Feedback et correction d'erreurs quantiques

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Underlying issues

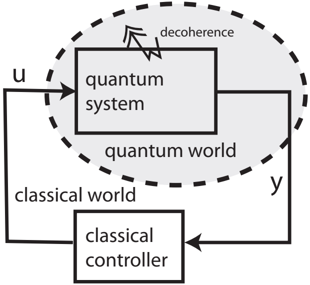
Quantum Error Correction (QEC) is based on a discrete-time feedback loop

- Current experiments: 10^{-3} is the typical error probability during elementary gates (manipulations) involving few physical qubits.
- High-order error-correcting codes with an important overhead; **more than 1000 physical qubits to encode a controllable logical qubit**[1].
- Today, no such controllable logical qubit has been built.
- Key issue: reduction by several magnitude orders of such error rates, far below the threshold required by actual QEC, to build a controllable logical qubit encoded in a reasonable number of physical qubits and protected by QEC.

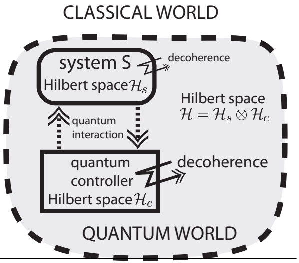
Control engineering can play a crucial role to build a controllable logical qubit protected by **adapted feedback schemes increasing precision and stability.**

[1] A.G. Fowler, M. Mariantoni, J.M. Martinis, A.N. Cleland: Surface codes: Towards practical large-scale quantum computation. Phys. Rev. A, 2012.

Two kinds of quantum feedback[2]



Measurement-based feedback: **controller is classical**; measurement back-action on the quantum system of Hilbert space $\mathcal{H}$ is stochastic (collapse of the wave-packet); the measured output y is a classical signal; the control input u is a classical variable appearing in some controlled Schrödinger equation; $u(t)$ depends on the past measurements $y(\tau)$, $\tau \leq t$.



Coherent/autonomous feedback and reservoir/dissipation engineering: the **system of Hilbert space $\mathcal{H}_s$** is coupled to **the controller, another quantum system**; the composite system of Hilbert space $\boldsymbol{\mathcal{H}_s} \otimes \boldsymbol{\mathcal{H}_c}$, is an open-quantum system relaxing to some target (separable) state. Relaxation behaviors in open quantum systems can be exploited: optical pumping of Alfred Kastler, physics Nobel prize 1966.

[2]Wiseman/Milburn: Quantum Measurement and Control, 2009, Cambridge University Press.

Outline

Quantum error correction with classical controllers

- Classical 3-bit code correcting bit-flip
- 9-qubit Shor code correcting bit-flip and phase-flip
- Dynamics based on Stochastic Master Equations (SME)

Quantum error correction with quantum controllers

- Quantum dissipation engineering
- Dynamics based on deterministic master equations
- Cat-qubit and autonomous correction of bit-flips

Stabilisation of GKP-qubit with quantum controllers

- Phase-space local errors and cat-qubit/GKP-qubit
- Autonomous correction of bit/phase-flips
- Towards implementation with super-conducting circuits

The simplest classical error correction code

• Single bit error model: the bit $b \in \{0,1\}$ flips with probability $p < 1/2$ during Δt (for usual DRAM: $p/\Delta t \leq 10^{-14}$ s^{-1}).
• Multi-bit error model: each bit $b_k \in \{0,1\}$ flips with probability $p < 1/2$ during Δt; no correlation between the bit flips.
•Use redundancy to construct with several physical bits b_k of flip probability p, a logical bit b_L with a flip probability $p_L < p$.
• The simplest solution, the 3-bit code (sampling time Δt):

$t = 0$: $b_L = [bbb]$ with $b \in \{0,1\}$

$t = \Delta t$: measure the three physical bits of $b_L = [b_1 b_2 b_3]$ (instantaneous) :

1. if all 3 bits coincide, nothing to do.
2. if one bit differs from the two other ones, flip this bit (instantaneous);

• Since the flip probability laws of the physical bits are independent, the probability that the logical bit b_L (protected with the above error correction code) flips during Δt is $\mathrm{p_L} = 3\mathrm{p}^2 - 2\mathrm{p}^3 < p$ since $p < 1/2$.

Outline

Dynamics of open quantum systems based on three quantum features [3]

1. Schrödinger ($\hbar = 1$): wave funct. $|\psi\rangle \in \mathcal{H}$, density op. $\widehat{\rho} \sim |\psi\rangle\langle\psi|$

$$\frac{d}{dt}|\psi\rangle = -i\widehat{H}|\psi\rangle, \quad \widehat{H} = \widehat{H}_0 + u\widehat{H}_1 = \widehat{H}^\dagger, \quad \frac{d}{dt}\widehat{\rho} = -i[\widehat{H}, \widehat{\rho}].$$

2. Origin of dissipation: collapse of the wave packet induced by the measurement of $\widehat{O} = \widehat{O}^\dagger$ with spectral decomp. $\sum_y \lambda_y \widehat{P}_y$:
   - measurement outcome y with proba. $\mathbb{P}_y = \langle\psi|\widehat{P}_y|\psi\rangle = \text{Tr}\left(\widehat{\rho}\widehat{P}_y\right)$ depending on $|\psi\rangle$, $\widehat{\rho}$ just before the measurement
   - measurement back-action if outcome y:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\widehat{P}_y|\psi\rangle}{\sqrt{\langle\psi|\widehat{P}_y|\psi\rangle}}, \quad \widehat{\rho} \mapsto \widehat{\rho}_+ = \frac{\widehat{P}_y\widehat{\rho}\widehat{P}_y}{\text{Tr}\left(\widehat{\rho}\widehat{P}_y\right)}$$

3. Tensor product for the description of composite systems (A, B):
   - Hilbert space $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$
   - Hamiltonian $\widehat{H} = \widehat{H}_a \otimes \widehat{I}_b + \widehat{H}_{ab} + \widehat{I}_a \otimes \widehat{H}_b$
   - observable on sub-system B only: $\widehat{O} = \widehat{I}_a \otimes \widehat{O}_b$.

[3] S. Haroche and J.M. Raimond (2006). *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts.

The 3-qubit bit-flip code (Peter Shor (1995))[5]

- Local bit-flip errors: each physical qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ becomes $\widehat{X}|\psi\rangle = \alpha|1\rangle + \beta|0\rangle$ [4] with probability $p < 1/2$ during Δt. (2023 best super-conducting qubit $p/\Delta t \sim 10^3$ s^{-1}).
- $t = 0$: $|\psi_L\rangle = \alpha|0_L\rangle + \beta|1_L\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \equiv \mathbb{C}^8$ with $|0_L\rangle = |000\rangle$ and $|1_L\rangle = |111\rangle$.
- $t = \Delta t$: $|\psi_L\rangle = \alpha|000\rangle + \beta|111\rangle$ becomes with

$$\text{1 flip:} \left\{ \begin{array}{l} \alpha|100\rangle + \beta|011\rangle \\ \alpha|010\rangle + \beta|101\rangle \\ \alpha|001\rangle + \beta|110\rangle \end{array} \right. ; \ \text{2 flips:} \left\{ \begin{array}{l} \alpha|110\rangle + \beta|001\rangle \\ \alpha|101\rangle + \beta|010\rangle \\ \alpha|011\rangle + \beta|100\rangle \end{array} \right. ; \ \text{3 flips: } \alpha|111\rangle + \beta|000\rangle.$$

- Key fact: 4 orthogonal planes $\mathcal{P}_c = \mathrm{span}(|000\rangle, |111\rangle)$, $\mathcal{P}_1 = \mathrm{span}(|100\rangle, |011\rangle)$, $\mathcal{P}_2 = \mathrm{span}(|010\rangle, |101\rangle)$ and $\mathcal{P}_3 = \mathrm{span}(|001\rangle, |110\rangle)$.
- Error syndromes: 3 commuting observables $\widehat{S}_1 = \widehat{I} \otimes \widehat{Z} \otimes \widehat{Z}$, $\widehat{S}_2 = \widehat{Z} \otimes \widehat{I} \otimes \widehat{Z}$ and $\widehat{S}_3 = \widehat{Z} \otimes \widehat{Z} \otimes \widehat{I}$ with spectrum $\{-1, +1\}$ and outcomes $(s_1, s_2, s_3) \in \{-1, +1\}$.

