Lake Como School of Advanced Studies

September 19-23, 2022

Quantum characterization and control of quantum complex systems

Quantum Stochastic Master Equations (SME)

based on preprint to appear in Annual Reviews in Control

https://arxiv.org/abs/2208.07416

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Outline

1. Schrödinger ($\hbar = 1$): wave funct. $|\psi\rangle \in \mathcal{H}$, density op. $\rho \sim |\psi\rangle\langle\psi|$

$$\frac{d}{dt}|\psi\rangle = -i\mathrm{H}|\psi\rangle, \quad \mathrm{H} = \mathrm{H}_0 + u\mathrm{H}_1 = \mathrm{H}^\dagger, \quad \frac{d}{dt}\rho = -i[\mathrm{H}, \rho].$$

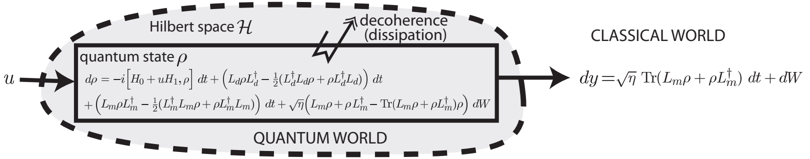
2. Origin of dissipation: collapse of the wave packet induced by the measurement of $\mathrm{O} = \mathrm{O}^\dagger$ with spectral decomp. $\sum_y \lambda_y \mathrm{P}_y$:
   - measurement outcome y with proba. $\mathbb{P}_y = \langle\psi|\mathrm{P}_y|\psi\rangle = \mathrm{Tr}\left(\rho \mathrm{P}_y\right)$ depending on $|\psi\rangle$, ρ just before the measurement
   - measurement back-action if outcome y:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\mathrm{P}_y|\psi\rangle}{\sqrt{\langle\psi|\mathrm{P}_y|\psi\rangle}}, \quad \rho \mapsto \rho_+ = \frac{\mathrm{P}_y \rho \mathrm{P}_y}{\mathrm{Tr}\left(\rho \mathrm{P}_y\right)}$$

3. Tensor product for the description of composite systems (S, C):
   - Hilbert space $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_c$
   - Hamiltonian $\mathrm{H} = \mathrm{H}_s \otimes \mathrm{I}_c + \mathrm{H}_{sc} + \mathrm{I}_s \otimes \mathrm{H}_c$
   - observable on sub-system C only: $\mathrm{O} = \mathrm{I}_s \otimes \mathrm{O}_c$.

[1] S. Haroche and J.M. Raimond (2006). *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts.

Diffusive stochastic master equation[2]



$t \mapsto \rho_t$ continuous time function (not differentiable), solution of

$$d\rho_t = -i\Big[H_0 + uH_1, \rho_t\Big]\ dt + \left(\sum_{\nu=d,m} L_\nu \rho_t L_\nu^\dagger - \tfrac{1}{2}(L_\nu^\dagger L_\nu \rho_t + \rho_t L_\nu^\dagger L_\nu)\right)\ dt + \dots$$

$$\dots + \sqrt{\eta}\Big(L_m\rho_t + \rho_t L_m^\dagger - \mathrm{Tr}(L_m\rho_t + \rho_t L_m^\dagger)\rho_t\Big)\ dW_t,$$

where $\eta \in [0,1]$ and the same Wiener process W_t is shared by the state dynamics and the output map

$$dy_t = \sqrt{\eta}\ \mathrm{Tr}(L_m\rho_t + \rho_t L_m^\dagger)\ dt + dW_t.$$

[2] A. Barchielli and M. Gregoratti. *Quantum Trajectories and Measurements in Continuous Time: the Diffusive Case*. Springer Verlag, 2009.

$t \mapsto \rho_t$ piecewise smooth time function, solution of

$$d\rho_t = \left(-i[\mathrm{H}, \rho_t] + \mathrm{V}\rho_t\mathrm{V}^\dagger - \tfrac{1}{2}(\mathrm{V}^\dagger\mathrm{V}\rho_t + \rho_t\mathrm{V}^\dagger\mathrm{V})\right) dt + \left(\frac{\bar{\theta}\rho_t + \bar{\eta}\mathrm{V}\rho_t\mathrm{V}^\dagger}{\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)} - \rho_t\right)\left(dy_t - \left(\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)\right) dt\right)$$

where $\bar{\theta} \geq 0$ (shot-noise rate) and $\bar{\eta} \in [0, 1]$ (detection efficiency) and where the counting detector outcome $dy_t \in \{0, 1\}$ with

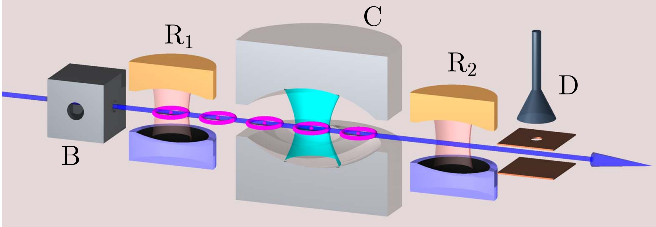
- $dy_t = 0$ with probability $1 - \left(\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)\right) dt$ and then

$$\rho_{t+dt} = \rho_t + \Big(- i[\mathrm{H}, \rho_t] + \mathrm{V}\rho_t\mathrm{V}^\dagger - \tfrac{1}{2}(\mathrm{V}^\dagger\mathrm{V}\rho_t + \rho_t\mathrm{V}^\dagger\mathrm{V}) + \bar{\eta}\left(\mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)\rho_t - \mathrm{V}\rho_t\mathrm{V}^\dagger\right)\Big) dt$$

- $dy_t = 1$ with probability $\left(\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)\right) dt$, and then

$$\rho_{t+dt} = \frac{\bar{\theta}\rho_t + \bar{\eta}\mathrm{V}\rho_t\mathrm{V}^\dagger}{\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)}.$$

[3] see, e.g., J. Dalibard, Y. Castin, and K. Mølmer. Wave-function approach to dissipative processes in quantum optics. *Phys. Rev. Lett.*, 68(5):580–583, February 1992.



- Dispersive qubit/photon interaction: $\mathbf{H}_{int} = -\chi\big(|e\rangle\langle e| - |g\rangle\langle g|\big) \otimes \mathbf{n}$ (with χ a constant parameter) yields $e^{-iT\mathbf{H}_{int}}$, the Schrödinger propagator during the time $T > 0$, given with $\theta = \chi T$ by

$$\mathbf{U}_\theta = |g\rangle\langle g| \otimes e^{-i\theta\mathbf{n}} + |e\rangle\langle e| \otimes e^{i\theta\mathbf{n}}.$$

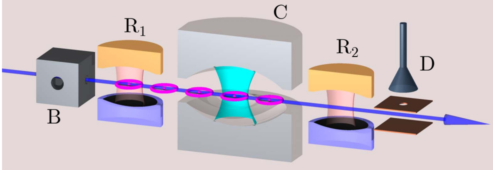
- resonant qubit/photon interaction: $\mathbf{H}_{int} = i\frac{\omega}{2}\Big(|g\rangle\langle e| \otimes \mathbf{a}^\dagger - |e\rangle\langle g| \otimes \mathbf{a}\Big)$ (with ω a constant parameter) yields $e^{-iT\mathbf{H}_{int}}$, the Schrödinger propagator during the time $T > 0$, given with $\theta = \omega T/2$ by

$$\mathbf{U}_\theta = |g\rangle\langle g| \otimes \cos(\theta\sqrt{\mathbf{n}}) + |e\rangle\langle e| \otimes \cos(\theta\sqrt{\mathbf{n}+\mathbf{I}}) + |g\rangle\langle e| \otimes \frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}\mathbf{a}^\dagger - |e\rangle\langle g| \otimes \mathbf{a}\frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}.$$

[4] LKB for Laboratoire Kastler Brossel.

Outline

Photons measured by dispersive qubits (1)



$$U = \left(\left(\left(\frac{|g\rangle - |e\rangle}{\sqrt{2}}\right)\langle g| + \left(\frac{|g\rangle + |e\rangle}{\sqrt{2}}\right)\langle e|\right) \otimes I\right)$$
$$\left(|g\rangle\langle g| \otimes e^{-i\theta \mathbf{n}} + |e\rangle\langle e| \otimes e^{i\theta \mathbf{n}}\right)$$
$$\left(\left(\left(\frac{|g\rangle + |e\rangle}{\sqrt{2}}\right)\langle g| + \left(\frac{-|g\rangle + |e\rangle}{\sqrt{2}}\right)\langle e|\right) \otimes I\right)$$

applied on $|\Psi\rangle = |g\rangle \otimes |\psi\rangle$ yields

$$U\ (|g\rangle|\psi\rangle) = |g\rangle\ \cos(\theta \mathbf{n})|\psi\rangle + |e\rangle\ i\sin(\theta \mathbf{n})|\psi\rangle.$$

Markov process induced by the passage of qubit number k:

$$|\psi_{k+1}\rangle = \begin{cases} \frac{\cos(\theta \mathbf{n})|\psi_k\rangle}{\sqrt{\langle\psi_k|\cos^2(\theta \mathbf{n})|\psi_k\rangle}} & \text{if } y_k = g \text{ with probability } \langle\psi_k|\cos^2(\theta \mathbf{n})|\psi_k\rangle\ ; \\ \frac{i\sin(\theta \mathbf{n})|\psi_k\rangle}{\sqrt{\langle\psi_k|\sin^2(\theta \mathbf{n})|\psi_k\rangle}} & \text{if } y_k = e \text{ with probability } \langle\psi_k|\sin^2(\theta \mathbf{n})|\psi_k\rangle\ ; \end{cases}$$

where $y_k \in \{g, e\}$ classical signal produced by measurement of qubit k.

