Control of open quantum systems by reservoir engineering

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based on collaborations with H. Amini, M. Brune, C. Le Bris, I. Dotsenko, Serge Haroche, Z. Leghtas, M. Mirrahimi, JM. Raimond, A.Sarlette, C. Sayrin, R. Sepulchre, R. Somaraju.

Outline

Three quantum features[1]

1. Schrödinger equation: wave function $|\psi\rangle \in \mathcal{H}$, density operator ρ

$$\frac{d}{dt}|\psi\rangle = -iH|\psi\rangle, \quad \frac{d}{dt}\rho = -i[H, \rho]$$

2. Origin of dissipation and irreversibility: collapse of the wave packet induced by the measure of observable $\mathcal{O}$ with spectral decomposition $\sum_\mu \lambda_\mu P_\mu$:
   - measure outcome λ_μ with proba. $p_\mu = \langle\psi|P_\mu|\psi\rangle = \text{Tr}\,(\rho P_\mu)$ depending $|\psi\rangle$, ρ just before the measurement
   - measure back-action if outcome λ_μ:

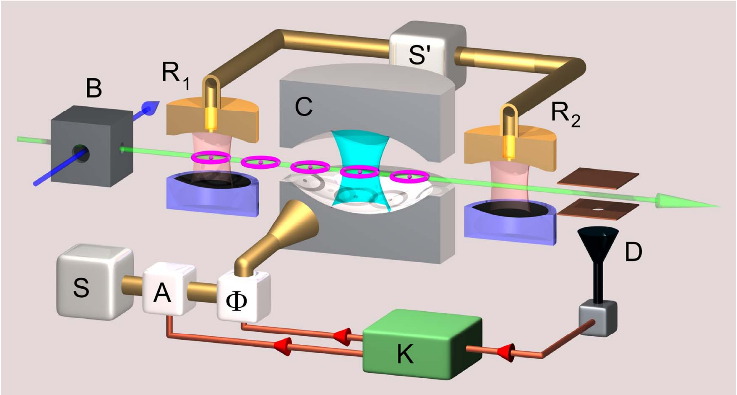
$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{P_\mu|\psi\rangle}{\sqrt{\langle\psi|P_\mu|\psi\rangle}}, \quad \rho \mapsto \rho_+ = \frac{P_\mu \rho P_\mu}{\text{Tr}\,(\rho P_\mu)}$$

3. Tensor product for the description of composite systems (S, M):
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $H = H_S \otimes \mathbb{I}_M + H_{int} + \mathbb{I}_S \otimes H_M$
   - observable on sub-system M only: $\mathcal{O} = \mathbb{I}_S \otimes \mathcal{O}_M$.

[1] S. Haroche and J.M. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts, 2006.

LKB photon Box: $\mathcal{H}_S$ cavity, $\mathcal{H}_M$ flying atom.

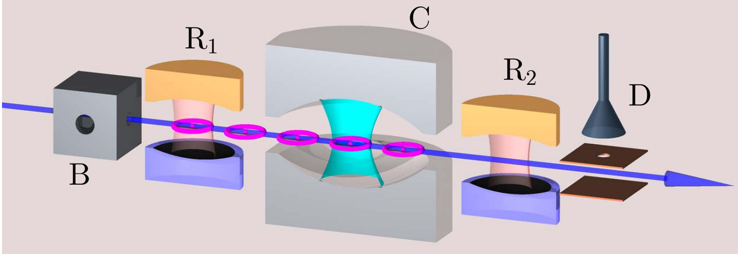
[2]



The experiment measuring and controlling the photons trapped into the cavity C (Cavity Quantum Electro-Dynamics group at Laboratoire Kastler-Brossel, ENS de Paris).

[2] Courtesy of Igor Dotsenko

The LKB Photon-Box: measuring photons with atoms



Atoms get out of box B one by one, undergo then a first Rabi pulse in Ramsey zone R_1, become entangled with electromagnetic field trapped in C, undergo a second Rabi pulse in Ramsey zone R_2 and finally are measured in the detector D.

The Markov chain model (1)

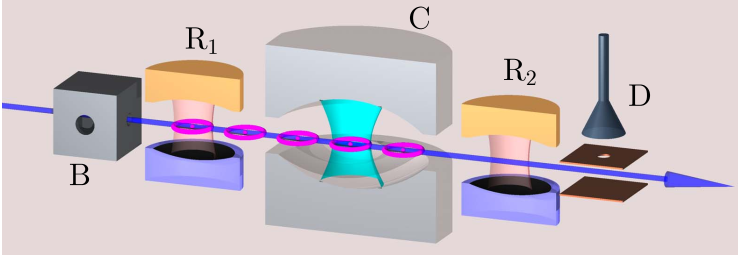
- System S corresponds to a quantized mode in C:

$$\mathcal{H}_S = \left\{ \sum_{n=0}^{\infty} \psi_n |n\rangle \mid (\psi_n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

  where $|n\rangle$ represents the Fock state associated to exactly n photons inside the cavity
- Meter M is associated to atoms : $\mathcal{H}_M = \mathbb{C}^2$, each atom admits two-level and is described by a wave function $c_g|g\rangle + c_e|e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$; atoms leaving B are all in state $|g\rangle$
- When atom comes out B, the state $|\Psi\rangle_B \in \mathcal{H}_S \otimes \mathcal{H}_M$ of the composite system atom/field is separable

$$|\Psi\rangle_B = |\psi\rangle \otimes |g\rangle.$$

The Markov chain model (2)



- When atom comes out B: $|\Psi\rangle_B = |\psi\rangle \otimes |g\rangle$.
- When atom comes out the first Ramsey zone R_1 the state remains separable but has changed to

