Motion planing and tracking for differentially flat systems

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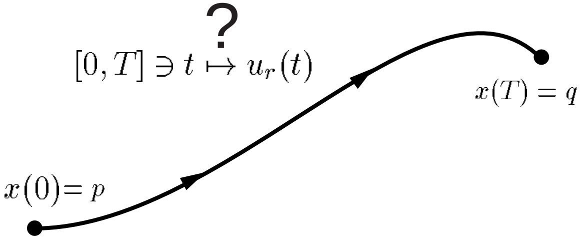
Outline

Interest of flat systems

1. History: ”integrability” for under-determinated systems of differential equations (Monge, Hilbert, Cartan, ...).
2. Control theory: flat systems admit simple solutions to the motion planing and tracking problems (Fliess and coworkers 1991 and later).
3. Books on differentially flat systems:
   - H. Sira-Ramirez and S.K. Agarwal: Differentially flat systems. CRC, 2004.
   - J. Lévine: Analysis and Control of Nonlinear Systems : A Flatness-Based Approach. Springer-Verlag, 2009.
   - J. Rudolph: Flatness Based Control of Distributed Parameter Systems. Shaker, Germany. 2003.

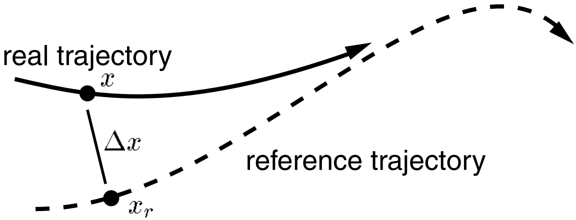
Motion planning: controllability.



Difficult problem due to integration of

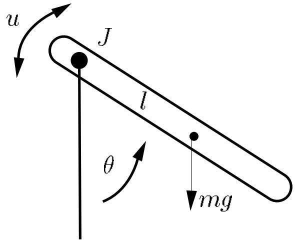
$$\frac{d}{dt}x = f(x, u_r(t)), \quad x(0) = p.$$

Tracking for $\frac{d}{dt}x = f(x, u)$: stabilization.



Compute Δu, $u = u_r + \Delta u$, depending Δx (feedback), such that $\Delta x = x - x_r$ tends to 0 (stabilization).

The simplest robot



- Newton ODE):

$$\frac{d^2}{dt^2}\theta = -p\sin(\theta) + u$$

  Non linear oscillator with scalar input u and parameter $p > 0$.

- Computed torque method: $u_r = \frac{d^2}{dt^2}\theta_r + p\sin\theta_r$ provides an explicit parameterization via KC^2 function: $t \mapsto \theta_r(t)$, the flat output.

Motion planing and tracking ($\xi, \omega_0 > 0$, two feedback gains)

$$u\left(t, \theta, \frac{d}{dt}\theta\right) = \frac{d^2}{dt^2}\theta_r + p\sin\theta - 2\xi\omega_0\left(\frac{d}{dt}\theta - \frac{d}{dt}\theta_r\right) - (\omega_0)^2\sin(\theta - \theta_r)$$

where $t \mapsto \theta_r(t)$ defines the reference trajectory (control goal).

Fully actuated mechanical systems

The computed torque method for

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] = \frac{\partial L}{\partial q} + M(q)u$$

consists in setting $t \mapsto q(t)$ to obtain u as a function of q, $\dot{q}$ and $\ddot{q}$.
(Fully actuated: $\dim q = \dim u$ and $M(q)$ invertible).

Oscillators and linear systems

System with 2 ODEs and 3 unknowns (x_1, x_2, u) ($a_1, a_2 > 0$ and $a_1 \neq a_2$)

$$\frac{d^2}{dt^2} x_1 = -a_1 (x_1 - u), \quad \frac{d^2}{dt^2} x_2 = -a_2 (x_2 - u)$$

defines a free module[1] with basis $y = \frac{a_2 x_1 - a_1 x_2}{a_2 - a_1}$:

$$\begin{cases} x_1 = y + \frac{d^2}{dt^2} y / a_2, \quad \frac{d}{dt} x_1 = \frac{d}{dt} y + \frac{d^3 y}{dt^3} / a_2 \\ x_2 = y + \frac{d^2}{dt^2} y / a_1, \quad \frac{d}{dt} x_2 = \frac{d}{dt} y + \frac{d^3 y}{dt^3} / a_1 \\ \\ u = y + \left(\frac{1}{a_1} + \frac{1}{a_2} \right) \frac{d^2}{dt^2} y + \left(\frac{1}{a_1 a_2} \right) \frac{d^4}{dt^4} y \end{cases}$$

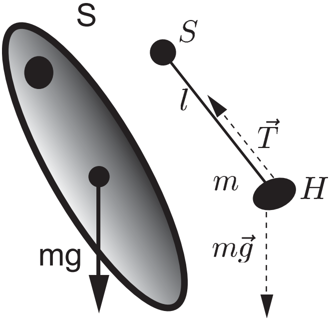
Reference trajectory for equilibrium $x_1 = x_2 = u = 0$ at $t = 0$ to equilibrium $x_1 = x_2 = u = D$ at $t = T > 0$:

$$y(t) = \quad 0 \text{ si } t \leq 0, \quad \frac{(t)^4}{t^4 + (T - t)^4} D \text{ si } t \in [0, T], \quad D \text{ si } t \geq T.$$

Generalization to n oscillators and any linear controllable system, $\frac{d}{dt} X = AX + Bu$.

[1] See the work of Alban Quadrat and co-workers....

$2k\pi$ juggling robot: prototype of implicit flat system



Isochronous punctual pendulum H (Huygens) :

$$\begin{aligned} m\frac{d^2}{dt^2}H &= \vec{T} + m\vec{g} \\ \vec{T} &\,//\, \vec{HS} \\ \|\vec{HS}\|^2 &= l \end{aligned}$$

- The suspension point $S \in \mathbb{R}^3$ stands for the control input
- The oscillation center $H \in \mathbb{R}^3$ is the flat output: since $\vec{T}/m = \frac{d^2}{dt^2}H - \vec{g}$ et $\vec{T} \,//\, \overrightarrow{HS}$, S is solution of the algebraic system:

$$\overrightarrow{HS} \,//\, \frac{d^2}{dt^2}H - \vec{g} \quad \text{and} \quad \|\vec{HS}\|^2 = l.$$

Return of the pendulum and smooth branch switch

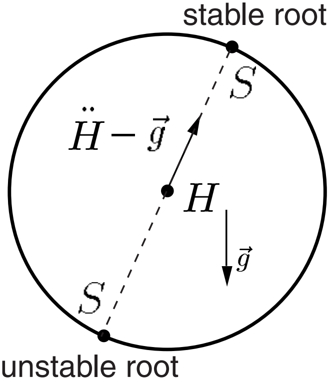
In a vertical plane: H of coordinates (y_1, y_2) and S of coordinates (u_1, u_2) satisfy

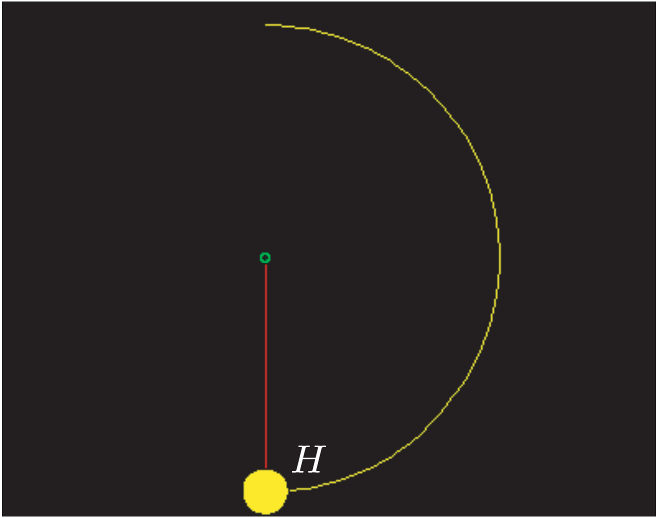
$$(y_1 - u_1)^2 + (y_2 - u_2)^2 = l, \quad (y_1 - u_1)\left(\frac{d^2}{dt^2} y_2 + g\right) = (y_2 - u_2)\frac{d^2}{dt^2} y_1 .$$

Find $[0, T] \ni t \mapsto y(t)$ C^2 such that $y(0) = (0, -l)$, $y(T) = (0, l)$ and $y^{(1,2)}(0, T) = 0$, and such that exists also $[0, T] \ni t \mapsto u(t)$ C^0 with $u(0) = u(T) = 0$ (switch between the stable and the unstable branches).