Photons measured by dispersive qubits (2)

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The density operator formulation ($\rho \equiv |\psi\rangle\langle\psi|$):

$$\rho_{k+1} = \begin{cases} \frac{M_g \rho_k M_g^\dagger}{\mathrm{Tr}\left(M_g \rho_k M_g^\dagger\right)} & \text{if } y_k = g \text{ with probability } \mathrm{Tr}\left(M_g \rho_k M_g^\dagger\right) ; \\ \frac{M_e \rho_k M_e^\dagger}{\mathrm{Tr}\left(M_e \rho_k M_e^\dagger\right)} & \text{if } y_k = e \text{ with probability } \mathrm{Tr}\left(M_e \rho_k M_e^\dagger\right) ; \end{cases}$$

with measurement Kraus operators $M_g = \cos(\theta N)$ and $M_e = \sin(\theta N)$. Notice that $M_g^\dagger M_g + M_e^\dagger M_e = I$.
For θ/π irrational, almost sure convergence towards a Fock state $|\bar{n}\rangle\langle\bar{n}|$ for some $\bar{n}$ based on the Lyapunov function (super-martingale)

$$V(\rho) = \sum_{0 \leq n_1 < n_2} \sqrt{\langle n_1|\rho|n_1\rangle\langle n_2|\rho|n_2\rangle}$$

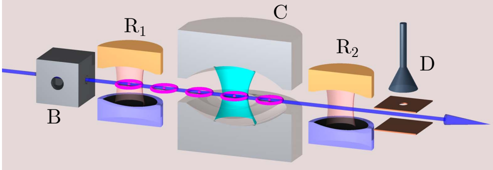
that converges in average towards 0 since

$$\mathbb{E}\left(V(\rho_{k+1}) \;\middle|\; \rho_k\right) \leq \left(\max_{0 \leq n_1 < n_2} |\cos(\theta(n_1 \pm n_2))|\right) \; V(\rho_k).$$

Probability that a realisation converges towards $|\bar{n}\rangle\langle\bar{n}|$ given by its initial population $\langle\bar{n}|\rho_0|\bar{n}\rangle$

Outline

Photons measured by resonant qubits (1)



Wave function $|\Psi\rangle$ of the composite qubit/photon system just before D:

$$\Bigg(|g\rangle\langle g|\cos(\theta\sqrt{\mathbf{n}}) + |e\rangle\langle e|\cos(\theta\sqrt{\mathbf{n}+\mathbf{I}})$$

$$+ |g\rangle\langle e|\frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}\mathbf{a}^\dagger - |e\rangle\langle g|\mathbf{a}\frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}\Bigg)|g\rangle|\psi\rangle$$

$$= |g\rangle\ \cos(\theta\sqrt{\mathbf{n}})|\psi\rangle - |e\rangle\ \mathbf{a}\frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}|\psi\rangle$$

Resulting Markov process associated to the measurement of the observable $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$ with classical output signal $y \in \{g, e\}$:

$$|\psi_{k+1}\rangle = \begin{cases} \frac{\cos(\theta\sqrt{\mathbf{n}})|\psi_k\rangle}{\sqrt{\langle\psi_k|\cos^2(\theta\sqrt{\mathbf{n}})|\psi_k\rangle}} & \text{if } y_k = g \text{ with probability } \langle\psi_k|\cos^2(\theta\sqrt{\mathbf{n}})|\psi_k\rangle\ ; \\ -\frac{\mathbf{a}\frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}|\psi_k\rangle}{\sqrt{\langle\psi_k|\sin^2(\theta\sqrt{\mathbf{n}})|\psi_k\rangle}} & \text{if } y_k = e \text{ with probability } \langle\psi_k|\sin^2(\theta\sqrt{\mathbf{n}})|\psi_k\rangle\ ; \end{cases}$$

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Density operator formulation;

$$\rho_{k+1} = \begin{cases} \frac{\mathrm{M}_g \rho_k \mathrm{M}_g^\dagger}{\mathrm{Tr}\left(\mathrm{M}_g \rho_k \mathrm{M}_g^\dagger\right)} & \text{if } y_k = g \text{ with probability } \mathrm{Tr}\left(\mathrm{M}_g \rho_k \mathrm{M}_g^\dagger\right) ; \\ \frac{\mathrm{M}_e \rho_k \mathrm{M}_e^\dagger}{\mathrm{Tr}\left(\mathrm{M}_e \rho_k \mathrm{M}_e^\dagger\right)} & \text{if } y_k = e \text{ with probability } \mathrm{Tr}\left(\mathrm{M}_e \rho_k \mathrm{M}_e^\dagger\right) ; \end{cases}$$

with measurement Kraus operators $\mathrm{M}_g = \cos(\theta\sqrt{\mathrm{n}})$ and $\mathrm{M}_e = \mathrm{a}\frac{\sin(\theta\sqrt{\mathrm{n}})}{\sqrt{\mathrm{n}}}$. Notice that, once again, $\mathrm{M}_g^\dagger \mathrm{M}_g + \mathrm{M}_e^\dagger \mathrm{M}_e = \mathrm{I}$.

For $\theta\sqrt{n}/\pi$ irrational for all n, almost surely towards vacuum state $|0\rangle\langle 0|$. Results from the following the Lyapunov function (super-martingale)

$$V(\rho) = \mathrm{Tr}\left(\mathrm{n}\rho\right)$$

since

$$\mathbb{E}\left(V(\rho_{k+1}) \,\middle|\, \rho_k\right) = V(\rho_k) - \mathrm{Tr}\left(\sin^2(\theta\sqrt{\mathrm{n}})\rho_k\right).$$

With measurement imperfections, use Bayes rule by taking as quantum state, the expectation value of ρ_{k+1} knowing ρ_k and the information provides by the imperfect measurement outcome.
Assume detector D broken. From

$$\rho_{k+1} = \begin{cases} \frac{M_g \rho_k M_g^\dagger}{\mathrm{Tr}\left(M_g \rho_k M_g^\dagger\right)} & \text{if } y_k = g \text{ with probability } \mathrm{Tr}\left(M_g \rho_k M_g^\dagger\right) ; \\ \frac{M_e \rho_k M_e^\dagger}{\mathrm{Tr}\left(M_e \rho_k M_e^\dagger\right)} & \text{if } y_k = e \text{ with probability } \mathrm{Tr}\left(M_e \rho_k M_e^\dagger\right) ; \end{cases}$$

we get the quantum channel:

$$\rho_{k+1} = \mathcal{K}(\rho_k) \triangleq \mathbb{E}\left(\rho_{k+1} \;\middle|\; \rho_k\right) = M_g \rho_k M_g^\dagger + M_e \rho_k M_e^\dagger.$$

When the qubit detector D, producing the classical measurement signal $y_k \in \{g, e\}$, has errors characterized by the error rate $\eta_e \in (0,1)$ (resp. $\eta_g \in (0,1)$) the probability of detector outcome g (resp. e) knowing that the perfect outcome is e (resp. g), Bayes law gives directly

$$\rho_{k+1} = \begin{cases} \mathbb{E}\left(\rho_{k+1} \;\middle|\; y_k = g, \rho_k\right) = \frac{(1-\eta_g)\mathsf{M}_g\rho_k\mathsf{M}_g^\dagger + \eta_e\mathsf{M}_e\rho_k\mathsf{M}_e^\dagger}{\mathrm{Tr}\left((1-\eta_g)\mathsf{M}_g\rho_k\mathsf{M}_g^\dagger + \eta_e\mathsf{M}_e\rho_k\mathsf{M}_e^\dagger\right)} \\ \qquad \text{with probability } \mathbb{P}(y_k = g|\rho_k) = \mathrm{Tr}\left((1-\eta_g)\mathsf{M}_g\rho_k\mathsf{M}_g^\dagger + \eta_e\mathsf{M}_e\rho_k\mathsf{M}_e^\dagger\right), \\ \mathbb{E}\left(\rho_{k+1} \;\middle|\; y_k = e, \rho_k\right) = \frac{\eta_g\mathsf{M}_g\rho_k\mathsf{M}_g^\dagger + (1-\eta_e)\mathsf{M}_e\rho_k\mathsf{M}_e^\dagger}{\mathrm{Tr}\left(\eta_g\mathsf{M}_g\rho_k\mathsf{M}_g^\dagger + (1-\eta_e)\mathsf{M}_e\rho_k\mathsf{M}_e^\dagger\right)} \\ \qquad \text{with probability } \mathbb{P}(y_k = e|\rho_k) = \mathrm{Tr}\left(\eta_g\mathsf{M}_g\rho_k\mathsf{M}_g^\dagger + (1-\eta_e)\mathsf{M}_e\rho_k\mathsf{M}_e^\dagger\right) \end{cases}$$

Notice that a broken detector corresponds to $\eta_e = \eta_g = 1/2$ and one recovers the above quantum channel.

General structure of discrete-time SME based on a quantum channel with the following Kraus decomposition (which is not unique)

$$\mathcal{K}(\rho) = \sum_\mu \mathsf{M}_\mu \rho \mathsf{M}_\mu^\dagger \quad \text{where} \sum_\mu \mathsf{M}_\mu^\dagger \mathsf{M}_\mu = \mathsf{I}$$

and a left stochastic matrix $(\eta_{y,\mu})$ where y corresponds to the different imperfect measurement outcomes. With $\mathcal{K}_y(\rho) = \sum_\mu \eta_{y,\mu} \mathsf{M}_\mu \rho \mathsf{M}_\mu^\dagger$, ones gets the following SME:

$$\rho_{k+1} = \frac{\mathcal{K}_{y_k}(\rho_k)}{\mathrm{Tr}\left(\mathcal{K}_{y_k}(\rho_k)\right)} \quad \text{where } y_k = y \text{ with probability } \mathrm{Tr}\left(\mathcal{K}_y(\rho_k)\right)$$

Notice that $\mathcal{K} = \sum_y \mathcal{K}_y$ since η is left stochastic.

