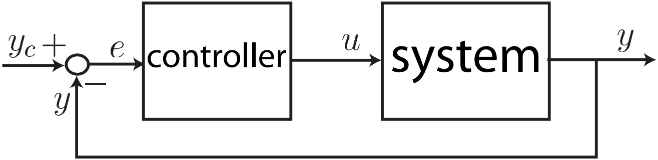
Stabilisation par feedback de systèmes quantiques ouverts
(Collège de France, 14 mars 2014)

Pierre Rouchon
Centre Automatique et Systèmes
Mines ParisTech

Feedback for quantum systems: the back-action of the measure.

A typical stabilizing feedback-loop for a classical system



Two kinds of stabilizing feedbacks for quantum systems

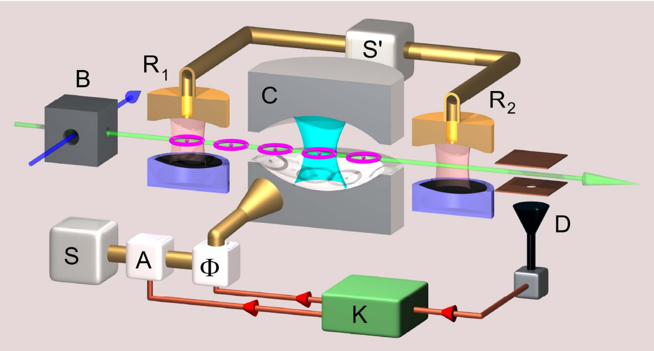
1. Measurement-based feedback: measurement back-action on $\mathcal{S}$ is stochastic (collapse of the wave-packet); controller is classical; the control input u is a classical variable appearing in some controlled Schrödinger equation; u depends on the past measures.

2. Coherent/autonomous feedback and reservoir engineering: the system $\mathcal{S}$ is coupled to another quantum system (the controller); the composite system, $\mathcal{H}_S \otimes \mathcal{H}_{controller}$, is an open-quantum system relaxing to some target (separable) state or decoherence free subspace.

Outline

The first experimental realization of a **quantum state feedback**

The LKB photon box: group of S.Haroche and J.M.Raimond [1]



Stabilization of a quantum state with exactly n photon(s) ($n = 0, 1, 2, 3, \ldots$).
Experiment: C. Sayrin et. al., Nature 477, 73-77, September 2011.
Theory: I. Dotsenko et al., Physical Review A, 80: 013805-013813, 2009.
R. Somaraju et al., Rev. Math. Phys., 25, 1350001, 2013.
H. Amini et. al., Automatica, 49 (9): 2683-2692, 2013.

[1] Courtesy of I. Dotsenko. Sampling period 80 μs.

Three quantum features emphasized by the LKB photon box[2]

1. Schrödinger equation: wave function $|\psi\rangle \in \mathcal{H}$, density operator ρ

$$\frac{d}{dt}|\psi\rangle = -\frac{i}{\hbar}\boldsymbol{H}|\psi\rangle, \quad \frac{d}{dt}\rho = -\frac{i}{\hbar}[\boldsymbol{H}, \rho], \quad \boldsymbol{H} = \boldsymbol{H}_0 + u\boldsymbol{H}_1$$

2. Origin of dissipation: collapse of the wave packet induced by the measure of observable $\boldsymbol{O}$ with spectral decomposition $\sum_\mu \lambda_\mu \boldsymbol{P}_\mu$:
   - measure outcome μ with proba. $p_\mu = \langle\psi|\boldsymbol{P}_\mu|\psi\rangle = \text{Tr}\left(\rho \boldsymbol{P}_\mu\right)$ depending on $|\psi\rangle$, ρ just before the measurement
   - measure back-action if outcome μ:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\boldsymbol{P}_\mu|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{P}_\mu|\psi\rangle}}, \quad \rho \mapsto \rho_+ = \frac{\boldsymbol{P}_\mu \rho \boldsymbol{P}_\mu}{\text{Tr}\left(\rho \boldsymbol{P}_\mu\right)}$$

3. Tensor product for the description of composite systems (S, M):
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $\boldsymbol{H} = \boldsymbol{H}_S \otimes \boldsymbol{I}_M + \boldsymbol{H}_{int} + \boldsymbol{I}_S \otimes \boldsymbol{H}_M$
   - observable on sub-system M only: $\boldsymbol{O} = \boldsymbol{I}_S \otimes \boldsymbol{O}_M$.

[2] S. Haroche and J.M. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts, 2006.

- System S corresponds to a quantized mode in C:

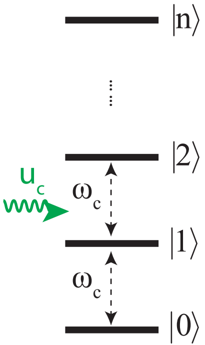
$$\mathcal{H}_S = \left\{ \sum_{n=0}^{\infty} \psi_n |n\rangle \;\middle|\; (\psi_n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

  where $|n\rangle$ represents the Fock state associated to exactly n photons inside the cavity
- Meter M is associated to atoms : $\mathcal{H}_M = \mathbb{C}^2$, each atom admits two energy levels and is described by a wave function $c_g|g\rangle + c_e|e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$; atoms leaving B are all in state $|g\rangle$
- When atom comes out B, the state $|\Psi\rangle_B \in \mathcal{H}_S \otimes \mathcal{H}_M$ of the composite system atom/field is separable

$$|\Psi\rangle_B = |\psi\rangle \otimes |g\rangle.$$

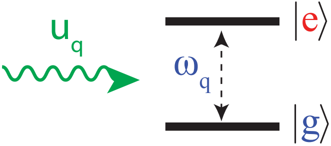
S: quantum harmonic oscillator

- Hilbert space:
  $\mathcal{H}_S = \{\sum_{n\geq 0} \psi_n |n\rangle,\ (\psi_n)_{n\geq 0} \in l^2(\mathbb{C})\}$.
- Quantum state space:
  $\mathcal{D} = \{\rho \in \mathcal{L}(\mathcal{H}_S), \rho^\dagger = \rho,\ \text{Tr}\,(\rho) = 1, \rho \geq 0\}$.
- Operators and commutations:
  $\boldsymbol{a}|n\rangle = \sqrt{n}|n-1\rangle$, $\boldsymbol{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$;
  $\boldsymbol{N} = \boldsymbol{a}^\dagger\boldsymbol{a}$, $\boldsymbol{N}|n\rangle = n|n\rangle$;
  $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = \boldsymbol{I}$, $\boldsymbol{a}f(\boldsymbol{N}) = f(\boldsymbol{N}+\boldsymbol{I})\boldsymbol{a}$;
  $\boldsymbol{D}_\alpha = e^{\alpha\boldsymbol{a}^\dagger - \alpha^\dagger\boldsymbol{a}}$.
  $\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right),\ [\boldsymbol{X}, \boldsymbol{P}] = \imath\boldsymbol{I}/2$.
- Hamiltonian: $\boldsymbol{H}_S/\hbar = \omega_c\boldsymbol{a}^\dagger\boldsymbol{a} + u_c(\boldsymbol{a} + \boldsymbol{a}^\dagger)$.
  (associated classical dynamics:
  $\frac{dx}{dt} = \omega_c p,\ \frac{dp}{dt} = -\omega_c x - \sqrt{2}u_c$).
- Coherent state of amplitude
  $\alpha \in \mathbb{C}$: $|\alpha\rangle = \sum_{n\geq 0}\left(e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\right)|n\rangle$; $|\alpha\rangle \equiv \frac{1}{\pi^{1/4}}e^{\imath\sqrt{2}x\Im\alpha}e^{-\frac{(x-\sqrt{2}\Re\alpha)^2}{2}}$
  $\boldsymbol{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\boldsymbol{D}_\alpha|0\rangle = |\alpha\rangle$.

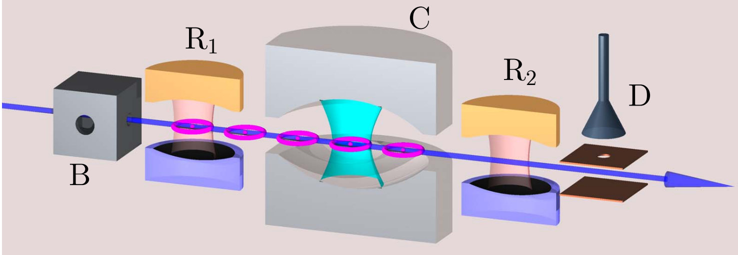


M: 2-level system, i.e. a qubit

- Hilbert space:
  $\mathcal{H}_M = \mathbb{C}^2 = \{c_g|g\rangle + c_e|e\rangle,\ c_g, c_e \in \mathbb{C}\}$.
- Quantum state space:
  $\mathcal{D} = \{\rho \in \mathcal{L}(\mathcal{H}_M), \rho^\dagger = \rho,\ \mathrm{Tr}(\rho) = 1, \rho \geq 0\}$.
- Operators and commutations:
  $\boldsymbol{\sigma_-} = |g\rangle\langle e|$, $\boldsymbol{\sigma_+} = \boldsymbol{\sigma_-}^\dagger = |e\rangle\langle g|$
  $\boldsymbol{\sigma_x} = \boldsymbol{\sigma_-} + \boldsymbol{\sigma_+} = |g\rangle\langle e| + |e\rangle\langle g|$;
  $\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_-} - i\boldsymbol{\sigma_+} = i|g\rangle\langle e| - i|e\rangle\langle g|$;
  $\boldsymbol{\sigma_z} = \boldsymbol{\sigma_+}\boldsymbol{\sigma_-} - \boldsymbol{\sigma_-}\boldsymbol{\sigma_+} = |e\rangle\langle e| - |g\rangle\langle g|$;
  $\boldsymbol{\sigma_x}^2 = \boldsymbol{I}$, $\boldsymbol{\sigma_x}\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_z}$, $[\boldsymbol{\sigma_x}, \boldsymbol{\sigma_y}] = 2i\boldsymbol{\sigma_z}$, ...



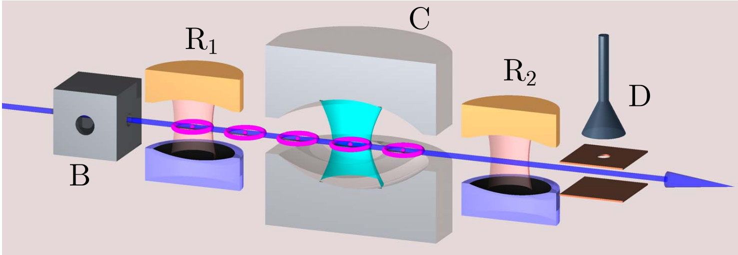
- Hamiltonian: $\boldsymbol{H}_M/\hbar = \omega_q\boldsymbol{\sigma_z}/2 + u_q\boldsymbol{\sigma_x}$.
- Bloch sphere representation:
  $\mathcal{D} = \left\{\frac{1}{2}\left(\boldsymbol{I} + x\boldsymbol{\sigma_x} + y\boldsymbol{\sigma_y} + z\boldsymbol{\sigma_z}\right) \mid (x, y, z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 \leq 1\right\}$



- When atom comes out B: $|\Psi\rangle_B = |\psi\rangle \otimes |g\rangle$.
- Just before the measurement in D, the state is in general entangled (not separable):

$$|\Psi\rangle_{R_2} = \boldsymbol{U}_{SM}\big(|\psi\rangle \otimes |g\rangle\big) = \big(\boldsymbol{M}_g|\psi\rangle\big) \otimes |g\rangle + \big(\boldsymbol{M}_e|\psi\rangle\big) \otimes |e\rangle$$

where $\boldsymbol{U}_{SM}$ is the total unitary transformation (Schrödinger propagator) defining the linear measurement operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ on $\mathcal{H}_S$. Since $\boldsymbol{U}_{SM}$ is unitary, $\boldsymbol{M}_g^\dagger \boldsymbol{M}_g + \boldsymbol{M}_e^\dagger \boldsymbol{M}_e = \boldsymbol{I}$.



The unitary propagator $\boldsymbol{U}_{SM}$ is derived from Jaynes-Cummings Hamiltonian $\boldsymbol{H}_{SM}$ in the interaction frame. Two kind of qubit/cavity Halmitonians:

resonant, $\boldsymbol{H}_{SM}/\hbar = i\big(\Omega(vt)/2\big)\ \big(\boldsymbol{a}^\dagger \otimes \boldsymbol{\sigma_-} - \boldsymbol{a} \otimes \boldsymbol{\sigma_+}\big)$,

dispersive, $\boldsymbol{H}_{SM}/\hbar = \big(\Omega^2(vt)/(2\delta)\big)\ \boldsymbol{N} \otimes \boldsymbol{\sigma_z}$,

where $\Omega(x) = \Omega_0 e^{-\frac{x^2}{w^2}}$, $x = vt$ with v atom velocity, Ω_0 vacuum Rabi pulsation, w radial mode-width and where $\delta = \omega_q - \omega_c$ is the detuning between qubit pulsation ω_q and cavity pulsation ω_c ($|\delta| \ll \Omega_0$).

