilizing feedback

    $$u_k = f(\widehat{\rho}^{\text{est}}_k, u_{k-1}, \ldots, u_{k-\tau})$$

    computed using $\widehat{\rho}^{\text{est}}$ instead of ρ.
  - $\mu'_k = \mu'$ with probability η_{μ',μ_k}.
- Filter stability: $F(\widehat{\rho}_k, \widehat{\rho}^{\text{est}}_k) \triangleq \left(\text{Tr}\left(\sqrt{\sqrt{\widehat{\rho}_k}\widehat{\rho}^{\text{est}}_k\sqrt{\widehat{\rho}_k}}\right)\right)^2$ is always a sub-martingale[15].
- If $\ker(\widehat{\rho}^{\text{est}}_0) \subset \ker(\rho_0)$, ρ_k and $\widehat{\rho}^{\text{est}}_k$ converge almost surely towards the target state $|\bar{n}\rangle\langle\bar{n}|$.

[15] P.R., IEEE Trans. Automatic Control, 2011.

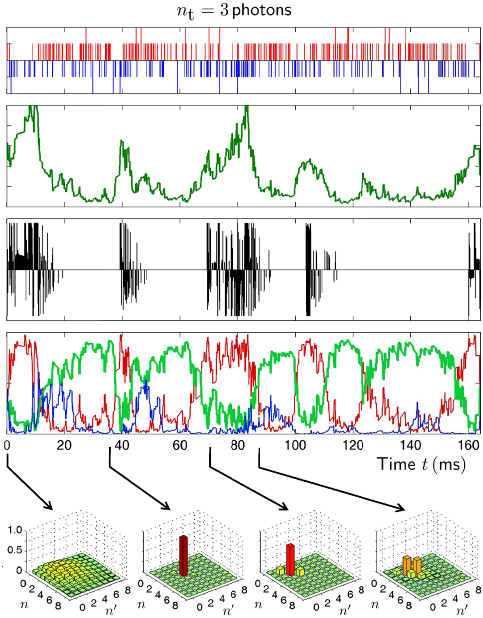
Closed-loop experimental data

- Initial state coherent state with $\bar{n} = 3$ photons
- State estimation via a quantum filter of state ρ_k^{est}.
- Lyapunov state feedback $u_k = f(\rho_k)$ stabilizing towards $|\bar{n}\rangle \langle \bar{n}|$
- ρ_k is replaced by its estimate ρ_k^{est} in the feedback (quantum separation principle)

Sampling period 80 μs
Experience imperfections:

- detection efficiency 40%
- detection error rate 10%
- delay 4 sampling periods
- truncation to 9 photons
- finite photon life time
- atom occupancy 30%

Stabilization around 3-photon state



The left stochastic matrix for the LKB photon box[16]

For each control input u,

- we have a total of $m = 3 \times 7 = 21$ Kraus operators. The jumps are labeled by $\mu = (\mu^a, \mu^c)$ with $\mu^a \in \{no, g, e, gg, ge, eg, ee\}$ labeling atom related jumps and $\mu^c \in \{o, +, -\}$ cavity decoherence jumps.
- we have only $m' = 6$ real detection possibilities $\mu' \in \{no, g, e, gg, ge, ee\}$ corresponding respectively to no detection, a single detection in g, a single detection in e, a double detection both in g, a double detection one in g and the other in e, and a double detection both in e.

$\mu' \setminus \mu$	(no, μ^c)	(g, μ^c)	(e, μ^c)	(gg, μ^c)	(ee, μ^c)	(ge, μ^c) or $(eg$
no	1	$1\text{-}\epsilon_d$	$1\text{-}\epsilon_d$	$(1\text{-}\epsilon_d)^2$	$(1\text{-}\epsilon_d)^2$	$(1\text{-}\epsilon_d)^2$
g	0	$\epsilon_d(1\text{-}\eta_g)$	$\epsilon_d\eta_e$	$2\epsilon_d(1\text{-}\epsilon_d)(1\text{-}\eta_g)$	$2\epsilon_d(1\text{-}\epsilon_d)\eta_e$	$\epsilon_d(1\text{-}\epsilon_d)(1\text{-}\eta_g +$
e	0	$\epsilon_d\eta_g$	$\epsilon_d(1\text{-}\eta_e)$	$2\epsilon_d(1\text{-}\epsilon_d)\eta_g$	$2\epsilon_d(1\text{-}\epsilon_d)(1\text{-}\eta_e)$	$\epsilon_d(1\text{-}\epsilon_d)(1\text{-}\eta_e +$
gg	0	0	0	$\epsilon_d^2(1\text{-}\eta_g)^2$	$\epsilon_d^2\eta_e^2$	$\epsilon_d^2\eta_e(1\text{-}\eta_g)$
ge	0	0	0	$2\epsilon_d^2\eta_g(1\text{-}\eta_g)$	$2\epsilon_d^2\eta_e(1\text{-}\eta_e)$	$\epsilon_d^2((1\text{-}\eta_g)(1\text{-}\eta_e) +$
ee	0	0	0	$\epsilon_d^2\eta_g^2$	$\epsilon_d^2(1\text{-}\eta_e)^2$	$\epsilon_d^2\eta_g(1\text{-}\eta_e)$

[16] R. Somaraju et al.: ACC 2012 (http://arxiv.org/abs/1109.5344)