Here the Hilbert space $\mathcal{H}$ is arbitrary and can be of infinite dimension, the Kraus operator M_μ are bounded operator on $\mathcal{H}$ and ρ is a density operator on $\mathcal{H}$ (Hermitian, trace-class with trace one, non-negative). When the index y or μ are continuous, discrete sums are replaced by integrals and probabilities by probability densities.

Outline

Probe photon in the coherent state $|i\frac{\alpha}{\sqrt{2}}\rangle$ with $\alpha > 0$. Just before D the composite qubit/photon wave function $|\Psi\rangle$ reads:

$$\Big(|g\rangle\langle g|e^{-i\theta \mathbf{n}} + |e\rangle\langle e|e^{i\theta \mathbf{n}}\Big)|\psi\rangle|i\tfrac{\alpha}{\sqrt{2}}\rangle = \langle g|\psi\rangle\,|g\rangle\,|ie^{-i\theta}\tfrac{\alpha}{\sqrt{2}}\rangle + \langle e|\psi\rangle\,|e\rangle\,|ie^{i\theta}\tfrac{\alpha}{\sqrt{2}}\rangle.$$

Measurement outcome $y \in \mathbb{R}$ corresponding to observable

$$\mathrm{Q} = \frac{\mathrm{a}+\mathrm{a}^\dagger}{\sqrt{2}} \equiv \int_{-\infty}^{+\infty} q|q\rangle\langle q|dq \text{ where } \langle q|q'\rangle = \delta(q-q').$$

Since $|ie^{\pm i\theta}\frac{\alpha}{\sqrt{2}}\rangle = \frac{1}{\pi^{1/4}}\int_{-\infty}^{+\infty} e^{iq\alpha\cos\theta}e^{-\frac{(q\pm\alpha\sin\theta)^2}{2}}|q\rangle dq$, we have

$$\langle g|\psi\rangle\,|g\rangle\,|ie^{-i\theta}\tfrac{\alpha}{\sqrt{2}}\rangle + \langle e|\psi\rangle\,|e\rangle\,|ie^{i\theta}\tfrac{\alpha}{\sqrt{2}}\rangle$$

$$= \frac{1}{\pi^{1/4}}\int_{-\infty}^{+\infty} e^{iq\alpha\cos\theta}\left(e^{-\frac{(q-\alpha\sin\theta)^2}{2}}\langle g|\psi\rangle\,|g\rangle + e^{-\frac{(q+\alpha\sin\theta)^2}{2}}\langle e|\psi\rangle\,|e\rangle\right)|q\rangle dq.$$

Thus

$$|\psi_{k+1}\rangle = e^{iy_k\alpha\cos\theta}\frac{e^{-\frac{(y_k-\alpha\sin\theta)^2}{2}}\langle g|\psi_k\rangle\,|g\rangle + e^{-\frac{(y_k+\alpha\sin\theta)^2}{2}}\langle e|\psi_k\rangle\,|e\rangle}{\sqrt{e^{-(y_k-\alpha\sin\theta)^2}|\langle g|\psi_k\rangle|^2 + e^{-(y_k+\alpha\sin\theta)^2}|\langle e|\psi_k\rangle|^2}}$$

where $y_k \in [y, y+dy]$ with prob. $\frac{e^{-(y-\alpha\sin\theta)^2}|\langle g|\psi_k\rangle|^2 + e^{-(y+\alpha\sin\theta)^2}|\langle e|\psi_k\rangle|^2}{\sqrt{\pi}}dy$.

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Density operator formulation

$$\rho_{k+1} = \frac{M_{y_k} \rho_k M_{y_k}^\dagger}{\mathrm{Tr}\left(M_{y_k} \rho_k M_{y_k}^\dagger \right)} \quad \text{where } y_k \in [y, y+dy] \text{ with probability } \mathrm{Tr}\left(M_y \rho_k M_y^\dagger \right) dy$$

and measurement Kraus operators

$$M_y = \tfrac{1}{\pi^{1/4}} e^{-\frac{(y-\alpha \sin\theta)^2}{2}} |g\rangle\langle g| + \tfrac{1}{\pi^{1/4}} e^{-\frac{(y+\alpha \sin\theta)^2}{2}} |e\rangle\langle e|.$$

Notice that

$$\mathrm{Tr}\left(M_y \rho M_y^\dagger \right) = \frac{1}{\sqrt{\pi}} e^{-(y-\alpha \sin\theta)^2} \langle g|\rho|g\rangle + \frac{1}{\sqrt{\pi}} e^{-(y+\alpha \sin\theta)^2} \langle e|\rho|e\rangle$$

and $\int_{-\infty}^{+\infty} M_y^\dagger M_y \, dy = |g\rangle\langle g| + |e\rangle\langle e| = I$.
For $\alpha \neq 0$, almost sure convergence towards $|g\rangle$ or $|e\rangle$ deduced from Lyapunov function

$$V(\rho) = \sqrt{\langle g|\rho|g\rangle \langle e|\rho|e\rangle} \text{ with } \mathbb{E}\left(V(\rho_{k+1}) \;\middle|\; \rho_k \right) = e^{-\alpha^2 \sin^2\theta} \; V(\rho_k).$$

Detection imperfections: probability density of y knowing perfect detection q is a Gaussian given by $\frac{1}{\sqrt{\pi\sigma}} e^{-\frac{(y-q)^2}{\sigma}}$ for some error parameter $\sigma > 0$. Then the above Markov process becomes

$$\rho_{k+1} = \frac{\mathcal{K}_{y_k}(\rho_k)}{\mathrm{Tr}\left(\mathcal{K}_{y_k}(\rho_k)\right)}$$

where

$$\mathcal{K}_y(\rho) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\sigma}} e^{-\frac{(y-q)^2}{\sigma}} \mathrm{M}_q \rho \mathrm{M}_q^\dagger \, dq$$

Standard computations using

$$\mathrm{M}_q = \frac{1}{\pi^{1/4}} e^{-\frac{(q-\alpha \sin\theta)^2}{2}} |g\rangle\langle g| + \frac{1}{\pi^{1/4}} e^{-\frac{(q+\alpha \sin\theta)^2}{2}} |e\rangle\langle e|$$

show that

$$\mathcal{K}_y(\rho) = \frac{1}{\sqrt{\pi(1+\sigma)}} \Bigg(e^{-\frac{(y-\alpha \sin\theta)^2}{1+\sigma}} \langle g|\rho|g\rangle |g\rangle\langle g| + e^{-\frac{(y+\alpha \sin\theta)^2}{1+\sigma}} \langle e|\rho|e\rangle |e\rangle\langle e| \\ + e^{-\frac{y^2}{1+\sigma} - (\alpha \sin\theta)^2} \left(\langle e|\rho|g\rangle |e\rangle\langle g| + \langle g|\rho|e\rangle |g\rangle\langle e| \right) \Bigg).$$

Outline

Density operator formulation (perfect detection)

$$\rho_{k+1} = \frac{\mathrm{M}_{y_k}\rho_k \mathrm{M}_{y_k}^\dagger}{\mathrm{Tr}\left(\mathrm{M}_{y_k}\rho_k \mathrm{M}_{y_k}^\dagger\right)} \quad \text{where } y_k \in [y, y+dy] \text{ with probability } \ \mathrm{Tr}\left(\mathrm{M}_{y}\rho_k \mathrm{M}_{y}^\dagger\right) dy$$

and measurement Kraus operators

$$\mathrm{M}_y = \tfrac{1}{\pi^{1/4}} e^{-\frac{(y-\alpha\sin\theta)^2}{2}} |g\rangle\langle g| + \tfrac{1}{\pi^{1/4}} e^{-\frac{(y+\alpha\sin\theta)^2}{2}} |e\rangle\langle e|.$$

Since

$$\mathbb{E}\left(y_k \;\middle|\; \rho_k=\rho\right) \triangleq \overline{y} = -\alpha\sin\theta \ \ \mathrm{Tr}\left(\sigma_z\rho\right), \ \mathbb{E}\left(y_k^2 \;\middle|\; \rho_k=\rho\right) \triangleq \overline{y^2} = 1/2+(\alpha\sin\theta)^2.$$

When $0 < \alpha\sin\theta = \epsilon \ll 1$, we have up-to third order terms versus ϵy,

$$\frac{\mathrm{M}_y\rho \mathrm{M}_y^\dagger}{\mathrm{Tr}\left(\mathrm{M}_y\rho \mathrm{M}_y^\dagger\right)} = \frac{(\cosh(\epsilon y)-\sinh(\epsilon y)\sigma_z)\rho(\cosh(\epsilon y)-\sinh(\epsilon y)\sigma_z)}{\cosh(2\epsilon y)-\sinh(2\epsilon y)\ \mathrm{Tr}\left(\sigma_z\rho\right)}$$

$$\approx \frac{\rho - \epsilon y(\sigma_z\rho+\rho\sigma_z) + (\epsilon y)^2(\rho+\sigma_z\rho\sigma_z)}{1-2\epsilon y\ \mathrm{Tr}\left(\sigma_z\rho\right) + 2(\epsilon y)^2}$$

$$= \rho + (\epsilon y)^2\left(\sigma_z\rho\sigma_z - \rho\right) + \left(\sigma_z\rho+\rho\sigma_z - 2\ \mathrm{Tr}\left(\sigma_z\rho\right)\rho\right)\left(-\epsilon y - 2(\epsilon y)^2\ \mathrm{Tr}\left(\sigma_z\rho\right)\right).$$

