Symmetry preserving asymptotic observers: theory and examples

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Outline

The linear case:

$$\frac{d}{dt}x = Ax + Bu \qquad y, u \text{ known signals}$$
$$y = Cx + Du$$

Linear observer

A stable filter mixing the input and output signals $u(t)$ and $y(t)$:

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu(t) - L(C\hat{x} + Du(t) - y(t))$$

Error system for $e := \hat{x} - x$

$$\frac{d}{dt}e = (A - LC)e$$

The error system is autonomous (separation principle...)

What about the nonlinear case?[1]

$$\frac{d}{dt}x = f(x,u), \quad y = h(x,u) \qquad y,u \text{ known signals}$$

Estimator, observer, filter, etc:

$$\frac{d}{dt}\hat{x} = f(\hat{x},u(t)) - L(\hat{x},y(t)) \cdot \big(h(\hat{x},u(t)) - y(t)\big)$$

- Luenberger observer, gain scheduling, high gains, ...
- Extended Kalman Filter (M, N “tuning" matrices)

$$A = \frac{\partial f}{\partial x}(\hat{x},u) \qquad L = -PC^T N$$

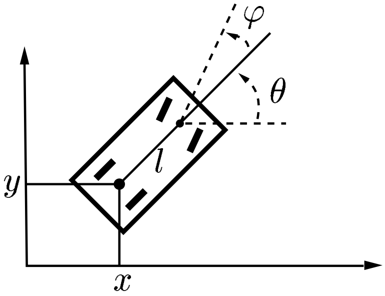
$$C = \frac{\partial h}{\partial x}(\hat{x},u) \qquad \frac{d}{dt}P = AP + PA^T + M^{-1} - PC^T NCP$$

- Tuning? Domain of convergence? Computational cost?

[1] See, e.g., G. Besançon (Ed.): Nonlinear Observers and Applications; Springer; Lecture Notes in Control and Information Sciences , Vol. 363, 2007. J.P. Gauthier, I. Kupka: Deterministic Observation Theory and Applications; Cambridge University Press; 2001.

Redemption by geometry: the nonholonomic car

$$\frac{d}{dt}x = u\cos\theta$$
$$\frac{d}{dt}y = u\sin\theta$$
$$\frac{d}{dt}\theta = uv$$
$$h(x,y,\theta) = (x,y)$$



$u, v = \tan\varphi / l$ known

A “geometric" observer

$$\frac{d}{dt}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{\theta}\end{pmatrix} = \begin{pmatrix}u\cos\hat{\theta}\\ u\sin\hat{\theta}\\ uv\end{pmatrix} + \begin{pmatrix}R_{-\hat{\theta}} & & 0\\ & & 0\\ 0 & 0 & 1\end{pmatrix} \cdot L \cdot R_{\hat{\theta}} \cdot \begin{pmatrix}\hat{x}-x\\ \hat{y}-y\end{pmatrix}$$

where L is a 3×2 gain matrix and $R_{\hat{\theta}} := \begin{pmatrix}\cos\hat{\theta} & \sin\hat{\theta}\\ -\sin\hat{\theta} & \cos\hat{\theta}\end{pmatrix}$

“Alternative" state error

$$\begin{pmatrix}\eta_x\\ \eta_y\end{pmatrix} = R_{\hat\theta}\cdot\begin{pmatrix}\hat x - x\\ \hat y - y\end{pmatrix} \qquad R_{\hat\theta} := \begin{pmatrix}\cos\hat\theta & \sin\hat\theta\\ -\sin\hat\theta & \cos\hat\theta\end{pmatrix}$$

Autonomous error system but for the “free" known u, v

$$\frac{d}{dt}\begin{pmatrix}\eta_x\\ \eta_y\\ \eta_\theta\end{pmatrix} = \begin{pmatrix}u(1-\cos\eta_\theta) + (uv + L_{31}\eta_x + L_{32}\eta_y)\eta_y\\ u\sin\eta_\theta - (uv + L_{31}\eta_x + L_{32}\eta_y)\eta_x\\ 0\end{pmatrix} + L\cdot\begin{pmatrix}\eta_x\\ \eta_y\end{pmatrix}$$