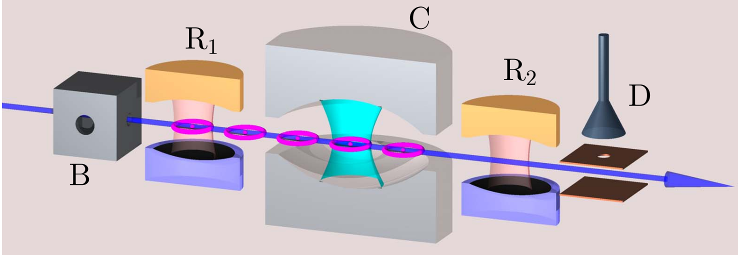
$$|\Psi\rangle_{R_1} = (\mathbb{I} \otimes U_{R_1})|\Psi\rangle_B = |\psi\rangle \otimes (U_{R_1}|g\rangle)$$

where the unitary transformation performed in R_1 only affects the atom:

$$U_{R_1} = e^{-i\frac{\theta_1}{2}(x_1\sigma_x+y_1\sigma_y+z_1\sigma_z)} = \cos(\tfrac{\theta_1}{2}) - i\sin(\tfrac{\theta_1}{2})(x_1\sigma_x+y_1\sigma_y+z_1\sigma_z)$$

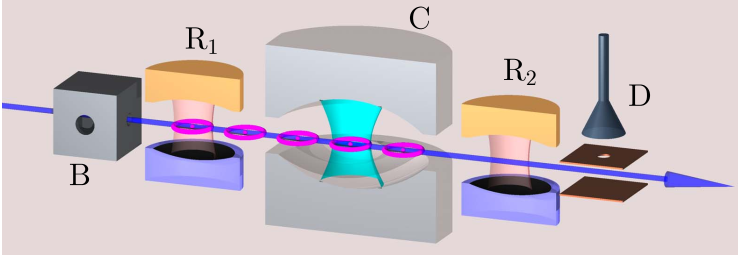
corresponds, in the Bloch sphere representation, to a rotation of angle θ_1 around $x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k}$ $(x_1^2 + y_1^2 + z_1^2 = 1)$

The Markov chain model (3)



- When atom comes out the first Ramsey zone R_1:
  $|\Psi\rangle_{R_1} = |\psi\rangle \otimes (U_{R_1}|g\rangle)$.
- When atom comes out cavity C, the state does not remain separable: atom and field becomes entangled and the state is described by
  $$|\Psi\rangle_C = U_C|\Psi\rangle_{R_1}$$
  where the unitary transformation U_C on $\mathcal{H}_S \otimes \mathcal{H}_M$ is associated to a Jaynes-Cumming Hamiltonian:
  $H_C = \frac{\Delta(t)}{2}\sigma_z + i\frac{\Omega(t)}{2}(a^\dagger \sigma_- - a\sigma_+)$ Parameters: $\Delta(t) = \omega_{eg} - \omega_c$, $\Omega(t)$ depend on time t.

The Markov chain model (4)



- When atom comes out cavity C: $|\Psi\rangle_C = U_C\big(|\psi\rangle \otimes (U_{R_1}|g\rangle)\big)$.
- When atom comes out second Ramsey zone R_2, the state becomes $|\Psi\rangle_{R_2} = (\mathbb{I} \otimes U_{R_2})|\Psi\rangle_C$ with $U_{R_2} = e^{-i\frac{\theta_2}{2}(x_2\sigma_x + y_2\sigma_y + z_2\sigma_z)}$.
- Just before the measurement in D, the state is given by

$$|\Psi\rangle_{R_2} = U_{SM}\big(|\psi\rangle \otimes |g\rangle\big) = \big(M_g|\psi\rangle\big) \otimes |g\rangle + \big(M_e|\psi\rangle\big) \otimes |e\rangle$$

where $U_{SM} = U_{R_2}U_C U_{R_1}$ is the total unitary transformation defining the linear measurement operators M_g and M_e on $\mathcal{H}_S$. Since U_{SM} is unitary, $M_g^\dagger M_g + M_e^\dagger M_e = \mathbb{I}$.

The Markov chain model (5)

Just before the measurement in D, the atom/field state is:

$$|g\rangle \otimes M_g|\psi\rangle + |e\rangle \otimes M_e|\psi\rangle$$

Denote by $\mu \in \{g, e\}$ the measurement outcome in detector D: with probability $p_\mu = \langle\psi|M_\mu^\dagger M_\mu|\psi\rangle$ we get μ. Just after the measurement outcome μ, the state becomes separable:

$$|\Psi\rangle_D = \frac{1}{\sqrt{p_\mu}}\ |\mu\rangle \otimes (M_\mu|\psi\rangle) = \frac{|\mu\rangle \otimes (M_\mu|\psi\rangle)}{\sqrt{\langle\psi|M_\mu^\dagger M_\mu|\psi\rangle}}.$$

Markov process (density matrix formulation)

$$\rho_+ = \begin{cases} \mathcal{M}_g(\rho) = \frac{M_g \rho M_g^\dagger}{\mathrm{Tr}\left(M_g \rho M_g^\dagger\right)}, & \text{with probability } p_g = \mathrm{Tr}\left(M_g \rho M_g^\dagger\right); \\ \mathcal{M}_e(\rho) = \frac{M_e \rho M_e^\dagger}{\mathrm{Tr}\left(M_e \rho M_e^\dagger\right)}, & \text{with probability } p_e = \mathrm{Tr}\left(M_e \rho M_e^\dagger\right). \end{cases}$$

Positive Operator Valued Measurement (POVM) (1)

System S of interest (a quantized electromagnetic field) interacts with the meter M (a probe atom), and the experimenter measures projectively the meter M (the probe atom). Need for a **composite system**: $\mathcal{H}_S \otimes \mathcal{H}_M$ where $\mathcal{H}_S$ and $\mathcal{H}_M$ are the Hilbert space of S and M.

Measurement process in three successive steps:

1. Initially the quantum state is separable

$$\mathcal{H}_S \otimes \mathcal{H}_M \ni |\Psi\rangle = |\psi_S\rangle \otimes |\theta_M\rangle$$

   with a well defined and known state $|\theta_M\rangle$ for M.

2. Then a Schrödinger evolution during a small time (unitary operator $U_{S,M}$) of the composite system from $|\psi_S\rangle \otimes |\theta_M\rangle$ and producing $U_{S,M}\big(|\psi_S\rangle \otimes |\theta_M\rangle\big)$, entangled in general.
3. Finally a projective measurement of the meter M: $\mathcal{O}_M = \mathbb{I}_S \otimes \left(\sum_\mu \lambda_\mu P_\mu\right)$ the measured observable for the meter. Projection operator P_μ is a rank-1 projection in $\mathcal{H}_M$ over the eigenstate $|\lambda_\mu\rangle \in \mathcal{H}_M$: $P_\mu = |\lambda_\mu\rangle\langle\lambda_\mu|$.