Replacing $\epsilon^2 y^2$ by its expectation value one gets, up to third order in ϵy and ϵ:

$$\frac{M_y \rho M_y^\dagger}{\mathrm{Tr}\left(M_y \rho M_y^\dagger\right)} \approx \rho + \frac{\epsilon^2}{2}\Big(\sigma_z \rho \sigma_z - \rho\Big) + \Big(\sigma_z \rho + \rho \sigma_z - 2\,\mathrm{Tr}\left(\sigma_z \rho\right)\rho\Big)\Big(-\epsilon y - \epsilon^2\,\mathrm{Tr}\left(\sigma_z \rho\right)\Big).$$

Set $\epsilon^2 = 2dt$ and $\epsilon y = -2\,\mathrm{Tr}\left(\sigma_z \rho\right) dt - dW$. Since by construction

$$\mathbb{E}\left(\epsilon y_k \;\middle|\; \rho_k = \rho\right) = -\epsilon^2\,\mathrm{Tr}\left(\sigma_z \rho\right) \text{ and } \mathbb{E}\left((\epsilon y_k)^2 \;\middle|\; \rho_k = \rho\right) = \epsilon^2 + \epsilon^4$$

one has $\mathbb{E}\left(dW \;\middle|\; \rho\right) = 0$ and $\mathbb{E}\left(dW^2 \;\middle|\; \rho\right) = dt$ up to order 4 versus ϵ. Thus for dt very small, we recover the following diffusive SME[5]

$$\rho_{t+dt} = \rho_t + dt\Big(\sigma_z \rho_t \sigma_z - \rho\Big) + \Big(\sigma_z \rho_t + \rho_t \sigma_z - 2\,\mathrm{Tr}\left(\sigma_z \rho_t\right)\rho\Big)\Big(dy_t - 2\,\mathrm{Tr}\left(\sigma_z \rho_t\right) dt\Big)$$

with $dy_t = 2\,\mathrm{Tr}\left(\sigma_z \rho_t\right) dt + dW_t$ replacing $-\epsilon y$ and $dy_t^2 = dW_t^2 = dt$ (Ito rules).

[5] Convergence in distribution when $dt \mapsto 0^+$: tightness property

$$\forall T > 0, \exists M > 0, \forall dt > 0, \forall k, k_1, k_2 \in \{0, \ldots, [T/dt]\}, \mathbb{E}\left(\|\rho_{k_1} - \rho_k\|^2 \|\; \|\rho_{k_2} - \rho_k\|^2 \;\middle|\; \rho_0\right) \leq M(k_1 - k_2)\,dt,$$

and (Markov generator) convergence of $\frac{\mathbb{E}\left(f(\rho_{k+1} \;\middle|\; \rho_k = \rho\right) - f(\rho)}{dt}$ towards $\mathbb{E}\left(df_t \;\middle|\; \rho_t = \rho\right)/dt$ for any C^2 real function f.

With measurement errors parameterized by $\sigma > 0$, the partial Kraus map

$$\mathcal{K}_y(\rho) = \frac{1}{\sqrt{\pi(1+\sigma)}} \Big(e^{-\frac{(y-\epsilon)^2}{1+\sigma}} \langle g|\rho|g\rangle |g\rangle\langle g| + e^{-\frac{(y+\epsilon)^2}{1+\sigma}} \langle e|\rho|e\rangle |e\rangle\langle e| + e^{-\frac{y^2}{1+\sigma}-\epsilon^2} \big(\langle e|\rho|g\rangle |e\rangle\langle g| + \langle g|\rho|e\rangle |g\rangle\langle e|\big)\Big)$$

yields $\mathbb{E}\left(y_k \;\middle|\; \rho_k\right) \triangleq \bar{y} = -\epsilon \operatorname{Tr}(\sigma_z\rho)$ and $\mathbb{E}\left(y_k^2 \;\middle|\; \rho_k\right) \triangleq \overline{y^2} = (1+\sigma)/2 + \epsilon^2$.

Similar approximations with $\epsilon^2 = 2dt$ and dt very small, yield an SME with detection efficiency $\eta = \frac{1}{1+\sigma}$:

$$\rho_{t+dt} = \rho_t + dt\Big(\sigma_z\rho_t\sigma_z - \rho\Big) + \sqrt{\eta}\Big(\sigma_z\rho_t + \rho_t\sigma_z - 2\operatorname{Tr}(\sigma_z\rho_t)\,\rho\Big)dW_t$$

with $dy_t = \sqrt{\eta}\operatorname{Tr}(\sigma_z\rho_t + \rho_t\sigma_z) + dW_t \sim -\epsilon y/\sqrt{1+\sigma}$.

Convergence towards either $|g\rangle$ or $|e\rangle$ (QND measurement of the qubit) based on Lyapunov fonction $V(\rho) = \sqrt{1 - \operatorname{Tr}(\sigma_z\rho)^2}$ and Ito rules:

$$dV = -\frac{zdz}{\sqrt{1-z^2}} - \frac{dz^2}{2(1-z^2)^{3/2}} = -\frac{zdz}{\sqrt{1-z^2}} - 2\eta^2 V dt$$

where $z = \operatorname{Tr}(\sigma_z\rho)$, $dz = 2\eta(1-z^2)dW$ and $dz^2 = 4\eta^2(1-z^2)^2 dt$. Since $\mathbb{E}\left(dz \;\middle|\; z\right) = 0$, $\bar{V}_t = \mathbb{E}\left(V(z_t) \;\middle|\; z_0\right)$ solution of $\frac{d}{dt}\bar{V}_t = -2\eta^2\bar{V}_t$.

General form of diffusive SME with Ito formulation:

$$d\rho_t = \left(-i[\mathsf{H}, \rho_t] + \sum_\nu \mathsf{L}_\nu \rho_t \mathsf{L}_\nu^\dagger - \frac{1}{2}(\mathsf{L}_\nu^\dagger \mathsf{L}_\nu \rho_t + \rho_t \mathsf{L}_\nu^\dagger \mathsf{L}_\nu) \right) dt$$
$$+ \sum_\nu \sqrt{\eta_\nu} \left(\mathsf{L}_\nu \rho_t + \rho_t \mathsf{L}_\nu^\dagger - \operatorname{Tr}\left((\mathsf{L}_\nu + \mathsf{L}_\nu^\dagger)\rho_t \right) \rho_t \right) dW_{\nu,t},$$
$$dy_{\nu,t} = \sqrt{\eta_\nu} \operatorname{Tr}\left(\mathsf{L}_\nu \rho_t + \rho_t \mathsf{L}_\nu^\dagger \right) dt + dW_{\nu,t}$$

with efficiencies $\eta_\nu \in [0,1]$ and $dW_{\nu,t}$ being independent Wiener processes. Equivalent formulation with Ito rules:

$$\rho_{t+dt} = \frac{\mathsf{M}_{dy_t} \rho_t \mathsf{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \mathsf{L}_\nu \rho_t \mathsf{L}_\nu^\dagger dt}{\operatorname{Tr}\left(\mathsf{M}_{dy_t} \rho_t \mathsf{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \mathsf{L}_\nu \rho_t \mathsf{L}_\nu^\dagger dt \right)}$$

with $\mathsf{M}_{dy_t} = \mathsf{I} + \left(-i\mathsf{H} - \frac{1}{2} \sum_\nu \mathsf{L}_\nu^\dagger \mathsf{L}_\nu \right) dt + \sum_\nu \sqrt{\eta_\nu} dy_{\nu,t} \mathsf{L}_\nu$. Moreover $dy_{\nu,t} = s_{\nu,t}\sqrt{dt}$ follows the following probability density knowing ρ_t:

$$\mathbb{P}\Big((s_{\nu,t} \in [s_\nu, s_\nu + ds_\nu])_\nu \mid \rho_t \Big) = \operatorname{Tr}\left(\mathsf{M}_{s\sqrt{dt}}\, \rho_t \mathsf{M}_{s\sqrt{dt}}^\dagger + \sum_\nu (1-\eta_\nu) \mathsf{L}_\nu \rho_t \mathsf{L}_\nu^\dagger dt \right) \prod_\nu \frac{e^{-\frac{s_\nu^2}{2}} ds_\nu}{\sqrt{2\pi}}.$$

[6] A. Barchielli and M. Gregoratti. *Quantum Trajectories and Measurements in Continuous Time: the Diffusive Case*. Springer Verlag, 2009.

Outline

Kraus maps and numerical schemes for diffusive SME[7]

Linearity/positivity/trace preserving numerical integration scheme for

$$d\rho_t = \left(-i[\mathrm{H}, \rho_t] + \sum_\nu \mathrm{L}_\nu \rho_t \mathrm{L}_\nu^\dagger - \frac{1}{2}(\mathrm{L}_\nu^\dagger \mathrm{L}_\nu \rho_t + \rho_t \mathrm{L}_\nu^\dagger \mathrm{L}_\nu) \right) dt + \sum_\nu \sqrt{\eta_\nu} \left(\mathrm{L}_\nu \rho_t + \rho_t \mathrm{L}_\nu^\dagger - \operatorname{Tr}\left((\mathrm{L}_\nu + \mathrm{L}_\nu^\dagger)\rho_t \right) \rho_t \right) dW_{\nu,t},$$

$$dy_{\nu,t} = \sqrt{\eta_\nu} \operatorname{Tr}\left(\mathrm{L}_\nu \rho_t + \rho_t \mathrm{L}_\nu^\dagger \right) dt + dW_{\nu,t}$$

With $\mathrm{M}_0 = \mathrm{I} + \left(-i\mathrm{H} - \frac{1}{2}\sum_\nu \mathrm{L}_\nu^\dagger \mathrm{L}_\nu \right) dt$, $\quad \mathrm{S} = \mathrm{M}_0^\dagger \mathrm{M}_0 + \left(\sum_\nu \mathrm{L}_\nu^\dagger \mathrm{L}_\nu \right) dt$ set

$$\widetilde{\mathrm{M}}_0 = \mathrm{M}_0 \mathrm{S}^{-1/2}, \quad \widetilde{\mathrm{L}}_\nu = \mathrm{L}_\nu \mathrm{S}^{-1/2}.$$