Just before the measurement in D, the atom/field state is:

$$\boldsymbol{M}_g|\psi\rangle \otimes |g\rangle + \boldsymbol{M}_e|\psi\rangle \otimes |e\rangle$$

Denote by $\mu \in \{g, e\}$ the measurement outcome in detector D: with probability $p_\mu = \langle\psi|\boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu|\psi\rangle$ we get μ. Just after the measurement outcome μ, the state becomes separable:

$$|\Psi\rangle_D = \frac{1}{\sqrt{p_\mu}} \left(\boldsymbol{M}_\mu|\psi\rangle\right) \otimes |\mu\rangle = \left(\frac{\boldsymbol{M}_\mu}{\sqrt{\langle\psi|\boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu|\psi\rangle}}|\psi\rangle\right) \otimes |\mu\rangle.$$

Markov process (density matrix formulation $\rho \sim |\psi\rangle\langle\psi|$)

$$\rho_+ = \begin{cases} \mathcal{M}_g(\rho) = \frac{\boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger\right)}, & \text{with probability } p_g = \mathrm{Tr}\left(\boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger\right); \\ \mathcal{M}_e(\rho) = \frac{\boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger\right)}, & \text{with probability } p_e = \mathrm{Tr}\left(\boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger\right). \end{cases}$$

Kraus map: $\mathbb{E}\left(\rho_+/\rho\right) = \boldsymbol{K}(\rho) = \boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger.$

Input u: classical amplitude of a coherent micro-wave pulse.
State ρ: the density operator of the photon(s) trapped in the cavity.
Output y: quantum projective measure of the probe atom.
The ideal model reads

$$\rho_{k+1} = \begin{cases} \dfrac{\boldsymbol{D}_{u_k} \boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger \boldsymbol{D}_{u_k}^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger\right)} & y_k = g \text{ with probability } p_{g,k} = \mathrm{Tr}\left(\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger\right) \\[2ex] \dfrac{\boldsymbol{D}_{u_k} \boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger \boldsymbol{D}_{u_k}^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger\right)} & y_k = e \text{ with probability } p_{e,k} = \mathrm{Tr}\left(\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger\right) \end{cases}$$

- Displacement unitary operator ($u \in \mathbb{R}$): $\boldsymbol{D}_u = e^{u\boldsymbol{a}^\dagger - u\boldsymbol{a}}$ with $\boldsymbol{a} = \text{upper diag}(\sqrt{1}, \sqrt{2}, \ldots)$ the photon annihilation operator.
- Measurement Kraus operators in the linear dispersive case $\boldsymbol{M}_g = \cos\left(\frac{\phi_0 \boldsymbol{N} + \phi_R}{2}\right)$ and $\boldsymbol{M}_e = \sin\left(\frac{\phi_0 \boldsymbol{N} + \phi_R}{2}\right)$: $\boldsymbol{M}_g^\dagger \boldsymbol{M}_g + \boldsymbol{M}_e^\dagger \boldsymbol{M}_e = \boldsymbol{I}$ with $\boldsymbol{N} = \boldsymbol{a}^\dagger \boldsymbol{a} = \text{diag}(0, 1, 2, \ldots)$ the photon number operator.

$$\rho_{k+1} = \begin{cases} \frac{\cos\left(\frac{\phi_0 \boldsymbol{N}+\phi_R}{2}\right)\rho_k \cos\left(\frac{\phi_0 \boldsymbol{N}+\phi_R}{2}\right)}{\text{Tr}\left(\cos^2\left(\frac{\phi_0 \boldsymbol{N}+\phi_R}{2}\right)\rho_k\right)} & \text{with prob. } \text{Tr}\left(\cos^2\left(\frac{\phi_0 \boldsymbol{N}+\phi_R}{2}\right)\rho_k\right) \\ \frac{\sin\left(\frac{\phi_0 \boldsymbol{N}+\phi_R}{2}\right)\rho_k \sin\left(\frac{\phi_0 \boldsymbol{N}+\phi_R}{2}\right)}{\text{Tr}\left(\sin^2\left(\frac{\phi_0 \boldsymbol{N}+\phi_R}{2}\right)\rho_k\right)} & \text{with prob. } \text{Tr}\left(\sin^2\left(\frac{\phi_0 \boldsymbol{N}+\phi_R}{2}\right)\rho_k\right) \end{cases}$$

Steady state: any Fock state $\rho = |\bar{n}\rangle\langle\bar{n}|$ ($\bar{n} \in \mathbb{N}$) is a steady-state (no other steady state when (ϕ_R, ϕ_0, π) are $\mathbb{Q}$-independent)
Martingales: for any real function g, $V_g(\rho) = \text{Tr}\left(g(\boldsymbol{N})\rho\right)$ is a martingale:

$$\mathbb{E}\left(V_g(\rho_{k+1}) \;/\; \rho_k\right) = V_g(\rho_k).$$

Convergence to a Fock state when (ϕ_R, ϕ_0, π) are $\mathbb{Q}$-independent:
$V(\rho) = -\frac{1}{2}\sum_n \langle n|\rho|n\rangle^2$ is a super-martingale with

$$\mathbb{E}\left(V(\rho_{k+1}) \;/\; \rho_k\right) = V(\rho_k) - Q(\rho_k)$$

where $Q(\rho) \geq 0$ and $Q(\rho) = 0$ iff, ρ is a Fock state.
For a realization starting from ρ_0, the probability to converge towards the Fock state $|\bar{n}\rangle\langle\bar{n}|$ is equal to $\text{Tr}\left(|\bar{n}\rangle\langle\bar{n}|\rho_0\right) = \langle\bar{n}|\rho_0|\bar{n}\rangle$.

Structure of the stabilizing quantum feedback scheme

With a sampling time of 80 μs, the controller is classical here

- Goal: stabilization of the steady-state $|\bar{n}\rangle\langle\bar{n}|$ (controller set-point).
- At each time step k:
  1. read y_k the measurement outcome for probe atom k.
  2. update the quantum state estimation $\boldsymbol{\rho}^{\text{est}}_{\boldsymbol{k-1}}$ to $\boldsymbol{\rho}^{\text{est}}_{\boldsymbol{k}}$ from y_k
  3. compute u_k as a function of $\boldsymbol{\rho}^{\text{est}}_{\boldsymbol{k}}$ (state feedback).
  4. apply the micro-wave pulse of amplitude u_k.

An observer/controller structure:

1. real-time state estimation based on asymptotic observer: here quantum filtering techniques;
2. state feedback stabilization towards a stationary regime: here control Lyapunov techniques based on open-loop martingales $\text{Tr}\,(g(\boldsymbol{N})\rho)$.

It takes into account imperfections, delays (5 sampling) and cavity decoherence.

In finite dimension (truncation to $n^{\max}$ photons), all the mathematical details and convergence proof are given in the Automatica 2013 paper

- With pure state $\rho = |\psi\rangle\langle\psi|$, we have

$$\rho_+ = |\psi_+\rangle\langle\psi_+| = \frac{1}{\text{Tr}\left(\boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger\right)} \boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger$$

  when the atom collapses in $\mu = g, e$ with proba. $\text{Tr}\left(\boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger\right)$.

- Detection error rates: $P(y = e/\mu = g) = \eta_g \in [0,1]$ the probability of erroneous assignation to e when the atom collapses in g; $P(y = g/\mu = e) = \eta_e \in [0,1]$ (given by the contrast of the Ramsey fringes).

Bayes law: expectation ρ_+ of $|\psi_+\rangle\langle\psi_+|$ knowing ρ and the imperfect detection y.

$$\rho_+ = \begin{cases} \frac{(1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + \eta_e \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger}{\text{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + \eta_e \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)} \text{ if } y = g, \text{ prob. } \text{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + \eta_e \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right); \\ \frac{\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + (1-\eta_e) \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger}{\text{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + (1-\eta_e) \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)} \text{ if } y = e, \text{ prob. } \text{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + (1-\eta_e) \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right). \end{cases}$$

ρ_+ does not remain pure: the quantum state ρ_+ becomes a mixed state; $|\psi_+\rangle$ becomes physically irrelevant (not numerically).

We get

$$\rho_+ = \begin{cases} \frac{(1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+\eta_e\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger}{\text{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+\eta_e\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)}, & \text{with prob. } \text{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+\eta_e\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right); \\ \frac{\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+(1-\eta_e)\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger}{\text{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+(1-\eta_e)\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)} & \text{with prob. } \text{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+(1-\eta_e)\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right). \end{cases}$$

Key point:

$$\text{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+\eta_e\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right) \text{ and } \text{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+(1-\eta_e)\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)$$

are the probabilities to detect $y = g$ and e, knowing ρ.
Generalization: with $(\eta_{\mu',\mu})$ a left stochastic matrix $\eta_{\mu',\mu} \geq 0$ and $\sum_{\mu'} \eta_{\mu',\mu} = 1$, we have

$$\rho_+ = \frac{\sum_\mu \eta_{\mu',\mu}\boldsymbol{M}_\mu\rho\boldsymbol{M}_\mu^\dagger}{\text{Tr}\left(\sum_\mu \eta_{\mu',\mu}\boldsymbol{M}_\mu\rho\boldsymbol{M}_\mu^\dagger\right)} \quad \text{when we detect } y = \mu'.$$

The probability to detect $y = \mu'$ knowing ρ is $\text{Tr}\left(\sum_\mu \eta_{\mu',\mu}\boldsymbol{M}_\mu\rho\boldsymbol{M}_\mu^\dagger\right)$.

Discrete-time models are Markov processes

$$\rho_{k+1} = \frac{\sum_{\mu=1}^m \eta_{\mu',\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger}{\mathrm{Tr}\left(\sum_{\mu=1}^m \eta_{\mu',\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)}, \text{ with proba. } p_{\mu'}(\rho_k) = \sum_{\mu=1}^m \eta_{\mu',\mu} \mathrm{Tr}\left(\boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)$$

associated to Kraus maps (ensemble average, quantum channel)

$$\mathbb{E}\left(\rho_{k+1}|\rho_k\right) = \boldsymbol{K}(\rho_k) = \sum_\mu \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger \quad \text{with} \quad \sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \boldsymbol{I}$$

Continuous-time models are stochastic differential systems

$$d\rho_t = \left(-\tfrac{i}{\hbar}[\boldsymbol{H}, \rho_t] + \sum_\nu \boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)\right) dt$$
$$+ \sum_\nu \sqrt{\eta_\nu}\left(\boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger - \mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\rho_t\right)\rho_t\right) dW_{\nu,t}$$

driven by Wiener process $dW_{\nu,t} = dy_{\nu,t} - \sqrt{\eta_\nu}\, \mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\,\rho_t\right) dt$
with measures $y_{\nu,t}$, detection efficiencies $\eta_\nu \in [0, 1]$ and
Lindblad-Kossakowski master equations ($\eta_\nu \equiv 0$):

$$\frac{d}{dt}\rho = -\tfrac{i}{\hbar}[\boldsymbol{H}, \rho] + + \sum_\nu \boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)$$

Continuous/discrete-time diffusive SME

With a single imperfect measure $dy_t = \sqrt{\eta}\,\text{Tr}\left((L + L^\dagger)\,\rho_t\right)\,dt + dW_t$ and detection efficiency $\eta \in [0, 1]$, the quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-\frac{i}{\hbar}[H, \rho_t] + L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L\rho_t + \rho_t L^\dagger L)\right) dt + \sqrt{\eta}\left(L\rho_t + \rho_t L^\dagger - \text{Tr}\left((L + L^\dagger)\rho_t\right)\rho_t\right) dW_t$$

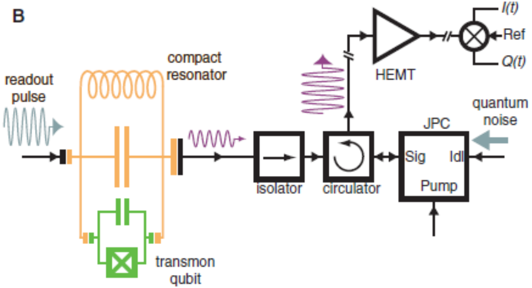
driven by the Wiener process dW_t

With Itō rules, it can be written as the following "discrete-time" Markov model

$$\rho_{t+dt} = \frac{M_{dy_t}\rho_t M_{dy_t}^\dagger + (1-\eta)L\rho_t L^\dagger dt}{\text{Tr}\left(M_{dy_t}\rho_t M_{dy_t}^\dagger + (1-\eta)L\rho_t L^\dagger dt\right)}$$

with $M_{dy_t} = I + \left(-\frac{i}{\hbar}H - \frac{1}{2}\left(L^\dagger L\right)\right) dt + \sqrt{\eta}\,dy_t L$.