Positive Operator Valued Measurement (POVM) (2)

Define the measurement operators M_μ via

$$\forall |\psi_S\rangle \in \mathcal{H}_S, \quad U_{S,M}\big(|\psi_S\rangle \otimes |\theta_M\rangle\big) = \sum_\mu \big(M_\mu |\psi_S\rangle\big) \otimes |\lambda_\mu\rangle.$$

Then $\sum_\mu M_\mu^\dagger M_\mu = \mathbb{I}_S$. The set $\{M_\mu\}$ defines a Positive Operator Valued Measurement (POVM).

In $\mathcal{H}_S \otimes \mathcal{H}_M$, projective measurement of $\mathcal{O}_M = \mathbb{I}_S \otimes \big(\sum_\mu \lambda_\mu P_\mu\big)$ with quantum state $U_{S,M}\big(|\psi_S\rangle \otimes |\theta_M\rangle\big)$:

1. The probability of obtaining the value λ_μ is given by $p_\mu = \langle \psi_S | M_\mu^\dagger M_\mu | \psi_S \rangle$
2. After the measurement, the conditional (a posteriori) state of the system, given the outcome λ_μ, is

$$|\psi_S\rangle_+ = \frac{M_\mu |\psi_S\rangle}{\sqrt{p_\mu}}.$$

For mixed state ρ (instead of pure state $|\psi_S\rangle$):

$p_\mu = \operatorname{Tr}\left(M_\mu \rho M_\mu^\dagger\right)$ and $\rho_+ = \dfrac{M_\mu \rho M_\mu^\dagger}{\operatorname{Tr}\left(M_\mu \rho M_\mu^\dagger\right)}$,

Markov chain et Kraus map

- To the POVM on $\mathcal{H}_S$ is attached a stochastic process of quantum state ρ, $\rho^\dagger = \rho \geq 0$, $\text{Tr}(\rho) = 1$ $\left(\sum_\mu M_\mu^\dagger M_\mu = \mathbb{I}\right)$

$$\rho_+ = \frac{M_\mu \rho M_\mu^\dagger}{\text{Tr}\left(M_\mu \rho M_\mu^\dagger\right)} \text{ with probability } p_\mu = \text{Tr}\left(M_\mu \rho M_\mu^\dagger\right).$$

- For any observable A on $\mathcal{H}_S$, its conditional expectation value after the transition knowing the state ρ

$$\mathbb{E}\left(\text{Tr}\left(A\, \rho_+\right) \,|\rho\right) = \text{Tr}\left(A\, \mathbf{K}(\rho)\right)$$

where the linear map $\rho \mapsto \mathbf{K}(\rho) = \sum_\mu M_\mu \rho M_\mu^\dagger$ is a Kraus map defining a quantum channel.

- If $\bar{A}$ is a stationary point of the adjoint Kraus map $\mathbf{K}^*$, $\mathbf{K}^*(\bar{A}) = \sum_\mu M_\mu^\dagger \bar{A} M_\mu$, then $\text{Tr}\left(\bar{A}\rho\right)$ is a martingale:

$$\mathbb{E}\left(\text{Tr}\left(\bar{A}\, \rho_+\right) \,|\, \rho\right) = \text{Tr}\left(\bar{A}\, \mathbf{K}(\rho)\right) = \text{Tr}\left(\rho\, \mathbf{K}^*(\bar{A})\right) = \text{Tr}\left(\rho \bar{A}\right).$$

Models of open quantum systems

Discrete-time models are Markov chains

$$\rho_{k+1} = \frac{1}{p_\mu(\rho_k)} M_\mu \rho_k M_\mu^\dagger \quad \text{with proba.} \quad p_\mu(\rho_k) = \text{Tr}\left(M_\mu \rho_k M_\mu^\dagger\right)$$

with measure μ and associated to Kraus maps (ensemble average, open quantum channels)

$$\mathbb{E}\left(\rho_{k+1}/\rho_k\right) = \mathbf{K}(\rho_k) = \sum_\mu M_\mu \rho_k M_\mu^\dagger \quad \text{with} \quad \sum_\mu M_\mu^\dagger M_\mu = \mathbb{I}$$

Continuous-time models are stochastic differential systems

$$d\rho = \left(-i[H,\rho] + L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)\right) dt + \left(L\rho + \rho L^\dagger - \text{Tr}\left((L+L^\dagger)\rho\right)\rho\right) dw$$

driven by Wiener processes[3] $dw = dy - \text{Tr}\left((L+L^\dagger)\rho\right) dt$ with measure y and associated to Lindbald master equations:

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho] + L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)$$

[3] Another common possibility not considered here: SDE driven by Poisson processes.

From discrete-time to continuous-time: heuristic connection

For Monte-Carlo simulations of

$$d\rho = \Big(-i[H,\rho] + L\rho L^\dagger - \tfrac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)\Big)\, dt \\ + \Big(L\rho + \rho L^\dagger - \operatorname{Tr}\Big((L+L^\dagger)\rho\Big)\,\rho\Big) dw$$

take a small sampling time dt, generate a random number dw_t according to a Gaussian law of standard deviation $\sqrt{dt}$, and set $\rho_{t+dt} = \mathcal{M}_{dy_t}(\rho_t)$ where the jump operator $\mathcal{M}_{dy_t}$ is labelled by the measurement outcome $dy_t = \operatorname{Tr}\left((L+L^\dagger)\,\rho_t\right)\, dt + dw_t$:

$$\mathcal{M}_{dy_t}(\rho_t) = \frac{\Big(I+(-iH-\frac{1}{2}L^\dagger L)dt+L dy_t\Big)\rho_t\Big(I+(iH-\frac{1}{2}L^\dagger L)dt+L^\dagger dy_t\Big)}{\operatorname{Tr}\Big(\Big(I+(-iH-\frac{1}{2}L^\dagger L)dt+L dy_t\Big)\rho_t\Big(I+(iH-\frac{1}{2}L^\dagger L)dt+L^\dagger dy_t\Big)\Big)}.$$

Then ρ_{t+dt} remains always a density operator and using the Ito rules (dw of order $\sqrt{dt}$ and $dw^2 \equiv dt$) we get the good $d\rho = \rho_{t+dt} - \rho_t$ up to $O((dt)^{3/2})$ terms.