Sampling of $dy_{\nu,t} = s_{\nu,t}\sqrt{dt}$ according to the following probability law:

$$\mathbb{P}\left((s_{\nu,t} \in [s_\nu, s_\nu + ds_\nu])_\nu \mid \rho_t \right) = \operatorname{Tr}\left(\widetilde{\mathrm{M}}_{s\sqrt{dt}} \rho_t \widetilde{\mathrm{M}}_{s\sqrt{dt}}^\dagger + \sum_\nu (1-\eta_\nu) \widetilde{\mathrm{L}}_\nu \rho_t \widetilde{\mathrm{L}}_\nu^\dagger dt \right) \prod_\nu \frac{e^{-\frac{s_\nu^2}{2}} ds_\nu}{\sqrt{2\pi}}.$$

where $\widetilde{\mathrm{M}}_{dy_t} = \widetilde{\mathrm{M}}_0 + \sum_\nu \sqrt{\eta_\nu} dy_{\nu,t} \widetilde{\mathrm{L}}_\nu$. Exact Kraus-map formulation:

$$\rho_{t+dt} = \frac{\widetilde{\mathrm{M}}_{dy_t} \rho_t \widetilde{\mathrm{M}}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \widetilde{\mathrm{L}}_\nu \rho_t \widetilde{\mathrm{L}}_\nu^\dagger dt}{\operatorname{Tr}\left(\widetilde{\mathrm{M}}_{dy_t} \rho_t \widetilde{\mathrm{M}}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \widetilde{\mathrm{L}}_\nu \rho_t \widetilde{\mathrm{L}}_\nu^\dagger dt \right)}.$$

[7] A. Jordan, A. Chantasri, PR, and B.Huard. Anatomy of fluorescence: quantum trajectory statistics from continuously measuring spontaneous emission. *Quantum Studies: Mathematics and Foundations*, 3(3):237–263, 2016.

Outline

Probe photon is in the vacuum state $|0\rangle$. Composite qubit/photon wave function $|\Psi\rangle$ before D:

$$\Big(|g\rangle\langle g|\cos(\theta\sqrt{\mathrm{n}}) + |e\rangle\langle e|\cos(\theta\sqrt{\mathrm{n}+\mathrm{I}}) \\ + |g\rangle\langle e|\frac{\sin(\theta\sqrt{\mathrm{n}})}{\sqrt{\mathrm{n}}}\mathrm{a}^\dagger - |e\rangle\langle g|\mathrm{a}\frac{\sin(\theta\sqrt{\mathrm{n}})}{\sqrt{\mathrm{n}}}\Big)|\psi\rangle|0\rangle \\ = \big(\langle g|\psi\rangle\,|g\rangle + \cos\theta\,\langle e|\psi\rangle\,|e\rangle\big)|0\rangle + \sin\theta\,\langle e|\psi\rangle\,|g\rangle|1\rangle.$$

With measurement observable $\mathrm{n} = \sum_{n\geq 0} n|n\rangle\langle n|$, outcome $y \in \{0,1\}$ reads (density operator formulation)

$$\rho_{k+1} = \begin{cases} \frac{\mathrm{M}_0\rho_k\mathrm{M}_0^\dagger}{\mathrm{Tr}\left(\mathrm{M}_0\rho_k\mathrm{M}_0^\dagger\right)} & \text{if } y_k = 0 \text{ with probability } \mathrm{Tr}\left(\mathrm{M}_0\rho_k\mathrm{M}_0^\dagger\right); \\ \frac{\mathrm{M}_1\rho_k\mathrm{M}_1^\dagger}{\mathrm{Tr}\left(\mathrm{M}_1\rho_k\mathrm{M}_1^\dagger\right)} & \text{if } y_k = 1 \text{ with probability } \mathrm{Tr}\left(\mathrm{M}_1\rho_k\mathrm{M}_1^\dagger\right); \end{cases}$$

measurement Kraus operators $\mathrm{M}_0 = |g\rangle\langle g| + \cos\theta|e\rangle\langle e|$ and $\mathrm{M}_1 = \sin\theta|g\rangle\langle e|$. Almost convergence analysis when $\cos^2(\theta) < 1$ towards $|g\rangle$ via the Lyapunov function (super martingale)

$$V(\rho) = \mathrm{Tr}\left(|e\rangle\langle e|\rho\right) \text{ since } \mathbb{E}\left(V(\rho_{k+1}) \,\middle|\, \rho_k\right) = \cos^2\theta\; V(\rho_k).$$

Since $\mathrm{Tr}\left(\mathrm{M}_0\rho\mathrm{M}_0^\dagger\right) = 1 - \sin^2\theta\ \mathrm{Tr}\left(\sigma_-\rho\sigma_+\right)$ and $\mathrm{Tr}\left(\mathrm{M}_1\rho\mathrm{M}_1^\dagger\right) = \sin^2\theta\ \mathrm{Tr}\left(\sigma_-\rho\sigma_+\right)$, one gets with $\sin^2\theta = dt$ and $y \sim dN$, an SME driven by Poisson process $dN_t \in \{0,1\}$ of expectation value $\mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right) dt$ knowing ρ_t:

$$d\rho_t = \left(\sigma_-\rho_t\sigma_+ - \tfrac{1}{2}(\sigma_+\sigma_-\rho_t + \rho_t\sigma_+\sigma_-)\right)\,dt + \left(\frac{\sigma_-\rho_t\sigma_+}{\mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)} - \rho_t\right)\left(dN_t - \left(\mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)\right)dt\right).$$

At each time-step, one has the following choice:

- with probabilty $1 - \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right) dt$, $dN_t = N_{t+dt} - N_t = 0$ and

  $$\rho_{t+dt} = \frac{\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger}{\mathrm{Tr}\left(\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger\right)}$$

  with $\mathrm{M}_0 = \mathrm{I} - \frac{dt}{2}\sigma_+\sigma_-$.
- with probability $\mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right) dt$, $dN_t = N_{t+dt} - N_t = 1$ and

  $$\rho_{t+dt} = \frac{\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger}{\mathrm{Tr}\left(\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right)}$$

  with $\mathrm{M}_1 = \sqrt{dt}\ \sigma_-$.

With left stochastic matrix $\begin{pmatrix} 1-\bar{\theta}dt & 1-\bar{\eta} \\ \bar{\theta}dt & \bar{\eta} \end{pmatrix}$ including shot noise of rate $\bar{\theta} \geq 0$ and detection efficiency $\bar{\eta} \in [0,1]$:

- $dN_t = N_{t+dt} - N_t = 0$ and

$$\rho_{t+dt} = \frac{(1-\bar{\theta}dt)\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar{\eta})\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger}{\mathrm{Tr}\left((1-\bar{\theta}dt)\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar{\eta})\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right)}$$
$$= \frac{\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar{\eta})\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger}{\mathrm{Tr}\left(\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar{\eta})\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right)} + O(dt^2).$$

  with probability

$$1-\left(\bar{\theta}+\bar{\eta}\,\mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)\right)dt = \mathrm{Tr}\left((1-\bar{\theta}dt)\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar{\eta})\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right) + O(dt^2)$$

  and where $\mathrm{M}_0 = \mathrm{I} - \frac{dt}{2}\sigma_+\sigma_-$ and $\mathrm{M}_1 = \sqrt{dt}\,\sigma_-$.

- $dN_t = N_{t+dt} - N_t = 1$ and

$$\rho_{t+dt} = \frac{\bar{\theta}\,dt\,\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + \bar{\eta}\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger}{\mathrm{Tr}\left(\bar{\theta}\,dt\,\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + \bar{\eta}\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right)} = \frac{\bar{\theta}\rho_t + \bar{\eta}\sigma_-\rho_t\sigma_+}{\bar{\theta} + \bar{\eta}\,\mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)} + O(dt)$$

  with probability

$$\left(\bar{\theta} + \bar{\eta}\,\mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)\right)dt = \mathrm{Tr}\left(\bar{\theta}\,dt\,\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + \bar{\eta}\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right) + O(dt^2)$$

Jump SME with shot noise rate $\bar\theta$ and detection efficiency $\bar\eta$

$$d\rho_t = \left(\sigma_- \rho_t \sigma_+ - \tfrac{1}{2}(\sigma_+\sigma_- \rho_t + \rho_t \sigma_+\sigma_-)\right)\,dt$$
$$+ \left(\frac{\bar\theta\rho_t + \bar\eta\sigma_-\rho_t\sigma_+}{\mathrm{Tr}\left(\bar\theta\rho_t + \bar\eta\sigma_-\rho_t\sigma_+\right)} - \rho_t\right)\left(dN_t - \left(\bar\theta + \bar\eta\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)\right)dt\right).$$

corresponds to the following choices

- $dN_t = N_{t+dt} - N_t = 0$

$$\rho_{t+dt} = \frac{\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar\eta)\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger}{\mathrm{Tr}\left(\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar\eta)\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right)}$$

  with probability $1 - \left(\bar\theta + \bar\eta\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)\right)dt$,

- $dN_t = N_{t+dt} - N_t = 1$ and

$$\rho_{t+dt} = \frac{\bar\theta\rho_t + \bar\eta\sigma_-\rho_t\sigma_+}{\bar\theta + \bar\eta\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)}$$

  with probability $1 - \left(\bar\theta + \bar\eta\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)\right)dt$,

where $\mathrm{M}_0 = \mathrm{I} - \frac{dt}{2}(\sigma_+\sigma_- + \mathrm{I})$ and $\mathrm{M}_1 = \sqrt{dt}\ \sigma_-$.