Planning the inversion trajectory

Any smooth trajectory connecting the stable to the unstable equilibrium is such that $\ddot{H}(t) = \vec{g}$ for at least one time t. During the motion there is a switch from the stable root to the unstable root (singularity crossing when $\ddot{H} = \vec{g}$)



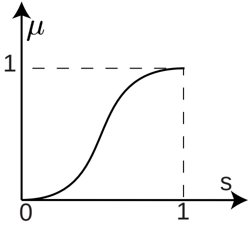

H

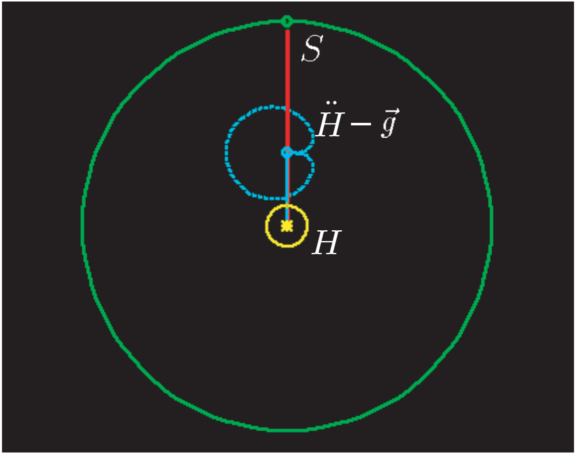
Crossing smoothly the singularity $\ddot{H} = \vec{g}$

The geometric path followed by H is a half-circle of radius $l$of center O:

$$H(t) = 0 + l \begin{bmatrix} \sin\theta(s) \\ -\cos\theta(s) \end{bmatrix} \text{ with } \theta(s) = \mu(s)\pi, \quad s = t/T \in [0, 1]$$

where T is the transition time and $\mu(s)$ a sigmoid function of the form:




S
$\ddot{H} - \vec{g}$
H

Time scaling and dilatation of $\ddot{H} - \vec{g}$

Denote by $'$ derivation with respect to s. From

$$H(t) = 0 + l \begin{bmatrix} \sin\theta(s) \\ -\cos\theta(s) \end{bmatrix}, \quad \theta(s) = \mu(t/T)\pi$$

we have

$$\ddot{H} = H''/T^2.$$

Changing T to αT yields to a dilation of factor $1/\alpha^2$ of the closed geometric path described by $\ddot{H} - \vec{g}$ for $t \in [0, T]$ ($\ddot{H}(0) = \ddot{H}(T) = 0$), the dilation center being $-\vec{g}$.
The inversion time is obtained when this closed path passes through 0. This construction holds true for generic μ.

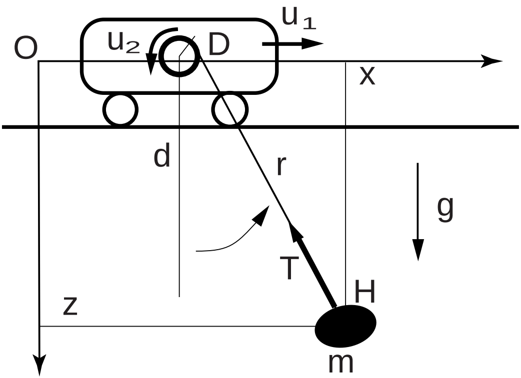
The crane

$$m\ddot{H} = \vec{T} + m\vec{g}$$

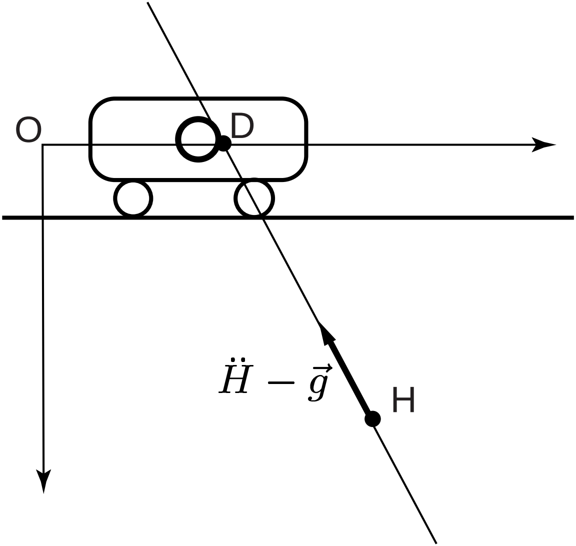
$$\vec{T} \,//\, \overrightarrow{HD}$$

$$HD = \mathrm{r}$$

$$\overrightarrow{0D} \cdot \vec{k} = 0$$

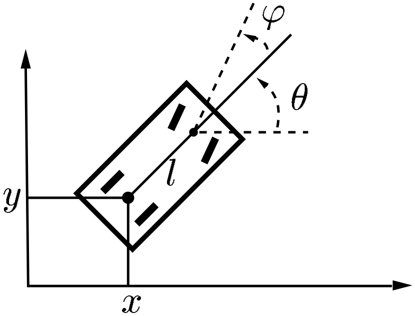


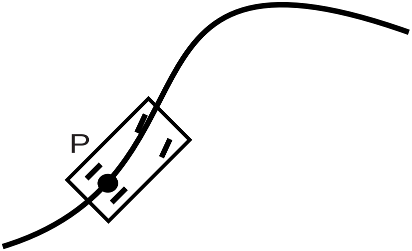
The geometric construction for the crane



Singularity when $\ddot{H} - \vec{g}$ is horizontal.

Single car [2]





$$\begin{cases} \frac{d}{dt}x = v\cos\theta \\ \frac{d}{dt}y = v\sin\theta \\ \frac{d}{dt}\theta = \frac{v}{l}\tan\varphi = \omega \end{cases}$$

$$\begin{cases} v = \pm\|\frac{d}{dt}P\| \\ \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} = \frac{\frac{d}{dt}P}{v} \\ \tan\varphi = \frac{l\det(\ddot{P},\dot{P})}{v\sqrt{|v|}} \end{cases}$$

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[2] For modeling and control of non-holonomic systems, see, e.g.,B. d'Andréa-Novel, G. Campion, G. Bastin: Control of Nonholonomic Wheeled Mobile Robots by State Feedback Linearization. International Journal of Robotics Research December 1995 vol. 14 no. 6 543-559.

The time scaling symmetry

For any $T \mapsto \sigma(T)$, the transformation

$$t = \sigma(T), \quad (x, y, \theta) = (X, Y, \Theta), \qquad (v, \omega) = (V, \Omega)/\sigma'(t)$$

leave the equations

$$\tfrac{d}{dt}x = v\cos\theta, \quad \tfrac{d}{dt}y = v\sin\theta, \quad \tfrac{d}{dt}\theta = \omega$$

unchanged:

$$\frac{d}{dT}X = V\cos\Theta, \quad \frac{d}{dT}Y = V\sin\Theta, \quad \frac{d}{dT}\Theta = \Omega.$$

$SE(2)$ invariance

For any (a, b, α), the transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X\cos\alpha - Y\sin\alpha + a \\ X\sin\alpha + Y\cos\alpha + b \end{bmatrix} \quad , \theta = \Theta - \alpha, \quad (v, \omega) = (V, \Omega)$$

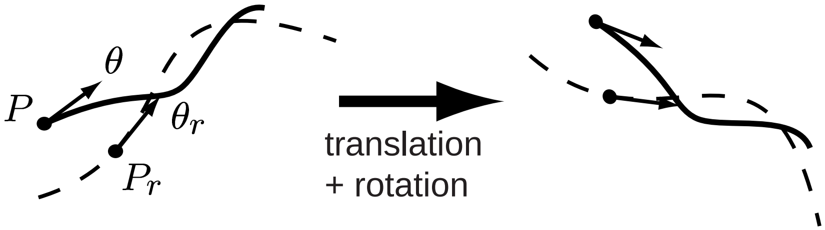
leave the equations

$$\tfrac{d}{dt}x = v\cos\theta, \quad \tfrac{d}{dt}y = v\sin\theta, \quad \tfrac{d}{dt}\theta = \omega$$

unchanged:

$$\frac{d}{dt}X = V\cos\Theta, \quad \frac{d}{dt}Y = V\sin\Theta, \quad \frac{d}{dt}\Theta = \Omega.$$

Invariant tracking[3]



[3] For a general setting see: Ph. Martin, P. R., J. Rudolph: Invariant tracking, ESAIM: Control, Optimisation and Calculus of Variations, 10:1–13,2004.

Invariant tracking for the car: goal

Given the reference trajectory

$$t \mapsto s_r \mapsto P_r(s_r), \quad \theta_r(s_r), \quad v_r = \dot{s}_r, \quad \omega_r = \dot{s}_r \kappa_r(s_r)$$

and the state (P, θ)
Find an invariant controller

$$v = v_r + \dots, \quad \omega = \omega_r + \dots$$

Invariant tracking for the car: time-scaling

Set

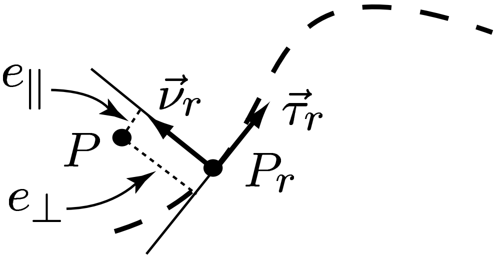
$$v = \bar{v}\ \dot{s}_r, \quad \omega = \bar{\omega}\ \dot{s}_r$$

and denote by $'$ derivation versus s_r.
Equations remain unchanged

$$P' = \bar{v}\ \vec{\tau}, \quad \vec{\tau}' = \bar{\omega}\ \vec{\nu}$$

with $P = (x, y)$, $\vec{\tau} = (\cos\theta, \sin\theta)$ and $\vec{\nu} = (-\sin\theta, \cos\theta)$.