From discrete-time to continuous-time: heuristic connection (end)

For the Lindblad equation

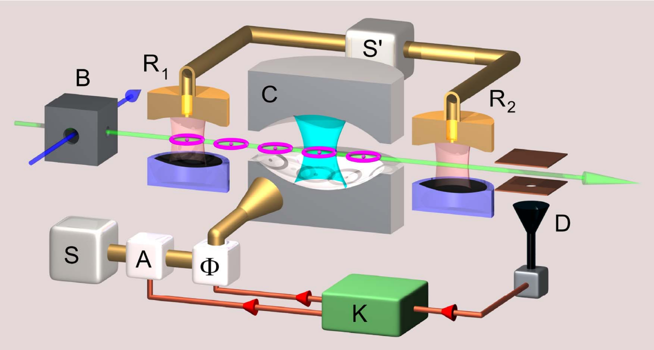
$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho] + L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)$$

take a small sampling time *dt* and set

$$\rho_{t+dt} = \frac{\left(I+(-iH-\frac{1}{2}L^\dagger L)dt\right)\rho_t\left(I+(iH-\frac{1}{2}L^\dagger L)dt\right)+dtL\rho_t L^\dagger}{\mathrm{Tr}\Big(\left(I+(-iH-\frac{1}{2}L^\dagger L)dt\right)\rho_t\left(I+(iH-\frac{1}{2}L^\dagger L)dt\right)+dtL\rho_t L^\dagger\Big)}.$$

Then ρ_{t+dt} remains always a density operator and $\frac{d}{dt}\rho = (\rho_{t+dt}-\rho_t)/dt$ up to $O(dt)$ terms.

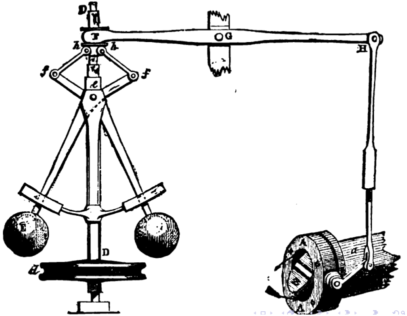
Measurement-based stabilization of photon-number state[4]



- Control input $u = Ae^{\imath\Phi}$; measure output $y \in \{g, e\}$.
- Sampling time 80 μs long enough for numerical computations in K.

[4] C. Sayrin, I. Dotsenko, X. Zhou, B. Peaudecerf, Th. Rybarczyk, S. Gleyzes, P. R., M. Mirrahimi, H. Amini, M. Brune, J.M. Raimond, S. Haroche: Real-time quantum feedback prepares and stabilizes photon number states. Nature, 477(7362),2011.

Watt regulator: a classical analogue of quantum coherent feedback. [5]



The first variations of speed $\delta\omega$ and governor angle $\delta\theta$ obey to

$$\frac{d}{dt}\delta\omega = -a\delta\theta$$
$$\frac{d^2}{dt^2}\delta\theta = -\Lambda\frac{d}{dt}\delta\theta - \Omega^2(\delta\theta - b\delta\omega)$$

with (a, b, Λ, Ω) positive parameters.

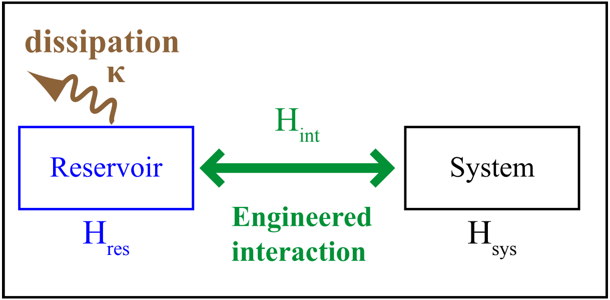
Third order system

$$\frac{d^3}{dt^3}\delta\omega = -\Lambda\frac{d^2}{dt^2}\delta\omega - \Omega^2\frac{d}{dt}\delta\omega - ab\Omega^2\delta\omega = 0$$

Characteristic polynomial $P(s) = s^3 + \Lambda s^2 + \Omega^2 s + ab\Omega^2$ with roots having negative real parts iff $\Lambda > ab$: governor damping must be strong enough to ensure asymptotic stability of the closed-loop system.

[5] J.C. Maxwell: On governors. Proc. of the Royal Society, No.100, 1868.

Reservoir Engineering[6] and coherent feedback[7]



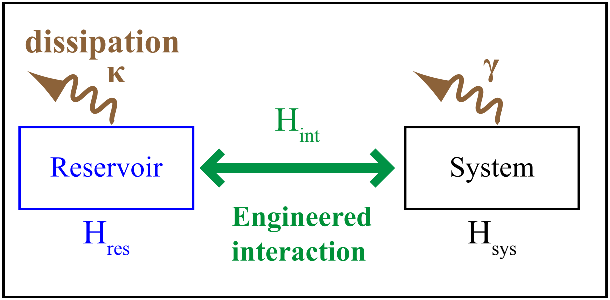
$$H = H_{\text{res}} + H_{\text{int}} + H_{\text{syst}}$$

if $\rho \underset{t\to\infty}{\to} |\bar{\psi}\rangle\langle\bar{\psi}| \otimes \rho_{res}$ exponentially on a time scale of $\tau \approx \kappa$ then . . . .

[6] Introduced by Poyatos, Cirac and Zoller, 1996.

[7] See, e.g., the lectures of H. Mabuchi delivered at the "école de physique des Houches", July 2011.

Reservoir Engineering[6] and coherent feedback[7]



$$H = H_{\text{res}} + H_{\text{int}} + H_{\text{syst}}$$

$$\ldots\ldots \quad \rho \underset{t\to\infty}{\to} |\bar{\psi}\rangle\langle\bar{\psi}| \otimes \rho_{res} + \Delta, \text{ if } \kappa \gg \gamma \text{ then } \|\Delta\| \ll 1$$

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[6] Introduced by Poyatos, Cirac and Zoller, 1996.

[7] See, e.g., the lectures of H. Mabuchi delivered at the "école de physique des Houches", July 2011.

Reservoir engineering (coherent feedback) v.s. measurement-based feedback

Advantages over measurement-based feedback

- Does not require knowing the measurement result.
- No external intervention on small time scale.