A key physical example in circuit QED[3]



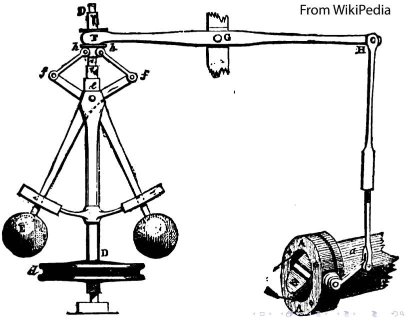
Superconducting qubit dispersively coupled to a cavity traversed by a microwave signal (input/output theory). The back-action on the qubit state of a single measurement of both output field quadratures I_t and Q_t is described by a simple SME for the qubit density operator.

$$d\rho_t = \left([u^* \boldsymbol{\sigma_-} - u\boldsymbol{\sigma_+}, \rho_t] + \gamma_t(\boldsymbol{\sigma_z}\rho\boldsymbol{\sigma_z} - \rho)\right)dt$$
$$+ \sqrt{\eta\gamma_t/2}\left(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\left(\boldsymbol{\sigma_z}\rho_t\right)\rho_t\right)\boldsymbol{dW_t^I} + \imath\sqrt{\eta\gamma_t/2}[\boldsymbol{\sigma_z}, \rho_t]\boldsymbol{dW_t^Q}$$

with I_t and Q_t given by $\boldsymbol{dI_t} = \sqrt{\eta\gamma_t/2}\,\mathrm{Tr}\left(2\boldsymbol{\sigma_z}\rho_t\right)dt + \boldsymbol{dW_t^I}$ and $\boldsymbol{dQ_t} = \boldsymbol{dW_t^Q}$, where $\gamma_t \geq 0$ is related by the read-out pulse shape and $\eta \in [0, 1]$ is the detection efficiency.

[3] M. Hatridge et al. Quantum Back-Action of an Individual Variable-Strength Measurement. Science, 2013, 339, 178-181.

Watt regulator: classical analogue of quantum coherent feedback.[4]



From WikiPedia

The first variations of speed $\delta\omega$ and governor angle $\delta\theta$ obey to

$$\frac{d}{dt}\delta\omega = -a\delta\theta$$

$$\frac{d^2}{dt^2}\delta\theta = -\Lambda\frac{d}{dt}\delta\theta - \Omega^2(\delta\theta - b\delta\omega)$$

with (a, b, Λ, Ω) positive parameters.

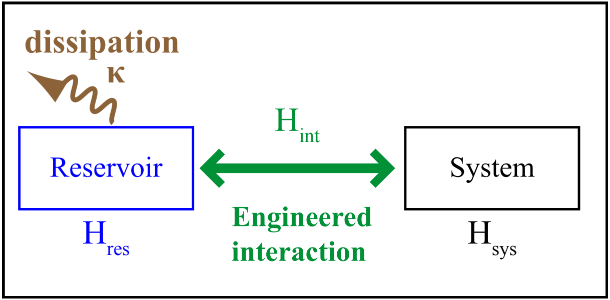
Third order system

$$\frac{d^3}{dt^3}\delta\omega + \Lambda\frac{d^2}{dt^2}\delta\omega + \Omega^2\frac{d}{dt}\delta\omega + ab\Omega^2\delta\omega = 0.$$

Characteristic polynomial $P(s) = s^3 + \Lambda s^2 + \Omega^2 s + ab\Omega^2$ with roots having negative real parts iff $\Lambda > ab$: governor damping must be strong enough to ensure asymptotic stability.

Key issues: asymptotic stability and convergence rates.

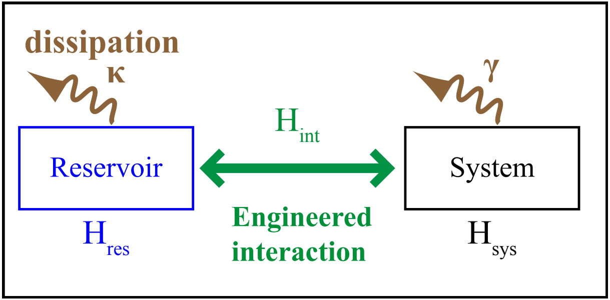
[4] J.C. Maxwell: On governors. Proc. of the Royal Society, No.100, 1868.



$$H = \boldsymbol{H}_{\text{res}} + \boldsymbol{H}_{\text{int}} + \boldsymbol{H}_{\text{syst}}$$

if $\rho \underset{t\to\infty}{\to} \rho_{res} \otimes |\bar{\psi}\rangle\langle\bar{\psi}|$ exponentially on a time scale of $\tau \approx 1/\kappa$ then . .

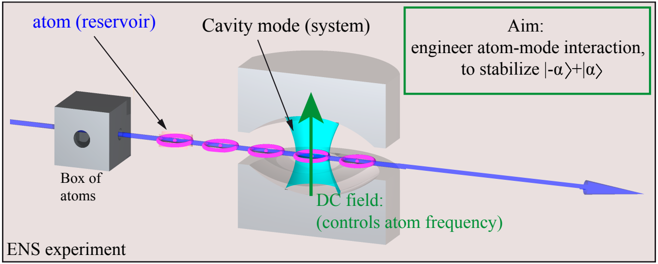
[5] See, e.g., the lectures of H. Mabuchi delivered at the "Ecole de physique des Houches", July 2011.



$$H = \boldsymbol{H}_{\text{res}} + \boldsymbol{H}_{\text{int}} + \boldsymbol{H}_{\text{syst}}$$

$\ldots\ldots \quad \rho \underset{t\to\infty}{\longrightarrow} \rho_{res} \otimes |\bar{\psi}\rangle\langle\bar{\psi}| + \Delta$, if $\kappa \gg \gamma$ then $\|\Delta\| \ll 1$

[5] See, e.g., the lectures of H. Mabuchi delivered at the "Ecole de physique des Houches", July 2011.

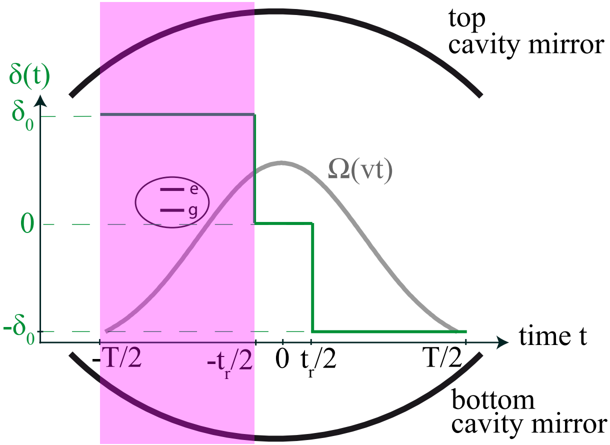


Jaynes-Cumming Hamiltionian

$$\boldsymbol{H}(t)/\hbar = \omega_c \boldsymbol{a}^\dagger \boldsymbol{a} \otimes \boldsymbol{I}_M + \boldsymbol{\omega_q(t)} \boldsymbol{I}_S \otimes \boldsymbol{\sigma_z}/2 + i\Omega(t)\left(\boldsymbol{a}^\dagger \otimes \boldsymbol{\sigma_-} - \boldsymbol{a} \otimes \boldsymbol{\sigma_+}\right)/2$$

with the open-loop control $t \mapsto \boldsymbol{\omega_q(t)}$ combining dispersive $\omega_q \neq \omega_c$ and resonant $\omega_q = \omega_c$ interactions.

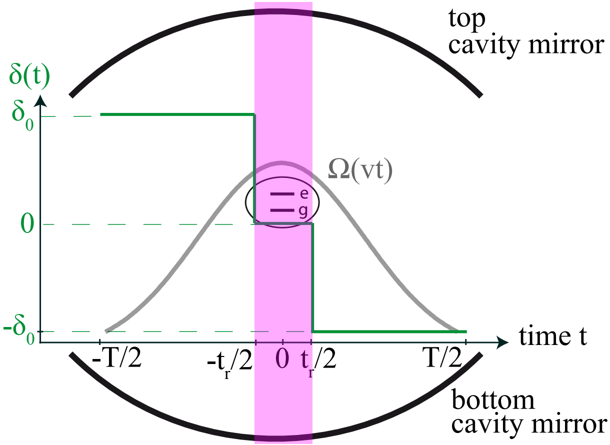
[6] A. Sarlette et al: Stabilization of Nonclassical States of the Radiation Field in a Cavity by Reservoir Engineering. Physical Review Letters, Volume 107, Issue 1, 2011.



$$\boldsymbol{U} = \boldsymbol{U}_{\text{off-resonant-1}}$$

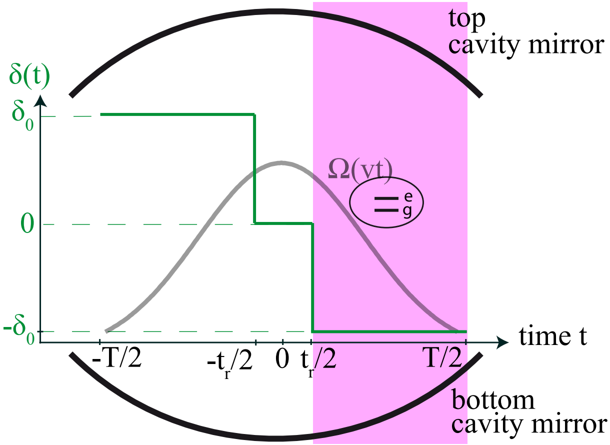
$$\boldsymbol{U} = \mathbf{X}(\xi_{\boldsymbol{N}})\,\mathbf{Z}(\phi_{\boldsymbol{N}}) \;,$$

Composite interaction with $\delta(t) = \omega_q(t) - \omega_c$



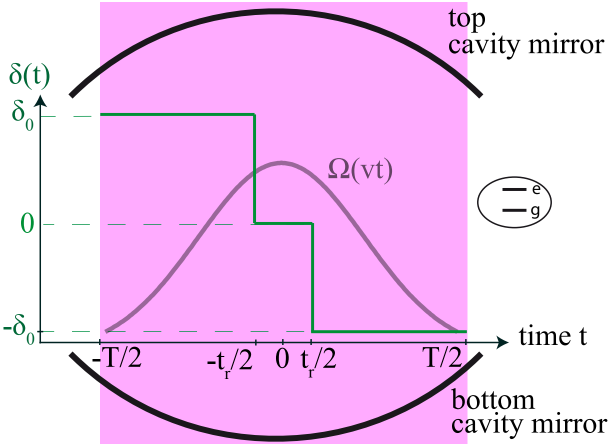
$$U = U_{\text{resonant}} U_{\text{off-resonant-1}}$$
$$U = \mathbf{Y}(\theta_N^r) \quad \mathbf{X}(\xi_N)\,\mathbf{Z}(\phi_N)\ ,$$

Composite interaction with $\delta(t) = \omega_q(t) - \omega_c$



$$U = U_{\text{off-resonant-2}} U_{\text{resonant}} U_{\text{off-resonant-1}}$$
$$U = \mathbf{Z}(-\phi_N)\,\mathbf{X}(\xi_N) \quad \mathbf{Y}(\theta_N^r) \quad \mathbf{X}(\xi_N)\,\mathbf{Z}(\phi_N)\ ,$$

Composite interaction with $\delta(t) = \omega_q(t) - \omega_c$



$$U \approx e^{-i\phi^{\text{Kerr}} N^2} U_{\text{resonant}} e^{i\phi^{\text{Kerr}} N^2}$$

Convergence of $\boldsymbol{K}$ iterates towards $(|\alpha_\infty\rangle + i|\text{-}\alpha_\infty\rangle)/\sqrt{2}$