Jump SME in continuous-time[8] (1)

General structure of a Jump SME in continuous time with counting process N_t with increment expectation value knowing ρ_t given by $\langle dN_t \rangle = \left(\bar\theta + \bar\eta \operatorname{Tr}\left(V\rho_t V^\dagger\right)\right) dt$, with $\bar\theta \geq 0$ (shot-noise rate) and $\bar\eta \in [0,1]$ (detection efficiency):

$$d\rho_t = \left(-i[\mathrm{H},\rho_t] + \mathrm{V}\rho_t\mathrm{V}^\dagger - \tfrac{1}{2}(\mathrm{V}^\dagger\mathrm{V}\rho_t + \rho_t\mathrm{V}^\dagger\mathrm{V})\right) dt$$
$$+ \left(\frac{\bar\theta\rho_t + \bar\eta\mathrm{V}\rho_t\mathrm{V}^\dagger}{\bar\theta + \bar\eta \operatorname{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)} - \rho_t\right)\left(dN_t - \left(\bar\theta + \bar\eta \operatorname{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)\right) dt\right).$$

Here H and V are operators on an underlying Hilbert space $\mathcal{H}$, H being Hermitian. At each time-step between t and $t + dt$, one has the following recipe

- $dN_t = 0$ with probability $1 - \left(\bar\theta + \bar\eta \operatorname{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)\right) dt$

$$\rho_{t+dt} = \frac{\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar\eta)\mathrm{V}\rho_t\mathrm{V}^\dagger dt}{\operatorname{Tr}\left(\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar\eta)\mathrm{V}\rho_t\mathrm{V}^\dagger dt\right)}$$

  where $\mathrm{M}_0 = I - \left(iH + \frac{1}{2}\mathrm{V}^\dagger\mathrm{V}\right) dt$.

- $dN_t = 1$ with probability $\left(\bar\theta + \bar\eta \operatorname{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)\right) dt$,

$$\rho_{t+dt} = \frac{\bar\theta\rho_t + \bar\eta\mathrm{V}\rho_t\mathrm{V}^\dagger}{\bar\theta + \bar\eta \operatorname{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)}.$$

[8] J. Dalibard, Y. Castin, and K. Mølmer. Wave-function approach to dissipative processes in quantum optics. *Phys. Rev. Lett.*, 68(5):580–583, 1992.

Outline

General mixed diffusive/jump SME (1)

Combine in a single SME Wiener and Poisson noises induced by diffusive and counting measurements:

$$d\rho_t = \left(-i[\mathsf{H},\rho_t] + \mathsf{L}\rho_t\mathsf{L}^\dagger - \tfrac{1}{2}(\mathsf{L}^\dagger\mathsf{L}\rho_t + \rho_t\mathsf{L}^\dagger\mathsf{L}) + \mathsf{V}\rho_t\mathsf{V}^\dagger - \tfrac{1}{2}(\mathsf{V}^\dagger\mathsf{V}\rho_t + \rho_t\mathsf{V}^\dagger\mathsf{V})\right)\,dt$$
$$+ \sqrt{\eta}\left(\mathsf{L}\rho_t + \rho_t\mathsf{L}^\dagger - \operatorname{Tr}\left((\mathsf{L}+\mathsf{L}^\dagger)\rho_t\right)\rho_t\right)dW_t$$
$$+ \left(\frac{\bar\theta\rho_t + \bar\eta\mathsf{V}\rho_t\mathsf{V}^\dagger}{\bar\theta + \bar\eta\operatorname{Tr}\left(\mathsf{V}\rho_t\mathsf{V}^\dagger\right)} - \rho_t\right)\left(dN_t - \left(\bar\theta + \bar\eta\operatorname{Tr}\left(\mathsf{V}\rho_t\mathsf{V}^\dagger\right)\right)dt\right)$$

With $dy_t = \sqrt{\eta}\operatorname{Tr}\left((\mathsf{L}+\mathsf{L}^\dagger)\rho_t\right)dt + dW_t$ and $dN_t = 0$ with probability $1 - \left(\bar\theta + \bar\eta\operatorname{Tr}\left(\mathsf{V}\rho_t\mathsf{V}^\dagger\right)\right)dt$. Kraus-map equivalent formulation:

- for $dN_t = 0$ of probability $1 - \left(\bar\theta + \bar\eta\operatorname{Tr}\left(\mathsf{V}\rho_t\mathsf{V}^\dagger\right)\right)dt$

$$\rho_{t+dt} = \frac{\mathsf{M}_{dy_t}\rho_t\mathsf{M}_{dy_t}^\dagger + (1-\eta)\mathsf{L}\rho_t\mathsf{L}^\dagger dt + (1-\bar\eta)\mathsf{V}\rho_t\mathsf{V}^\dagger dt}{\operatorname{Tr}\left(\mathsf{M}_{dy_t}\rho_t\mathsf{M}_{dy_t}^\dagger + (1-\eta)\mathsf{L}\rho_t\mathsf{L}^\dagger dt + (1-\bar\eta)\mathsf{V}\rho_t\mathsf{V}^\dagger dt\right)}$$

  with $\mathsf{M}_{dy_t} = I - \left(i\mathsf{H} + \tfrac{1}{2}\mathsf{L}^\dagger\mathsf{L} + \tfrac{1}{2}\mathsf{V}^\dagger\mathsf{V}\right)dt + \sqrt{\eta}dy_t\mathsf{L}$.

- for $dN_t = 1$ of probability $\left(\bar\theta + \bar\eta\operatorname{Tr}\left(\mathsf{V}\rho_t\mathsf{V}^\dagger\right)\right)dt$:

$$\rho_{t+dt} = \frac{\mathsf{M}_{dy_t}\tilde\rho_t\mathsf{M}_{dy_t}^\dagger + (1-\eta)\mathsf{L}\tilde\rho_t\mathsf{L}^\dagger dt + (1-\bar\eta)\mathsf{V}\tilde\rho_t\mathsf{V}^\dagger dt}{\operatorname{Tr}\left(\mathsf{M}_{dy_t}\tilde\rho_t\mathsf{M}_{dy_t}^\dagger + (1-\eta)\mathsf{L}\tilde\rho_t\mathsf{L}^\dagger dt + (1-\bar\eta)\mathsf{V}\tilde\rho_t\mathsf{V}^\dagger dt\right)} \quad \text{with } \tilde\rho_t = \frac{\bar\theta\rho_t + \bar\eta\mathsf{V}\rho_t\mathsf{V}^\dagger}{\bar\theta + \bar\eta\operatorname{Tr}\left(\mathsf{V}\rho_t\mathsf{V}^\dagger\right)}$$

$$d\rho_t = \left(-i[\mathbf{H},\rho_t] + \sum_\nu \mathbf{L}_\nu \rho_t \mathbf{L}_\nu^\dagger - \tfrac{1}{2}(\mathbf{L}_\nu^\dagger \mathbf{L}_\nu \rho_t + \rho_t \mathbf{L}_\nu^\dagger \mathbf{L}_\nu) + \sum_\mu \mathbf{V}_\mu \rho_t \mathbf{V}_\mu^\dagger - \tfrac{1}{2}(\mathbf{V}_\mu^\dagger \mathbf{V}_\mu \rho_t + \rho_t \mathbf{V}_\mu^\dagger \mathbf{V}_\mu)\right) dt$$

$$+ \sum_\nu \sqrt{\eta_\nu}\left(\mathbf{L}_\nu \rho_t + \rho_t \mathbf{L}_\nu^\dagger - \operatorname{Tr}\left((\mathbf{L}_\nu + \mathbf{L}_\nu^\dagger)\rho_t\right)\rho_t\right) dW_{\nu,t}$$

$$+\sum_\mu \left(\frac{\bar\theta_\mu \rho_t + \sum_{\mu'} \bar\eta_{\mu,\mu'} \mathbf{V}_{\mu'} \rho_t \mathbf{V}_{\mu'}^\dagger}{\bar\theta_\mu + \sum_{\mu'} \bar\eta_{\mu,\mu'} \operatorname{Tr}\left(\mathbf{V}_{\mu'} \rho_t \mathbf{V}_{\mu'}^\dagger\right)} - \rho_t\right)\left(dN_{\mu,t} - \left(\bar\theta_\mu + \sum_{\mu'} \bar\eta_{\mu,\mu'} \operatorname{Tr}\left(\mathbf{V}_{\mu'} \rho_t \mathbf{V}_{\mu'}^\dagger\right)\right) dt\right)$$

where $\eta_\nu \in [0,1]$, $\bar\theta_\mu, \bar\eta_{\mu,\mu'} \geq 0$ with $\bar\eta_{\mu'} = \sum_\mu \bar\eta_{\mu,\mu'} \leq 1$. The equivalent Kraus-map formulation

- When $\forall \mu,\ dN_{\mu,t} = 0$ (probability $1 - \sum_\mu \left(\bar\theta_\mu + \bar\eta_\mu \operatorname{Tr}\left(\mathbf{V}_\mu \rho_t \mathbf{V}_\mu^\dagger\right)\right) dt$) we have

$$\rho_{t+dt} = \frac{\mathbf{M}_{dy_t} \rho_t \mathbf{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \mathbf{L}_\nu \rho_t \mathbf{L}_\nu^\dagger dt + \sum_\mu (1-\bar\eta_\mu) \mathbf{V}_\mu \rho_t \mathbf{V}_\mu^\dagger dt}{\operatorname{Tr}\left(\mathbf{M}_{dy_t} \rho_t \mathbf{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \mathbf{L}_\nu \rho_t \mathbf{L}_\nu^\dagger dt + \sum_\mu (1-\bar\eta_\mu) \mathbf{V}_\mu \rho_t \mathbf{V}_\mu^\dagger dt\right)}$$

  with $\mathbf{M}_{dy_t} = I - \left(iH + \frac{1}{2}\sum_\nu \mathbf{L}_\nu^\dagger \mathbf{L}_\nu + \frac{1}{2}\sum_\mu \mathbf{V}_\mu^\dagger \mathbf{V}_\mu\right) dt + \sum_\nu \sqrt{\eta_\nu} dy_{\nu t} \mathbf{L}_\nu$ and where $dy_{\nu,t} = \sqrt{\eta_\nu} \operatorname{Tr}\left((\mathbf{L}_\nu + \mathbf{L}_\nu^\dagger)\rho_t\right) dt + dW_{\nu,t}$.