Difficulty

- For each target state $|\bar{\psi}\rangle$, engineer a coupling to the reservoir which drives ρ to $\rho_{res} \otimes |\bar{\psi}\rangle\langle\bar{\psi}|$, compatible with lab constraints.

Reservoir engineering for discrete-time systems

Data: $\mathcal{H}_S$ with Hamiltonian H_S, a pure goal state $\bar\rho_S = |\bar\psi_S\rangle\langle\bar\psi_S|$.
Find a "realistic" meter system of Hilbert space $\mathcal{H}_M$ with initial state $|\theta_M\rangle$, with Hamiltonian H_M and interaction Hamiltonian H_{int} such that

1. the propagator $U_{S,M} = U(T)$ between 0 and time T ($\frac{d}{dt}U = -i(H_S + H_M + H_{int})U$, $U(0) = \mathbb{I}$) reads:

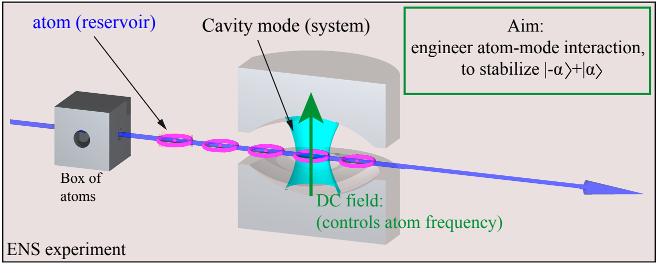
$$\forall |\psi_S\rangle \in \mathcal{H}_S, \quad U_{S,M}\big(|\psi_S\rangle \otimes |\theta_M\rangle\big) = \sum_\mu \big(M_\mu|\psi_S\rangle\big) \otimes |\lambda_\mu\rangle$$

   where $|\lambda_\mu\rangle$ is a ortho-normal basis of $\mathcal{H}_M$.
2. the resulting measurement operators M_μ admit $|\bar\psi_S\rangle$ as common eigen-vector, i.e., $\bar\rho_S$ is a fixed point of the Kraus map $\mathbf{K}(\rho) = \sum_\mu M_\mu \rho M_\mu^\dagger$: $\qquad \mathbf{K}(\bar\rho_S) = \bar\rho_S$.
3. iterates of $\mathbf{K}$ converge to $\bar\rho_S$ for any initial condition ρ_0:

$$\lim_{k\mapsto+\infty} \rho_k = \bar\rho_S \text{ where } \rho_k = \mathbf{K}(\rho_{k-1}).$$

Here the reservoir is made of the infinite set of identical meter systems with initial state $|\theta_M\rangle$ at $t = (k-1)T$ and interacting with $\mathcal{H}_S$ during $[(k-1)T, kT]$, $k = 1, 2, \ldots$.

Reservoir stabilizing "Schrödinger cats" [8]



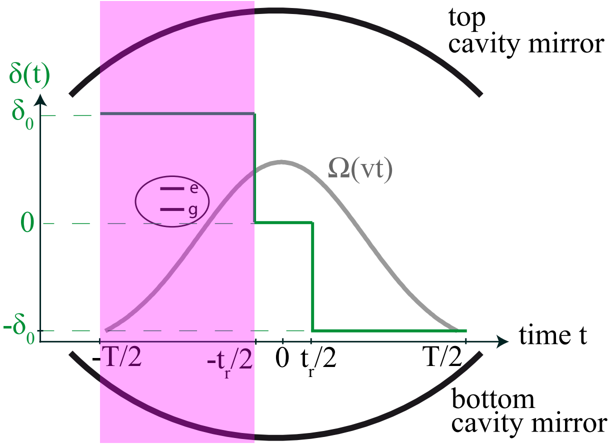
Here $\mathcal{H}_S = \mathcal{H}_c = \{\sum_{n\geq 0} \psi_n |n\rangle,\ (\psi_n)_{n\geq 0} \in l^2(\mathbb{C})\}$ and
$\mathcal{H}_M = \mathcal{H}_q = \{c_g|g\rangle + c_e|e\rangle,\ c_g, c_e \in \mathbb{C}\}$.
$H_S + H_M + H_{int}$ is the Jaynes-Cumming Hamiltionian

$$\boldsymbol{H}(t) = \omega_c \mathbf{a}^\dagger \mathbf{a} + \frac{\delta(t)}{2} \boldsymbol{\sigma_z} + i\frac{\Omega(vt)}{2}(\boldsymbol{a}^\dagger |g\rangle\langle e| - \boldsymbol{a}|e\rangle\langle g|)$$

which is time varying with control $\delta(t) = \omega_q(t) - \omega_c$ and Gaussian radial profile $\Omega(x) = \Omega_0 e^{-\frac{x^2}{w^2}}$, $x = vt$ with v atom velocity.

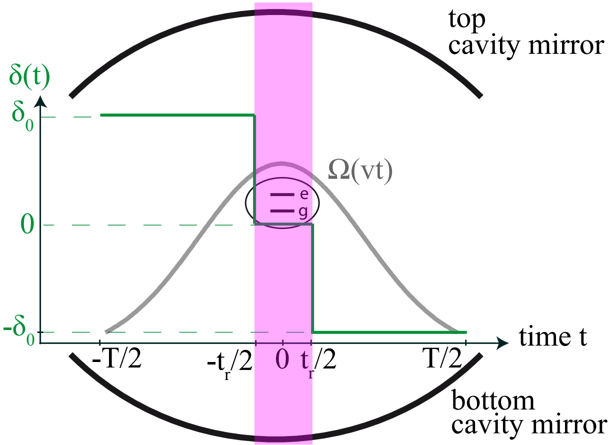
[8] A. Sarlette, Z. Leghtas, M. Brune, J.M. Raimond, P.R.: Stabilization of nonclassical states of one and two-mode radiation fields by reservoir engineering. Phys. Rev. A 86, 012114 (2012)