Iterations $\rho_{k+1} = \boldsymbol{K}(\rho_k) = \boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger$ in the Kerr frame $\boldsymbol{\rho = e^{-ih_N^{\text{Kerr}}} \rho^{\text{Kerr}} e^{ih_N^{\text{Kerr}}}}$ yields

$$\rho_{k+1}^{\text{Kerr}} = \boldsymbol{K}^{\text{Kerr}}(\rho_k^{\text{Kerr}}) = \boldsymbol{M}_g^{\text{Kerr}} \rho_k^{\text{Kerr}} (\boldsymbol{M}_g^{\text{Kerr}})^\dagger + \boldsymbol{M}_e^{\text{Kerr}} \rho_k^{\text{Kerr}} (\boldsymbol{M}_e^{\text{Kerr}})^\dagger.$$

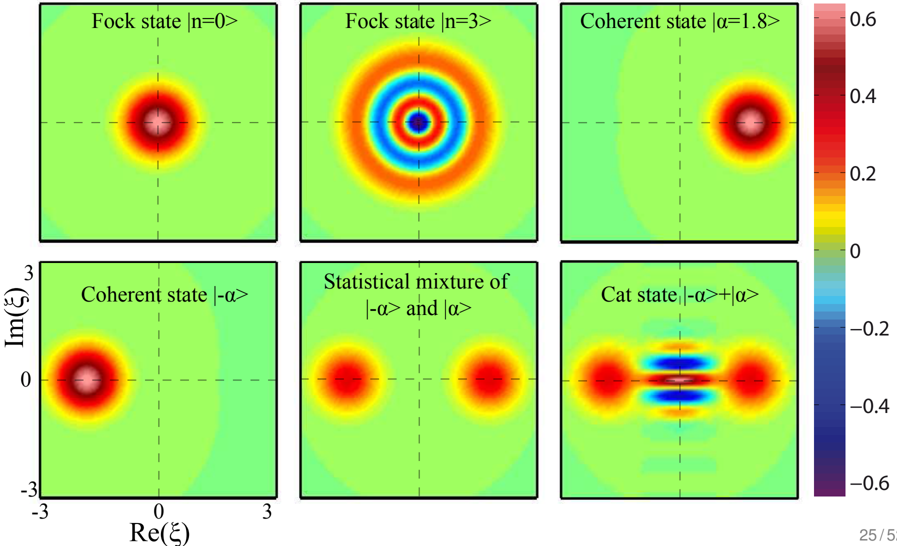
with $\boldsymbol{M}_g^{\text{Kerr}} = \cos(\frac{u}{2})\cos(\theta_{\boldsymbol{N}}/2) + \sin(\frac{u}{2}) \frac{\sin(\theta_{\boldsymbol{N}}/2)}{\sqrt{\boldsymbol{N}}} \boldsymbol{a}^\dagger$ and
$\boldsymbol{M}_e^{\text{Kerr}} = \sin(\frac{u}{2})\cos(\theta_{\boldsymbol{N}+1}/2) - \cos(\frac{u}{2}) \boldsymbol{a} \frac{\sin(\theta_{\boldsymbol{N}}/2)}{\sqrt{\boldsymbol{N}}}$.
Assume $|u| \leq \pi/2$, $\theta_0 = 0$, $\theta_n \in]0, \pi[$ for $n > 0$ and $\lim_{n \mapsto +\infty} \theta_n = \pi/2$, then (Zaki Leghtas, PhD thesis (2012))

- exists a unique common eigen-state $|\psi^{\text{Kerr}}\rangle$ of $\boldsymbol{M}_g^{\text{Kerr}}$ and $\boldsymbol{M}_e^{\text{Kerr}}$: $\rho_\infty^{\text{Kerr}} = |\psi^{\text{Kerr}}\rangle\langle\psi^{\text{Kerr}}|$ fixed point of $\boldsymbol{K}^{\text{Kerr}}$.
- if, moreover $n \mapsto \theta_n$ is increasing, $\lim_{k \mapsto +\infty} \rho_k^{\text{Kerr}} = \rho_\infty^{\text{Kerr}}$.

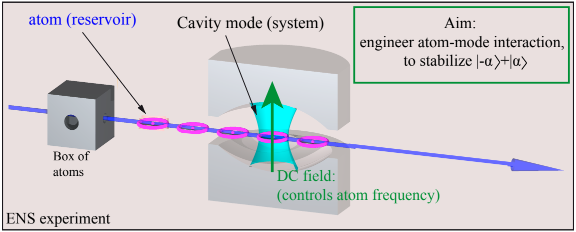
For well chosen experimental parameters, $\boldsymbol{\rho_\infty^{\text{Kerr}} \approx |\alpha_\infty\rangle\langle\alpha_\infty|}$ and $h_{\boldsymbol{N}}^{\text{Kerr}} \approx \pi \boldsymbol{N}^2/2$. Since $e^{-i\frac{\pi}{2}\boldsymbol{N}^2}|\alpha_\infty\rangle = \frac{e^{-i\pi/4}}{\sqrt{2}}\left(|\alpha_\infty\rangle + i|\text{-}\alpha_\infty\rangle\right)$:

$$\lim_{k \mapsto +\infty} \rho_k = \tfrac{1}{2}\Big(|\alpha_\infty\rangle + i|\text{-}\alpha_\infty\rangle\Big)\Big(\langle\alpha_\infty| + i\langle\text{-}\alpha_\infty|\Big)$$
$$\neq \tfrac{1}{2}|\alpha_\infty\rangle\langle\alpha_\infty| + \tfrac{1}{2}|\text{-}\alpha_\infty\rangle\langle\text{-}\alpha_\infty|.$$

$$W^\rho : \mathbb{C} \ni \xi \to \frac{2}{\pi} \text{Tr}\left(e^{i\pi \boldsymbol{N}} \boldsymbol{D}_{-\xi} \rho \boldsymbol{D}_\xi\right) \in [-2/\pi, 2/\pi]$$



An approximation by a continuous-time model[7]



In the Kerr frame $\rho = e^{-i\pi/2\ \boldsymbol{N}^2} \rho^{\text{Kerr}} e^{i\pi/2\ \boldsymbol{N}^2}$:

$$\frac{d}{dt}\rho^{\text{Kerr}} = u[\boldsymbol{a}^\dagger - \boldsymbol{a}, \rho^{\text{Kerr}}] + \kappa\left(\boldsymbol{a}\rho^{\text{Kerr}}\boldsymbol{a}^\dagger - (\boldsymbol{N}\rho^{\text{Kerr}} + \rho^{\text{Kerr}}\boldsymbol{N})/2\right)$$

Identical to the Lindbald master equation of a damped harmonic oscillator ($\kappa > 0$) driven by a coherent input field of amplitude u.
Simulations: convergence from vacuum in **ideal** and **realistic** cases.

[7] A. Sarlette et al: Stabilization of nonclassical states of one and two-mode radiation fields by reservoir engineering. Phys. Rev. A 86, 012114 (2012).

Lemma: the solutions of

$$\frac{d}{dt}\rho = u[\boldsymbol{a}^\dagger - \boldsymbol{a}, \rho] + \kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - (\boldsymbol{N}\rho + \rho\boldsymbol{N})/2\right)$$

converge exponentially towards $|\alpha_\infty\rangle\langle\alpha_\infty|$ with $\alpha_\infty = 2u/\kappa$.
Elementary proof: under the unitary change of frame

$$\rho = e^{(\alpha_\infty \boldsymbol{a}^\dagger - \alpha_\infty \boldsymbol{a})}\ \xi\ e^{-(\alpha_\infty \boldsymbol{a}^\dagger - \alpha_\infty \boldsymbol{a})}$$

the new density operator ξ is governed by

$$\frac{d}{dt}\xi = \kappa\left(\boldsymbol{a}\xi\boldsymbol{a}^\dagger - (\boldsymbol{N}\xi + \xi\boldsymbol{N})/2\right);$$

its energy $E = \text{Tr}\left(\boldsymbol{N}\xi\right) = \text{Tr}\left(\boldsymbol{a}^\dagger\boldsymbol{a}\xi\right)$ converges exponentially to 0 since it obeys to $\frac{d}{dt}E = -\kappa E$; thus ξ converges exponentially to $|0\rangle\langle 0|$.
Computation only based on commutation relations:

$$[\boldsymbol{a}, \boldsymbol{a}^\dagger] = 1, \quad \boldsymbol{a}\ f(\boldsymbol{N}) = f(\boldsymbol{N} + \boldsymbol{I})\ \boldsymbol{a}, \quad e^{-(\alpha\boldsymbol{a}^\dagger + \alpha^*\boldsymbol{a})}\ \boldsymbol{a}\ e^{(\alpha\boldsymbol{a}^\dagger - \alpha^*\boldsymbol{a})} = \boldsymbol{a} + \alpha.$$

$$\frac{d}{dt}\rho = u[\boldsymbol{a}^\dagger - \boldsymbol{a}, \rho] + \kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - (\boldsymbol{N}\rho + \rho\boldsymbol{N})/2\right)$$

ρ can be represented by its Wigner function W^ρ defined by

$$\mathbb{C} \ni \xi = x + ip \mapsto W^\rho(\xi) = \tfrac{2}{\pi}\,\mathrm{Tr}\left(e^{i\pi\boldsymbol{N}} e^{-\xi\boldsymbol{a}^\dagger + \xi^*\boldsymbol{a}}\rho e^{\xi\boldsymbol{a}^\dagger - \xi^*\boldsymbol{a}}\right)$$

With the correspondences

$$\frac{\partial}{\partial\xi} = \tfrac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial p}\right), \quad \frac{\partial}{\partial\xi^*} = \tfrac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial p}\right)$$

$$W^{\rho\boldsymbol{a}} = \left(\xi - \tfrac{1}{2}\frac{\partial}{\partial\xi^*}\right)W^\rho, \quad W^{\boldsymbol{a}\rho} = \left(\xi + \tfrac{1}{2}\frac{\partial}{\partial\xi^*}\right)W^\rho$$

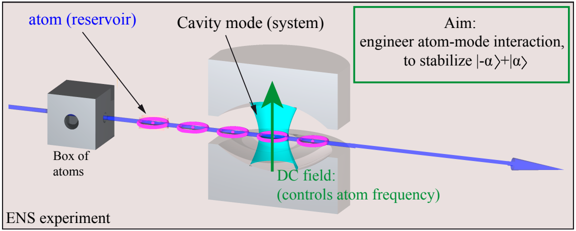
$$W^{\rho\boldsymbol{a}^\dagger} = \left(\xi^* + \tfrac{1}{2}\frac{\partial}{\partial\xi}\right)W^\rho, \quad W^{\boldsymbol{a}^\dagger\rho} = \left(\xi^* - \tfrac{1}{2}\frac{\partial}{\partial\xi}\right)W^\rho$$

we get the following PDE for W^ρ ($\alpha_\infty = 2u/\kappa$):

$$\frac{\partial W^\rho}{\partial t} = \tfrac{\kappa}{2}\left(\frac{\partial}{\partial x}\left((x-\alpha_\infty)W^\rho\right) + \frac{\partial}{\partial p}\left(pW^\rho\right) + \tfrac{1}{4}\Delta W^\rho\right)$$

converging toward the Gaussian $W^{\rho_\infty}(x,p) = \frac{2}{\pi}e^{-2(x-\alpha_\infty)^2 - 2p^2}$.

Reservoir with the cavity relaxation ($1/\kappa_c$ photon life-time)[8]



In the Kerr representation frame $\rho = e^{-i\pi/2\ \boldsymbol{N}^2} \rho^{\text{Kerr}} e^{i\pi/2\ \boldsymbol{N}^2}$:

$$\frac{d}{dt}\rho^{\text{Kerr}} = \overbrace{u[\boldsymbol{a}^\dagger - \boldsymbol{a}, \rho^{\text{Kerr}}] + \kappa\left(\boldsymbol{a}\rho^{\text{Kerr}}\boldsymbol{a}^\dagger - (\boldsymbol{N}\rho^{\text{Kerr}} + \rho^{\text{Kerr}}\boldsymbol{N})/2\right)}^{\text{reservoir relaxation}} \\ + \underbrace{\kappa_c\left(e^{i\pi\boldsymbol{N}}\boldsymbol{a}\rho^{\text{Kerr}}\boldsymbol{a}^\dagger e^{-i\pi\boldsymbol{N}} - (\boldsymbol{N}\rho^{\text{Kerr}} + \rho^{\text{Kerr}}\boldsymbol{N})/2\right)}_{\text{cavity decoherence}}.$$

[8] A. Sarlette et al: Stabilization of nonclassical states of one and two-mode radiation fields by reservoir engineering. Phys. Rev. A 86, 012114 (2012).