Invariant errors



Construct the decoupling and/or linearizing controller with the two following invariant errors

$$e_\parallel = (P - P_r) \cdot \vec{\tau}_r, \quad e_\perp = (P - P_r) \cdot \vec{\nu}_r.$$

Computations of $e_{\parallel}$ and $e_{\perp}$ derivatives

Since $e_{\parallel} = (P - P_r) \cdot \vec{\tau}_r$ and $e_{\perp} = (P - P_r) \cdot \vec{\nu}_r$ we have (remember that $' = d/ds_r$)

$$e_{\parallel}' = (P' - P_r') \cdot \vec{\tau}_r + (P - P_r) \cdot \vec{\tau}_r'.$$

But $P' = \bar{v}\vec{\tau}$, $P_r' = \vec{\tau}_r$ and $\vec{\tau}_r' = \kappa_r \vec{\nu}_r$, thus

$$e_{\parallel}' = \bar{v}\vec{\tau} \cdot \vec{\tau}_r - 1 + \kappa_r (P - P_r) \cdot \vec{\nu}_r.$$

Similar computations for $e_{\perp}'$ yield:

$$e_{\parallel}' = \bar{v}\cos(\theta - \theta_r) - 1 + \kappa_r e_{\perp}, \quad e_{\perp}' = \bar{v}\sin(\theta - \theta_r) - \kappa_r e_{\parallel}.$$

Computations of $e_\parallel$ and $e_\perp$ second derivatives

Derivation of

$$e_\parallel' = \bar{v}\cos(\theta - \theta_r) - 1 + \kappa_r e_\perp, \quad e_\perp' = \bar{v}\sin(\theta - \theta_r) - \kappa_r e_\parallel$$

with respect to s_r gives

$$\begin{aligned} e_\parallel'' &= \bar{v}'\cos(\theta - \theta_r) - \bar{\omega}\bar{v}\sin(\theta - \theta_r) \\ &\quad + 2\kappa_r\bar{v}\sin(\theta - \theta_r) + \kappa_r' e_\perp - \kappa_r^2 e_\parallel \end{aligned}$$

$$\begin{aligned} e_\perp'' &= \bar{v}'\sin(\theta - \theta_r) + \bar{\omega}\bar{v}\cos(\theta - \theta_r) \\ &\quad - 2\kappa_r\bar{v}\cos(\theta - \theta_r) - \kappa_r' e_\parallel + \kappa_r + \kappa_r^2 e_\parallel. \end{aligned}$$

The dynamics feedback in s_r time-scale

We have obtain

$$\begin{aligned} e_{\|}'' &= \bar{v}' \cos(\theta - \theta_r) - \bar{\omega}\bar{v}\sin(\theta - \theta_r) \\ &+ 2\kappa_r \bar{v}\sin(\theta - \theta_r) + \kappa_r' e_{\perp} - \kappa_r^2 e_{\|} \end{aligned}$$

$$\begin{aligned} e_{\perp}'' &= \bar{v}' \sin(\theta - \theta_r) + \bar{\omega}\bar{v}\cos(\theta - \theta_r) \\ &- 2\kappa_r \bar{v}\cos(\theta - \theta_r) - \kappa_r' e_{\|} + \kappa_r + \kappa_r^2 e_{\|}. \end{aligned}$$

Choose $\bar{v}'$ and $\bar{\omega}$ such that

$$e_{\|}'' = -\left(\frac{1}{L_{\|}^1} + \frac{1}{L_{\|}^2}\right) e_{\|}' - \left(\frac{1}{L_{\|}^1 L_{\|}^2}\right) e_{\|}$$
$$e_{\perp}'' = -\left(\frac{1}{L_{\perp}^1} + \frac{1}{L_{\perp}^2}\right) e_{\perp}' - \left(\frac{1}{L_{\perp}^1 L_{\perp}^2}\right) e_{\perp}$$

Possible around a large domain around the reference trajectory since the determinant of the decoupling matrix is $\bar{v} \approx 1$.

The dynamics feedback in physical time-scale

In the s_r scale, we have the following dynamic feedback

$$\bar{v}' = \Phi(\bar{v}, P, P_r, \theta, \theta_r, \kappa_r, \kappa_r')$$
$$\bar{\omega} = \Psi(\bar{v}, P, P_r, \theta, \theta_r, \kappa_r, \kappa_r')$$

Since $' = d/ds_r = d/(\dot{s}_r dt)$ we have

$$\frac{d\bar{v}}{dt} = \Phi(\bar{v}, P, P_r, \theta, \theta_r, \kappa_r, \kappa_r')\ \dot{s}_r(t)$$
$$\bar{\omega} = \Psi(\bar{v}, P, P_r, \theta, \theta_r, \kappa_r, \kappa_r')$$

and the real control is

$$v = \bar{v}\dot{s}_r(t), \quad \tan\phi = \frac{l\bar{\omega}}{\bar{v}}$$

Nothing blows up when $\dot{s}_r(t)$ tends to 0: the controller is well defined around steady-state via a simple use of time-scaling symmetry

Conversion into chained form destroys $SE(2)$ invariance

The car model

$$\tfrac{d}{dt}x = v\cos\theta, \quad \tfrac{d}{dt}y = v\sin\theta, \quad \tfrac{d}{dt}\theta = \frac{v}{l}\tan\varphi$$

can be transformed into chained form

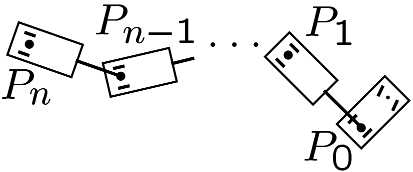
$$\tfrac{d}{dt}x_1 = u_1, \quad \tfrac{d}{dt}x_2 = u_2, \quad \tfrac{d}{dt}x_3 = x_2 u_1$$

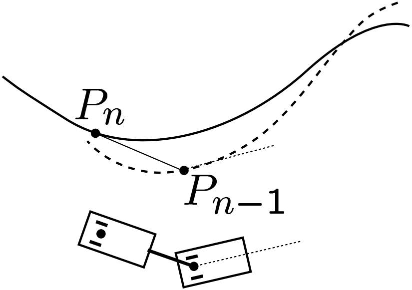
via change of coordinates and static feedback

$$x_1 = x, \quad x_2 = \frac{dy}{dx} = \tan\theta, \quad x_3 = y.$$

But the symmetries are not preserved in such coordinates: one privileges axis x versus axis y without any good reason. The behavior of the system seems to depend on the origin you take to measure the angle (artificial singularity when $\theta = \pm\pi/2$).

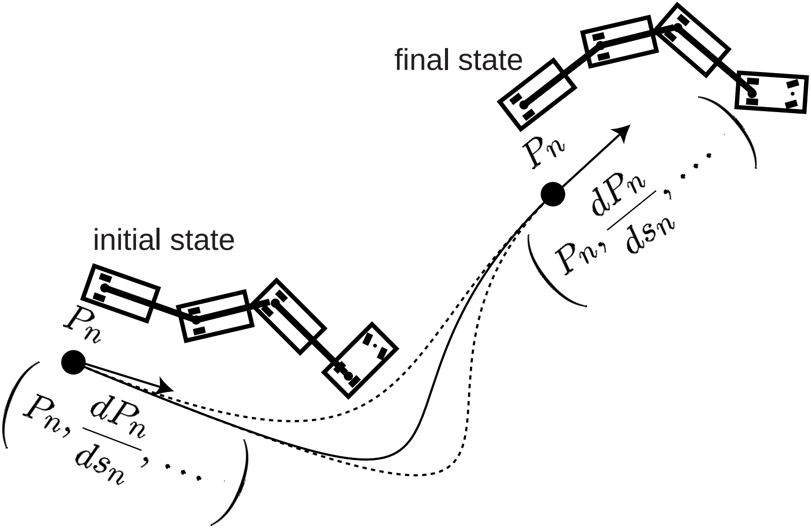
The standard *n*-trailers system



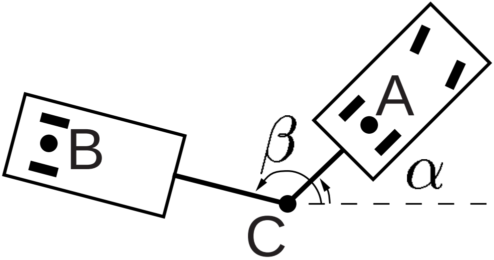


$$P_{n-1} = P_n + d_n \frac{dP_n}{ds_n}$$

Motion planning for the standard n trailers system

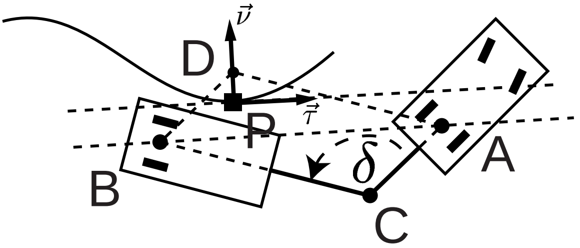


The general 1-trailer system (CDC93)



Rolling without slipping conditions ($A = (x, y)$, $u = (v, \varphi)$):

$$
\begin{aligned}
\frac{d}{dt}x &= v\ \cos\alpha \\
\frac{d}{dt}y &= v\ \sin\alpha \\
\frac{d}{dt}\alpha &= \frac{v}{l}\tan\varphi \\
\frac{d}{dt}\beta &= \frac{v}{b}\left(\frac{a}{l}\tan\varphi\cos(\beta-\alpha)+\sin(\beta-\alpha)\right).
\end{aligned}
$$