-1- $s_1 = s_2 = s_3$: $\mathcal{P}_c \ni |\psi_L\rangle = \left\{ \begin{array}{l} \alpha|000\rangle + \beta|111\rangle \text{ 0 flip} \\ \beta|000\rangle + \alpha|111\rangle \text{ 3 flips} \end{array} \right.$; no correction

-2- $s_1 \neq s_2 = s_3$: $\mathcal{P}_1 \ni |\psi_L\rangle = \left\{ \begin{array}{l} \alpha|100\rangle + \beta|011\rangle \text{ 1 flip} \\ \beta|100\rangle + \alpha|011\rangle \text{ 2 flips} \end{array} \right.$; $(\widehat{X} \otimes \widehat{I} \otimes \widehat{I})|\psi_L\rangle \in \mathcal{P}_c$.

-3- $s_2 \neq s_3 = s_1$: $\mathcal{P}_2 \ni |\psi_L\rangle = \left\{ \begin{array}{l} \alpha|010\rangle + \beta|101\rangle \text{ 1 flip} \\ \beta|010\rangle + \alpha|101\rangle \text{ 2 flips} \end{array} \right.$; $(\widehat{I} \otimes \widehat{X} \otimes \widehat{I})|\psi_L\rangle \in \mathcal{P}_c$.

-4- $s_3 \neq s_1 = s_2$: $\mathcal{P}_3 \ni |\psi_L\rangle = \left\{ \begin{array}{l} \alpha|001\rangle + \beta|110\rangle \text{ 1 flip} \\ \beta|001\rangle + \alpha|110\rangle \text{ 2 flips} \end{array} \right.$; $(\widehat{I} \otimes \widehat{I} \otimes \widehat{X})|\psi_L\rangle \in \mathcal{P}_c$.

[4] $\widehat{X} = |1\rangle\langle 0| + |0\rangle\langle 1|$ and $\widehat{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

[5] M.A Nielsen, I.L. Chuang (2000): Quantum Computation and Quantum Information.Cambridge University Press.

The 3-qubit phase-flip code

• Local phase-flip error: each physical qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ becomes $\widehat{Z}|\psi\rangle = \alpha|0\rangle - \beta|1\rangle$ [6] with probability $p < 1/2$ during Δt.
• Since $\widehat{X} = \widehat{H}\widehat{Z}\widehat{H}$ and $\widehat{Z} = \widehat{H}\widehat{X}\widehat{H}$ ($\widehat{H}^2 = \widehat{I}$), use the 3-qubit bit flip code in the frame defined by $\widehat{H}$:

$$|0\rangle \mapsto \frac{|0\rangle + |1\rangle}{\sqrt{2}} \triangleq |+\rangle, \quad |1\rangle \mapsto \frac{|0\rangle - |1\rangle}{\sqrt{2}} \triangleq |-\rangle, \quad \widehat{X} \mapsto \widehat{H}\widehat{X}\widehat{H} = \widehat{Z} = |+\rangle\langle -| + |-\rangle\langle +|.$$

• $t = +$: $|\psi_L\rangle = \alpha|+_L\rangle + \beta|-_L\rangle$ with $|+_L\rangle = |+++\rangle$ and $|-_L\rangle = |---\rangle$.
• $t = \Delta t$: $|\psi_L\rangle$ becomes with

$$\text{1 flip: } \left\{ \begin{array}{l} \alpha|-++\rangle + \beta|+--\rangle \\ \alpha|+-+\rangle + \beta|-+-\rangle \\ \alpha|++-\rangle + \beta|--+\rangle \end{array} \right. ; \text{ 2 flips: } \left\{ \begin{array}{l} \alpha|--+\rangle + \beta|++-\rangle \\ \alpha|-+-\rangle + \beta|+-+\rangle \\ \alpha|+--\rangle + \beta|-++\rangle \end{array} \right. ; \text{ 3 flips: } \alpha|---\rangle + \beta|+++\rangle.$$

• Key fact: 4 orthogonal planes $\mathcal{P}_c = \text{span}(|+++\rangle, |---\rangle)$, $\mathcal{P}_1 = \text{span}(|-++\rangle, |+--\rangle)$, $\mathcal{P}_2 = \text{span}(|+-+\rangle, |-+-\rangle)$ and $\mathcal{P}_3 = \text{span}(|++-\rangle, |--+\rangle)$.
• Error syndromes: 3 commuting observables $\widehat{S}_1 = \widehat{I} \otimes \widehat{X} \otimes \widehat{X}$, $\widehat{S}_2 = \widehat{X} \otimes \widehat{I} \otimes \widehat{X}$ and $\widehat{S}_3 = \widehat{X} \otimes \widehat{X} \otimes \widehat{I}$ with spectrum $\{-1, +1\}$ and outcomes $(s_1, s_2, s_3) \in \{-1, +1\}$.

-1- $s_1 = s_2 = s_3$: $\mathcal{P}_c \ni |\psi_L\rangle = \left\{ \begin{array}{ll} \alpha|+++\rangle + \beta|---\rangle & \text{0 flip} \\ \beta|+++\rangle + \alpha|---\rangle & \text{3 flips} \end{array} \right.$; no correction

-2- $s_1 \neq s_2 = s_3$: $\mathcal{P}_1 \ni |\psi_L\rangle = \left\{ \begin{array}{ll} \alpha|-++\rangle + \beta|+--\rangle & \text{1 flip} \\ \beta|-++\rangle + \alpha|+--\rangle & \text{2 flips} \end{array} \right.$; $(\widehat{Z} \otimes \widehat{I} \otimes \widehat{I})|\psi_L\rangle \in \mathcal{P}_c$.

-3- $s_2 \neq s_3 = s_1$: $\mathcal{P}_2 \ni |\psi_L\rangle = \left\{ \begin{array}{ll} \alpha|+-+\rangle + \beta|-+-\rangle & \text{1 flip} \\ \beta|+-+\rangle + \alpha|-+-\rangle & \text{2 flips} \end{array} \right.$; $(\widehat{I} \otimes \widehat{Z} \otimes \widehat{I})|\psi_L\rangle \in \mathcal{P}_c$.

-4- $s_3 \neq s_1 = s_2$: $\mathcal{P}_3 \ni |\psi_L\rangle = \left\{ \begin{array}{ll} \alpha|++-\rangle + \beta|--+\rangle & \text{1 flip} \\ \beta|++-\rangle + \alpha|--+\rangle & \text{2 flips} \end{array} \right.$; $(\widehat{I} \otimes \widehat{I} \otimes \widehat{Z})|\psi_L\rangle \in \mathcal{P}_c$.

[6] $\widehat{X} = |1\rangle\langle 0| + |0\rangle\langle 1|$, $\widehat{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|$ and $\widehat{H} = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\langle 0| + \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\langle 1|$.

The 9-qubit bit-flip and phase-flip code (Shor code (1995))

• Take the phase flip code $|+++\rangle$ and $|---\rangle$. Replace each $|+\rangle$ (resp. $|-\rangle$) by $\frac{|000\rangle+|111\rangle}{\sqrt{2}}$ (resp. $\frac{|000\rangle-|111\rangle}{\sqrt{2}}$).

New logical qubit $|\psi_L\rangle = \alpha|0_L\rangle + \beta|1_L\rangle \in \mathbb{C}^{2^9} \equiv \mathbb{C}^{512}$:

$$|0_L\rangle = \frac{\big(|000\rangle+|111\rangle\big)\big(|000\rangle+|111\rangle\big)\big(|000\rangle+|111\rangle\big)}{2\sqrt{2}},\ |1_L\rangle = \frac{\big(|000\rangle-|111\rangle\big)\big(|000\rangle-|111\rangle\big)\big(|000\rangle-|111\rangle\big)}{2\sqrt{2}}$$

• Local errors: each of the 9 physical qubits can have a bit-flip $\widehat{X}$, a phase flip $\widehat{Z}$ or a bit flip followed by a phase flip $\widehat{Z}\widehat{X} = i\widehat{Y}$ [7] with probability p during Δt.