Easy tuning on the linearized error system with $a, b, c > 0$.

$$\frac{d}{dt}\begin{pmatrix}\delta\eta_x\\ \delta\eta_y\\ \delta\eta_\theta\end{pmatrix} = \begin{pmatrix}uv\delta\eta_y\\ u\delta\eta_\theta - uv\delta\eta_x\\ 0\end{pmatrix} + \begin{pmatrix}-|u|a & -uv\\ uv & -|u|c\\ 0 & -ub\end{pmatrix}\begin{pmatrix}\delta\eta_x\\ \delta\eta_y\end{pmatrix}$$

Transformation groups and invariance[2]

Let G be a Lie Group with identity e and Σ be an open set of $\mathbb{R}^\sigma$. A (local) transformation group $(\phi_g)_{g\in G}$ on Σ is a (smooth) map

$$(g,\xi) \in G \times \Sigma \mapsto \phi_g(\xi) \in \Sigma$$

such that:

- $\phi_e(\xi) = \xi$ for all ξ
- $\phi_{g_2} \circ \phi_{g_1}(\xi) = \phi_{g_2 g_1}(\xi)$ for all g_1, g_2, ξ

The map $\xi \mapsto J(\xi)$ is *invariant* if

$$J(\phi_g(\xi)) = J(\xi) \quad \text{for all } g, \xi$$

The vector field $\xi \mapsto w(\xi)$ is *invariant* if

$$w(\phi_g(\xi)) = D\phi_g(\xi) \cdot w(\xi) \quad \text{for all } g, \xi$$

[2] See, e.g., P.J. Olver: Equivalence, Invariants and Symmetry; Cambridge University Press; 1995.

The moving frame method for computing all sorts of invariants[3]

When $\dim g \leq \dim \xi = \sigma$:

we can assume $\phi_g = (\phi_g^a, \phi_g^b)$ with ϕ_g^a invertible wrt g

- **Normalization:**
  solve $\phi_g^a(\xi) = c$ for $g \quad \Rightarrow \quad g = \gamma(\xi) \quad$ (ie $\phi_{\gamma(\xi)}^a(\xi) = c$)

- the map $\xi \mapsto J(\xi) := \phi_{\gamma(\xi)}^b(\xi)$ is invariant
- any other invariant is a function of J

- the vector field $\xi \mapsto w(\xi) := \left[D\varphi_{\gamma(\xi)}(\xi)\right]^{-1} \cdot \Upsilon$, Υ constant vector in $\mathbb{R}^\sigma$, is invariant
- in particular $w_i(\xi) := \left[D\varphi_{\gamma(\xi)}(\xi)\right]^{-1} \cdot e_i$, $i = 1, \ldots, \sigma = \dim \Sigma$, form an invariant frame

[3] See P.J. Olver: Classical Invariant Theory; Cambridge University Press; 1999.

Invariant systems and invariant preobservers[4]

$$\frac{d}{dt}x = f(x,u), \qquad x \in \mathcal{X} \subset \mathbb{R}^n,\ u \in \mathcal{U} \subset \mathbb{R}^m$$
$$y = h(x,u), \qquad y \in \mathcal{Y} \subset \mathbb{R}^p$$

Transformation groups $(X, U, Y) = \big(\varphi_g(x), \psi_g(u), \varrho_g(y)\big)$

The system $\frac{d}{dt}x = f(x,u)$ is *invariant* if

$$f\big(\varphi_g(x), \psi_g(u)\big) = D\varphi_g(x) \cdot f(x,u) \quad \text{for all } g, x, u$$