With $\delta = \widehat{BCA}$ we have

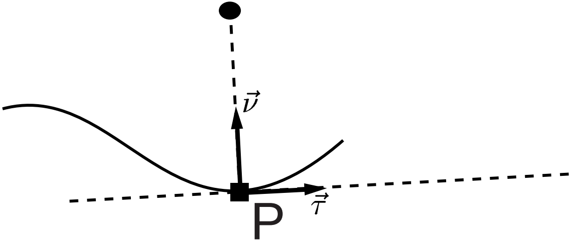
$$D = P - L(\delta)\vec{\nu} \quad \text{with} \quad L(\delta) = ab \int_0^{\pi+\delta} \frac{-\cos\sigma}{\sqrt{a^2 + b^2 + 2ab\cos\sigma}}\, d\sigma$$

Curvature is given by

$$K(\delta) = \frac{\sin\delta}{\cos\delta\,\sqrt{a^2 + b^2 - 2ab\cos\delta} - L(\delta)\,\sin\delta}$$

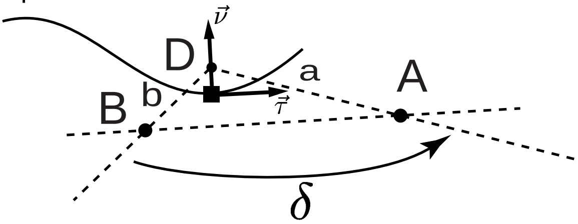
The geometric construction

Assume that $s \mapsto P(s)$ is known. Let us show how to deduce (A, B, α, β) the system configuration.
We know thus P, $\vec{\tau} = dP/ds$ and $\kappa = d\theta/ds$ (θ is the angle of $\vec{\tau}$:



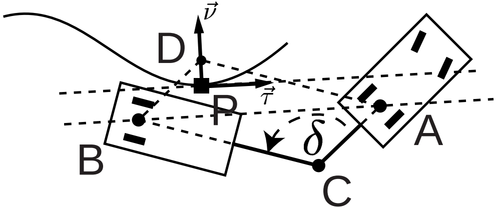
The geometric construction

From κ we deduce $\delta = \widehat{BCA} = \widehat{BDA}$ by inverting $\kappa = K(\delta)$.
D is then known since $D = P - L(\delta)\vec{\nu}$.
Finally $\vec{\tau}$ is parallel to AB and $DB = a$ and $DA = b$.



The complete construction

One to one correspondence between P, $\vec{\tau}$ and κ and (A, α, β).



Differential forms

Eliminate v from

$$\frac{d}{dt}x = v\ \cos\alpha, \quad \frac{d}{dt}y = v\ \sin\alpha, \quad \frac{d}{dt}\alpha = \frac{v}{l}\tan\varphi, \quad \frac{d}{dt}\beta = \dots$$

to have 3 equations with 5 variables

$$\begin{aligned}
&\sin\alpha\ \tfrac{d}{dt}x - \cos\alpha\ \tfrac{d}{dt}y = 0\\
&\tfrac{d}{dt}\alpha - \left(\frac{\tan\varphi\cos\alpha}{l}\right)\ \tfrac{d}{dt}x - \left(\frac{\tan\varphi\sin\alpha}{l}\right)\ \tfrac{d}{dt}y = 0\\
&\tfrac{d}{dt}\beta \dots
\end{aligned}$$

defining a module of differential forms, $I = \{\eta_1, \eta_2, \eta_3\}$

$$\begin{aligned}
\eta_1 &= \sin\alpha\ dx - \cos\alpha\ dy\\
\eta_2 &= d\alpha - \left(\frac{\tan\varphi\cos\alpha}{l}\right)\ dx - \left(\frac{\tan\varphi\sin\alpha}{l}\right)\ dy\\
\eta_3 &= d\beta - \dots
\end{aligned}$$

Following [4], compute the sequence $I = I^{(0)} \supseteq I^{(1)} \supseteq I^{(2)} \dots$ where

$$I^{(k+1)} = \{\eta \in I^{(k)} \mid d\eta = 0 \mod (I^{(k)})\}$$

and find that

$$\dim I^{(0)} = 3, \quad \dim I^{(1)} = 2, \quad \dim I^{(2)} = 1, \quad \dim I^{(3)} = 0.$$

The Cartesian coordinates (X, Y) of P are obtained via the Pfaff normal form of the differential form μ generating $I^{(2)}$

$$\mu = f(\alpha, \beta)\, dX + g(\alpha, \beta)\, dY.$$

(X, Y) is not unique; $SE(2)$ invariance simplifies computations.

[4] E. Cartan: Sur l'intégration de certains systèmes indéterminés d'équations différentielles. J. für reine und angew. Math. Vol. 145, 1915.

Contact systems:

The driftless system $\frac{d}{dt}x = f_1(x)u_1 + f_2(x)u_2$ is also a Pfaffian system of codimension 2

$$\omega_i \equiv \sum_{j=1}^{n} a_i^j(x)\, dx_j = 0, \quad i = 1, \ldots, n-2.$$

Pfaffian systems equivalent via changes of x-coordinates to contact systems (related to chained-form, Murray-Sastry 1993)

$$dx_2 - x_3 dx_1 = 0, \quad dx_3 - x_4 dx_1 = 0, \quad \ldots dx_{n-1} - x_n dx_1 = 0$$

are mainly characterized by the derived flag (Weber(1898), Cartan(1916), Goursat (1923), Giaro-Kumpera-Ruiz(1978), Murray (1994), Pasillas-Respondek (2000), ...).

Interest of contact systems (chained form):

$$dx_2 - x_3 dx_1 = 0, \quad dx_3 - x_4 dx_1 = 0, \quad \ldots dx_{n-1} - x_n dx_1 = 0$$

The general solution reads in terms of $z \mapsto w(z)$ and its derivatives,

$$x_1 = z, \quad x_2 = w(z), \quad , x_3 = \frac{dw}{dz}, \quad \ldots \quad , x_n = \frac{d^{n-2}w}{dz^{n-2}}.$$

In this case, the general solution of $\frac{d}{dt}x = f_1(x)u_1 + f_2 u_2$ reads in terms of $t \mapsto z(t)$ any C^1 time function and any C^{n-2} function of z, $z \mapsto w(z)$. The quantities $x_1 = z(t)$ and $x_2 = w(z(t))$ play here a special role. We call them the flat output.

An elementary definition based on inversion

- Explicit control systems: $\frac{d}{dt}x = f(x,u)$ ($x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$) is flat, iff, exist $\alpha \in \mathbb{N}$ and $h(x,u,\dots,u^{(\alpha)}) \in \mathbb{R}^m$ such that the generic solution of

$$\tfrac{d}{dt}x = f(x,u), \quad y = h(x,u,\dots,u^{(\alpha)})$$

reads ($\beta \in \mathbb{N}$)

$$x = \mathcal{A}(y,\dots,y^{(\beta)}), \quad u = \mathcal{B}(y,\dots,y^{(\beta+1)})$$

- Under-determined systems: $F(x,\dots,x^{(r)}) = 0$ ($x \in \mathbb{R}^n$, $F \in \mathbb{R}^{n-m}$) is flat, iff, exist $\alpha \in \mathbb{N}$ and $h(x,\dots,x^{(\alpha)}) \in \mathbb{R}^m$ such that the generic solution of

$$F(x,\dots,x^{(r)}) = 0, \quad y = h(x,\dots,x^{(\alpha)}) \quad \text{reads} \quad x = \mathcal{A}(y,\dots,y^{(\beta)})$$

y is called a flat output: Fliess and co-workers 1991, ...
Integrable under-determined differential systems: Monge (1784), Darboux, Goursat, Hilbert (1912), Cartan (1914).

Flat systems (Fliess-et-al, 1992,…,1999)

A basic definition extending remark of Isidori-Moog-DeLuca (CDC86) on dynamic feedback linearization (Charlet-Lévine-Marino (1989)):

$$\frac{d}{dt}x = f(x, u)$$

is flat, iff, exist $m = \dim(u)$ output functions $y = h(x, u, \ldots, u^{(p)})$, $\dim(h) = \dim(u)$, such that the inverse of $u \mapsto y$ has no dynamics, i.e.,

$$x = \Lambda\left(y, \dot{y}, \ldots, y^{(q)}\right), \quad u = \Upsilon\left(y, \dot{y}, \ldots, y^{(q+1)}\right).$$

Behind this: an equivalence relationship exchanging trajectories (absolute equivalence of Cartan and dynamic feedback: Shadwick (1990), Sluis (1992), Nieuwstadt-et-al (1994), Pomet et al (1992), Pomet (1995),… Lévine (2011)).