• Denote by $\widehat{X}_k$ (resp. $\widehat{Y}_k$ and $\widehat{Z}_k$), the local operator $\widehat{X}$ (resp. $\widehat{Y}$ and $\widehat{Z}$) acting on physical qubit no $k \in \{1,\dots,9\}$. Denote by $\mathcal{P}_c = \text{span}(|0_L\rangle, |1_L\rangle)$ the code space. One get a family of the $1 + 3 \times 9 = 28$ **orthogonal planes:**

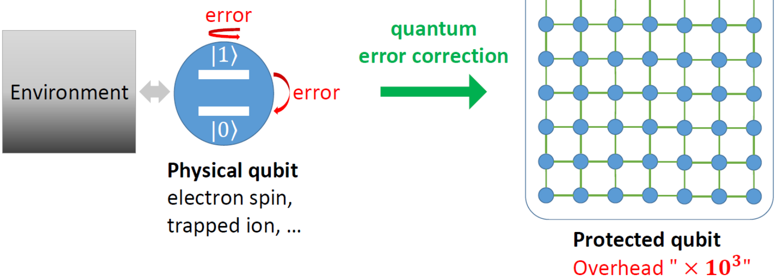
$$\mathcal{P}_c, \quad \Big(\widehat{X}_k\mathcal{P}_c\Big)_{k=1,\dots,9}, \quad \Big(\widehat{Y}_k\mathcal{P}_c\Big)_{k=1,\dots,9}, \quad \Big(\widehat{Z}_k\mathcal{P}_c\Big)_{k=1,\dots,9}.$$

• One can always construct error syndromes to obtain, when there is only one error among the 9 qubits during Δt, the number k of the qubit and the error type it has undergone ($\widehat{X}$, $\widehat{Y}$ or $\widehat{Z}$). These 28 planes are then eigen-planes by the syndromes.

• If the physical qubit k is subject to **any kind of local errors** associated to arbitrary operator $\widehat{M}_k = g\widehat{I} + a\widehat{X}_k + b\widehat{Y}_k + c\widehat{Z}_k$ $(g,a,b,c \in \mathbb{C})$, $|\psi_L\rangle \mapsto \frac{\widehat{M}_k|\psi_L\rangle}{\sqrt{\langle\psi_L|\widehat{M}_k^\dagger\widehat{M}_k|\psi_L\rangle}}$, the syndrome measurements will project the corrupted logical qubit on one of the 4 planes $\mathcal{P}_c$, $\widehat{X}_k\mathcal{P}_c$, $\widehat{Y}_k\mathcal{P}_c$ or $\widehat{Z}_k\mathcal{P}_c$. It is then simple by using either $\widehat{I}$, $\widehat{X}_k$, $\widehat{Y}_k$ or $\widehat{Z}_k$, to recover up to a global phase the original logical qubit $|\psi_L\rangle$.

[7] $\widehat{X} = |1\rangle\langle 0| + |0\rangle\langle 1|$, $\widehat{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|$ and $\widehat{Y} = i|1\rangle\langle 0| - i|0\rangle\langle 1|$.

QEC: 2D redundancy to correct bit-flip and phase-flip errors[8]



[8] Daniel Lidar and Todd Brun, ed. (2013). Quantum Error Correction. Cambridge University Press.

Outline

Structure of discrete-time dynamical models

Four modeling features[9]:

1. Schrödinger equations defining unitary transformations.
2. **Randomness**, irreversibility and dissipation induced by the measurement of observables with degenerate spectra.
3. Entanglement and tensor product for composite systems.
4. Classical probability (e.g. Bayes law) to include classical noises, measurement errors and uncertainties.

$\Rrightarrow$ **Hidden-state controlled Markov system**

Control input $\boldsymbol{u}$, state $\widehat{\rho}$ (density op.), measured output $\boldsymbol{y}$:

$$\widehat{\boldsymbol{\rho}}_{t+1} = \frac{\mathcal{K}_{\boldsymbol{u}_t,\boldsymbol{y}_t}(\widehat{\rho}_t)}{\mathrm{Tr}\left(\mathcal{K}_{\boldsymbol{u}_t,\boldsymbol{y}_t}(\widehat{\rho}_t)\right)}, \text{ with proba. } \mathbb{P}\Big(\boldsymbol{y}_t \;/\widehat{\rho}_t, \boldsymbol{u}_t\Big) = \mathrm{Tr}\left(\mathcal{K}_{\boldsymbol{u}_t,\boldsymbol{y}_t}(\widehat{\rho}_t)\right)$$

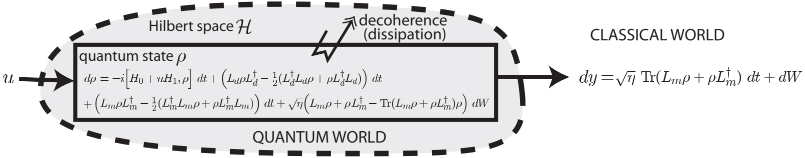
where $\mathcal{K}_{\boldsymbol{u},\boldsymbol{y}}(\widehat{\rho}) = \sum_{\mu=1}^{m} \eta_{\boldsymbol{y},\mu} \widehat{M}_{\boldsymbol{u},\mu}\widehat{\rho}\widehat{M}_{\boldsymbol{u},\mu}^{\dagger}$ with **left stochastic matrix** $(\eta_{\boldsymbol{y},\mu})$ and Kraus operators $\widehat{M}_{\boldsymbol{u},\mu}$ satisfying $\sum_{\mu} \widehat{M}_{\boldsymbol{u},\mu}^{\dagger}\widehat{M}_{\boldsymbol{u},\mu} = \widehat{I}$.

Kraus map $\mathcal{K}_{\boldsymbol{u}}$ (ensemble average, quantum channel)

$$\mathbb{E}\left(\widehat{\rho}_{t+1}|\widehat{\rho}_t\right) = \mathcal{K}_{\boldsymbol{u}}(\widehat{\rho}_t) = \sum_{\boldsymbol{y}} \mathcal{K}_{\boldsymbol{u},\boldsymbol{y}}(\widehat{\rho}_t) = \sum_{\mu} \widehat{M}_{\boldsymbol{u},\mu}\widehat{\rho}_t\widehat{M}_{\boldsymbol{u},\mu}^{\dagger}.$$

[9] See, e.g., books: E.B Davies in 1976; S. Haroche with J.M. Raimond in 2006; C. Gardiner with P. Zoller in 2014/2015.

Continuous dynamical models relying on Stochastic Master Equation (SME) [10]



Continuous-time models: stochastic differential systems (Itō formulation)
Control input $\boldsymbol{u}$, state $\widehat{\rho}$ (density op.), measured output $\boldsymbol{y}$:

$$d\widehat{\rho}_t = \Big(-i[\widehat{H}_0 + \boldsymbol{u_t} \widehat{H}_1, \widehat{\rho}_t] + \sum_{\nu=d,m} \widehat{L}_\nu \widehat{\rho}_t \widehat{L}_\nu^\dagger - \frac{1}{2}(\widehat{L}_\nu^\dagger \widehat{L}_\nu \widehat{\rho}_t + \widehat{\rho}_t \widehat{L}_\nu^\dagger \widehat{L}_\nu)\Big) dt$$
$$+ \sqrt{\eta_m}\Big(\widehat{L}_m \widehat{\rho}_t + \widehat{\rho}_t \widehat{L}_m^\dagger - \mathrm{Tr}\left((\widehat{L}_m + \widehat{L}_m^\dagger)\widehat{\rho}_t\right) \widehat{\rho}_t\Big) \boldsymbol{dW_t}$$

driven by the Wiener process $\boldsymbol{W_t}$, with measurement $\boldsymbol{y_t}$,

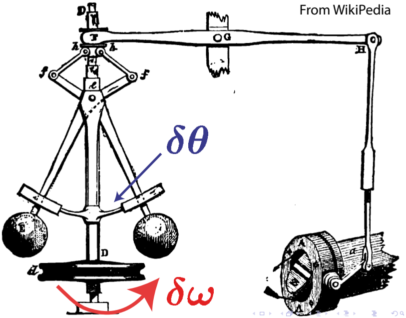
$$\boldsymbol{dy_t} = \sqrt{\eta_m}\ \mathrm{Tr}\left((\widehat{L}_m + \widehat{L}_m^\dagger)\, \widehat{\rho}_t\right)\ dt + \boldsymbol{dW_t} \quad \text{detection efficiencies } \eta_m \in [0, 1].$$

Measurement backaction: $d\widehat{\rho}_t$ **and** $\boldsymbol{dy_t}$ **share the same noises** $\boldsymbol{dW_t}$. Very different from Kalman I/O state-space description used in control engineering.