The output $y = h(x,u)$ is *equivariant* if

$$h\big(\varphi_g(x), \psi_g(u)\big) = \varrho_g\big(h(x,u)\big) \quad \text{for all } g, x, u$$

The system $\frac{d}{dt}\hat{x} = F(\hat{x}, u, y)$ is a *preobserver* if

$$F\big(x, u, h(x,u)\big) = f(x,u) \quad \text{for all } x, u$$

The preobserver $\frac{d}{dt}\hat{x} = F(\hat{x}, u, y)$ is *invariant* if

$$F\big(\varphi_g(\hat{x}), \psi_g(u), \varrho_g(y)\big) = D\varphi_g(\hat{x}) \cdot F(\hat{x}, u, y) \quad \text{for all } g, \hat{x}, u, y$$

[4] Bonnabel, Martin, R: Invariant asymptotic observers. IEEE-TAC, 2008.

Structure of invariant preobservers

Every invariant preobserver reads

$$\frac{d}{dt}\hat{x} = f(\hat{x}, u) + W(\hat{x}) L\Big(I(\hat{x}, u), E(\hat{x}, u, y) \Big) E(\hat{x}, u, y)$$

- $E(\hat{x}, u, y)$ invariant output error
- $W(\hat{x}) = \big(w_1(\hat{x}), .., w_n(\hat{x})\big)$ invariant frame
- $I(\hat{x}, u)$ invariant
- $L(I, E)$ freely chosen $n \times p$ gain matrix

Explicit construction by the moving frame method

$(\hat{x}, u, y) \mapsto E(\hat{x}, u, y) \in \mathcal{Y}$ is an *invariant output error* if

- $y \mapsto E(\hat{x}, u, y)$ is invertible for all $\hat{x}, u$
- $E\big(\hat{x}, u, h(\hat{x}, u)\big) = 0$ for all $\hat{x}, u$
- $E\big(\varphi_g(\hat{x}), \psi_g(u), \varrho_g(y)\big) = E(\hat{x}, u, y)$ for all $\hat{x}, u, y$

So what?

The error system is “nearly" autonomous:

$$\frac{d}{dt}\eta = \Upsilon\big(\eta, I(\hat{x}, u)\big)$$, where η is an invariant state error

When $\dim G = \dim \mathcal{X}$, autonomous error system but for the “free" known invariant I!

$(\hat{x}, x) \mapsto \eta(\hat{x}, x) \in \mathcal{X}$ is an *invariant state error* if

- $x \mapsto \eta(\hat{x}, x)$ is invertible for all $\hat{x}$ and vice versa
- $\eta(x, x) = 0$ for all x
- $\eta\big(\varphi_g(\hat{x}), \varphi_g(x)\big) = \eta(\hat{x}, x)$ for all $\hat{x}, x$

Benefit of the symmetry-preserving approach:

when $\dim G = \dim \mathcal{X}$, the observer is easily tuned for (at least) local convergence around every “permanent" trajectory, with a good local behavior (interesting practical property)

Back to the nonholonomic car with $G = SE(2)$

Invariance by rotation and translation $g = (x_g, y_g, \theta_g) \in G$

$$\begin{pmatrix} x_g \\ y_g \\ \theta_g \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} = \begin{pmatrix} x\cos\theta_g - y\sin\theta_g + x_g \\ x\sin\theta_g + y\cos\theta_g + y_g \\ \theta + \theta_g \end{pmatrix}$$

$$\varphi_{(x_g,y_g,\theta_g)}(x, y, \theta) = \begin{pmatrix} x_g \\ y_g \\ \theta_g \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}, \quad \psi_{(x_g,y_g,\theta_g)}(u, v) = \begin{pmatrix} u \\ v \end{pmatrix}$$