Equivalence and flatness (intrinsic point of view, IEEE-AC 1999)

Take $\frac{d}{dt}x = f(x,u),\ (x,u) \in X \times U \subset \mathbb{R}^n \times \mathbb{R}^m$. It generates a system $(F, \mathfrak{M})$, (D-variety) where

$$\mathfrak{M} := X \times U \times \mathbb{R}_m^\infty$$

with the vector field $F(x,u,u^1,\dots) := (f(x,u), u^1, u^2, \dots)$. $(F, \mathfrak{M})$ is equivalent to $(G, \mathfrak{N})$ ($\dot{z} = g(z,v)$: $\mathfrak{N} := Z \times V \times \mathbb{R}_m^\infty$ with the vector field $G(z,v,v^1,\dots) := (g(z,v), v^1, v^2, \dots)$) iff exists an invertible transformation $\Phi : \mathfrak{M} \longmapsto \mathfrak{N}$ such that

$$\forall \xi := (x,u,u^1,\dots) \in \mathfrak{M}, \quad G(\Phi(\xi)) = D\Phi(\xi) \cdot F(\xi).$$

Equivalence and flatness (extrinsic point of view)

Elimination of u from the n state equations $\frac{d}{dt}x = f(x,u)$ provides an under-determinate system of $n-m$ equations with n unknowns

$$F\left(x, \frac{d}{dt}x\right) = 0.$$

An endogenous transformation $x \mapsto z$ is defined by

$$z = \Phi(x, \dot{x}, \dots, x^{(p)}), \quad x = \Psi(z, \dot{z}, \dots, z^{(q)})$$

(nonlinear analogue of uni-modular matrices, the "integral free" transformations of Hilbert).
Two systems are equivalents, iff, exists an endogenous transformation exchanging the equations.
A system equivalent to the trivial equation $z_1 = 0$ with $z = (z_1, z_2)$ is flat with z_2 the flat output.

The time dependent definition

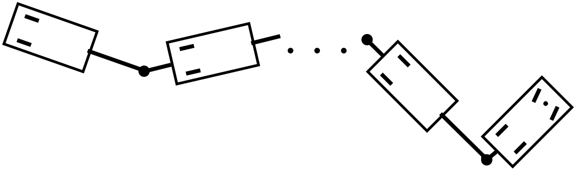
We present here the simplest version of this definition (Murray and co-workers (SIAM JCO 1998)):

$$\frac{d}{dt}x = f(t, x, u)$$

is flat, iff, exist $m = \dim(u)$ output functions $y = h(t, x, u, \dots, u^{(p)})$, $\dim(h) = \dim(u)$, such that the inverse of $u \mapsto y$ has no dynamics, i.e.,

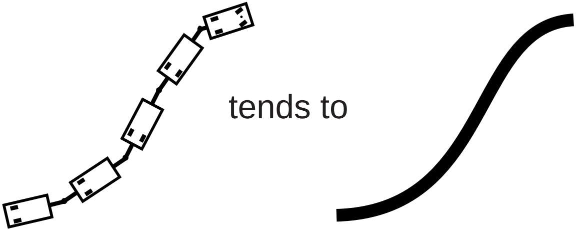
$$x = \Lambda\left(t, y, \dot{y}, \dots, y^{(q)}\right), \quad u = \Upsilon\left(t, y, \dot{y}, \dots, y^{(q+1)}\right).$$

The general *n*-trailer system for $n \geq 2$ is not flat.

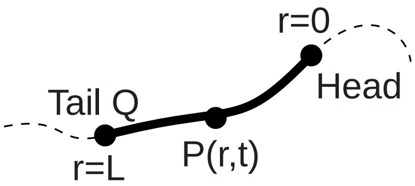


Proof: by pure chance, the characterization of codimension 2 contact systems is also a characterization of drifless flat systems (Cartan 1914, Martin-R. 1994) (adding integrator, endogenous or exogenous or singular dynamic feedbacks are useless here).

When the number n of trailers becomes large…



The nonholonomic snake: a trivial delay system.



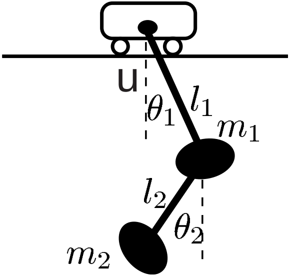
Implicit partial differential nonlinear system:

$$\left\| \frac{\partial P}{\partial r} \right\| = 1, \quad \frac{\partial P}{\partial r} \wedge \frac{\partial P}{\partial t} = 0.$$

General solution via $s \mapsto Q(s)$ arbitrary smooth:

$$P(r,t) = Q(s(t) + L - r) \equiv \sum_{k \geq 0} \frac{(L-r)^k}{k!} \frac{dQ^k}{ds^k}(s(t)).$$

Two linearized pendulum in series

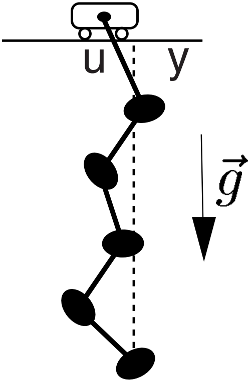


Flat output $y = u + l_1\theta_1 + l_2\theta_2$:

$$\theta_2 = -\frac{\ddot{y}}{g}, \quad \theta_1 = -\frac{m_1 \ddot{\overbrace{(y - l_2\theta_2)}}}{(m_1 + m_2)g} + \frac{m_2}{m_1 + m_2}\theta_2$$

and $u = y - l_1\theta_1 - l_2\theta_2$ is a linear combination of $(y, y^{(2)}, y^{(4)})$.

n pendulum in series

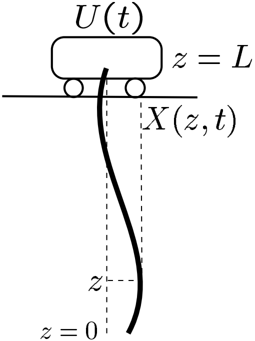


Flat output $y = u + l_1\theta_1 + \ldots + l_n\theta_n$:

$$u = y + a_1 y^{(2)} + a_2 y^{(4)} + \ldots + a_n y^{(2n)}.$$

When n tends to ∞ the system tends to a partial differential equation.

The heavy chain [5]



$$\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right)$$
$$X(L,t) = U(t)$$

Flat output $y(t) = X(0,t)$ with

$$U(t) = \frac{1}{2\pi} \int_0^{2\pi} y\left(t - 2\sqrt{L/g}\,\sin\zeta\right)\, d\zeta$$

[5] N. Petit, P. R.: motion planning for heavy chain systems. SIAM J. Control and Optim., 41:475-495, 2001.

With the same flat output, for a discrete approximation (n pendulums in series, n large) we have

$$u(t) = y(t) + a_1 \ddot{y}(t) + a_2 y^{(4)}(t) + \ldots + a_n y^{(2n)}(t),$$

for a continuous approximation (the heavy chain) we have

$$U(t) = \frac{1}{2\pi} \int_0^{2\pi} y\left(t + 2\sqrt{L/g}\,\sin\zeta\right)\, d\zeta.$$

Why? Because formally

$$y(t + 2\sqrt{L/g}\,\sin\zeta) = y(t) + \ldots + \frac{\left(2\sqrt{L/g}\,\sin\zeta\right)^n}{n!}\, y^{(n)}(t) + \ldots$$

But integral formula is preferable (divergence of the series…).

The general solution of the PDE

$$\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right)$$

is

$$X(z,t) = \frac{1}{2\pi} \quad \int_0^{2\pi} y\left(t - 2\sqrt{z/g}\ \sin\zeta\right)\ d\zeta$$

where $t \mapsto y(t)$ is any time function.
Proof: replace $\frac{d}{dt}$ by s, the Laplace variable, to obtain a singular second order ODE in z with bounded solutions. Symbolic computations and operational calculus on

$$s^2 X = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right).$$

Symbolic computations in the Laplace domain

Thanks to $x = 2\sqrt{\frac{z}{g}}$, we get

$$x\frac{\partial^2 X}{\partial x^2}(x,t) + \frac{\partial X}{\partial x}(x,t) - x\frac{\partial^2 X}{\partial t^2}(x,t) = 0.$$

Use Laplace transform of X with respect to the variable t

$$x\frac{\partial^2 \hat{X}}{\partial x^2}(x,s) + \frac{\partial \hat{X}}{\partial x}(x,s) - xs^2\hat{X}(x,s) = 0.$$

This is a the Bessel equation defining J_0 and Y_0:

$$\hat{X}(z,s) = A(s)\ J_0(2\imath s\sqrt{z/g}) + B(s)\ Y_0(2\imath s\sqrt{z/g}).$$

Since we are looking for a bounded solution at $z = 0$ we have $B(s) = 0$ and (remember that $J_0(0) = 1$):

$$\hat{X}(z,s) = J_0(2\imath s\sqrt{z/g})\hat{X}(0,s).$$

$$\hat{X}(z,s) = J_0(2\imath s\sqrt{z/g})\hat{X}(0,s).$$

Using Poisson's integral representation of J_0

$$J_0(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \exp(\imath\zeta \sin\theta)\, d\theta, \quad \zeta \in \mathbb{C}$$

we have

$$J_0(2\imath s\sqrt{x/g}) = \frac{1}{2\pi}\int_0^{2\pi} \exp(2s\sqrt{x/g}\sin\theta)\, d\theta.$$

In terms of Laplace transforms, this last expression is a combination of delay operators:

$$X(z,t) = \frac{1}{2\pi}\int_0^{2\pi} y(t + 2\sqrt{z/g}\sin\theta)\, d\theta$$

with $y(t) = X(0,t)$.

Explicit parameterization of the heavy chain

The general solution of

$$\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right), \quad U(t) = X(L,t)$$

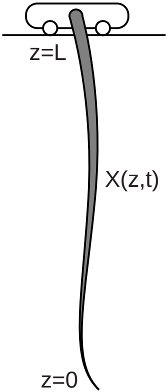
reads

$$X(z,t) = \frac{1}{2\pi}\int_0^{2\pi} y(t + 2\sqrt{z/g}\sin\theta)\, d\theta$$

There is a one to one correspondence between the (smooth) solutions of the PDE and the (smooth) functions $t \mapsto y(t)$.