[10] A. Barchielli, M. Gregoratti (2009): Quantum Trajectories and Measurements in Continuous Time: the Diffusive Case. Springer Verlag.

Outline

Watt regulator: classical analogue of a quantum controller. [11]



Third order system

The first variations of speed $\delta\omega$ and governor angle $\delta\theta$ obey to

$$\frac{d}{dt}\delta\omega = -a\delta\theta$$
$$\frac{d^2}{dt^2}\delta\theta = -\Lambda\frac{d}{dt}\delta\theta - \Omega^2(\delta\theta - b\delta\omega)$$

with (a, b, Λ, Ω) positive parameters.

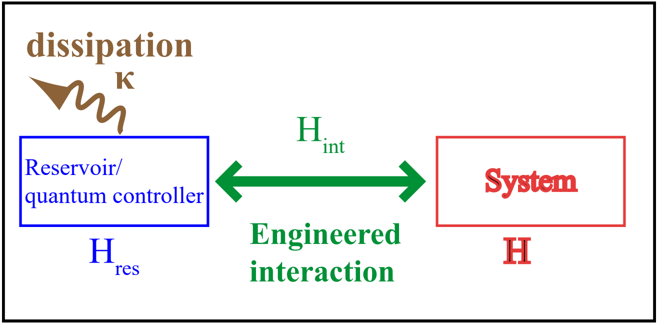
$$\frac{d^3}{dt^3}\delta\omega + \Lambda\frac{d^2}{dt^2}\delta\omega + \Omega^2\frac{d}{dt}\delta\omega + ab\Omega^2\delta\omega = 0.$$

Characteristic polynomial $P(s) = s^3 + \Lambda s^2 + \Omega^2 s + ab\Omega^2$ with roots having negative real parts iff $\Lambda > ab$: governor damping must be strong enough to ensure asymptotic stability.

Key issues: asymptotic stability and convergence rates.

[11] J.C. Maxwell: On governors. Proc. of the Royal Society, No.100, 1868.

Reservoir/dissipation engineering and quantum controller (1) [12]

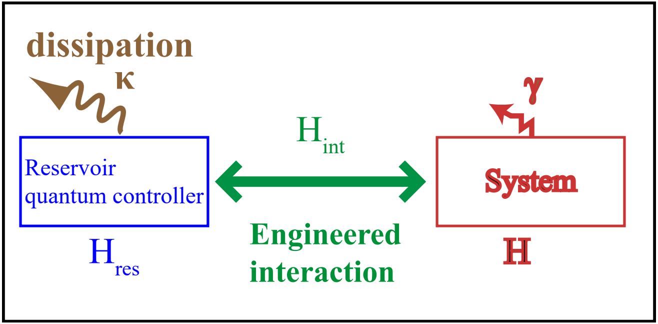


$$\widehat{H} = \widehat{H}_{res} + \widehat{H}_{int} + \widehat{H}$$

If $\widehat{\rho} \underset{t\to\infty}{\longrightarrow} \widehat{\rho}_{res} \otimes |\bar{\psi}\rangle\langle\bar{\psi}|$ exponentially with rate $1/\tau > 0$ then . . . . . .

[12] See, e.g., the lectures of H. Mabuchi delivered at the "Ecole de physique des Houches", July 2011.

Reservoir/dissipation engineering and quantum controller (2)



$$\widehat{H} = \widehat{H}_{\text{res}} + \widehat{H}_{\text{int}} + \widehat{H}$$

$$\ldots\ldots \quad \widehat{\rho} \underset{t\to\infty}{\rightarrow} \widehat{\rho}_{res} \otimes |\bar{\psi}\rangle\langle\bar{\psi}| + \overline{\delta\widehat{\rho}}, \text{ if } \tau\gamma \ll 1 \text{ then } |\overline{\delta\widehat{\rho}}| \ll 1$$

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Stabilisation of GKP-qubit with quantum controllers

- Phase-space local errors and cat-qubit/GKP-qubit
- Autonomous correction of bit/phase-flips
- Towards implementation with super-conducting circuits

Quantum dynamics with dissipation (decoherence)

Gorini–Kossakowski –Sudarshan–Lindblad (GKSL) master equation:

$$\frac{d}{dt}\widehat{\rho} = -i[\widehat{H}_0 + u\widehat{H}_1, \widehat{\rho}] + \sum_{\nu} \left(\widehat{L}_\nu \widehat{\rho} \widehat{L}_\nu^\dagger - \frac{1}{2}(\widehat{L}_\nu^\dagger \widehat{L}_\nu \widehat{\rho} + \widehat{\rho} \widehat{L}_\nu^\dagger \widehat{L}_\nu) \right)$$

- Preservation of trace, hermiticity and positivity: $\widehat{\rho}$ lies in the set of Hermitian and trace-class operators that are non-negative and of trace one.
- Invariance under unitary transformations.
  A time-varying change of frame $\widehat{\rho} \mapsto \widehat{U}_t^\dagger \widehat{\rho} \widehat{U}_t$ with $\widehat{U}_t$ unitary.
  The new density operator obeys to a similar master equation where $\widehat{H}_0 + u\widehat{H}_1 \mapsto \widehat{U}_t^\dagger(\widehat{H}_0 + u\widehat{H}_0)\widehat{U}_t + i\widehat{U}_t^\dagger \left(\frac{d}{dt}\widehat{U}_t\right)$ and $\widehat{L}_\nu \mapsto \widehat{U}_t^\dagger \widehat{L}_\nu \widehat{U}_t$.
- "L^1-contraction" properties. Such master equations generate contraction semi-groups for many distances (nuclear distance[13], Hilbert metric on the cone of non negative operators[14]).
- If the Hermitian operator $\widehat{A}$ satisfies the operator inequality

$$i[\widehat{H}_0 + u\widehat{H}_1, \widehat{A}] + \sum_{\nu} \left(\widehat{L}_\nu^\dagger \widehat{A} \widehat{L}_\nu - \frac{1}{2}(\widehat{L}_\nu^\dagger \widehat{L}_\nu \widehat{A} + \widehat{A} \widehat{L}_\nu^\dagger \widehat{L}_\nu) \right) \leq 0$$

  then $V(\widehat{\rho}) = \text{Tr}\left(\widehat{A}\widehat{\rho}\right)$ is a Lyapunov function when $\widehat{A} \geq 0$.

[13] D.Petz (1996). Monotone metrics on matrix spaces. Linear Algebra and its Applications

[14] R. Sepulchre, A. Sarlette, PR (2010). Consensus in non-commutative spaces. IEEE-CDC.

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- Dynamics based on Stochastic Master Equations (SME)

Quantum error correction with quantum controllers

- Quantum dissipation engineering
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- Cat-qubit and autonomous correction of bit-flips

Stabilisation of GKP-qubit with quantum controllers

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- Towards implementation with super-conducting circuits

Bosonic code with cat-qubits

- Quantum error corrrection requires redundancy.
- Bosonic code: instead of encoding a logical qubit in N physical qubits living in $\mathbb{C}^{2^N}$, encode a logical qubit in an harmonic oscillator living in Fock space $\text{span}\{|0\rangle, |1\rangle, \ldots, |n\rangle, \ldots\} \sim L^2(\mathbb{R}, \mathbb{C})$ of infinite dimension.
- Cat-qubit [15]: $|\psi_L\rangle \in \text{span}\{|\alpha\rangle, |\text{-}\alpha\rangle\}$ where $|\alpha\rangle$ is the coherent state of real amplitude α: $\widehat{a}|\alpha\rangle = \alpha|\alpha\rangle$ with $\widehat{a} = (\widehat{q} + i\widehat{p})/\sqrt{2}$ and $[\widehat{q}, \widehat{p}] = i$:

$$|\psi\rangle \sim \psi(q) \in L^2(\mathbb{R}, \mathbb{C}),\ \widehat{q}|\psi\rangle \sim q\psi(q),\ \widehat{p}|\psi\rangle \sim -i\frac{d\psi}{dq}(q),\ |\alpha\rangle \sim \frac{\exp\left(-\frac{(q-\alpha\sqrt{2})^2}{2}\right)}{\sqrt{2\pi}}.$$

- Stabilisation of cat-qubit via a single Lindblad dissipator $\widehat{L} = \widehat{a}^2 - \alpha^2$. For any initial density operator $\widehat{\rho}(0)$, the solution $\rho(t)$ of

$$\frac{d}{dt}\widehat{\rho} = \widehat{L}\widehat{\rho}\widehat{L}^\dagger - \frac{1}{2}(\widehat{L}^\dagger\widehat{L}\widehat{\rho} + \widehat{\rho}\widehat{L}^\dagger\widehat{L})$$

converges exponentially towards a steady-state density operator since

$$\frac{d}{dt}\ \text{Tr}\left(\widehat{L}^\dagger\widehat{L}\widehat{\rho}\right) \leq -2\ \text{Tr}\left(\widehat{L}^\dagger\widehat{L}\widehat{\rho}\right), \quad \ker\widehat{L} = \text{span}\{|\alpha\rangle, |\text{-}\alpha\rangle\}.$$

Any density operator with support in $\text{span}\{|\alpha\rangle, |\text{-}\alpha\rangle\}$ is a steady-state.