An invariant observer

$$\begin{pmatrix} \frac{d}{dt}\hat{x} \\ \frac{d}{dt}\hat{y} \\ \frac{d}{dt}\hat{\theta} \end{pmatrix} = \begin{pmatrix} u\cos\hat{\theta} \\ u\sin\hat{\theta} \\ uv \end{pmatrix} + \begin{pmatrix} R_{-\hat{\theta}} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot L \cdot R_{\hat{\theta}} \cdot \begin{pmatrix} \hat{x} - x \\ \hat{y} - y \end{pmatrix}$$

where L is a 3×2 gain matrix and $R_{\hat{\theta}} := \begin{pmatrix} \cos\hat{\theta} & \sin\hat{\theta} \\ -\sin\hat{\theta} & \cos\hat{\theta} \end{pmatrix}$

Velocity aided-AHRS : the considered problem[5]

- 3 gyros give $\omega_m := \omega$ (body frame)
- 3 acceleros give $a_m := a$ (body frame)
- 3 magnetos gives the Earth magnetic field (body frame)
- An air data system gives the velocity vector (body frame)
- No biases

"Quaternion" model

$$\frac{d}{dt}q = \frac{1}{2}q * \omega$$

$$\frac{d}{dt}v = v \times \omega + q^{-1} * \mathcal{A} * q + a$$

$$y = \begin{pmatrix} y_v \\ y_b \end{pmatrix} = \begin{pmatrix} v \\ q^{-1} * \mathcal{B} * q \end{pmatrix}$$

"Natural" transformation group

$$\varphi_{\begin{pmatrix} q_g \\ v_g \end{pmatrix}} \begin{pmatrix} q \\ v \end{pmatrix} = \begin{pmatrix} q * q_g \\ q_g^{-1} * v * q_g + v_g \end{pmatrix}$$

$$\psi_{\begin{pmatrix} q_g \\ v_g \end{pmatrix}} \begin{pmatrix} \omega \\ a \end{pmatrix} = \begin{pmatrix} q_g^{-1} * \omega * q_g \\ q_g^{-1} * a * q_g - \ldots \\ v_g \times (q_g^{-1} * \omega * q_g) \end{pmatrix}$$

$$\rho_{\begin{pmatrix} q_g \\ v_g \end{pmatrix}} \begin{pmatrix} y_v \\ y_b \end{pmatrix} = \begin{pmatrix} q_g^{-1} * y_v * q_g + v_g \\ q_g^{-1} * y_b * q_g \end{pmatrix}$$

[5] See PhD Thesis of Erwan Salaün where such kinds of observers are extensively developed and tested experimentally on micro-controllers.

A simplified Velocity aided AHRS

- Model with v, a and ω measured in the body frame. Constant gravity and magnetic vectors, $\mathcal{A}$ and $\mathcal{B}$ in the Earth Frame.

$$\frac{d}{dt}q = \frac{1}{2}q * \omega$$
$$\frac{d}{dt}v = v \times \omega + q^{-1} * \mathcal{A} * q + a$$
$$y = \begin{pmatrix} y_v \\ y_b \end{pmatrix} = \begin{pmatrix} v \\ q^{-1} * \mathcal{B} * q \end{pmatrix}$$

  and (ω, a) are inputs. We want to estimate (q, v).
- In the body Frame: invariance versus rotation q_g and velocity translation v_g.

Natural transformation group SE(3)

Here $x = \begin{pmatrix} q \\ v \end{pmatrix}$, $g = \begin{pmatrix} q_g \\ v_g \end{pmatrix}$ ($G \sim SE(3)$) and we take:

$$\varphi_{\begin{pmatrix} q_g \\ v_g \end{pmatrix}} \begin{pmatrix} q \\ v \end{pmatrix} = \begin{pmatrix} q_g \\ v_g \end{pmatrix} \cdot \begin{pmatrix} q \\ v \end{pmatrix} = \begin{pmatrix} q * q_g \\ q_g^{-1} * v * q_g + v_g \end{pmatrix}$$