Composite interaction



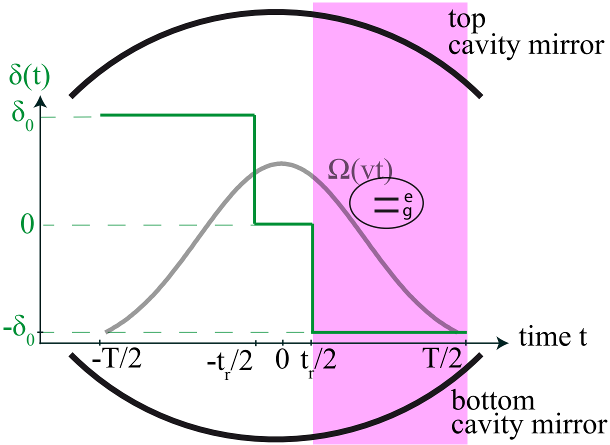
$$U = U_{\text{off-resonant}}$$

Composite interaction



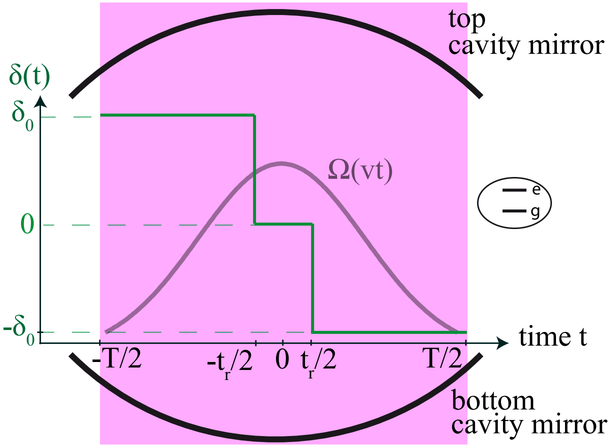
$$U = U_{\text{resonant}} U_{\text{off-resonant}}$$

Composite interaction



$$U = U^{\dagger}_{\text{off-resonant}} U_{\text{resonant}} U_{\text{off-resonant}}$$

Composite interaction



$$U \approx e^{-i\phi_{\text{Kerr}}\mathbf{n}^2} U_{\text{resonant}} e^{i\phi_{\text{Kerr}}\mathbf{n}^2}$$

Simulation: convergence toward the cat from vacuum $\rho_0 = |0\rangle\langle 0|$

Simulation: convergence after a cat jump

Reservoir engineering for continuous time systems

Data: $\mathcal{H}_S$ with Hamiltonian H_S, a pure goal state $\bar{\rho}_S = |\bar{\psi}_S\rangle\langle\bar{\psi}_S|$.
Find a "realistic" controller of Hilbert space $\mathcal{H}_M$, with Hamiltonian H_M and interaction Hamiltonian H_{int} and Lindblad operators L_μ acting only on $\mathcal{H}_M$ such that

1. the Lindblad master equation of the composite system $\mathcal{H}_S \otimes \mathcal{H}_M$ governing the density operator ρ evolution

$$\frac{d}{dt}\rho = -i\Big[H_S + H_M + H_{int}, \rho\Big] + \sum_\mu L_\mu \rho L_\mu^\dagger - \tfrac{1}{2} L_\mu^\dagger L_\mu \rho - \tfrac{1}{2}\rho L_\mu^\dagger L_\mu$$

   admits a separable steady state $\bar{\rho} = \bar{\rho}_S \otimes \bar{\rho}_M$ for some density operator $\bar{\rho}_M$ on $\mathcal{H}_M$.

2. For any initial condition $\rho(0)$, $\lim_{t\mapsto+\infty} \rho(t) = \bar{\rho}$.

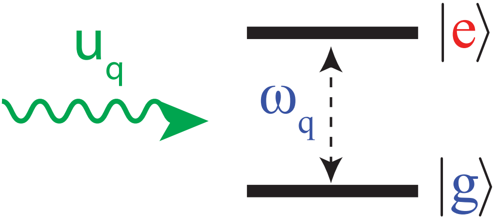
Example: cavity cooling towards $\bar{\rho}_S = |0\rangle\langle 0|$ with a qubit-controller

$$H_S = \omega_c \mathbf{a}^\dagger \mathbf{a},\;\; H_M = \tfrac{\delta}{2}(|e\rangle\langle e| - |g\rangle\langle g|),\;\; H_{int} = \tfrac{\Omega}{2}(\mathbf{a}^\dagger\, |g\rangle\langle e| + \mathbf{a}\, |e\rangle\langle g|)$$

when $|e\rangle$ is unstable of life-time T_q: $L = \sqrt{\frac{1}{T_q}}|g\rangle\langle e|$, $\bar{\rho} = |0\rangle\langle 0| \otimes |g\rangle\langle g|$
and $\frac{d}{dt}\,\mathrm{Tr}\left(\bar{\rho}\rho\right) = \frac{\mathrm{Tr}(|0\rangle\langle 0|\otimes|e\rangle\langle e|\;\rho)}{T_q} \geq 0$ as Lyapunov function .

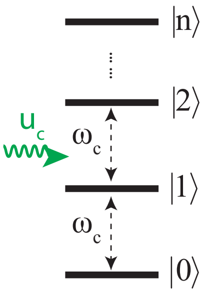
A qubit: 2 level system

- State space: $\mathcal{H}_q = \{c_g|g\rangle + c_e|e\rangle,\ c_g, c_e \in \mathbb{C}\}$.
- Operators: $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$, $\sigma_x = |e\rangle\langle g| + |g\rangle\langle e|$, $\sigma_y = -i|e\rangle\langle g| + i|g\rangle\langle e|$.
- Hamiltonian: $H_q = \omega_q\sigma_z/2 + u_q\sigma_x$.