[15] M. Mirrahimi, Z. Leghtas, ..., **M. Devoret**: Dynamically protected cat-qubits: a new paradigm for universal quantum computation. 2014, New Journal of Physics.

Asymmetrically Threaded SQUID (ATS) stabilizing a cat-qubit [16]

9

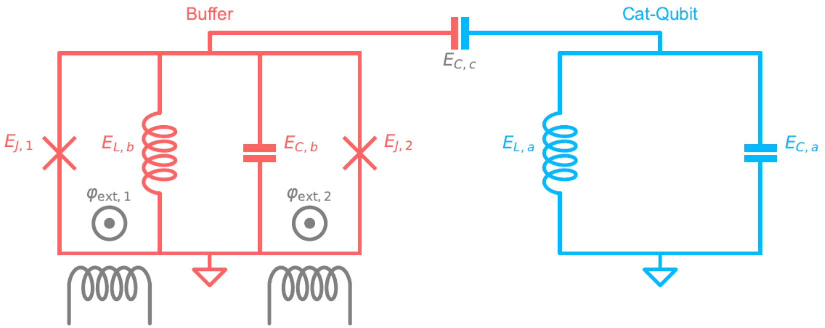


Figure S3. Equivalent circuit diagram. The cat-qubit (blue), a linear resonator, is capacitively coupled to the buffer (red). One recovers the circuit of Fig. 2 by replacing the buffer inductance with a 5-junction array and by setting $\varphi_\Sigma = (\varphi_{\text{ext},1} + \varphi_{\text{ext},2})/2$ and $\varphi_\Delta = (\varphi_{\text{ext},1} - \varphi_{\text{ext},2})/2$. Not shown here: the buffer is capacitively coupled to a transmission line, the cat-qubit resonator is coupled to a transmon qubit

[16] R. Lescanne, M. Villiers, Th. Peronnin, . . . , M. Mirrahimi and Z. Leghtas: Exponential suppression of bit-flips in a qubit encoded in an oscillator. 2020, Nature Physics

Master equations of the ATS super-conducting circuit

Oscillator $\widehat{a}$ with **quantum controller based on a damped oscillator** $\widehat{b}$:

$$\frac{d}{dt}\widehat{\rho} = g_2\left[(\widehat{a}^2 - \alpha^2)\widehat{b}^\dagger - ((\widehat{a}^\dagger)^2 - \alpha^2)\widehat{b}\ ,\ \widehat{\rho}\right] + \kappa_b\left(\widehat{b}\widehat{\rho}\widehat{b}^\dagger - (\widehat{b}^\dagger\widehat{b}\widehat{\rho} + \widehat{\rho}\widehat{b}^\dagger\widehat{b})/2\right)$$

with $\alpha \in \mathbb{R}$ such that $\alpha^2 = u/g_2$, the drive amplitude $u \in \mathbb{R}$ applied to mode $\widehat{b}$ and $1/\kappa_b > 0$ the life-time of photon in mode $\widehat{b}$.
Any density operators $\bar{\rho} = \bar{\rho}_a \otimes |0\rangle\langle 0|_b$ **is a steady-state as soon as the support of $\bar{\rho}_a$ belongs to the two dimensional vector space spanned by the quasi-classical wave functions $|\alpha\rangle$ and $|\text{-}\alpha\rangle$** (range$(\bar{\rho}_a) \subset$ span$\{|\alpha\rangle, |\text{-}\alpha\rangle\}$)

Usually $\kappa_b \gg |g_2|$, mode $\widehat{b}$ relaxes rapidly to vaccuum $|0\rangle\langle 0|_b$, can be eliminated adiabatically (**singular perturbations**, second order corrections) to provides the slow evolution of mode $\widehat{a}$

$$\frac{d}{dt}\widehat{\rho}_a = \tfrac{4|g_2|^2}{\kappa_b}\left(\widehat{L}\widehat{\rho}\widehat{L}^\dagger - \frac{1}{2}(\widehat{L}^\dagger\widehat{L}\widehat{\rho} + \widehat{\rho}\widehat{L}^\dagger\widehat{L})\right) \text{ with } \widehat{L} = \widehat{a}^2 - \alpha^2.$$

Convergence via the exponential Lyapunov function $V(\widehat{\rho}) = \operatorname{Tr}\left(\widehat{L}^\dagger\widehat{L}\widehat{\rho}\right)$ [17]

[17] For a mathematical proof of convergence analysis in an adapted Banach space, see :R. Azouit, A. Sarlette, PR: Well-posedness and convergence of the Lindblad master equation for a quantum harmonic oscillator with multi-photon drive and damping. 2016, ESAIM: COCV.

Cat-qubit: exponential suppression of bit-flip for large α.

Since $\langle\alpha|\text{-}\alpha\rangle = e^{-2\alpha^2} \approx 0$:

$$|0_L\rangle \approx |\alpha\rangle,\ |1_L\rangle \approx |\text{-}\alpha\rangle,\ |+_L\rangle \propto \tfrac{|\alpha\rangle+|\text{-}\alpha\rangle}{\sqrt{2}},\ |-_L\rangle \propto \tfrac{|\alpha\rangle-|\text{-}\alpha\rangle}{\sqrt{2}}.$$

Photon loss as dominant error channel (dissipator $\widehat{a}$ with $0 < \kappa_1 \ll 1$):

$$\frac{d}{dt}\widehat{\rho}_a = \mathcal{D}_{\widehat{a}^2-\alpha^2}(\widehat{\rho}) + \kappa_1 \mathcal{D}_{\widehat{a}}(\widehat{\rho})$$

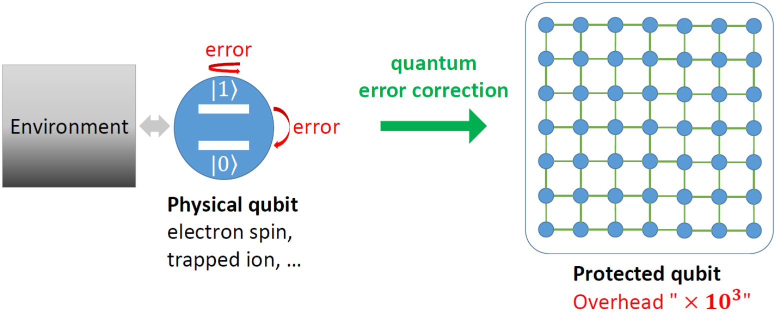
with $\mathcal{D}_{\widehat{L}}(\widehat{\rho}) = \widehat{L}\widehat{\rho}\widehat{L}^\dagger - \frac{1}{2}(\widehat{L}^\dagger\widehat{L}\widehat{\rho} + \widehat{\rho}\widehat{L}^\dagger\widehat{L})$.