Here $u = \begin{pmatrix} a \\ \omega \end{pmatrix}$ and we take:

$$\psi_{\begin{pmatrix} q_g \\ v_g \end{pmatrix}} \begin{pmatrix} a \\ \omega \end{pmatrix} = \begin{pmatrix} q_g^{-1} * a * q_g - v_g \times (q_g^{-1} * \omega * q_g) \\ q_g^{-1} * \omega * q_g \end{pmatrix}$$

Here $y = \begin{pmatrix} y_v \\ y_b \end{pmatrix} = \begin{pmatrix} v \\ q^{-1} * \mathcal{B} * q \end{pmatrix}$ and we take:

$$\varrho_{\begin{pmatrix} q_g \\ v_g \end{pmatrix}} \begin{pmatrix} y_v \\ y_b \end{pmatrix} = \begin{pmatrix} q_g^{-1} * y_v * q_g + v_g \\ q_g^{-1} * y_b * q_g \end{pmatrix}$$

$SE(3)$ invariance of the system

The set of equations characterizing the system

$$\frac{d}{dt}q = \frac{1}{2}q * \omega, \qquad \frac{d}{dt}v = v \times \omega + q^{-1} * \mathcal{A} * q + a$$

$$y = \begin{pmatrix} y_v \\ y_b \end{pmatrix} = \begin{pmatrix} v \\ q^{-1} * \mathcal{B} * q \end{pmatrix}$$

is unchanged if, for any $\begin{pmatrix} q_g \\ v_g \end{pmatrix}$, we set

$$\varphi_{\begin{pmatrix} q_g \\ v_g \end{pmatrix}} \begin{pmatrix} q \\ v \end{pmatrix} = \begin{pmatrix} Q \\ V \end{pmatrix}, \quad \psi_{\begin{pmatrix} q_g \\ v_g \end{pmatrix}} \begin{pmatrix} a \\ \omega \end{pmatrix} = \begin{pmatrix} A \\ \Omega \end{pmatrix}, \quad \varrho_{\begin{pmatrix} q_g \\ v_g \end{pmatrix}} \begin{pmatrix} y_v \\ y_b \end{pmatrix} = \begin{pmatrix} Y_v \\ Y_b \end{pmatrix}$$

We have

$$\frac{d}{dt}Q = \frac{1}{2}Q * \Omega, \quad \frac{d}{dt}V = V \times \Omega + Q^{-1} * \mathcal{A} * Q + A$$

$$Y = \begin{pmatrix} Y_v \\ Y_b \end{pmatrix} = \begin{pmatrix} V \\ Q^{-1} * \mathcal{B} * Q \end{pmatrix}$$

The invariant nonlinear observer

$$\frac{d}{dt}\hat{q} = \frac{1}{2}\hat{q} * \omega(t) + (\bar{\mathcal{L}}^q_v E_v + \bar{\mathcal{L}}^q_b E_b) * \hat{q}$$
$$\frac{d}{dt}\hat{v} = \hat{v} \times \omega(t) + \hat{q}^{-1} * \mathcal{A} * \hat{q} + a(t) + \hat{q}^{-1} * (\bar{\mathcal{L}}^v_v E_v + \bar{\mathcal{L}}^v_b E_b) * \hat{q}$$

with the following invariant errors:

$$E_v = \hat{q} * (\hat{v} - y_v(t)) * \hat{q}^{-1}, \quad E_b = \mathcal{B} - \hat{q} * y_b(t) * \hat{q}^{-1}$$

and $\bar{\mathcal{L}}^q_v$, $\bar{\mathcal{L}}^v_v$, $\bar{\mathcal{L}}^q_b$ and $\bar{\mathcal{L}}^v_b$ are constant matrices.
Autonomous errors dynamics: tune the gains to ensure local exponential convergence around any system trajectory with uniform Lyapunov exponents.[6]

[6] See Bonnabel, Martin, R.: Invariant asymptotic observers;IEEE-TAC, 2008. See also Mahony, Hamel, Pflimlin: Nonlinear Complementary Filters on the Special Orthogonal Group, IEEE-TAC 2008.