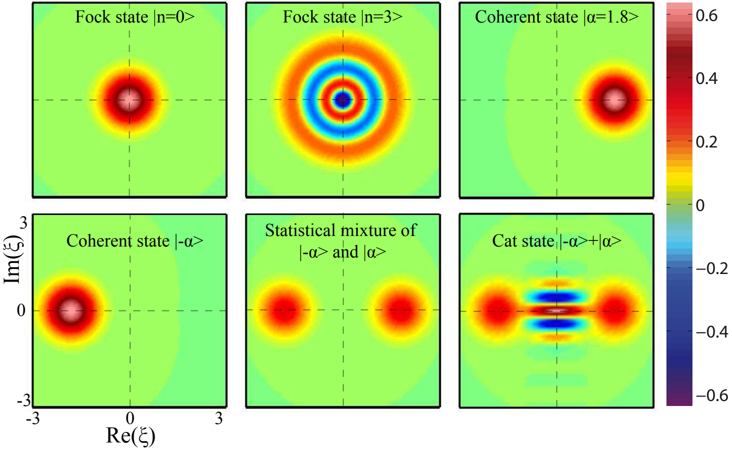
A cavity: quantum harmonic oscillator

- State space: $\mathcal{H}_c = \{\sum_{n\geq 0} \psi_n |n\rangle,\ (\psi_n)_{n\geq 0} \in l^2(\mathbb{C})\}$.
- $\mathcal{D} = \{\rho \in \mathcal{L}(\mathcal{H}_c), \rho^\dagger = \rho,\ \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}$.
- Operators:
  $\mathbf{a}|n\rangle = \sqrt{n}|n-1\rangle,\ \mathbf{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle,$
  $\mathbf{n}|n\rangle = \mathbf{a}^\dagger\mathbf{a}|n\rangle = n|n\rangle,\ \mathbf{D}_\alpha = e^{\alpha\mathbf{a}^\dagger - \alpha^\dagger\mathbf{a}}.$
- Hamiltonian: $\mathbf{H}_c = \omega_c\mathbf{a}^\dagger\mathbf{a} + u_c(\mathbf{a} + \mathbf{a}^\dagger)$.
- Coherent state of amplitude $\alpha \in \mathbb{C}$:
  $|\alpha\rangle = \sum_{n\geq 0}\left(e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\right)|n\rangle.$
- $\mathbf{a}|\alpha\rangle = \alpha|\alpha\rangle$.
- $\mathbf{D}_\alpha|0\rangle = |\alpha\rangle$.



Cavity state representation: the Wigner function

$$W_\rho : \mathbb{C} \ni \xi \to \frac{2}{\pi} \operatorname{Tr}\left(e^{i\pi \mathbf{a}^\dagger \mathbf{a}} \mathbf{D}_{-\xi} \rho \mathbf{D}_\xi \right) \in \mathbb{R}$$



Harmonic oscillator[9] (1): quantization and correspondence principle

Classical Hamiltonian formulation of $\frac{d^2}{dt^2}x = -\omega^2 x$

$$\frac{d}{dt}x = \omega p = \frac{\partial \mathbb{H}}{\partial p}, \quad \frac{d}{dt}p = -\omega x = -\frac{\partial \mathbb{H}}{\partial x}, \quad \mathbb{H} = \frac{\omega}{2}(p^2 + x^2).$$

Quantization: probability wave function $|\psi\rangle_t \sim (\psi(x,t))_{x\in\mathbb{R}}$ with $|\psi\rangle_t \sim \psi(.,t) \in L^2(\mathbb{R},\mathbb{C})$ obeys to the Schrödinger equation ($\hbar = 1$ in all the lectures)

$$i\frac{d}{dt}|\psi\rangle = H|\psi\rangle, \quad H = \omega(P^2 + X^2) = -\frac{\omega}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega}{2}x^2$$

where H results from $\mathbb{H}$ by replacing x by position operator $\sqrt{2}X$ and p by impulsion operator $\sqrt{2}P = -i\frac{\partial}{\partial x}$.

PDE model: $i\frac{\partial \psi}{\partial t}(x,t) = -\frac{\omega}{2}\frac{\partial^2 \psi}{\partial x^2}(x,t) + \frac{\omega}{2}x^2\psi(x,t), \quad x \in \mathbb{R}$.

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[9] Two references: C. Cohen-Tannoudji, B. Diu, and F. Laloë. *Mécanique Quantique*, volume I& II. Hermann, Paris, 1977.
M. Barnett and P. M. Radmore. *Methods in Theoretical Quantum Optics*. Oxford University Press, 2003.

Harmonic oscillator (2): annihilation and creation operators

Averaged position $\langle X \rangle_t = \langle \psi | X | \psi \rangle$ and impulsion $\langle P \rangle_t = \langle \psi | P | \psi \rangle$ [10]:

$$\langle X \rangle_t = \frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} x|\psi|^2 dx, , \quad \langle P \rangle_t = -\frac{i}{\sqrt{2}} \int_{-\infty}^{+\infty} \psi^* \frac{\partial \psi}{\partial x} dx.$$

Annihilation a and creation operators $a^\dagger$:

$$a = X + iP = \frac{1}{\sqrt{2}} \left(x + \frac{\partial}{\partial x} \right), \quad a^\dagger = X - iP = \frac{1}{\sqrt{2}} \left(x - \frac{\partial}{\partial x} \right)$$

Commutation relationships:

$$[X, P] = \frac{i}{2}, \quad [a, a^\dagger] = 1, \quad H = \omega(P^2 + X^2) = \omega \left(a^\dagger a + \frac{1}{2} \right).$$

Set $X_\lambda = \frac{1}{2} \left(e^{-i\lambda} a + e^{i\lambda} a^\dagger \right)$ for any angle λ:

$$\left[X_\lambda, X_{\lambda + \frac{\pi}{2}} \right] = \frac{i}{2}.$$

[10] We assume everywhere that for each t, $x \mapsto \psi(x, t)$ is of the Schwartz class (fast decay at infinity + smooth).

Harmonic oscillator (3): spectral decomposition and Fock states

$[a, a^\dagger] = 1$ and Ker(a) of dimension one imply that the spectrum of $N = a^\dagger a$ is non-degenerate and is $\mathbb{N}$. More we have the useful commutations for any entire function f:

$$a\, f(N) = f(N + I)\, a, \quad f(N)\, a^\dagger = a^\dagger\, f(N + I).$$

Fock state with n photon(s): the eigen-state of N associated to the eigen-value n:

$$N|n\rangle = n|n\rangle, \quad a|n\rangle = \sqrt{n}\,|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle.$$

The ground state $|0\rangle$ (0 photon state or vacuum state) satisfies $a|0\rangle = 0$ and corresponds to the Gaussian function:

$$|0\rangle \sim \psi_0(x) = \frac{1}{\pi^{1/4}} \exp(-x^2/2).$$

The operator a (resp. $a^\dagger$) is the annihilation (resp. creation) operator since it transfers $|n\rangle$ to $|n-1\rangle$ (resp. $|n+1\rangle$) and thus decreases (resp. increases) the quantum number n by one unit.