The steady state $\rho_\infty^{\text{Kerr}}$ in the Kerr frame

$$0 = u[\boldsymbol{a}^\dagger - \boldsymbol{a}, \rho_\infty^{\text{Kerr}}] + \kappa\left(\boldsymbol{a}\rho_\infty^{\text{Kerr}}\boldsymbol{a}^\dagger - (\boldsymbol{N}\rho_\infty^{\text{Kerr}} + \rho_\infty^{\text{Kerr}}\boldsymbol{N})/2\right)$$
$$+ \kappa_c\left(e^{i\pi\boldsymbol{N}}\boldsymbol{a}\rho_\infty^{\text{Kerr}}\boldsymbol{a}^\dagger e^{-i\pi\boldsymbol{N}} - (\boldsymbol{N}\rho_\infty^{\text{Kerr}} + \rho_\infty^{\text{Kerr}}\boldsymbol{N})/2\right)$$

is unique

$$\rho_\infty^{\text{Kerr}} = \int_{-\alpha_\infty^c}^{\alpha_\infty^c} \mu(x)|x\rangle\langle x|\, dx.$$

The positive weight function μ (Glauber-Shudarshan P distribution) is given by

$$\mu(x) = \mu_0 \frac{\left(((\alpha_\infty^c)^2 - x^2)^{(\alpha_\infty^c)^2}\ e^{x^2}\right)^{r_c}}{\alpha_\infty^c - x},$$

with $r_c = 2\kappa_c/(\kappa + \kappa_c)$ and $\alpha_\infty^c = 2u/(\kappa + \kappa_c)$. The normalization factor $\mu_0 > 0$ ensures that $\int_{-\alpha_\infty^c}^{\alpha_\infty^c} \mu(x)dx = 1$.

Conjecture: global (exponential) convergence towards $\rho_\infty^{\text{Kerr}}$ of $\rho^{\text{Kerr}}(t)$ as $t \mapsto +\infty$.

Robustness of the reservoir stabilizing the two-leg cat.

Since $W^{e^{i\pi N}\rho^{\text{Kerr}}e^{-i\pi N}}(\xi) = W^{\rho^{\text{Kerr}}}(-\xi)$ the master Lindblad equation

$$\frac{d}{dt}\rho^{\text{Kerr}} = \overbrace{u[a^\dagger - a, \rho^{\text{Kerr}}] + \kappa\left(a\rho^{\text{Kerr}}a^\dagger - (N\rho^{\text{Kerr}} + \rho^{\text{Kerr}}N)/2\right)}^{\text{reservoir relaxation}}$$
$$+ \underbrace{\kappa_c\left(ae^{i\pi N}\rho^{\text{Kerr}}e^{-i\pi N}a^\dagger - (N\rho^{\text{Kerr}} + \rho^{\text{Kerr}}N)/2\right)}_{\text{cavity decoherence}}.$$

yields to the following non local diffusion PDE (quantum Fokker-Planck equation):

$$\left.\frac{\partial W^{\rho^{\text{Kerr}}}}{\partial t}\right|_{(x,p)} = \tfrac{\kappa+\kappa_c}{2}\left(\frac{\partial}{\partial x}\left((x-\alpha_\infty)W^{\rho^{\text{Kerr}}}\right) + \frac{\partial}{\partial p}\left(pW^{\rho^{\text{Kerr}}}\right) + \tfrac{1}{4}\Delta W^{\rho^{\text{Kerr}}}\right)_{(x,p)}$$
$$+\kappa_c\left((x^2+p^2+\tfrac{1}{2})\left(\left.W^{\rho^{\text{Kerr}}}\right|_{(-x,-p)} - \left.W^{\rho^{\text{Kerr}}}\right|_{(x,p)}\right) + \frac{1}{16}\left(\left.\Delta W^{\rho^{\text{Kerr}}}\right|_{(-x,-p)} - \left.\Delta W^{\rho^{\text{Kerr}}}\right|_{(x,p)}\right)\right)$$
$$-\kappa_c\left(\frac{x}{2}\left(\left.\frac{\partial W^{\rho^{\text{Kerr}}}}{\partial x}\right|_{(-x,-p)} + \left.\frac{\partial W^{\rho^{\text{Kerr}}}}{\partial x}\right|_{(x,p)}\right) + \frac{p}{2}\left(\left.\frac{\partial W^{\rho^{\text{Kerr}}}}{\partial p}\right|_{(-x,-p)} + \left.\frac{\partial W^{\rho^{\text{Kerr}}}}{\partial p}\right|_{(x,p)}\right)\right)$$

Convergence towards $W^{\rho^{\text{Kerr}}_\infty}(x,p) = \int_{-\alpha^c_\infty}^{\alpha^c_\infty} \frac{2\mu(\alpha)}{\pi} e^{-2(x-\alpha)^2-2p^2}\, d\alpha$ remains to be proved.

Quantum information processing with cat-qubits [9]

It is possible with circuit QED to design an open quantum system governed by

$$\frac{d}{dt}\rho = u[(\boldsymbol{a}^2)^\dagger - \boldsymbol{a}^2, \rho] + \kappa\left(\boldsymbol{a}^2\rho(\boldsymbol{a}^2)^\dagger - ((\boldsymbol{a}^2)^\dagger\boldsymbol{a}^2\rho + \rho(\boldsymbol{a}^2)^\dagger\boldsymbol{a}^2)/2\right)$$

where $\boldsymbol{a}$ is replaced by $\boldsymbol{a}^2$. The supports of all solutions $\rho(t)$ converge to the decoherence free space spanned by the even and odd cat-state;

$$|C^+_{\alpha_\infty}\rangle \propto |\alpha_\infty\rangle + |\text{-}\alpha_\infty\rangle, \quad |C^-_{\alpha_\infty}\rangle \propto |\alpha_\infty\rangle - |\text{-}\alpha_\infty\rangle \text{ with } \alpha_\infty = \sqrt{2u/\kappa}.$$

The corresponding PDE for W^ρ is of order 4 in x and p.
A similar system where $\boldsymbol{a}$ is replaced now with $\boldsymbol{a}^4$ could be very interesting for quantum information processing where the logical qubit is encoded in the planes spanned by even and odd cat-states:

$$\left\{|C^+_{\alpha_\infty}\rangle, |C^+_{i\alpha_\infty}\rangle\right\}, \quad \left\{|C^-_{\alpha_\infty}\rangle, |C^-_{i\alpha_\infty}\rangle\right\}. \text{ with } \alpha_\infty = \sqrt[4]{2u/\kappa}.$$

The corresponding PDE for W^ρ is of order 8 in x and p.

[9] M. Mirrahimi et al: Dynamically protected cat-qubits: a new paradigm for universal quantum computation, `arXiv:1312.2017v1`, 2014.

Conclusion: convergence issues of open-quantum systems

Discrete time model (Kraus maps):

$$\rho_{k+1} = K(\rho_k) = \sum_\nu M_\nu \rho_k M_\nu^\dagger \quad \text{with} \quad \sum_\nu M_\nu^\dagger M_\nu = \boldsymbol{I}$$

Continuous-time model (Lindbald, Fokker-Planck eq.):

$$\frac{d}{dt}\rho = -\tfrac{i}{\hbar}[H,\rho] + \sum_\nu \left(L_\nu \rho L_\nu^\dagger - (L_\nu^\dagger L_\nu \rho + \rho L_\nu^\dagger L_\nu)/2 \right),$$

Stability induces by contraction for a lot of metrics (nuclear norm $\text{Tr}\left(|\rho-\sigma|\right)$, fidelity $\text{Tr}\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)$, see the work of D. Petz).
Open issues motivated by robust quantum information processing:

1. characterization of the Ω-limit support of ρ: decoherence free spaces are affine spaces where the dynamics are of Schrödinger types; they can be reduced to a point (pointer-state);
2. Estimation of convergence rate and robustness.
3. Reservoir engineering: design of realistic M_ν and L_ν to achieve rapid convergence towards prescribed affine spaces (protection against decoherence).

- **Former PhDs and PostDocs:** Hadis Amini, Hector Bessa Silveira, Zaki Leghtas, Alain Sarlette, Ram Somaraju.
- **LKB Physicists:** Michel Brune, Igor Dotsenko, Sébastien Gleyzes, Serge Haroche, Jean-Michel Raimond, Bruno Peaudecerf, Clément Sayrin, Xingxing Zhou.
- **Quantic project (INRIA/ENS/MINES):** Benjamin Huard, François Mallet, Mazyar Mirrahimi, Landry Bretheau, Philippe Campagne, Joachim Cohen, Emmanuel Flurin, Ananda Roy, Pierre Six.
- **Mathematicians:** Karine Beauchard, Jean-Michel Coron, Thomas Chambrion, Bernard Bonnard, Ugo Boscain, Sylvain Ervedoza, Stéphane Gaubert, Andrea Grigoriu, Claude Le Bris, Yvon Maday, Vahagn Nersesyan, Clément Pellégrini, Paulo Sergio Pereira da Silva, Jean-Pierre Puel, Lionel Rosier, Julien Salomon, Rodolphe Sepulchre, Mario Sigalotti, Gabriel Turinici.
- **Centre Automatique et Systèmes:** François Chaplais, Florent Di Méglio, Jean Lévine, Philippe Martin, Nicolas Petit, Laurent Praly.
- **And also:** Lectures at Collège de France, ANR projects CQUID and EMAQS, UPS-COFECUB, . . .

The Lyapunov feedback scheme is based on a strict control Lyapunov function:

$$V_\epsilon(\rho) = \sum_n \left(-\epsilon\langle n\,|\rho|\,n\rangle^2 + \sigma_n\langle n|\rho|n\rangle\right)$$

where $\epsilon > 0$ is small enough and

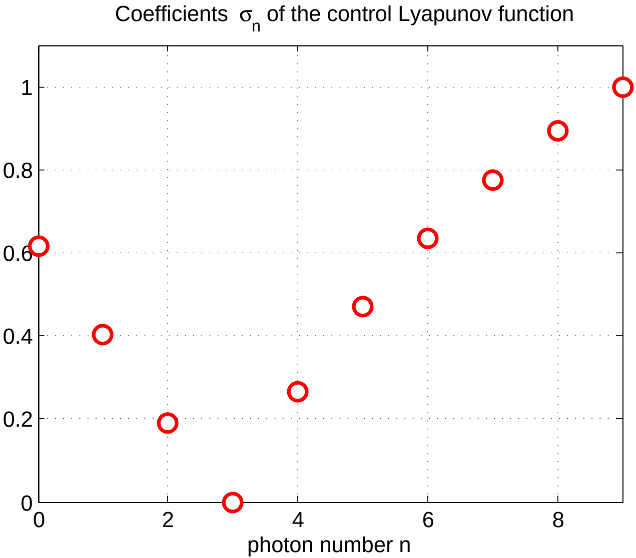
$$\sigma_n = \begin{cases} \frac{1}{4} + \sum_{\nu=1}^{\bar{n}} \frac{1}{\nu} - \frac{1}{\nu^2}, & \text{if } n = 0; \\ \sum_{\nu=n+1}^{\bar{n}} \frac{1}{\nu} - \frac{1}{\nu^2}, & \text{if } n \in [1, \bar{n}-1]; \\ 0, & \text{if } n = \bar{n}; \\ \sum_{\nu=\bar{n}+1}^{n} \frac{1}{\nu} + \frac{1}{\nu^2}, & \text{if } n \in [\bar{n}+1, +\infty] \end{cases}$$

Feedback law: $u = f(\rho) =: \underset{\upsilon\in[-\bar{u},\bar{u}]}{\text{Argmin}} \quad V_\epsilon\Big(\boldsymbol{D}_\upsilon\left(\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)\boldsymbol{D}_\upsilon^\dagger\Big).$

Achieve global stabilization since the decrease is **strict**

$$\forall\rho\neq|\bar{n}\rangle\langle\bar{n}|,\quad V_\epsilon\Big(\boldsymbol{D}_{f(\rho)}\left(\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)\boldsymbol{D}_{f(\rho)}^\dagger\Big) < V_\epsilon\Big(\rho\Big).$$

The control Lyapunov function used for experiment.



$$V_\epsilon(\rho) = \sum_n \left(-\epsilon\langle n\,|\rho|\,n\rangle^2 + \sigma_n\langle n|\rho|n\rangle\right) \text{ for } \bar{n} = 3.$$

$\sigma_n \sim \log(n)$: key issue to avoid trajectories escaping to $n = +\infty$.