Autonomous errors dynamics

Consider the invariant state-estimation errors

$$\eta_q = \hat{q} * q^{-1}, \quad \eta_v = q * (\hat{v} - v) * q^{-1}.$$

Their dynamics satisfy the following autonomous equation:

$$\frac{d}{dt}\eta_q = \left[\bar{\mathcal{L}}_v^q(\eta_q * \eta_v * \eta_q^{-1}) + \bar{\mathcal{L}}_b^q(\mathcal{B} - \eta_q * \mathcal{B} * \eta_q^{-1})\right] * \eta_q$$
$$\frac{d}{dt}\eta_v = \eta_q^{-1} * \left[\mathcal{A} + \bar{\mathcal{L}}_v^v(\eta_q * \eta_v * \eta_q^{-1}) + \bar{\mathcal{L}}_b^v(\mathcal{B} - \eta_q * \mathcal{B} * \eta_q^{-1})\right] * \eta_q - \mathcal{A}$$

Convergence of the linearized error system

Indeed for $\hat{q}$ and $\hat{v}$ close to q and v we have

$$\delta E_v = \delta\eta_v$$

and

$$\delta E_b = -\delta\eta_q * \mathcal{B} + \mathcal{B} * \delta\eta_q = 2\mathcal{B} \times \delta\eta_q$$

The linearized error equation writes:

$$\frac{d}{dt}\delta\eta_q = \bar{\mathcal{L}}_v^q \delta\eta_v + 2\bar{\mathcal{L}}_b^q(\mathcal{B} \times \delta\eta_q)$$
$$\frac{d}{dt}\delta\eta_v = 2\mathcal{A} \times \delta\eta_q + \bar{\mathcal{L}}_v^v \delta\eta_v + 2\bar{\mathcal{L}}_b^v(\mathcal{B} \times \delta\eta_q)$$

Let us choose

$$\bar{\mathcal{L}}_v^q = \begin{pmatrix} 0 & -M_{12} & 0 \\ M_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \bar{\mathcal{L}}_v^v = -\begin{pmatrix} N_{11} & 0 & 0 \\ 0 & N_{22} & 0 \\ 0 & 0 & N_{33} \end{pmatrix}$$

$$\bar{\mathcal{L}}_b^q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\lambda\mathcal{B}^2 & \lambda\mathcal{B}^1 & 0 \end{pmatrix} \qquad \bar{\mathcal{L}}_b^v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In (Earth-fixed) coordinates,

$$\delta\eta_q := \begin{pmatrix} 0 \\ \delta\eta_q^1 \\ \delta\eta_q^2 \\ \delta\eta_q^3 \end{pmatrix}, \quad \delta\eta_v := \begin{pmatrix} \delta\eta_v^1 \\ \delta\eta_v^2 \\ \delta\eta_v^3 \end{pmatrix} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} 0 \\ 0 \\ \mathcal{A}^3 \end{pmatrix},$$

The error system breaks in four decoupled subsystems:

- the longitudinal subsystem

$$\frac{d}{dt}\begin{pmatrix}\delta\eta_q^2\\ \delta\eta_v^1\end{pmatrix} = \begin{pmatrix}0 & M_{21}\\ -2\mathcal{A}^3 & -N_{11}\end{pmatrix}\begin{pmatrix}\delta\eta_q^2\\ \delta\eta_v^1\end{pmatrix}$$

- the lateral subsystem

$$\frac{d}{dt}\begin{pmatrix}\delta\eta_q^1\\ \delta\eta_v^2\end{pmatrix} = \begin{pmatrix}0 & -M_{12}\\ 2\mathcal{A}^3 & -N_{22}\end{pmatrix}\begin{pmatrix}\delta\eta_q^1\\ \delta\eta_v^2\end{pmatrix}$$