- if $\widehat{\rho}(0) = |0_L\rangle\langle 0_L|$ or $|1_L\rangle\langle 1_L|$, $\widehat{\rho}(t)$ converges to a statistical mixture of quasi-classical states close to $\frac{1}{2}|\alpha\rangle\langle\alpha| + \frac{1}{2}|\text{-}\alpha\rangle\langle\text{-}\alpha|$ in a time
$$T_{bit-flip} \sim \frac{e^{2\alpha^2}}{\kappa_1}$$
since $\widehat{a}|0_L\rangle \approx \alpha|0_L\rangle$ and $\widehat{a}|1_L\rangle \approx -\alpha|1_L\rangle$.
- if $\widehat{\rho}(0) = |+_L\rangle\langle +_L|$ or $|-_L\rangle\langle -_L|$, $\widehat{\rho}(t)$ converges also to the same statistical mixture in a time
$$T_{phase-flip} \sim \frac{1}{\kappa_1\alpha^2}$$
since $\widehat{a}|+_L\rangle = \alpha|-L\rangle$ and $\widehat{a}|-_L\rangle = \alpha|+L\rangle$.

Take α large to ignore bit-flip and to correct only the phase-flip with $1D$ repetition code: important overhead reduction investigated by the startup **Alice&Bob** and also by **AWS**.

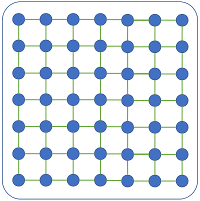
Outline

QEC: 2D redundancy to correct bit-flip and phase-flip errors



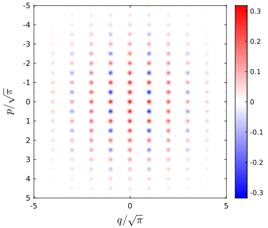
Local noise assumption (1)

DV-QEC



$$\mathcal{H} = \left(\mathbb{C}^2\right)^{\otimes N}$$
Dimension: 2^N

CV-QEC



$$\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})$$
Dimension: $+\infty$

Wave function $|\psi\rangle : \mathbb{R} \ni q \mapsto \psi(q) \in \mathbb{C}$, and Wigner function

$$\mathbb{R}^2 \ni (q,p) \mapsto W^{|\psi\rangle\langle\psi|}(q,p) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \psi^*\left(q - \frac{u}{2}\right)\psi\left(q + \frac{u}{2}\right)e^{-2ipu}du.$$

Local error operators $\hat{q}$ and $\hat{p}$ ($[\hat{q},\hat{p}] = i$) on $|\psi\rangle$: small random shifts along q ($e^{i\pm\epsilon\hat{p}} \equiv e^{\pm\epsilon d/dq}$) and p ($e^{i\pm\epsilon\hat{q}} \equiv e^{\mp\epsilon d/dp}$) similar to diffusion along q and p axis for $W^{|\psi\rangle\langle\psi|}$.

Local noise assumption (2)

For a density operator $\widehat{\rho}$, its Wigner function

$$\mathbb{R}^2 \ni (q,p) \mapsto W^{\widehat{\rho}}(q,p) \in \mathbb{R}$$

reads ($\widehat{a} = \frac{\widehat{q}+i\widehat{p}}{\sqrt{2}}$)

$$W^{\widehat{\rho}}(q,p) = \tfrac{1}{\pi}\ \mathrm{Tr}\left(e^{i\pi\widehat{a}^\dagger\widehat{a}}\ e^{i(p\widehat{q}-q\widehat{p})}\ \widehat{\rho}\ e^{-i(p\widehat{q}-q\widehat{p})}\right)$$

Since

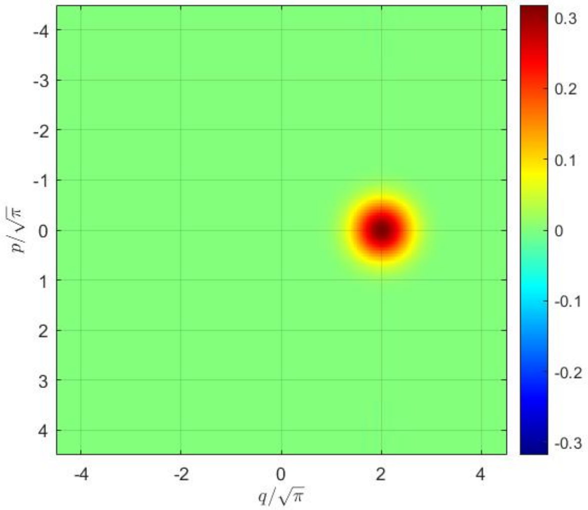
$$W^{\mathcal{D}_{\widehat{q}}(\widehat{\rho})} = \frac{1}{2}\frac{\partial^2}{\partial p^2}W^{\widehat{\rho}}, \quad W^{\mathcal{D}_{\widehat{p}}(\widehat{\rho})} = \frac{1}{2}\frac{\partial^2}{\partial q^2}W^{\widehat{\rho}}$$

and

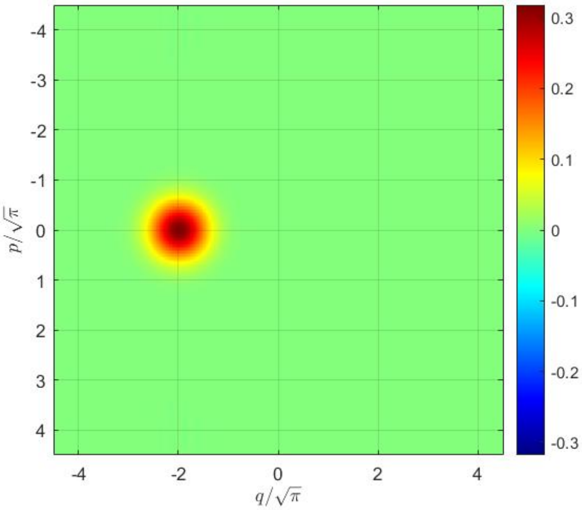
$$W^{\mathcal{D}_{\widehat{a}}(\widehat{\rho})} = \frac{1}{2}\frac{\partial}{\partial q}(qW^{\widehat{\rho}}) + \frac{1}{2}\frac{\partial}{\partial p}(pW^{\widehat{\rho}}) + \frac{1}{2}\frac{\partial^2}{\partial q^2}W^{\widehat{\rho}} + \frac{1}{2}\frac{\partial^2}{\partial p^2}W^{\widehat{\rho}}$$

dominant errors on $\widehat{\rho}$ correspond to local differential operators in the phase-space (q,p).

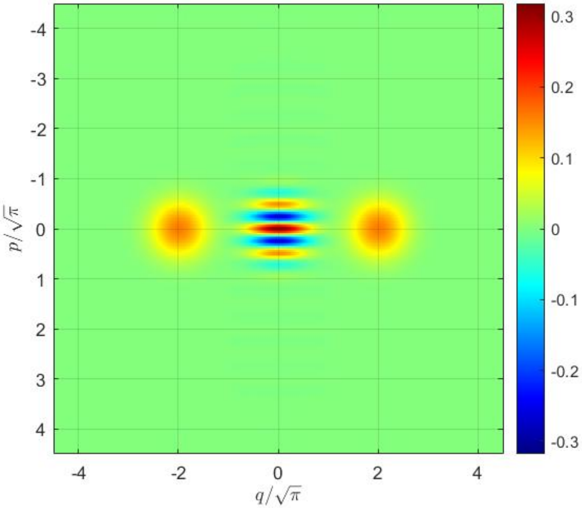
Wigner function of coherent state $|\sqrt{2\pi}\rangle \equiv \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(q-2\sqrt{\pi})^2}{2}\right) \approx |0_L\rangle$



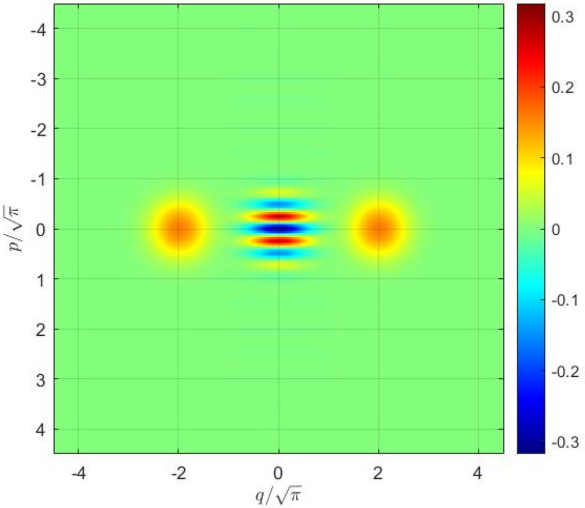
Wigner function of coherent state $|\text{-}\sqrt{2\pi}\rangle \equiv \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(q-2\sqrt{\pi})^2}{2}\right) \approx |1_L\rangle$



Wigner function of $|+_L\rangle \propto \frac{|\sqrt{2\pi}\rangle+|\text{-}\sqrt{2\pi}\rangle}{\sqrt{2}}$ ("Schrödinger phase cat")



Wigner function of $|-_L\rangle \propto \frac{|\sqrt{2\pi}\rangle - |\text{-}\sqrt{2\pi}\rangle}{\sqrt{2}}$ ("Schrödinger phase cat")



Grid-states and GKP-qubits

- Poisson summation formula: the Fourier transform of Dirac comb $f(q)$ of period T is a Dirac comb $g(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(q) e^{-iqp} dq$ of period $2\pi/T$.