- If, for some μ, $dN_{\mu,t} = 1$ (probability $\left(\bar\theta_\mu + \sum_{\mu'} \bar\eta_{\mu,\mu'} \operatorname{Tr}\left(\mathbf{V}_{\mu'} \rho_t \mathbf{V}_{\mu'}^\dagger\right)\right) dt$) we have

$$\rho_{t+dt} = \frac{\mathbf{M}_{dy_t} \tilde\rho_t \mathbf{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \mathbf{L}_\nu \tilde\rho_t \mathbf{L}_\nu^\dagger dt + \sum_{\mu'} (1-\bar\eta_{\mu'}) \mathbf{V}_{\mu'} \tilde\rho_t \mathbf{V}_{\mu'}^\dagger dt}{\operatorname{Tr}\left(\mathbf{M}_{dy_t} \tilde\rho_t \mathbf{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \mathbf{L}_\nu \tilde\rho_t \mathbf{L}_\nu^\dagger dt + \sum_{\mu'} (1-\bar\eta_{\mu'}) \mathbf{V}_{\mu'} \tilde\rho_t \mathbf{V}_{\mu'}^\dagger dt\right)}$$

  with $\tilde\rho_t = \dfrac{\bar\theta_\mu \rho_t + \sum_{\mu'} \bar\eta_{\mu,\mu'} \mathbf{V}_{\mu'} \rho_t \mathbf{V}_{\mu'}^\dagger}{\bar\theta_\mu + \sum_{\mu'} \bar\eta_{\mu,\mu'} \operatorname{Tr}\left(\mathbf{V}_{\mu'} \rho_t \mathbf{V}_{\mu'}^\dagger\right)}$.

[9] H. Amini, C. Pellegrini, and PR. Stability of continuous-time quantum filters with measurement imperfections. *Russian Journal of Mathematical Physics*, 21(3):297–315–, 2014.

Four features[10]:

1. Bayes law: $\mathbb{P}(\mu'/\mu) = \mathbb{P}(\mu/\mu')\mathbb{P}(\mu') \,/\, \left(\sum_{\nu'} \mathbb{P}(\mu/\nu')\mathbb{P}(\nu')\right)$,
2. Schrödinger equations defining unitary transformations.
3. Randomness, irreversibility and dissipation induced by the measurement of observables with degenerate spectra.
4. Entanglement and tensor product for composite systems.

$\Rrightarrow$ **Discrete-time models with parameter** p

Take a set of operators M^{p}_μ satisfying $\sum_\mu (M^{\mathsf{p}}_\mu)^\dagger M^{\mathsf{p}}_\mu = I$ and a left stochastic matrices $(\eta^{\mathsf{p}}_{y_t,\mu})$. Consider the following Markov process of state ρ (density op.)and measured output y:

$$\rho_{t+1} = \frac{\mathcal{K}^{\mathsf{p}}_{y_t}(\rho_t)}{\mathrm{Tr}\left(\mathcal{K}^{\mathsf{p}}_{y_t}(\rho_t)\right)}, \text{ with proba. } \mathbb{P}_{y_t}(\rho_t) = \mathrm{Tr}\left(\mathcal{K}^{\mathsf{p}}_{y_t}(\rho_t)\right)$$

with $\mathcal{K}^{\mathsf{p}}_y(\rho) = \sum_{\mu=1}^m \eta^{\mathsf{p}}_{y,\mu} M^{\mathsf{p}}_\mu \rho (M^{\mathsf{p}}_\mu)^\dagger$. It is associated to the Kraus map (ensemble average, quantum channel)

$$\mathbb{E}\left(\rho_{t+1}|\rho_t \;\Big|\; =\right) \mathcal{K}^{\mathsf{p}}(\rho_t) = \sum_y \mathcal{K}^{\mathsf{p}}_y(\rho_t) = \sum_\mu M^{\mathsf{p}}_\mu \rho_t (M^{\mathsf{p}}_\mu)^\dagger.$$

[10] See the book of S. Haroche and J.M. Raimond.

- Denote by $\mathbb{P}_n(\rho, \mathrm{p})$ the probability of getting measurement trajectory n, $(\mathrm{y}_\mathrm{t}^{(\mathrm{n})})_{t=0,\dots,T}$, knowing the initial state $\rho_0^{(n)} = \rho$ and parameter p.
- Since $\rho_{t+1}^{(n)} = \dfrac{\mathcal{K}^{\mathrm{p}}_{\mathrm{y}_\mathrm{t}^{(\mathrm{n})}}\left(\rho_t^{(n)}\right)}{\mathrm{Tr}\left(\mathcal{K}^{\mathrm{p}}_{\mathrm{y}_\mathrm{t}^{(\mathrm{n})}}\left(\rho_t^{(n)}\right)\right)}$ with $\mathrm{Tr}\left(\mathcal{K}^{\mathrm{p}}_{\mathrm{y}_\mathrm{t}^{(\mathrm{n})}}\left(\rho_t^{(n)}\right)\right)$ the probability of having detected $\mathrm{y}_\mathrm{t}^{(\mathrm{n})}$ knowing $\rho_t^{(n)}$ and p, a direct use of Bayes law yields

$$\mathbb{P}_n(\rho, \mathrm{p}) = \prod_{t=0}^{T} \mathrm{Tr}\left(\mathcal{K}^{\mathrm{p}}_{\mathrm{y}_\mathrm{t}^{(\mathrm{n})}}\left(\rho_t^{(n)}\right)\right) = \mathrm{Tr}\left(\mathcal{K}^{\mathrm{p}}_{\mathrm{y}_\mathrm{T}^{(\mathrm{n})}} \circ \ldots \circ \mathcal{K}^{\mathrm{p}}_{\mathrm{y}_0^{(\mathrm{n})}}(\rho)\right).$$

- With adjoint map $\mathcal{K}_{y}^{p*}$ $(\forall A, B,\ \text{Tr}\left(\mathcal{K}_{y}^{p}(A)\ B\right) \equiv\ \text{Tr}\left(A\ \mathcal{K}_{y}^{p*}(B)\right))$:

$$\mathbb{P}_n(\rho, p) = \ \text{Tr}\left(\mathcal{K}^{p}_{y_T^{(n)}} \circ \ldots \circ \mathcal{K}^{p}_{y_0^{(n)}}(\rho) \quad I\right) = \ \text{Tr}\left(\rho \quad \mathcal{K}^{p*}_{y_0^{(n)}} \circ \ldots \circ \mathcal{K}^{p*}_{y_T^{(n)}}(I)\right).$$

- Normalized adjoint quantum filter[11] $E_t^{(n)} = \dfrac{\mathcal{K}^{p*}_{y_t^{(n)}}\left(E_{t+1}^{(n)}\right)}{\text{Tr}\left(\mathcal{K}^{p*}_{y_t^{(n)}}\left(E_{t+1}^{(n)}\right)\right)}$ with $E_{T+1}^{(n)} = I/\ \text{Tr}\,(I)$, we get

$$\mathbb{P}_n(\rho, p) = \prod_{t=T}^{0} \text{Tr}\left(\mathcal{K}^{p*}_{y_t^{(n)}}\left(E_{t+1}^{(n)}\right)\right)\ \text{Tr}\left(\rho E_0^{(n)}\right) \triangleq g_n(Y, p)\ \text{Tr}\left(\rho E_0^{(n)}\right).$$

- A simple expression of the gradients:

$$\nabla_{\rho} \log \mathbb{P}_n = \frac{E_0^{(n)}}{\text{Tr}\left(\rho E_0^{(n)}\right)}, \quad \nabla_{p} \log \mathbb{P}_n \cdot \delta p = \sum_{t=0}^{T} \frac{\text{Tr}\left(E_{t+1}^{(n)}\ \left(\nabla_{p} \mathcal{K}^{p}_{y_t^{(n)}}\left(\rho_t^{(n)}\right) \cdot \delta p\right)\right)}{\text{Tr}\left(E_{t+1}^{(n)}\ \mathcal{K}^{p}_{y_t^{(n)}}\left(\rho_t^{(n)}\right)\right)},$$

[11] M. Tsang. Time-symmetric quantum theory of smoothing. PRL 2009. S. Gammelmark, B. Julsgaard, and K. Mølmer. Past quantum states of a monitored system. PRL 2013.

From $\mathbb{P}_{\mathrm{n}}(\rho, \mathrm{p}) = g_n(\mathrm{Y}, \mathrm{p})\ \mathrm{Tr}\left(\rho E_0^{(n)}\right)$ we have

$$\mathbb{P}(\rho, \mathrm{p}) \triangleq \prod_{\mathrm{n}=1}^{N} \mathbb{P}_{\mathrm{n}}(\rho, \mathrm{p}) = \left(\prod_{n=1}^{N} g_n(\mathrm{Y}, \mathrm{p})\right)\left(\prod_{n=1}^{N} \mathrm{Tr}\left(\rho E_0^{(n)}\right)\right).$$

- MaxLike **state tomography:** p is known and ρ_{ML} maximizes

$$\rho \mapsto \sum_{n=1}^{N} \log\left(\mathrm{Tr}\left(\rho E_0^{(n)}\right)\right)$$

  a concave function on the convex set of density operators ρ:
  a well structured convex optimization problem.