- the vertical subsystem

$$\frac{d}{dt}\delta\eta_v^3 = -N_{33}\delta\eta_v^3$$

- the heading subsystem

$$\frac{d}{dt}\delta\eta_q^3 = \lambda\mathcal{B}^3(\mathcal{B}^1\delta\eta_q^1 - \mathcal{B}^2\delta\eta_q^2) - \lambda((\mathcal{B}^1)^2 + (\mathcal{B}^2)^2)\delta\eta_q^3$$

Invariant dynamics and observers on a Lie group[7]

Consider a left-invariant dynamics on a Lie group G

$$\frac{d}{dt}g = DL_g \omega_s(t)$$
$$y = h(g)$$

with h an G-équivariant output : for all $g_1 \in G$ there exists ρ_{g_1} such that for all $g_2 \in G$:

$$h(g_1 g_2) = \rho_{g_1}(h(g_2))$$

with $\rho_{g_1} \circ \rho_{g_2} = \rho_{g_1 g_2}$. Any invariant observer writes

$$\frac{d}{dt}\hat{g} = DL_{\hat{g}} \omega_s(t) + DL_{\hat{g}} \left(\sum_{i=1}^{n} \mathcal{L}_i(\rho_{\hat{g}^{-1}}(y)) W_i \right)$$

avec $(W_1, ..W_n)$ basis of the Lie algebra $\mathcal{L}_i(h(e))) = 0$

[7] Bonnabel, Martin, R.: Non-linear Symmetry preserving observers on Lie groups. IEEE-TAC, June 2009.

The invariant state error is

$$\eta = g^{-1}\hat{g}$$

The error equation

$$\frac{d}{dt}\eta = DL_\eta \omega_s - DR_\eta \omega_s + DL_\eta \Big(\sum_{i=1}^{n} \mathcal{L}_i \circ h(\eta^{-1})\ W_i\Big)$$

reminds

$$\frac{d}{dt}e = (A(t) + LC)e$$

- What if the condition on the output is $h(g_1 g_2) = \rho_{g_2}(h(g_1))$?
- $\eta = \hat{g}g^{-1} \Rightarrow$ AUTONOMOUS error equation ![8]
- reminds $\frac{d}{dt}e = LCe$

[8]This autonomous character resulting form left translations on the state and right translations on the output was first noticed in Bonnabel, Martin, R.: Groupe de Lie et observateur non-linéaire; CIFA 2006 (Conférence Internationale Francophone d'Automatique), Bordeaux, France, 2006. It has been also noticed in 2008 and independently by Mahony and co-workers: http://arxiv.org/abs/0810.0748 and http://arxiv.org/abs/0805.0828

Conclusion

Invariance is just a way to "put physics" in nonlinear estimation and filtering processes.

- Chemical reactors where the group is just associated to changes of units and invariance relies on the fact that material and energy balance equations do not depend on chosen units.
- Quantum systems described by Lindblad-Kossakovski differential equation

$$\frac{d}{dt}\rho = \frac{-\imath}{\hbar}[H, \rho] + 2L\rho L^\dagger - L^\dagger L\rho - \rho L^\dagger L, \quad y = \text{tr}\left(L\rho L^\dagger\right)$$

  where ρ is the density operator (symmetric, semi-positive with $\text{tr}\,(\rho) = 1$), H and L are operators. All these operators are defined up to a change of frames $U \in G = U(n)$ via:

$$(\rho, H, L) \mapsto (U\rho U^\dagger, UHU^\dagger, ULU^\dagger)$$

- Infinite dimensional systems described by PDE's such as the Saint-Venant equation `http://arxiv.org/abs/0809.1400`.
- For inertial navigation and data fusion between camera, acceleros, gyros and magnetos... Galilean invariance could certainly be exploited in a much more systematic way....