Take $|\psi_{k+1}\rangle\langle\psi_{k+1}| = \frac{1}{\mathrm{Tr}\left(\boldsymbol{M}_{\mu_k}|\psi_k\rangle\langle\psi_k|\boldsymbol{M}_{\mu_k}^\dagger\right)}\left(\boldsymbol{M}_{\mu_k}|\psi_k\rangle\langle\psi_k|\boldsymbol{M}_{\mu_k}^\dagger\right)$ with measure imperfections and decoherence described by the left stochastic matrix η: $\eta_{\mu',\mu} \in [0,1]$ is the probability of having the imperfect outcome $\mu' \in \{1,\ldots,m'\}$ knowing that the perfect one is $\mu \in \{1,\ldots,m\}$.

The optimal Belavkin filter: $\rho_k = \mathbb{E}\left(|\psi_k\rangle\langle\psi_k| \Big| |\psi_0\rangle, \mu'_0,\ldots,\mu'_{k-1}\right)$ can be computed efficiently via the following recurrence

$$\rho_{k+1} = \frac{1}{\mathrm{Tr}\left(\sum_{\mu=1}^m \eta_{\mu'_k,\mu}\boldsymbol{M}_\mu\rho_k\boldsymbol{M}_\mu^\dagger\right)}\left(\sum_{\mu=1}^m \eta_{\mu'_k,\mu}\boldsymbol{M}_\mu\rho_k\boldsymbol{M}_\mu^\dagger\right)$$

where the detector outcome μ'_k takes values μ' in $\{1,\cdots,m'\}$ with probability $p_{\mu',\rho_k} = \mathrm{Tr}\left(\sum_{\mu=1}^m \eta_{\mu'_k,\mu}\boldsymbol{M}_\mu\rho_k\boldsymbol{M}_\mu^\dagger\right)$.

- The quantum state $\rho_k = \mathbb{E}\left(|\psi_k\rangle\langle\psi_k| \,\Big|\, |\psi_0\rangle, \mu'_0, \ldots, \mu'_{k-1} \right)$ is given by the following optimal Belavkin filtering process

$$\rho_{k+1} = \frac{1}{\mathrm{Tr}\left(\sum_{\mu=1}^{m} \eta_{\mu'_k,\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)} \left(\sum_{\mu=1}^{m} \eta_{\mu'_k,\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger \right)$$

with the perfect initialization: $\rho_0 = |\psi_0\rangle\langle\psi_0|$.

- Its estimate $\boldsymbol{\rho}^{\text{est}}$ follows the same recurrence

$$\boldsymbol{\rho}^{\text{est}}_{\boldsymbol{k+1}} = \frac{1}{\mathrm{Tr}\left(\sum_{\mu=1}^{m} \eta_{\mu'_k,\mu} \boldsymbol{M}_\mu \boldsymbol{\rho}^{\text{est}}_{\boldsymbol{k}} \boldsymbol{M}_\mu^\dagger\right)} \left(\sum_{\mu=1}^{m} \eta_{\mu'_k,\mu} \boldsymbol{M}_\mu \boldsymbol{\rho}^{\text{est}}_{\boldsymbol{k}} \boldsymbol{M}_\mu^\dagger \right)$$

but with imperfect initialization $\boldsymbol{\rho}^{\text{est}}_{\boldsymbol{0}} \neq |\psi_0\rangle\langle\psi_0|$.

A natural question : $\boldsymbol{\rho}^{\text{est}}_{\boldsymbol{k}} \mapsto \rho_k$ when $k \mapsto +\infty$?

Markov process of state $(\rho_k, \boldsymbol{\rho_k^{\text{est}}})$

$$\rho_{k+1} = \frac{\sum_{\mu=1}^{m} \eta_{\mu'_k,\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger}{\text{Tr}\left(\sum_{\mu=1}^{m} \eta_{\mu'_k,\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)}, \quad \boldsymbol{\rho_{k+1}^{\text{est}}} = \frac{\sum_{\mu=1}^{m} \eta_{\mu'_k,\mu} \boldsymbol{M}_\mu \boldsymbol{\rho_k^{\text{est}}} \boldsymbol{M}_\mu^\dagger}{\text{Tr}\left(\sum_{\mu=1}^{m} \eta_{\mu'_k,\mu} \boldsymbol{M}_\mu \boldsymbol{\rho_k^{\text{est}}} \boldsymbol{M}_\mu^\dagger\right)}$$

Proba. to get μ'_k at step k, $\text{Tr}\left(\sum_{\mu=1}^{m} \eta_{\mu'_k,\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)$, depends on ρ_k.

- Convergence of $\boldsymbol{\rho_k^{\text{est}}}$ towards ρ_k when $k \mapsto +\infty$ is an open problem.
  A partial result (continuous-time) due to R. van Handel: The stability of quantum Markov filters. Infin. Dimens. Anal. Quantum Probab. Relat. Top. , 2009, 12, 153-172.
- Stability[10]: the fidelity $F(\rho_k, \boldsymbol{\rho_k^{\text{est}}}) = \text{Tr}^2\left(\sqrt{\sqrt{\rho_k}\boldsymbol{\rho_k^{\text{est}}}\sqrt{\rho_k}}\right)$ is a sub-martingale for any η and $\boldsymbol{M}_\mu$:

$$\mathbb{E}\left(F(\rho_{k+1}, \boldsymbol{\rho_{k+1}^{\text{est}}})/\rho_k\right) \geq F(\rho_k, \boldsymbol{\rho_k^{\text{est}}}).$$

Fidelity: $0 \leq F(\rho, \rho^e) \leq 1$ and $F(\rho, \rho^e) = 1$ iff $\rho = \rho^e$.

[10] A. Somaraju et al: Design and Stability of Discrete-Time Quantum Filters with Measurement Imperfections. American Control Conference, 2012, 5084-5089.

The key inequality underlying $F(\rho, \rho^e)$ is sub-martingale[11]

For

- any set of m matrices $\boldsymbol{M}_\mu$ with $\sum_{\mu=1}^m \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = 1$,
- any partition of $\{1, \ldots, m\}$ into $p \geq 1$ sub-sets $\mathcal{P}_\nu$,
- any Hermitian non-negative matrices ρ and σ of trace one,

the following inequality holds

$$\sum_{\nu=1}^{\nu=p} \text{Tr}\left(\sum_{\mu\in\mathcal{P}_\nu} \boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger\right) F\left(\frac{\sum_{\mu\in\mathcal{P}_\nu} \boldsymbol{M}_\mu \sigma \boldsymbol{M}_\mu^\dagger}{\text{Tr}\left(\sum_{\mu\in\mathcal{P}_\nu} \boldsymbol{M}_\mu \sigma \boldsymbol{M}_\mu^\dagger\right)}, \frac{\sum_{\mu\in\mathcal{P}_\nu} \boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger}{\text{Tr}\left(\sum_{\mu\in\mathcal{P}_\nu} \boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger\right)}\right) \geq F(\sigma, \rho)$$

where $F(\sigma, \rho) = \text{Tr}^2\left(\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}\right)$.
Proof combines on a lifting procedure with Ulhmann's theorem.

[11] PR. Fidelity is a Sub-Martingale for Discrete-Time Quantum Filters. IEEE Transactions on Automatic Control, 2011, 56, 2743-2747.

With Poisson process $\boldsymbol{N(t)}$, $\langle \boldsymbol{dN(t)} \rangle = \left(\overline{\theta} + \overline{\eta}\,\text{Tr}\left(\boldsymbol{V}\rho_t\boldsymbol{V}^\dagger\right)\right) dt$, and detection imperfections modeled by $\overline{\theta} \geq 0$ and $\overline{\eta} \in [0, 1]$, the quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-\tfrac{i}{\hbar}[\boldsymbol{H}, \rho_t] + \boldsymbol{V}\rho_t\boldsymbol{V}^\dagger - \frac{1}{2}(\boldsymbol{V}^\dagger\boldsymbol{V}\rho_t + \rho_t\boldsymbol{V}^\dagger\boldsymbol{V})\right) dt$$
$$+ \left(\frac{\overline{\theta}\rho_t + \overline{\eta}\boldsymbol{V}\rho_t\boldsymbol{V}^\dagger}{\overline{\theta} + \overline{\eta}\,\text{Tr}\left(\boldsymbol{V}\rho_t\boldsymbol{V}^\dagger\right)} - \rho_t\right)\left(\boldsymbol{dN(t)} - \left(\overline{\theta} + \overline{\eta}\,\text{Tr}\left(\boldsymbol{V}\rho_t\boldsymbol{V}^\dagger\right)\right) dt\right)$$

For $\boldsymbol{N(t+dt) - N(t) = 1}$ we have $\rho_{t+dt} = \dfrac{\overline{\theta}\rho_t + \overline{\eta}\boldsymbol{V}\rho_t\boldsymbol{V}^\dagger}{\overline{\theta} + \overline{\eta}\,\text{Tr}\left(\boldsymbol{V}\rho_t\boldsymbol{V}^\dagger\right)}$.

For $\boldsymbol{dN(t) = 0}$ we have

$$\rho_{t+dt} = \frac{\boldsymbol{M_0}\rho_t\boldsymbol{M_0^\dagger} + (1-\overline{\eta})\boldsymbol{V}\rho_t\boldsymbol{V}^\dagger dt}{\text{Tr}\left(\boldsymbol{M_0}\rho_t\boldsymbol{M_0^\dagger} + (1-\overline{\eta})\boldsymbol{V}\rho_t\boldsymbol{V}^\dagger dt\right)}$$

with $\boldsymbol{M_0} = \boldsymbol{I} + \left(-\tfrac{i}{\hbar}\boldsymbol{H} + \tfrac{1}{2}\left(\overline{\eta}\,\text{Tr}\left(\boldsymbol{V}\rho_t\boldsymbol{V}^\dagger\right)\boldsymbol{I} - \boldsymbol{V}^\dagger\boldsymbol{V}\right)\right) dt$.

Continuous/discrete-time diffusive-jump SME

The quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-\frac{i}{\hbar}[\boldsymbol{H},\rho_t] + \boldsymbol{L}\rho_t\boldsymbol{L}^\dagger - \frac{1}{2}(\boldsymbol{L}^\dagger\boldsymbol{L}\rho_t + \rho_t\boldsymbol{L}^\dagger\boldsymbol{L}) + \boldsymbol{V}\rho_t\boldsymbol{V}^\dagger - \frac{1}{2}(\boldsymbol{V}^\dagger\boldsymbol{V}\rho_t + \rho_t\boldsymbol{V}^\dagger\boldsymbol{V})\right) dt$$
$$+ \sqrt{\eta}\left(\boldsymbol{L}\rho_t + \rho_t\boldsymbol{L}^\dagger - \operatorname{Tr}\left((\boldsymbol{L}+\boldsymbol{L}^\dagger)\rho_t\right)\rho_t\right) \boldsymbol{dW_t}$$
$$+ \left(\frac{\overline{\theta}\rho_t + \overline{\eta}\boldsymbol{V}\rho_t\boldsymbol{V}^\dagger}{\overline{\theta} + \overline{\eta}\operatorname{Tr}\left(\boldsymbol{V}\rho_t\boldsymbol{V}^\dagger\right)} - \rho_t\right)\left(\boldsymbol{dN(t)} - \left(\overline{\theta} + \overline{\eta}\operatorname{Tr}\left(\boldsymbol{V}\rho_t\boldsymbol{V}^\dagger\right)\right) dt\right)$$

For $\boldsymbol{N(t+dt) - N(t) = 1}$ we have $\rho_{t+dt} = \dfrac{\overline{\theta}\rho_t + \overline{\eta}\boldsymbol{V}\rho_t\boldsymbol{V}^\dagger}{\overline{\theta} + \overline{\eta}\operatorname{Tr}\left(\boldsymbol{V}\rho_t\boldsymbol{V}^\dagger\right)}$.

For $\boldsymbol{dN(t) = 0}$ we have

$$\rho_{t+dt} = \frac{\boldsymbol{M}_{\boldsymbol{dy_t}}\rho_t\boldsymbol{M}_{\boldsymbol{dy_t}}^\dagger + (1-\eta)\boldsymbol{L}\rho_t\boldsymbol{L}^\dagger dt + (1-\overline{\eta})\boldsymbol{V}\rho_t\boldsymbol{V}^\dagger dt}{\operatorname{Tr}\left(\boldsymbol{M}_{\boldsymbol{dy_t}}\rho_t\boldsymbol{M}_{\boldsymbol{dy_t}}^\dagger + (1-\eta)\boldsymbol{L}\rho_t\boldsymbol{L}^\dagger dt + (1-\overline{\eta})\boldsymbol{V}\rho_t\boldsymbol{V}^\dagger dt\right)}$$

with $\boldsymbol{M}_{\boldsymbol{dy_t}} = \boldsymbol{I} + \left(-\frac{i}{\hbar}\boldsymbol{H} - \frac{1}{2}\boldsymbol{L}^\dagger\boldsymbol{L} + \frac{1}{2}\left(\overline{\eta}\operatorname{Tr}\left(\boldsymbol{V}\rho_t\boldsymbol{V}^\dagger\right)\boldsymbol{I} - \boldsymbol{V}^\dagger\boldsymbol{V}\right)\right) dt + \sqrt{\eta}\,\boldsymbol{dy_t}\boldsymbol{L}$.