- MaxLike **process tomography:** ρ is known
  and p_{ML} maximizes $\mathrm{p} \mapsto f(\mathrm{p}) = \log \mathbb{P}(\rho, \mathrm{p})$ those gradient is given by

$$\nabla_{\mathrm{p}} f(\mathrm{p}) \cdot \delta\mathrm{p} = \sum_{n=1}^{N} \sum_{t=0}^{T} \frac{\mathrm{Tr}\left(E_{t+1}^{(n)}\ \left(\nabla_{\mathrm{p}} \mathcal{K}_{\mathrm{y}_{\mathrm{t}}^{(\mathrm{n})}}^{\mathrm{p}}\left(\rho_t^{(n)}\right) \cdot \delta\mathrm{p}\right)\right)}{\mathrm{Tr}\left(E_{t+1}^{(n)}\ \mathcal{K}_{\mathrm{y}_{\mathrm{t}}^{(\mathrm{n})}}^{\mathrm{p}}\left(\rho_t^{(n)}\right)\right)},$$

  The Hessian $\nabla_{\mathrm{p}}^2 f$ can be computed similarly (Fisher information).

- For $\rho_{k+1} = \mathcal{K}(\rho_k)$, contraction for many distances[12] (nuclear norm, fidelity,...)
- Adjoint map (unital map) $A_{k+1} = \mathcal{K}^*(A_k)$ contracts spectrum[13]:

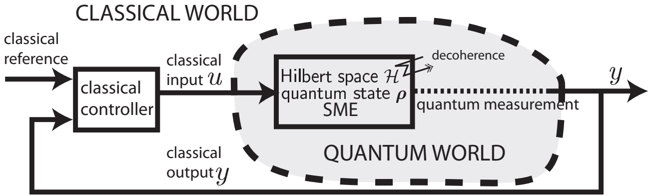
$$\lambda_{min}(A_k) \leq \lambda_{min}(A_{k+1}) \leq \lambda_{max}(A_{k+1}) \leq \lambda_{max}(A_k).$$

- Quantum filter $\hat{\rho}_{k+1} = \frac{\mathcal{K}_{y_k}(\hat{\rho}_k)}{\mathrm{Tr}\left(\mathcal{K}_{y_k}(\hat{\rho}_k)\right)}$ where y_k is governed by $\rho_{k+1} = \frac{\mathcal{K}_{y_k}(\rho_k)}{\mathrm{Tr}\left(\mathcal{K}_{y_k}(\rho_k)\right)}$ with $\hat{\rho}_0 \neq \rho_0$: fidelity $\mathrm{Tr}^2\left(\sqrt{\sqrt{\hat{\rho}_k}\rho_k\sqrt{\hat{\rho}_k}}\right)$ is always a sub-martingale[14]
- Convergence issues around filtering and parameter estimation along quantum trajectories: seminal works of Belavkin in continuous-time, Van-Handel thesis at Caltech 2007. See also recent works of Nina Amini, Maël Bompais, Tristan Benoit and Clément Pellegrini.

[12] D. Petz. Monotone metrics on matrix spaces. *Linear Algebra and its Applications*, 244:81–96, 1996.

[13] R. Sepulchre, A. Sarlette, and PR.. Consensus in non-commutative spaces. In *Decision and Control (CDC), 2010 49th IEEE Conference on*, pages 6596–6601, 2010.

[14] PR. Fidelity is a sub-martingale for discrete-time quantum filters. *IEEE Transactions on Automatic Control*, 56(11):2743–2747, 2011.

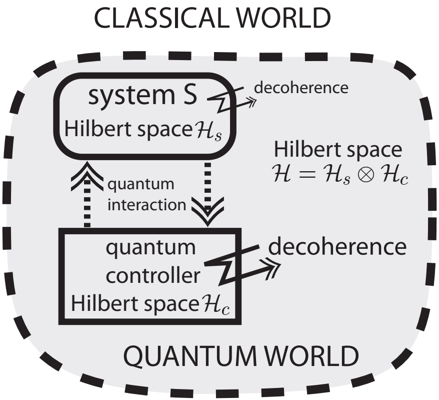


- P-controller (Markovian feedback[15]) for $u_t\ dt = k\ dy_t$, the ensemble average closed-loop dynamics of ρ remains governed by a linear Lindblad master equation.
- PID controller: no Lindblad master equation in closed-loop for dynamics output feedback
- Nonlinear hidden-state stochastic systems: Lyapunov state-feedback[16]; many open issues on convergence rates, delays, robustness, ...
- **Short sampling times limit feedback complexity**

[15] H. Wiseman, G. Milburn (2009). Quantum Measurement and Control. Cambridge University Press.

[16] See e.g.: C. Ahn et. al (2002): Continuous quantum error correction via quantum feedback control. Phys. Rev. A 65;
M. Mirrahimi, R. Handel (2007): Stabilizing feedback controls for quantum systems. SIAM Journal on Control and Optimization, 46(2), 445-467;
G. Cardona, A. Sarlette, PR (2019): Continuous-time quantum error correction with noise-assisted quantum feedback. IFAC Mechatronics & Nolcos Conf.

Quantum analogue of Watt speed governor: a **dissipative** mechanical system controls another mechanical system [17]



Optical pumping (Kastler 1950), coherent population trapping (Arimondo 1996)

Dissipation engineering, autonomous feedback: (Zoller, Cirac, Wolf, Verstraete, Devoret, Schoelkopf, Siddiqi, Martinis, Mølmer, Raimond, Brune,. . . , Lloyd, Viola, Ticozzi, Leghtas, Mirrahimi, Sarlette, PR, ...)

(S,L,H) theory and linear quantum systems: quantum feedback networks based on stochastic Schrödinger equation, Heisenberg picture (Gardiner, Yurke, Mabuchi, Genoni, Serafini, Milburn, Wiseman, Doherty, . . . , Gough, James, Petersen, Nurdin, Yamamoto, Zhang, Dong, . . .)

Stability analysis: Kraus maps and Lindblad propagators are always contractions (non commutative diffusion and consensus).

[17] J.C. Maxwell (1868): On governors. Proc. of the Royal Society, No.100.

The closed-loop Lindblad master equation on $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_c$:

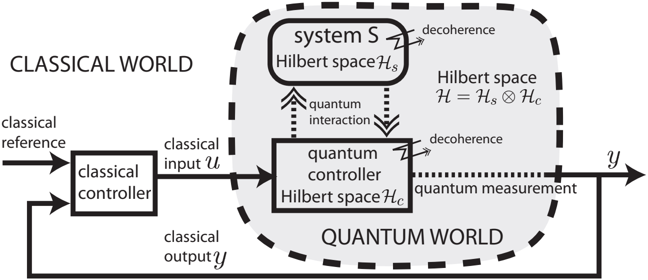
$$\frac{d}{dt}\rho = -i\Big[\mathsf{H}_s \otimes \mathsf{I}_c + \mathsf{I}_s \otimes \mathsf{H}_c + \mathsf{H}_{sc}\ ,\ \rho\Big] + \sum_{\nu} \mathbb{D}_{\mathsf{L}_{s,\nu}\otimes \mathsf{I}_c}(\rho) + \sum_{\nu'} \mathbb{D}_{\mathsf{I}_s \otimes \mathsf{L}_{c,\nu'}}(\rho)$$

with $\mathbb{D}_{\mathsf{L}}(\rho) = \mathsf{L}\rho\mathsf{L}^\dagger - \frac{1}{2}\left(\mathsf{L}^\dagger\mathsf{L}\rho + \rho\mathsf{L}^\dagger\mathsf{L}\right)$ and operators made of tensor products.

• Consider a convex subset $\overline{\mathcal{D}}_s$ of steady-states for original system S: each density operator $\overline{\rho}_s$ on $\mathcal{H}_s$ belonging to $\overline{\mathcal{D}}_s$ satisfy $i[\mathsf{H}_s, \overline{\rho}_s] = \sum_\nu \mathbb{D}_{\mathsf{L}_{s,\nu}}(\overline{\rho}_s)$.

• Designing a **realistic** quantum controller C $(\mathsf{H}_c, \mathsf{L}_{c,\nu'})$ and coupling Hamiltonian H_{sc} stabilizing $\overline{\mathcal{D}}_s$ is non trivial. **Realistic** means in particular relying on physical time-scales and constraints:

- Fastest time-scales attached to H_s and H_c (Bohr frequencies) and averaging approximations: $\|\mathsf{H}_s\|, \|\mathsf{H}_c\| \gg \|\mathsf{H}_{sc}\|$,
- High-quality oscillations: $\|\mathsf{H}_s\| \gg \|\mathsf{L}_{s,\nu}^\dagger \mathsf{L}_{s,\nu}\|$ and $\|\mathsf{H}_c\| \gg \|\mathsf{L}_{c,\nu'}^\dagger \mathsf{L}_{c,\nu'}\|$.
- Decoherence rates of S much slower than those of C: $\|\mathsf{L}_{s,\nu}^\dagger \mathsf{L}_{s,\nu}\| \ll \|\mathsf{L}_{c,\nu'}^\dagger \mathsf{L}_{c,\nu'}\|$: model reduction by quasi-static approximations (adiabatic elimination, singular perturbations).



To protect quantum information stored in system S (alternative to usual QEC):

- fast stabilization and protection mainly achieved by a **quantum controller** (coherent feedback stabilizing decoherence-free sub-spaces);
- slow decoherence and perturbations mainly tackled by a **classical controller** (measurement-based feedback "finishing the job")

Underlying mathematical methods for high-precision dynamical modeling and control based on stochastic master equations (SME):

- High-order averaging methods and geometric singular perturbations for coherent feedback.
- Stochastic control Lyapunov methods for exponential stabilization via measurement-based feedback.

Quantic research group ENS/Inria/Mines/CNRS, March 2022