Harmonic oscillator (4): displacement operator

Quantization of $\frac{d^2}{dt^2}x = -\omega^2 x - \omega\sqrt{2}u$

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right) + u(a + a^\dagger).$$

The associated controlled PDE

$$i\frac{\partial\psi}{\partial t}(x,t) = -\frac{\omega}{2}\frac{\partial^2\psi}{\partial x^2}(x,t) + \left(\frac{\omega}{2}x^2 + \sqrt{2}ux\right)\psi(x,t).$$

Glauber displacement operator D_α (unitary) with $\alpha \in \mathbb{C}$:

$$D_\alpha = e^{\alpha a^\dagger - \alpha^* a} = e^{2i\Im\alpha X - 2\imath\Re\alpha P}$$

From Baker-Campbell Hausdorf formula valid for any operators A and B,

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \ldots$$

we get the Glauber formula when $[A,[A,B]] = [B,[A,B]] = 0$:

$$e^{A+B} = e^A\, e^B\, e^{-\frac{1}{2}[A,B]}.$$

(show that $C_t = e^{t(A+B)} - e^{tA}\, e^{tB}\, e^{-\frac{t^2}{2}[A,B]}$ satisfies $\frac{d}{dt}C = (A+B)C$)

Harmonic oscillator (5): identities resulting from Glauber formula

With $A = \alpha a^\dagger$ and $B = -\alpha^* a$, Glauber formula gives:

$$D_\alpha = e^{-\frac{|\alpha|^2}{2}}\ e^{\alpha a^\dagger} e^{-\alpha^* a} = e^{+\frac{|\alpha|^2}{2}}\ e^{-\alpha^* a} e^{\alpha a^\dagger}$$

$$D_{-\alpha} a D_\alpha = a + \alpha \quad \text{and} \quad D_{-\alpha} a^\dagger D_\alpha = a^\dagger + \alpha^*.$$

With $A = 2i\Im\alpha X \sim i\sqrt{2}\Im\alpha x$ and $B = -2\imath\Re\alpha P \sim -\sqrt{2}\Re\alpha\frac{\partial}{\partial x}$, Glauber formula gives[11]:

$$D_\alpha = e^{-i\Re\alpha\Im\alpha}\ e^{i\sqrt{2}\Im\alpha x} e^{-\sqrt{2}\Re\alpha\frac{\partial}{\partial x}}$$

$$(D_\alpha|\psi\rangle)_{x,t} = e^{-i\Re\alpha\Im\alpha}\ e^{i\sqrt{2}\Im\alpha x}\psi(x - \sqrt{2}\Re\alpha, t)$$

Exercice

For any $\alpha, \beta, \epsilon \in \mathbb{C}$, prove that

$$D_{\alpha+\beta} = e^{\frac{\alpha^*\beta - \alpha\beta^*}{2}} D_\alpha D_\beta$$

$$D_{\alpha+\epsilon} D_{-\alpha} = \left(1 + \frac{\alpha\epsilon^* - \alpha^*\epsilon}{2}\right)\mathbb{I} + \epsilon a^\dagger - \epsilon^* a + O(|\epsilon|^2)$$

$$\left(\frac{d}{dt}D_\alpha\right) D_{-\alpha} = \left(\frac{\alpha\frac{d}{dt}\alpha^* - \alpha^*\frac{d}{dt}\alpha}{2}\right)\mathbb{I} + \left(\frac{d}{dt}\alpha\right)a^\dagger - \left(\frac{d}{dt}\alpha^*\right)a.$$

[11] Remember that a time-delay of r corresponds to the operator $e^{-r\frac{d}{dt}}$.

Harmonic oscillator (6): lack of controllability

Take $|\psi\rangle$ solution of the controlled Schrödinger equation $i\frac{d}{dt}|\psi\rangle = \left(\omega\left(a^\dagger a + \frac{1}{2}\right) + u(a + a^\dagger)\right)|\psi\rangle$. Set $\langle a\rangle = \langle\psi|a\psi\rangle$. Then

$$\frac{d}{dt}\langle a\rangle = -i\omega\langle a\rangle - iu.$$

From $a = X + iP$, we have $\langle a\rangle = \langle X\rangle + i\langle P\rangle$ where $\langle X\rangle = \langle\psi|X|\psi\rangle \in \mathbb{R}$ and $\langle P\rangle = \langle\psi|P|\psi\rangle \in \mathbb{R}$. Consequently:

$$\frac{d}{dt}\langle X\rangle = \omega\langle P\rangle, \quad \frac{d}{dt}\langle P\rangle = -\omega\langle X\rangle - u.$$

Consider the change of frame $|\psi\rangle = e^{-i\theta_t} D_{\langle a\rangle_t} |\chi\rangle$ with

$$\theta_t = \int_0^t \left(|\langle a\rangle|^2 + u\Re(\langle a\rangle)\right), \quad D_{\langle a\rangle_t} = e^{\langle a\rangle_t a^\dagger - \langle a\rangle_t^* a},$$

Then $|\chi\rangle$ obeys to autonomous Schrödinger equation

$$i\frac{d}{dt}|\chi\rangle = \omega a^\dagger a|\chi\rangle.$$

The dynamics of $|\psi\rangle$ can be decomposed into two parts:

- a controllable part of dimension two for $\langle a\rangle$
- an uncontrollable part of infinite dimension for $|\chi\rangle$.

Harmonic oscillator (7): coherent states as reachable ones from $|0\rangle$

Coherent states

$$|\alpha\rangle = D_\alpha|0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{+\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \quad \alpha \in \mathbb{C}$$

are the states reachable from vacuum set. They are also the eigen-state of a: $a|\alpha\rangle = \alpha|\alpha\rangle$.
A widely known result in quantum optics[12]: classical currents and sources (generalizing the role played by u) only generate classical light (quasi-classical states of the quantized field generalizing the coherent state introduced here)
We just propose here a control theoretic interpretation in terms of reachable set from vacuum[13]

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[12] See complement B_{III}, page 217 of C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg. *Photons and Atoms: Introduction to Quantum Electrodynamics*.Wiley, 1989.

[13] see also: MM-PR, IEEE Trans. Automatic Control, 2004 and MM-PR, CDC-ECC, 2005.