Continuous/discrete-time general diffusive-jump SME

The quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-\frac{i}{\hbar}[\boldsymbol{H}, \rho_t] + \sum_\nu \boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu) + \sum_\mu \boldsymbol{V}_\mu \rho_t \boldsymbol{V}_\mu^\dagger - \frac{1}{2}(\boldsymbol{V}_\mu^\dagger \boldsymbol{V}_\mu \rho_t + \rho_t \boldsymbol{V}_\mu^\dagger \boldsymbol{V}_\mu)\right) dt$$

$$+ \sum_\nu \sqrt{\eta_\nu}\left(\boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger - \operatorname{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\rho_t\right)\rho_t\right) \boldsymbol{dW_{\nu,t}}$$

$$+ \sum_\mu \left(\frac{\overline{\theta}_\mu \rho_t + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \boldsymbol{V}_\mu \rho_t \boldsymbol{V}_\mu^\dagger}{\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \operatorname{Tr}\left(\boldsymbol{V}_{\mu'} \rho_t \boldsymbol{V}_{\mu'}^\dagger\right)} - \rho_t\right)\left(\boldsymbol{dN_\mu(t)} - \left(\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \operatorname{Tr}\left(\boldsymbol{V}_{\mu'} \rho_t \boldsymbol{V}_{\mu'}^\dagger\right)\right) dt\right)$$

where $\eta_\nu \in [0,1]$, $\overline{\theta}_\mu, \overline{\eta}_{\mu,\mu'} \geq 0$ with $\overline{\eta}_{\mu'} = \sum_\mu \overline{\eta}_{\mu,\mu'} \leq 1$ are parameters modelling measurements imperfections.

If, for some μ, $\boldsymbol{N_\mu(t+dt) - N_\mu(t) = 1}$, we have $\rho_{t+dt} = \dfrac{\overline{\theta}_\mu \rho_t + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \boldsymbol{V}_{\mu'} \rho_t \boldsymbol{V}_{\mu'}^\dagger}{\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \operatorname{Tr}\left(\boldsymbol{V}_{\mu'} \rho_t \boldsymbol{V}_{\mu'}^\dagger\right)}$.

When $\forall \mu$, $\boldsymbol{dN_\mu(t) = 0}$, we have

$$\rho_{t+dt} = \frac{\boldsymbol{M_{dy_t}} \rho_t \boldsymbol{M_{dy_t}^\dagger} + \sum_\nu (1-\eta_\nu) \boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger dt + \sum_\mu (1-\overline{\eta}_\mu) \boldsymbol{V}_\mu \rho_t \boldsymbol{V}_\mu^\dagger dt}{\operatorname{Tr}\left(\boldsymbol{M_{dy_t}} \rho_t \boldsymbol{M_{dy_t}^\dagger} + \sum_\nu (1-\eta_\nu) \boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger dt + \sum_\mu (1-\overline{\eta}_\mu) \boldsymbol{V}_\mu \rho_t \boldsymbol{V}_\mu^\dagger dt\right)}$$

with $\boldsymbol{M_{dy_t}} = \boldsymbol{I} + \left(-\frac{i}{\hbar}\boldsymbol{H} - \frac{1}{2}\sum_\nu \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu + \frac{1}{2}\sum_\mu \left(\overline{\eta}_\mu \operatorname{Tr}\left(\boldsymbol{V}_\mu \rho_t \boldsymbol{V}_\mu^\dagger\right)\boldsymbol{I} - \boldsymbol{V}_\mu^\dagger \boldsymbol{V}_\mu\right)\right) dt + \sum_\nu \sqrt{\eta_\nu}\, \boldsymbol{dy_{\nu,t}} \boldsymbol{L}_\nu$
and where $\boldsymbol{dy_{\nu,t}} = \sqrt{\eta_\nu} \operatorname{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\rho_t\right) dt + \boldsymbol{dW_{\nu,t}}$.

Could be used as a numerical integration scheme that preserves the positiveness of ρ.

Continuous-time diffusive SME and quantum filtering

For clarity'sake, take a single measure y_t associated to operator $\boldsymbol{L}$ and detection efficiency $\eta \in [0,1]$. Then ρ_t obeys to the following diffusive SME

$$d\rho_t = -\frac{i}{\hbar}[\boldsymbol{H}, \rho_t]\, dt + \left(\boldsymbol{L}\rho_t\boldsymbol{L}^\dagger - \frac{1}{2}(\boldsymbol{L}^\dagger\boldsymbol{L}\rho_t + \rho_t\boldsymbol{L}^\dagger\boldsymbol{L})\right)\, dt$$
$$+ \sqrt{\eta}\left(\boldsymbol{L}\rho_t + \rho_t\boldsymbol{L}^\dagger - \operatorname{Tr}\left((\boldsymbol{L}+\boldsymbol{L}^\dagger)\rho_t\right)\rho_t\right)\, \boldsymbol{dW_t}$$

driven by the Wiener processes W_t,

Since $\boldsymbol{dy_t} = \sqrt{\eta}\operatorname{Tr}\left((\boldsymbol{L}+\boldsymbol{L}^\dagger)\,\rho_t\right)\, dt + \boldsymbol{dW_t}$, the estimate $\boldsymbol{\rho_t^{\text{est}}}$ is given by

$$d\boldsymbol{\rho_t^{\text{est}}} = -\frac{i}{\hbar}[\boldsymbol{H}, \boldsymbol{\rho_t^{\text{est}}}]\, dt + \left(\boldsymbol{L}\boldsymbol{\rho_t^{\text{est}}}\boldsymbol{L}^\dagger - \frac{1}{2}(\boldsymbol{L}^\dagger\boldsymbol{L}\boldsymbol{\rho_t^{\text{est}}} + \boldsymbol{\rho_t^{\text{est}}}\boldsymbol{L}^\dagger\boldsymbol{L})\right)\, dt$$
$$+\sqrt{\eta}\left(\boldsymbol{L}\boldsymbol{\rho_t^{\text{est}}} + \boldsymbol{\rho_t^{\text{est}}}\boldsymbol{L}^\dagger - \operatorname{Tr}\left((\boldsymbol{L}+\boldsymbol{L}^\dagger)\boldsymbol{\rho_t^{\text{est}}}\right)\rho_t^e\right)\left(\boldsymbol{dy_t} - \sqrt{\eta}\operatorname{Tr}\left((\boldsymbol{L}+\boldsymbol{L}^\dagger)\boldsymbol{\rho_t^{\text{est}}}\right)\, dt\right)$$

initialized to any density matrix $\boldsymbol{\rho_0^{\text{est}}}$.

Stability of diffusive quantum filtering[12]

Assume that $(\rho, \boldsymbol{\rho}^{\text{est}})$ obey to

$$d\rho_t = -\frac{i}{\hbar}[\boldsymbol{H}, \rho_t]\,dt + \left(\boldsymbol{L}\rho_t\boldsymbol{L}^\dagger - \frac{1}{2}(\boldsymbol{L}^\dagger\boldsymbol{L}\rho_t + \rho_t\boldsymbol{L}^\dagger\boldsymbol{L})\right)\,dt$$
$$+\sqrt{\eta}\left(\boldsymbol{L}\rho_t + \rho_t\boldsymbol{L}^\dagger - \operatorname{Tr}\left((\boldsymbol{L}+\boldsymbol{L}^\dagger)\rho_t\right)\rho_t\right)\,\boldsymbol{dW_t}$$

$$d\boldsymbol{\rho}_t^{\text{est}} = -\frac{i}{\hbar}[\boldsymbol{H}, \boldsymbol{\rho}_t^{\text{est}}]\,dt + \left(\boldsymbol{L}\boldsymbol{\rho}_t^{\text{est}}\boldsymbol{L}^\dagger - \frac{1}{2}(\boldsymbol{L}^\dagger\boldsymbol{L}\boldsymbol{\rho}_t^{\text{est}} + \boldsymbol{\rho}_t^{\text{est}}\boldsymbol{L}^\dagger\boldsymbol{L})\right)\,dt$$
$$+\sqrt{\eta}\left(\boldsymbol{L}\boldsymbol{\rho}_t^{\text{est}} + \boldsymbol{\rho}_t^{\text{est}}\boldsymbol{L}^\dagger - \operatorname{Tr}\left((\boldsymbol{L}+\boldsymbol{L}^\dagger)\boldsymbol{\rho}_t^{\text{est}}\right)\boldsymbol{\rho}_t^{\text{est}}\right)\,\boldsymbol{dW_t}$$
$$+\underbrace{\eta\left(\boldsymbol{L}\boldsymbol{\rho}_t^{\text{est}} + \boldsymbol{\rho}_t^{\text{est}}\boldsymbol{L}^\dagger - \operatorname{Tr}\left((\boldsymbol{L}+\boldsymbol{L}^\dagger)\boldsymbol{\rho}_t^{\text{est}}\right)\boldsymbol{\rho}_t^{\text{est}}\right)\operatorname{Tr}\left((\boldsymbol{L}+\boldsymbol{L}^\dagger)(\boldsymbol{\rho_t} - \boldsymbol{\rho}_t^{\text{est}})\right)\,dt}_{\text{correction terms vanishing when } \rho_t = \boldsymbol{\rho}_t^{\text{est}}}.$$

Then for any $\boldsymbol{H}$, $\boldsymbol{L}$ and $\eta \in [0,1]$, $F(\rho_t, \boldsymbol{\rho}_t^{\text{est}}) = \operatorname{Tr}^2\left(\sqrt{\sqrt{\rho_t}\boldsymbol{\rho}_t^{\text{est}}\sqrt{\rho_t}}\right)$ is a sub-martingale:

$$t \mapsto \mathbb{E}\left(F(\rho_t, \boldsymbol{\rho}_t^{\text{est}})\right) \text{ is non-decreasing.}$$

[12] H. Amini et al: Stability of continuous-time quantum filters with measurement imperfections. http://arxiv.org/abs/1312.0418, 2013

Cat states obtained via Kerr transformations of coherent states[13]

Take a coherent state $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$ of complex amplitude α. Depending on ϕ^{Kerr}, the Kerr-propagated state

$$e^{-i\phi^{\text{Kerr}} \boldsymbol{N}^2} |\alpha\rangle$$

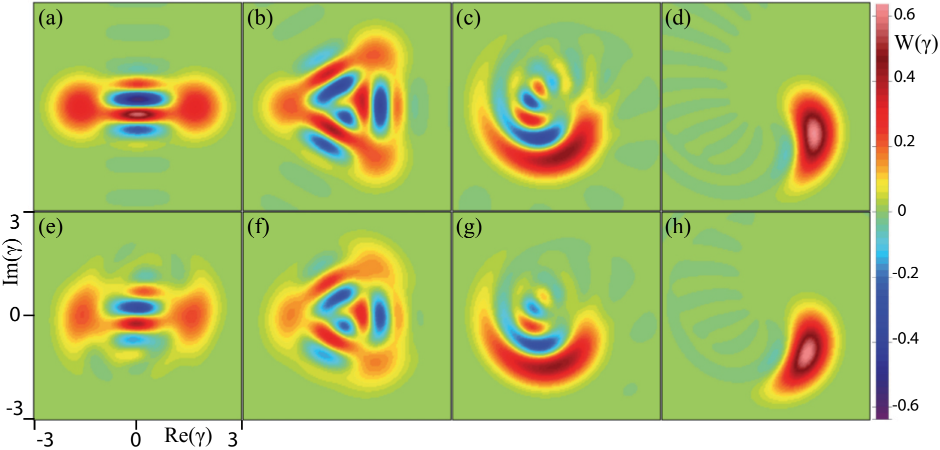
can take a number of nonclassical forms:

1. squeezed states for $\phi^{\text{Kerr}} \ll \pi$;
2. states with 'banana'-shaped Wigner function for slightly larger ϕ^{Kerr};
3. mesoscopic field state superpositions $|k_\alpha\rangle$ with k equally spaced components for $t_K \gamma_K = \pi/k$.
4. in particular, for $\phi^{\text{Kerr}} = \frac{\pi}{2}$, a superposition of two coherent states with opposite amplitudes:

$$|c_\alpha\rangle = (|\alpha\rangle + i\,|\text{-}\alpha\rangle)/\sqrt{2}.$$

[13] S. Haroche and J.M. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts, 2006.