Heavy chain with a variable section

$$\begin{cases} \dfrac{\tau'(z)}{g} \quad \dfrac{\partial^2 X}{\partial t^2} = \dfrac{\partial}{\partial z}\left(\tau(z)\dfrac{\partial X}{\partial z}\right) \\ \\ \qquad X(L,t) = u(t) \end{cases}$$



The general solution of

$$\begin{cases} \dfrac{\tau'(z)}{g} \quad \dfrac{\partial^2 X}{\partial t^2} = \dfrac{\partial}{\partial z}\left(\tau(z)\dfrac{\partial X}{\partial z}\right) \\ \\ X(L,t) = u(t) \end{cases}$$

where $\tau(z) \geq 0$ is the tension in the rope, can be parameterized by an arbitrary time function $y(t)$, the position of the free end of the system $y = X(0,t)$, via delay and advance operators with compact support.

Sketch of the proof.

Main difficulty: $\tau(0) = 0$. The bounded solution $B(z, s)$ of

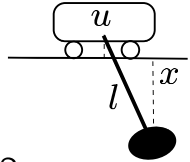
$$\frac{\partial}{\partial z}\left(\tau(z)\frac{\partial X}{\partial z}\right) = \frac{s^2\tau'(z)}{g}\, X$$

is an entire function of s, is of exponential type and

$$\mathbb{R} \ni \omega \mapsto B(z, \imath\omega)$$

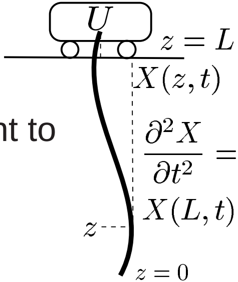
is L^2 modulo some J_0. By the Paley-Wiener theorem $B(z, s)$ can be described via

$$\int_a^b K(z, \zeta) \exp(s\zeta)\, d\zeta.$$



$$\frac{d^2x}{dt^2} = \frac{g}{l}(u - x)$$

is equivalent to



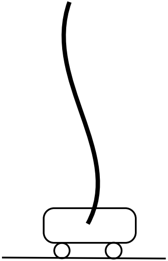
$$\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right)$$

$$X(L,t) = U$$

The following maps exchange the trajectories:

$$\begin{cases} x(t) = X(0,t) \\ u(t) = \frac{\partial^2 X}{\partial t^2}(0,t) \end{cases} \quad \begin{cases} X(z,t) = \frac{1}{2\pi}\int_0^{2\pi} x\left(t - 2\sqrt{z/g}\sin\zeta\right) d\zeta \\ U(t) = \frac{1}{2\pi}\int_0^{2\pi} x\left(t - 2\sqrt{L/g}\sin\zeta\right) d\zeta \end{cases}$$

The Indian rope.



$$
\begin{aligned}
\frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right) + \frac{\partial^2 X}{\partial t^2} &= 0 \\
X(L,t) &= U(t)
\end{aligned}
$$

The equation becomes elliptic and the Cauchy problem is not well posed in the sense of Hadamard. Nevertheless formulas are still valid with a complex time and y holomorphic

$$X(z,t) = \frac{1}{2\pi} \int_0^{2\pi} y\left(t - (2\sqrt{z/g}\ \sin\zeta)\ \sqrt{-1}\right)\ d\zeta.$$

A computation due to Holmgren[6]

Take the 1D-heat equation, $\frac{\partial\theta}{\partial t}(x,t) = \frac{\partial^2\theta}{\partial x^2}(x,t)$ for $x \in [0,1]$ and set, formally, $\theta = \sum_{i=0}^{\infty} a_i(t)\frac{x^i}{i!}$. Since,

$$\frac{\partial\theta}{\partial t} = \sum_{i=0}^{\infty} \frac{da_i}{dt}\left(\frac{x^i}{i!}\right), \quad \frac{\partial^2\theta}{\partial x^2} = \sum_{i=0}^{\infty} a_{i+2}\left(\frac{x^i}{i!}\right)$$

the heat equation $\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2}$ reads $\frac{d}{dt}a_i = a_{i+2}$ and thus

$$a_{2i+1} = a_1^{(i)}, \quad a_{2i} = a_0^{(i)}$$

With two arbitrary smooth time-functions $f(t)$ and $g(t)$, playing the role of a_0 and a_1, the general solution reads:

$$\theta(x,t) = \sum_{i=0}^{\infty} f^{(i)}(t)\left(\frac{x^{2i}}{(2i)!}\right) + g^{(i)}(t)\left(\frac{x^{2i+1}}{(2i+1)!}\right).$$

Convergence issues ?

[6] E. Holmgren, Sur l'équation de la propagation de la chaleur. Arkiv für Math. Astr. Physik, t. 4, (1908), p. 1-4

Gevrey functions[7]

- A C^∞-function $[0,T] \ni t \mapsto f(t)$ is of Gevrey-order α when,

  $$\exists\, M, A > 0, \quad \forall t \in [0,T], \forall i \geq 0, \quad |f^{(i)}(t)| \leq M A^i \Gamma(1+\alpha i)$$

  where Γ is the gamma function with $n! = \Gamma(n+1)$, $\forall n \in \mathbb{N}$.
- Analytic functions correspond to Gevrey-order ≤ 1.
- When $\alpha > 1$, the set of C^∞-functions with Gevrey-order α contains non-zero functions with compact supports. Prototype of such functions:

$$t \mapsto f(t) = \begin{cases} \exp\left(-\left(\frac{1}{t(1-t)}\right)^{\frac{1}{\alpha-1}}\right) & \text{if } t \in]0,1[\\ 0 & \text{otherwise.} \end{cases}$$

[7] M. Gevrey: La nature analytique des solutions des équations aux dérivées partielles, Ann. Sc. Ecole Norm. Sup., vol.25, pp:129–190, 1918.

Gevrey functions and exponential decay[8]

- Take, in the complex plane, the open bounded sector $\mathcal{S}$ those vertex is the origin. Assume that f is analytic on $\mathcal{S}$ and admits an exponential decay of order $\sigma > 0$ and type A in $\mathcal{S}$:

$$\exists C, \rho > 0, \quad \forall z \in \mathcal{S}, \quad |f(z)| \leq C|z|^{\rho} \exp\left(\frac{-1}{A|z|^{\sigma}}\right)$$

  Then in any closed sub-sector $\tilde{\mathcal{S}}$ of $\mathcal{S}$ with origin as vertex, exists $M > 0$ such that

$$\forall z \in \tilde{\mathcal{S}}/\{0\}, \quad |f^{(i)}(z)| \leq MA^{i}\,\Gamma\left(1 + i\left(\frac{1}{\sigma} + 1\right)\right)$$

- Rule of thumb: if a piece-wise analytic f admits an exponential decay of order σ then it is of Gevrey-order $\alpha = \frac{1}{\sigma} + 1$.

[8] J.P. Ramis: Dévissage Gevrey. Astérisque, vol:59-60, pp:173–204, 1978. See also J.P. Ramis: Séries Divergentes et Théories Asymptotiques; SMF, Panoramas et Synthèses, 1993.

Gevrey space and ultra-distributions[9]

Denote by $\mathcal{D}_\alpha$ the set of functions $\mathbb{R} \mapsto \mathbb{R}$ of order $\alpha > 1$ and with compact supports. As for the class of C^∞ functions, most of the usual manipulations remain in $\mathcal{D}_\alpha$:

- $\mathcal{D}_\alpha$ is stable by addition, multiplication, derivation, integration, ...
- if $f \in \mathcal{D}_\alpha$ and F is an analytic function on the image of f, then $F(f)$ remains in $\mathcal{D}_\alpha$.
- if $f \in \mathcal{D}_\alpha$ and $F \in L^1_{loc}(\mathbb{R})$ then the convolution $f * F$ is of Gevrey-order α on any compact interval.

As for the construction of $\mathcal{D}'$, the space of distributions (the dual of $\mathcal{D}$ the space of C^∞ functions of compact supports), one can construct $\mathcal{D}'_\alpha \supset \mathcal{D}'$, a space of ultra-distributions, the dual of $\mathcal{D}_\alpha \subset \mathcal{D}$.

[9] See, e.g., I.M. Guelfand and G.E. Chilov: Les Distributions, tomes 2 et 3. Dunod, Paris,1964.

Symbolic computations: $s := d/dt$, $s \in \mathbb{C}$

The general solution of $\theta'' = s\theta$ reads ($' := d/dx$)

$$\theta = \cosh(x\sqrt{s})\ f(s) + \frac{\sinh(x\sqrt{s})}{\sqrt{s}}\ g(s)$$

where $f(s)$ and $g(s)$ are the two constants of integration. Since cosh and sinh gather the even and odd terms of the series defining exp, we have

$$\cosh(x\sqrt{s}) = \sum_{i\geq 0} s^i \frac{x^{2i}}{(2i)!}, \quad \frac{\sinh(x\sqrt{s})}{\sqrt{s}} = \sum_{i\geq 0} s^i \frac{x^{2i+1}}{(2i+1)!}$$

and we recognize $\theta = \sum_{i=0}^{\infty} f^{(i)}(t)\left(\frac{x^{2i}}{(2i)!}\right) + g^{(i)}(t)\left(\frac{x^{2i+1}}{(2i+1)!}\right)$.