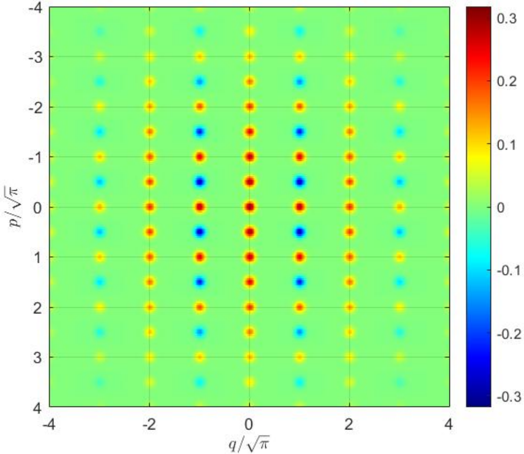
infinite energy grid-states	q representation	p representation
$\|0_L\rangle$	$\sum_k \delta(q - 2k\sqrt{\pi})$	$\sum_k \delta(p - k\sqrt{\pi})$
$\|1_L\rangle$	$\sum_k \delta(q - 2(k+1)\sqrt{\pi})$	$\sum_k (-1)^k \delta(p - k\sqrt{\pi})$
$\|+_L\rangle \sim \|0_L\rangle + \|1_L\rangle$	$\sum_k \delta(q - k)\sqrt{\pi})$	$\sum_k \delta(p - 2k\sqrt{\pi})$
$\|-_L\rangle \sim \|0_L\rangle - \|1_L\rangle$	$\sum_k (-1)^k \delta(q - k)\sqrt{\pi})$	$\sum_k \delta(p - 2(k+1)\sqrt{\pi})$

- Pauli operators of a GKP-qubit[18] with Bloch coordinates $(x, y, z) \in \mathbb{R}^3$:
$$\widehat{Z} = \text{sign}(\cos(\sqrt{\pi}\widehat{q})),\ \widehat{X} = \text{sign}(\cos(\sqrt{\pi}\widehat{p})) \text{ and } \widehat{Y} = -i\widehat{Z}\widehat{X}.$$
- 4 stabilizer operators $\widehat{S}$ relying on commuting modular operators in $\widehat{q}$ and $\widehat{p}$:
$\forall \widehat{S} \in \{e^{i2\sqrt{\pi}\widehat{q}}, e^{i2\sqrt{\pi}\widehat{p}}, e^{-i2\sqrt{\pi}\widehat{q}}, e^{-i2\sqrt{\pi}\widehat{p}}\}$ and $\forall |\psi_L\rangle \in \text{span}\{|0_L\rangle, |1_L\rangle\}$: $\widehat{S}|\psi_L\rangle = |\psi_L\rangle$
- Finite energy regularization with $0 < \epsilon \ll 1$,
$$|0_\epsilon\rangle \approx e^{-\epsilon \widehat{a}^\dagger \widehat{a}} |0_L\rangle, \quad |1_\epsilon\rangle \approx e^{-\epsilon \widehat{a}^\dagger \widehat{a}} |0_L\rangle,$$
where $\widehat{a}^\dagger \widehat{a} = \frac{1}{2}(\widehat{q}^2 + \widehat{p}^2) \sim \frac{1}{2}(q^2 + \partial^2/\partial q^2)$, provides a finite-energy code space where **any small local error can be corrected**[19].

[18] D. Gottesman, A. Kitaev and J. Preskill: Encoding a qubit in an oscillator. Physical Review A, 2001.

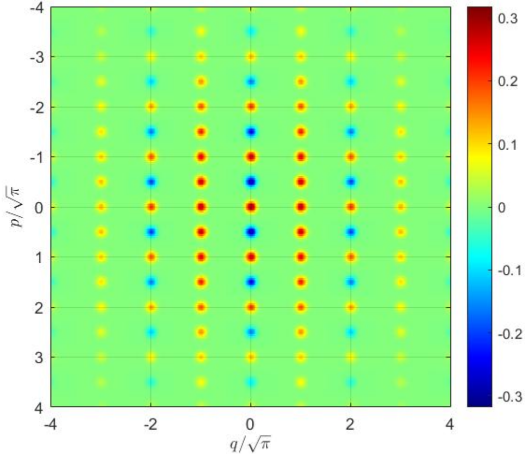
[19] 3 recent experiments stabilizing GKP-qubits via classical controllers: Ph. Campagne-Ibarcq et al. "Quantum error correction of a qubit encoded in grid states of an oscillator" Nature (2020); B. de Neeve et al. "Error correction of a logical grid state qubit by dissipative pumping" Nature (2022); V. Sivak et al. "Real-Time Quantum Error Correction beyond Break-Even" Nature (2023).

Wigner function of the GKP finite energy grid-state $|0_\epsilon\rangle$ [20]



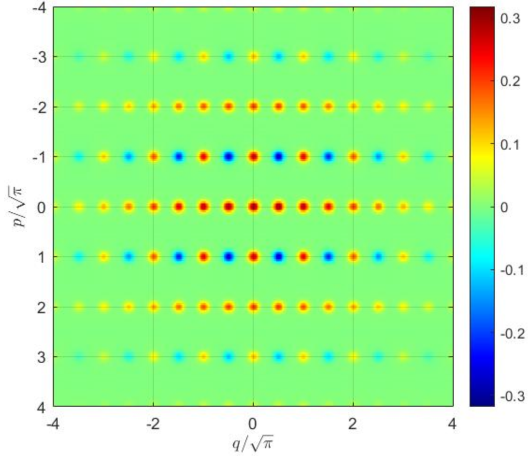
[20] $|0_\epsilon\rangle \approx e^{-\epsilon q^2} \sum_k \exp\left(-\frac{(q-2k\sqrt{\pi})^2}{\epsilon}\right)$ with $\epsilon = \frac{1}{30}$.

Wigner function of the GKP finite energy grid-state $|1_\epsilon\rangle$ [21]



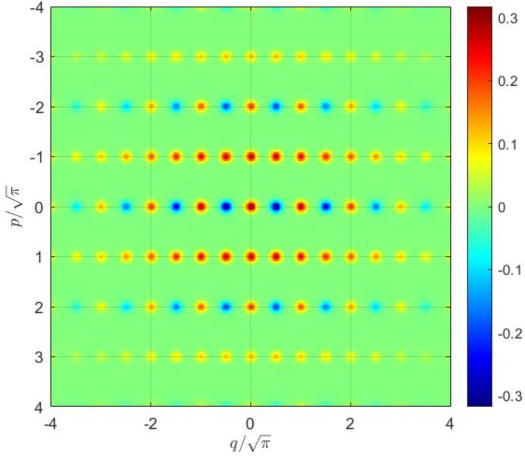
[21] $|1_\epsilon\rangle \approx e^{-\epsilon q^2} \sum_k \exp\left(-\frac{(q-(2k+1)\sqrt{\pi})^2}{\epsilon}\right)$ with $\epsilon = \frac{1}{30}$.

Wigner function of the GKPfinite energy grid-state $|+_\epsilon\rangle$ [22]



[22] $|+_\epsilon\rangle \approx e^{-\epsilon q^2} \sum_k \exp\left(\frac{(q-k\sqrt{\pi})^2}{\epsilon}\right) \equiv e^{-\epsilon p^2} \sum_k \exp\left(\frac{(p-2k\sqrt{\pi})^2}{\epsilon}\right)$.