Wigner functions of $e^{-i\phi^{\text{Kerr}} \boldsymbol{N}^2} |\alpha\rangle$ for different values of ϕ^{Kerr}.



(a): $\phi^{\text{Kerr}} = \pi/2$; (b): $\phi^{\text{Kerr}} = \pi/3$; (c): $\phi^{\text{Kerr}} = 0.28$; (d): $\phi^{\text{Kerr}} = 0.08$

(e-h): similar states stabilized, despite decoherence, by the atomic reservoir onto which we focus in this talk.

Data: $\mathcal{H}_S$ with Hamiltonian $\boldsymbol{H}_S$, a pure goal state $\bar{\rho}_S = |\bar{\psi}_S\rangle\langle\bar{\psi}_S|$.
Find a "realistic" meter system of Hilbert space $\mathcal{H}_M$ with initial state $|\theta_M\rangle$, with Hamiltonian $\boldsymbol{H}_M$ and **interaction Hamiltonian $\boldsymbol{H}_{int}$** such that

1. the propagator $\boldsymbol{U}_{S,M} = \boldsymbol{U}(T)$ between 0 and time T $(\frac{d}{dt}\boldsymbol{U} = -\frac{i}{\hbar}(\boldsymbol{H}_S + \boldsymbol{H}_M + \boldsymbol{H}_{int})\boldsymbol{U}$, $\boldsymbol{U}(0) = \boldsymbol{I})$ reads:
$$\forall|\psi_S\rangle \in \mathcal{H}_S, \quad \boldsymbol{U}_{S,M}\big(|\psi_S\rangle \otimes |\theta_M\rangle\big) = \sum_\mu \big(\boldsymbol{M}_\mu|\psi_S\rangle\big) \otimes |\lambda_\mu\rangle$$
where $|\lambda_\mu\rangle$ is an ortho-normal basis of $\mathcal{H}_M$.
2. the resulting **measurement operators $\boldsymbol{M}_\mu$ admit $|\bar{\psi}_S\rangle$ as common eigen-vector**, i.e., $\bar{\rho}_S$ is a fixed point of the Kraus map $\boldsymbol{K}(\rho) = \sum_\mu \boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger$: $\quad \boldsymbol{K}(\bar{\rho}_S) = \bar{\rho}_S$.
3. iterates of $\boldsymbol{K}$ converge to $\bar{\rho}_S$ for any initial condition ρ_0:
$$\lim_{k \mapsto +\infty} \rho_k = \bar{\rho}_S \text{ where } \rho_k = \boldsymbol{K}(\rho_{k-1}) \quad \textbf{(asymptotic stability)}.$$

Here the reservoir is made of the infinite set of identical meter systems with initial state $|\theta_M\rangle$ at $t = (k-1)T$ and interacting with $\mathcal{H}_S$ during $[(k-1)T, kT]$, $k = 1, 2, \ldots$.

$\boldsymbol{U} = \mathbf{Z}(-\phi_{\boldsymbol{N}})\,\mathbf{X}(\xi_{\boldsymbol{N}})\quad \mathbf{Y}(\theta^r_{\boldsymbol{N}})\quad \mathbf{X}(\xi_{\boldsymbol{N}})\,\mathbf{Z}(\phi_{\boldsymbol{N}})$,[14]

Generalized rotations around Bloch spheres labeled with n:

$$\begin{aligned}
\mathbf{X}(f_{\boldsymbol{N}}) &= \cos(f_{\boldsymbol{N}}/2) \otimes |g\rangle\langle g| + \cos(f_{\boldsymbol{N+I}}/2) \otimes |e\rangle\langle e| \\
&\quad - i\boldsymbol{a}\frac{\sin(f_{\boldsymbol{N}}/2)}{\sqrt{\boldsymbol{N}}} \otimes |e\rangle\langle g| - i\frac{\sin(f_{\boldsymbol{N}}/2)}{\sqrt{\boldsymbol{N}}}\,\boldsymbol{a}^\dagger \otimes |g\rangle\langle e| \\
\mathbf{Y}(f_{\boldsymbol{N}}) &= \cos(f_{\boldsymbol{N}}/2) \otimes |g\rangle\langle g| + \cos(f_{\boldsymbol{N+I}}/2) \otimes |e\rangle\langle e| \\
&\quad - \boldsymbol{a}\frac{\sin(f_{\boldsymbol{N}}/2)}{\sqrt{\boldsymbol{N}}} \otimes |e\rangle\langle g| + \frac{\sin(f_{\boldsymbol{N}}/2)}{\sqrt{\boldsymbol{N}}}\,\boldsymbol{a}^\dagger \otimes |g\rangle\langle e| \\
\mathbf{Z}(f_{\boldsymbol{N}}) &= e^{i\,f_{\boldsymbol{N}}\,/2} \otimes |g\rangle\langle g| + e^{-i\,f_{\boldsymbol{N+I}}\,/2} \otimes |e\rangle\langle e|\ .
\end{aligned}$$

The different angles depending on the photon-numbers:

$$\theta^r_n = \sqrt{n}\int_{-t_r/2}^{t_r/2} \Omega(vt)\,dt, \quad \phi_n = \delta_0 \int_{-T/2}^{-t_r/2} \sqrt{1 + n\,(\Omega(vt)/\delta_0)^2}\,dt$$

$$\tan\xi_n = \tfrac{\Omega(vt_r/2)\sqrt{n}}{\delta_0} \quad \text{with } \xi_n \in \left(\tfrac{-\pi}{2}, \tfrac{\pi}{2}\right).$$

[14] A. Sarlette et al: Stabilization of nonclassical states of one and two-mode radiation fields by reservoir engineering. Phys. Rev. A 86, 012114 (2012).

1. With $\theta_n \in [0, 2\pi)$ defined by $\cos(\theta_n/2) = \cos(\theta^r_n/2)\ \cos\xi_n$ and $\phi^c_{\boldsymbol{N}} = \phi_{\boldsymbol{N}} + \text{angle}[\,\sin(\theta^r_{\boldsymbol{N}}/2) - i\cos(\theta^r_{\boldsymbol{N}}/2)\sin\xi_{\boldsymbol{N}}\,]$:

$$\boldsymbol{U} = \cos(\theta_{\boldsymbol{N}}/2) \otimes |g\rangle\langle g| + \cos(\theta_{\boldsymbol{N}+\boldsymbol{I}}/2) \otimes |e\rangle\langle e|$$
$$- \boldsymbol{a}\frac{\sin(\theta_{\boldsymbol{N}}/2)}{\sqrt{\boldsymbol{N}}}\ e^{i\phi^c_{\boldsymbol{N}}} \otimes |e\rangle\langle g| + \frac{\sin(\theta^c_{\boldsymbol{N}}/2)}{\sqrt{\boldsymbol{N}}}\ e^{-i\phi^c_{\boldsymbol{N}}}\,\boldsymbol{a}^\dagger \otimes |g\rangle\langle e|.$$

2. Using $\boldsymbol{a}\,f(\boldsymbol{N}) \equiv f(\boldsymbol{N}+\boldsymbol{I})\,\boldsymbol{a}$ we get

$$\boldsymbol{U} = e^{-ih^{\text{Kerr}}_{\boldsymbol{N}}}\ \mathbf{Y}(\theta^c_{\boldsymbol{N}})\ e^{ih^{\text{Kerr}}_{\boldsymbol{N}}}$$

   with $h^{\text{Kerr}}_{n+1} - h^{\text{Kerr}}_n = \phi^c_{n+1}$ defining "Kerr Hamiltonian" $h^{\text{Kerr}}_{\boldsymbol{N}}$.

3. With $|u_{\text{at}}\rangle = \cos(u/2)|g\rangle + \sin(u/2)|e\rangle$,

$$\boldsymbol{U}\,(|\psi\rangle \otimes |u_{\text{at}}\rangle) = \boldsymbol{M}_g|\psi\rangle \otimes |g\rangle + \boldsymbol{M}_e|\psi\rangle \otimes |e\rangle$$

   where

$$\boldsymbol{M}_g = e^{-ih^{\text{Kerr}}_{\boldsymbol{N}}}\boldsymbol{M}^{\text{Kerr}}_g e^{ih^{\text{Kerr}}_{\boldsymbol{N}}}, \quad \boldsymbol{M}_e = e^{-ih^{\text{Kerr}}_{\boldsymbol{N}}}\boldsymbol{M}^{\text{Kerr}}_e e^{ih^{\text{Kerr}}_{\boldsymbol{N}}}$$

Convergence issues in $\{\sum_{n\geq 0} \psi_n|n\rangle,\ (\psi_n)_{n\geq 0} \in l^2(\mathbb{C})\}$

The two measurement operators in the Kerr frame

$$\boldsymbol{M}_g^{\text{Kerr}} = \cos(\tfrac{u}{2})\cos(\theta_{\boldsymbol{N}}/2) + \sin(\tfrac{u}{2})\,\frac{\sin(\theta_{\boldsymbol{N}}/2)}{\sqrt{\boldsymbol{N}}}\,\boldsymbol{a}^\dagger$$

$$\boldsymbol{M}_e^{\text{Kerr}} = \sin(\tfrac{u}{2})\cos(\theta_{\boldsymbol{N}+1}/2) - \cos(\tfrac{u}{2})\,\boldsymbol{a}\,\frac{\sin(\theta_{\boldsymbol{N}}^r/2)}{\sqrt{\boldsymbol{N}}}$$

and the Kraus map:

$$\rho_{k+1}^{\text{Kerr}} = \boldsymbol{K}^{\text{Kerr}}(\rho_k) = \boldsymbol{M}_g^{\text{Kerr}}\rho_k^{\text{Kerr}}(\boldsymbol{M}_g^{\text{Kerr}})^\dagger + \boldsymbol{M}_e^{\text{Kerr}}\rho_k^{\text{Kerr}}(\boldsymbol{M}_e^{\text{Kerr}})^\dagger.$$

When $|u| < \pi/2$, $\theta_0 = 0$, $\theta_n \in]0,\pi[$ for $n > 0$ and $\lim_{n\mapsto+\infty}\theta_n = \pi/2$:

- exists a unique common eigen-state $|\psi^{\text{Kerr}}\rangle$ of $\boldsymbol{M}_g^{\text{Kerr}}$ and $\boldsymbol{M}_e^{\text{Kerr}}$: $\rho_\infty^{\text{Kerr}} = |\psi^{\text{Kerr}}\rangle\langle\psi^{\text{Kerr}}|$ fixed point of $\boldsymbol{K}^{\text{Kerr}}$.
- if $n \mapsto \theta_n$ is increasing, Zaki Leghtas has proved in his PhD thesis (2012) global convergence (Lyapunov function $\text{Tr}\left(\rho_\infty^{\text{Kerr}}\rho_k^{\text{Kerr}}\right)$, precompacity of the trajectories, Lassalle invariance principle,...).

Conjecture: global convergence without $n \mapsto \theta_n$ increasing.

Books on open quantum systems

1. E. Davies: Quantum Theory of Open Systems. Academic Press, 1976.
2. K. Parthasarathy: An Introduction to Quantum Stochastic Calculus. Birkhäuser, 1992.
3. V. Braginsky, F. Khalili: Quantum Measurement. Cambridge University Press, 1995.
4. H. Carmichael: Statistical Methods in Quantum Optics 1: Master Equations and Fokker-Planck Equations. Springer 1999.
5. M. Nielsen, I. Chuang: Quantum Computation and Quantum Information. Cambridge University Press, 2000.
6. S.M.Barnett, P.M. Radmore: Methods in Theoretical Quantum Optics. Oxford University Press, 2003.
7. H.P. Breuer, F. Petruccione: The Theory of Open Quantum Systems. Clarendon-Press, Oxford, 2006.
8. S. Haroche, J.M. Raimond: Exploring the Quantum: Atoms, Cavities and Photons. Oxford University Press, 2006.
9. H. Carmichael: Statistical Methods in Quantum Optics 2: Non-Classical Fields. Springer, 2007.
10. H. Wiseman, G. Milburn: Quantum Measurement and Control. Cambridge University Press, 2009.
11. A. Barchielli , M. Gregoratti: Quantum Trajectories and Measurements in Continuous Time: the Diffusive Case, volume 782. Springer Verlag, 2009.
12. C. Gardiner, P. Zoller: Quantum Noise. 3rd Edition, Springer, 2010.