For each x, the operators $\cosh(x\sqrt{s})$ and $\sinh(x\sqrt{s})/\sqrt{s}$ are ultra-distributions of $\mathcal{D}'_{2^-}$:

$$\sum_{i\geq 0} \frac{(-1)^i x^{2i}}{(2i)!}\delta^{(i)}(t), \quad \sum_{i\geq 0} \frac{(-1)^i x^{2i+1}}{(2i+1)!}\delta^{(i)}(t)$$

with δ, the Dirac distribution.

Entire functions of $s = d/dt$ as ultra-distributions

- $\mathbb{C} \ni s \mapsto P(s) = \sum_{i \geq 0} a_i s^i$ is an entire function when the radius of convergence is infinite.
- If its order at infinity is $\sigma > 0$ and its type is finite, i.e., $\exists M, K > 0$ such that $\forall s \in \mathbb{C}$, $|P(s)| \leq M \exp(K|s|^\sigma)$, then

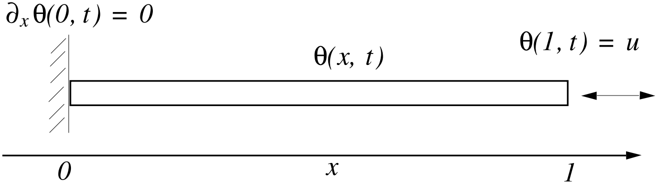
$$\exists A, B > 0 \mid \forall i \geq 0, \quad |a_i| \leq A \frac{B^i}{\Gamma(i/\sigma + 1)}.$$

  $\cosh(\sqrt{s})$ and $\sinh(\sqrt{s})/\sqrt{s}$ are entire functions of order $\sigma = 1/2$ and of type 1.
- Take $P(s)$ of order $\sigma < 1$ with $s = d/dt$. Then $P \in \mathcal{D}'_{\frac{1}{\sigma}-}$: $P(s)f(s)$ corresponds, in the time domain, to absolutely convergent series

$$P(s)y(s) \equiv \sum_{i=0}^{\infty} a_i\, f^{(i)}(t)$$

  when $t \mapsto f(t)$ is a C^∞-function of Gevrey-order $\alpha < 1/\sigma$.

Motion planning for the 1D heat equation



The data are:

1. the model relating the control input $u(t)$ to the state, $(\theta(x,t))_{x\in[0,1]}$:

$$\frac{\partial\theta}{\partial t}(x,t) = \frac{\partial^2\theta}{\partial x^2}(x,t), \quad x \in [0,1]$$
$$\frac{\partial\theta}{\partial x}(0,t) = 0 \qquad \theta(1,t) = u(t).$$

2. A transition time $T > 0$, the initial (resp. final) state: $[0,1] \ni x \mapsto p(x)$ (resp. $q(x)$)

The goal is to find the open-loop control $[0,T] \ni t \mapsto u(t)$ steering $\theta(x,t)$ from the initial profile $\theta(x,0) = p(x)$ to the final profile $\theta(x,T) = q(x)$.

Series solutions

Set, formally

$$\theta = \sum_{i=0}^{\infty} a_i(t)\frac{x^i}{i!}, \quad \frac{\partial \theta}{\partial t} = \sum_{i=0}^{\infty} \frac{da_i}{dt}\left(\frac{x^i}{i!}\right), \quad \frac{\partial^2 \theta}{\partial x^2} = \sum_{i=0}^{\infty} a_{i+2}\left(\frac{x^i}{i!}\right)$$

and $\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}$ reads $\frac{d}{dt}a_i = a_{i+2}$. Since $a_1 = \frac{\partial \theta}{\partial x}(0,t) = 0$ and $a_0 = \theta(0,t)$ we have

$$a_{2i+1} = 0, \quad a_{2i} = a_0^{(i)}$$

Set $y := a_0 = \theta(0,t)$ we have, in the time domain,

$$\theta(x,t) = \sum_{i=0}^{\infty}\left(\frac{x^{2i}}{(2i)!}\right) y^{(i)}(t), \quad u(t) = \sum_{i=0}^{\infty}\left(\frac{1}{(2i)!}\right) y^{(i)}(t)$$

that also reads in the Laplace domain ($s = d/dt$):

$$\theta(x,s) = \cosh(x\sqrt{s})\, y(s), \quad u(s) = \cosh(\sqrt{s})y(s).$$

An explicit parameterization of trajectories

For any C^∞-function $y(t)$ of Gevrey-order $\alpha < 2$, the time function

$$u(t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!}$$

is well defined and smooth. The (x, t)-function

$$\theta(x,t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!}\ x^{2i}$$

is also well defined (entire versus x and smooth versus t). More over for all t and $x \in [0,1]$, we have, whatever $t \mapsto y(t)$ is,

$$\frac{\partial \theta}{\partial t}(x,t) = \frac{\partial^2 \theta}{\partial x^2}(x,t), \quad \frac{\partial \theta}{\partial x}(0,t) = 0, \quad \theta(1,t) = u(t)$$

An infinite dimensional analogue of differential flatness.[10]

[10] Fliess et al: Flatness and defect of nonlinear systems: introductory theory and examples, International Journal of Control. vol.61, pp:1327–1361. 1995.

Motion planning of the heat equation[11]

Take $\sum_{i\geq 0} a_i \frac{\xi^i}{i!}$ and $\sum_{i\geq 0} b_i \frac{\xi^i}{i!}$ entire functions of ξ. With $\sigma > 1$

$$y(t) = \left(\sum_{i\geq 0} a_i \frac{t^i}{i!}\right)\left(\frac{e^{\frac{-T^\sigma}{(T-t)^\sigma}}}{e^{\frac{-T^\sigma}{t^\sigma}} + e^{\frac{-T^\sigma}{(T-t)^\sigma}}}\right) + \left(\sum_{i\geq 0} b_i \frac{t^i}{i!}\right)\left(\frac{e^{\frac{-T^\sigma}{t^\sigma}}}{e^{\frac{-T^\sigma}{t^\sigma}} + e^{\frac{-T^\sigma}{(T-t)^\sigma}}}\right)$$

the series

$$\theta(x,t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!} x^{2i}, \quad u(t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!}.$$

are convergent and provide a trajectory from

$$\theta(x,0) = \sum_{i\geq 0} a_i \frac{x^{2i}}{(2i)!} \ \text{ to } \ \theta(x,T) = \sum_{i\geq 0} b_i \frac{x^{2i}}{(2i)!}$$

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[11] B. Laroche, Ph. Martin, P. R.: Motion planning for the heat equation. Int. Journal of Robust and Nonlinear Control. Vol.10, pp:629–643, 2000.

Real-time motion planning for the heat equation

Take $\sigma > 1$ and $\epsilon > 0$. Consider the positive function

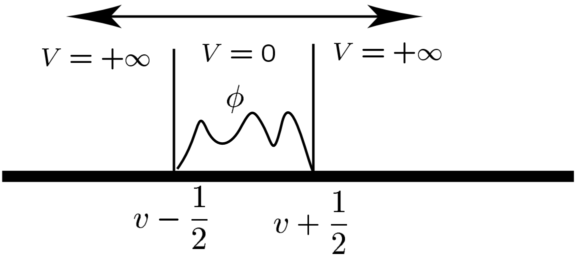
$$\phi_\epsilon(t) = \frac{\exp\left(\frac{-\epsilon^{2\sigma}}{(-t(t+\epsilon))^\sigma}\right)}{A_\epsilon} \quad \text{for} \quad t \in [-\epsilon, 0]$$

prolonged by 0 outside $[-\epsilon, 0]$ and where the normalization constant $A_\epsilon > 0$ is such that $\int \phi_\epsilon = 1$.
For any L^1_{loc} signal $t \mapsto Y(t)$, set $y_r = \phi_\epsilon * Y$: its order $1 + 1/\sigma$ is less than 2. Then $\theta_r = \cosh(x\sqrt{s})y_r$ reads

$$\theta_r(x,t) = \Phi_{x,\epsilon} * Y(t), \quad u_r(t) = \Phi_{1,\epsilon} * Y(t),$$

where for each x, $\Phi_{x,\epsilon} = \cosh(x\sqrt{s})\phi_\epsilon$ is a smooth time function with support contained in $[-\epsilon, 0]$. Since $u_r(t)$ and the profile $\theta_r(\cdot, t)$ depend only on the values of Y on $[t-\epsilon, t]$, such computations are well adapted to real-time generation of reference trajectories $t \mapsto (\theta_r, u_r)$ (see matlab code `heat.m`).

Quantum particle inside a moving box[12]



Schrödinger equation in a Galilean frame:

$$\imath \frac{\partial \phi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \phi}{\partial z^2}, \quad z \in [v - \frac{1}{2}, v + \frac{1}{2}],$$

$$\phi(v - \frac{1}{2}, t) = \phi(v + \frac{1}{2}, t) = 0$$

[12] P.R.: Control of a quantum particle in a moving potential well. IFAC 2nd Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control, 2003. See, for the proof of nonlinear controllability, K. Beauchard and J.-M. Coron: Controllability of a quantum particle in a moving potential well; J. of Functional Analysis, vol.232, pp:328–389, 2006.

Particle in a moving box of position v

- In a Galilean frame

$$\imath \frac{\partial \phi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \phi}{\partial z^2}, \quad z \in [v - \frac{1}{2}, v + \frac{1}{2}],$$
$$\phi(v - \frac{1}{2}, t) = \phi(v + \frac{1}{2}, t) = 0$$

  where v is the position of the box and z is an absolute position.