Wigner function of the GKP finite energy grid-state $|-_\epsilon\rangle$ [23]



[23] $|-_\epsilon\rangle \approx e^{-\epsilon q^2} \sum_k (-1)^k \exp\left(-\frac{(q-k\sqrt{\pi})^2}{\epsilon}\right) \equiv e^{-\epsilon p^2} \sum_k \exp\left(-\frac{(p-(2k+1)\sqrt{\pi})^2}{\epsilon}\right)$.

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Exponential stabilisation of finite energy GKP-qubits[24]

- 4 regularized stabilizers:

$$\widehat{S}_{\epsilon,k} \triangleq e^{-(\epsilon - i\frac{k\pi}{2})\widehat{a}^\dagger \widehat{a}}\ e^{i2\sqrt{\pi}\widehat{q}}\ e^{(\epsilon - i\frac{k\pi}{2})\widehat{a}^\dagger \widehat{a}}, \quad k = 0,1,2,3.$$

- Master equation with 4 dissipators $\widehat{M}_{\epsilon,k} = \widehat{S}_{\epsilon,k} - \widehat{I}$

$$\frac{d}{dt}\widehat{\rho} = \sum_{k=0}^{3} \mathcal{D}_{\widehat{M}_{\epsilon,k}}(\widehat{\rho})$$

- Lyapunov function:

$$V(\widehat{\rho}) = \sum_k \mathrm{Tr}\left(\widehat{M}_{\epsilon,k}^\dagger \widehat{M}_{\epsilon,k}\widehat{\rho}\right) \text{ with } \frac{d}{dt}V \leq -\left(32\pi^2\epsilon^2 + O(\epsilon^3)\right)V$$

ensuring exponential convergence towards the finite-energy code space

$$\mathrm{span}\left\{e^{-\epsilon\widehat{a}^\dagger\widehat{a}}|0_L\rangle, e^{-\epsilon\widehat{a}^\dagger\widehat{a}}|1_L\rangle\right\}$$

[24] L.A. Sellem, Ph. Campagne-Ibarcq, M. Mirrahimi, A. Sarlette, PR: Exponential convergence of a dissipative quantum system towards finite-energy grid states of an oscillator: IEEE CDC 2022 (arXiv:2203.16836).

Approximated Lindblad dissipators with exponentially small decoherence rates [25]

Replace the ideal dissipators $\widehat{M}_{\epsilon,k}$ by more realistic dissipators $\widehat{L}_{\epsilon,k}$ derived from a first-order approximation in ϵ:

$$\widehat{L}_{\epsilon,k} \triangleq e^{i\frac{k\pi}{2}\widehat{a}^\dagger\widehat{a}} \left(e^{-2\pi\epsilon} e^{i2\sqrt{\pi}\widehat{q}}(\widehat{I} - 2\epsilon\sqrt{\pi}\widehat{p}) - \widehat{I} \right) e^{-i\frac{k\pi}{2}\widehat{a}^\dagger\widehat{a}}$$

For $\widehat{\rho}$ governed by master equation $\frac{d}{dt}\widehat{\rho} = \sum_{k=0}^{3} \mathcal{D}_{\widehat{L}_{\epsilon,k}}(\widehat{\rho})$:

- Energy $\operatorname{Tr}\left(\widehat{a}^\dagger\widehat{a}\widehat{\rho}\right)$ remains finite and for t large is less than $\frac{1}{2\epsilon} + 0(1)$.
- For any 2π periodic function $f(\theta)$, one has

$$\frac{d}{dt}\operatorname{Tr}\left(f(\sqrt{\pi}\widehat{q})\rho\right) = -4\epsilon\pi e^{-2\pi\epsilon} \operatorname{Tr}\left(\left(\sin(2\sqrt{\pi}\widehat{q})f'(\sqrt{\pi}\hat{q}) - \epsilon\pi e^{-2\pi\epsilon} f''(\sqrt{\pi}\widehat{q})\right)\widehat{\rho}\right).$$

- Spect. $(\lambda_n)_{n\geq 0}$ of Witten Laplacian$\mathcal{L}_\sigma(f(\theta)) = \sin(2\theta)f'(\theta) - \sigma f''(\theta)$ with 2π-periodic function $f(\theta)$ and $0 < \sigma \ll 1$:
$$\lambda_0 = 0 < \boldsymbol{\lambda_1 \sim \frac{4}{\pi} e^{-1/\sigma}} < 1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_n \leq \ldots$$
with eigenfunction $f_1(\theta) \approx \operatorname{sign}(\cos\theta)$ corresponding to λ_1. Thus $z \approx \operatorname{Tr}\left(f_1(\sqrt{\pi}\widehat{q})\widehat{\rho}\right)$ is almost constant: $\frac{d}{dt}z \approx \boldsymbol{-16\epsilon \exp\left(-\frac{1}{\epsilon\pi}\right)}\ z$.
Similar exponentially small decays for (x, y, z) with quadrature noises,
i.e. when $\frac{d}{dt}\widehat{\rho} = \sum_{k=0}^{3} \mathcal{D}_{\widehat{L}_{\epsilon,k}}(\widehat{\rho}) + \kappa_q \mathcal{D}_{\widehat{q}}(\widehat{\rho}) + \kappa_p \mathcal{D}_{\widehat{p}}(\widehat{\rho})$ $(\kappa_q, \kappa_q \ll 1)$

[25] L.A. Sellem, R. Robin, Ph. Campagne-Ibarcq, PR: Stability and decoherence rates of a GKP qubit protected by dissipation. IFAC WC 2023 (arXiv:2304.03806).

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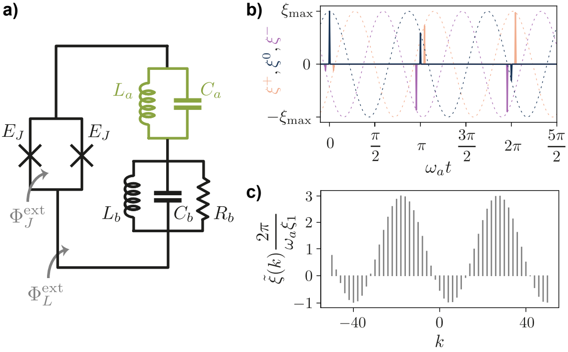
Engineering modular dissipation with super-conducting Josephson circuits [26]



High impedance $\sqrt{L_a/C_a}$ and low pulsation $1/\sqrt{L_a C_a}$ for storage mode $\widehat{a}$.

High pulsation $1/\sqrt{L_b C_b}$ of damped mode $\widehat{b}$ (quantum controller $R_b > 0$).

Josephson energy E_J between $\hbar/\sqrt{L_a C_a}$ and $\hbar/\sqrt{L_b C_b}$.

Classical open-loop control signals $\Phi_J^{ext}(t)$ and $\Phi_L^{ext}(t)$ made of short pulses.

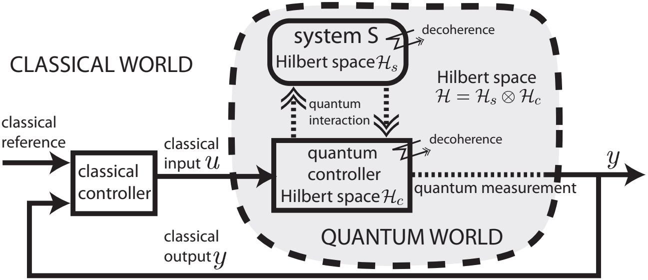
Mathematical analysis to recover master equation with dissipators $\widehat{L}_k$.

Numerical simulations to test robustness versus experimental imperfections.

---

[26] L.A. Sellem, A. Salette, Z. Leghtas, M. Mirrahimi, PR, Ph. Campagne-Ibarcq: A GKP qubit protected by dissipation in a high-impedance superconducting circuit driven by a microwave frequency comb. Submitted 2023 (arXiv:2304.01425).

Quantum feedback engineering for robust quantum information processing



To protect quantum information stored in system S:

- fast stabilization and protection mainly achieved by **quantum controllers** (autonomous feedback stabilizing decoherence-free sub-spaces);
- slow decoherence and perturbations, parameter estimation mainly tackled by **classical controllers and estimation algorithms** (measurement-based feedback and estimation "finishing the job")

Need of adapted mathematical and numerical methods for high-precision dynamical modeling and control with (stochastic) master equations.

Quantic research group ENS/Inria/Mines/CNRS, June 2023