- In the box frame $x = z - v$:

$$\imath \frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} + \ddot{v} x \psi, \quad x \in [-\frac{1}{2}, \frac{1}{2}],$$
$$\psi(-\frac{1}{2}, t) = \psi(\frac{1}{2}, t) = 0$$

Tangent linearization around state $\bar{\psi}$ of energy $\bar{\omega}$

With[13] $-\frac{1}{2}\frac{\partial^2\bar{\psi}}{\partial x^2} = \bar{\omega}\bar{\psi}$, $\bar{\psi}(-\frac{1}{2}) = \bar{\psi}(\frac{1}{2}) = 0$ and with

$$\psi(x,t) = \exp(-\imath\bar{\omega}t)(\bar{\psi}(x) + \Psi(x,t))$$

Ψ satisfies

$$\imath\frac{\partial\Psi}{\partial t} + \bar{\omega}\Psi = -\frac{1}{2}\frac{\partial^2\Psi}{\partial x^2} + \ddot{v}x(\bar{\psi} + \Psi)$$
$$0 = \Psi(-\frac{1}{2}, t) = \Psi(\frac{1}{2}, t).$$

Assume Ψ and $\ddot{v}$ small and neglecte the second order term $\ddot{v}x\Psi$:

$$\imath\frac{\partial\Psi}{\partial t} + \bar{\omega}\Psi = -\frac{1}{2}\frac{\partial^2\Psi}{\partial x^2} + \ddot{v}x\bar{\psi}, \quad \Psi(-\frac{1}{2}, t) = \Psi(\frac{1}{2}, t) = 0.$$

[13] Remember that $\int_{-1/2}^{1/2}\bar{\psi}^2(x)dx = 1$.

Operational computations $s = d/dt$

The general solution of ($'$ stands for d/dx)

$$(\imath s + \bar{\omega})\Psi = -\frac{1}{2}\Psi'' + s^2 v x \bar{\psi}$$

is

$$\Psi = A(s,x)a(s) + B(s,x)b(s) + C(s,x)v(s)$$

where

$$A(s,x) = \cos\left(x\sqrt{2\imath s + 2\bar{\omega}}\right)$$

$$B(s,x) = \frac{\sin\left(x\sqrt{2\imath s + 2\bar{\omega}}\right)}{\sqrt{2\imath s + 2\bar{\omega}}}$$

$$C(s,x) = (-\imath s x \bar{\psi}(x) + \bar{\psi}'(x)).$$

Case $x \mapsto \bar{\phi}(x)$ even

The boundary conditions imply

$$A(s,1/2)a(s) = 0, \quad B(s,1/2)b(s) = -\psi'(1/2)v(s).$$

$a(s)$ is a torsion element: the system is not controllable.
Nevertheless, for steady-state controllability, we have

$$b(s) = -\bar{\psi}'(1/2)\frac{\sin\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right)}{\sqrt{-2\imath s + 2\bar{\omega}}}\ y(s)$$

$$v(s) = \frac{\sin\left(\frac{1}{2}\sqrt{2\imath s + 2\bar{\omega}}\right)}{\sqrt{2\imath s + 2\bar{\omega}}}\frac{\sin\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right)}{\sqrt{-2\imath s + 2\bar{\omega}}}y(s)$$

$$\Psi(s,x) = B(s,x)b(s) + C(s,x)v(s)$$

Series and convergence

$$v(s) = \frac{\sin\left(\frac{1}{2}\sqrt{2\imath s + 2\bar{\omega}}\right)}{\sqrt{2\imath s + 2\bar{\omega}}} \frac{\sin\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right)}{\sqrt{-2\imath s + 2\bar{\omega}}} y(s) = F(s)y(s)$$

where the entire function $s \mapsto F(s)$ is of order $1/2$,

$$\exists K, M > 0, \forall s \in \mathbb{C}, \quad |F(s)| \leq K \exp(M|s|^{1/2}).$$

Set $F(s) = \sum_{n \geq 0} a_n s^n$ where $|a_n| \leq K^n / \Gamma(1 + 2n)$ with $K > 0$ independent of n. Then $F(s)y(s)$ corresponds, in the time domain, to

$$\sum_{n \geq 0} a_n y^{(n)}(t)$$

that is convergent when $t \mapsto y(t)$ is C^∞ of Gevrey-order $\alpha < 2$.

Steady state controllability

Steering from $\Psi = 0$, $v = 0$ at time $t = 0$, to $\Psi = 0$, $v = D$ at $t = T$ is possible with the following C^∞-function of Gevrey-order $\sigma + 1$:

$$[0, T] \ni t \mapsto y(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ \bar{D} \dfrac{\exp\left(-\left(\frac{T}{t}\right)^{\frac{1}{\sigma}}\right)}{\exp\left(-\left(\frac{T}{t}\right)^{\frac{1}{\sigma}}\right) + \exp\left(-\left(\frac{T}{T-t}\right)^{\frac{1}{\sigma}}\right)} & \text{for } 0 < t < T \\ \bar{D} & \text{for } t \geq T \end{cases}$$

with $\bar{D} = \frac{2\bar{\omega}D}{\sin^2(\sqrt{\bar{\omega}/2})}$. The fact that this C^∞-function is of Gevrey-order $\sigma + 1$ results from its exponential decay of order $1/\sigma$ around 0 and T.

Practical computations via Cauchy formula

Using the "magic" Cauchy formula

$$y^{(n)}(t) = \frac{\Gamma(n+1)}{2\imath\pi} \oint_\gamma \frac{y(t+\xi)}{\xi^{n+1}}\, d\xi$$

where γ is a closed path around zero, $\sum_{n\geq 0} a_n y^{(n)}(t)$ becomes

$$\sum_{n\geq 0} a_n \frac{\Gamma(n+1)}{2\imath\pi} \oint_\gamma \frac{y(t+\xi)}{\xi^{n+1}}\, d\xi = \frac{1}{2\imath\pi} \oint_\gamma \left(\sum_{n\geq 0} a_n \frac{\Gamma(n+1)}{\xi^{n+1}} \right) y(t+\xi)\, d\xi.$$

But

$$\sum_{n\geq 0} a_n \frac{\Gamma(n+1)}{\xi^{n+1}} = \int_{D_\delta} F(s) \exp(-s\xi) ds = B_1(F)(\xi)$$

is the Borel/Laplace transform of F in direction $\delta \in [0, 2\pi]$.

Practical computations via Cauchy formula (end)

(matlab code `Qbox.m`)

In the time domain $F(s)y(s)$ corresponds to

$$\frac{1}{2\imath\pi}\oint_\gamma B_1(F)(\xi)y(t+\xi)\,d\xi$$

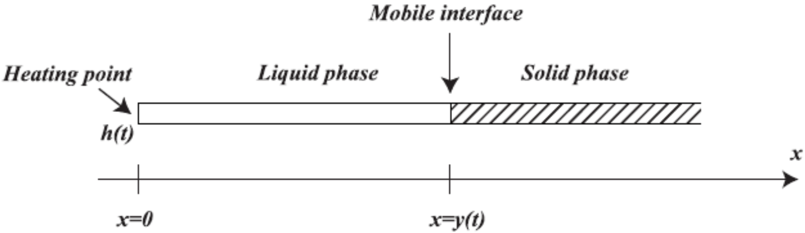
where γ is a closed path around zero. Such integral representation is very useful when y is defined by convolution with a real signal Y,

$$y(\zeta)=\frac{1}{\varepsilon\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(-(\zeta-t)^2/2\varepsilon^2)Y(t)\,dt$$

where $\mathbb{R}\ni t\mapsto Y(t)\in\mathbb{R}$ is any measurable and bounded function. Approximate motion planning with:

$$v(t)=\int_{-\infty}^{+\infty}\left[\frac{1}{\imath\varepsilon(2\pi)^{\frac{3}{2}}}\oint_\gamma B_1(F)(\xi)\exp(-(\xi-\tau)^2/2\varepsilon^2)\,d\xi\right]Y(t-\tau)\,d\tau.$$

A free-boundary Stefan problem[14]



$$\frac{\partial\theta}{\partial t}(x,t) = \frac{\partial^2\theta}{\partial x^2}(x,t) - \nu\frac{\partial\theta}{\partial x}(x,t) - \rho\theta^2(x,t), \quad x \in [0, y(t)]$$

$$\theta(0,t) = u(t), \quad \theta(y(t),t) = 0$$

$$\frac{\partial\theta}{\partial x}(y(t),t) = -\tfrac{d}{dt}y(t)$$

with $\nu, \rho \geq 0$ parameters.

[14] W. Dunbar, N. Petit, P. R., Ph. Martin. Motion planning for a non-linear Stefan equation. ESAIM: Control, Optimisation and Calculus of Variations, 9:275–296, 2003.

Series solutions

- Set $\theta(x,t) = \sum_{i=0}^{\infty} a_i(t)\frac{(x-y(t))^i}{i!}$ in

  $$\frac{\partial\theta}{\partial t}(x,t) = \frac{\partial^2\theta}{\partial x^2}(x,t) - \nu\frac{\partial\theta}{\partial x}(x,t) - \rho\theta^2(x,t), \quad x \in [0, y(t)]$$
  $$\theta(0,t) = u(t), \quad \theta(y(t),t) = 0, \quad \frac{\partial\theta}{\partial x}(y(t),t) = -\tfrac{d}{dt}y(t)$$

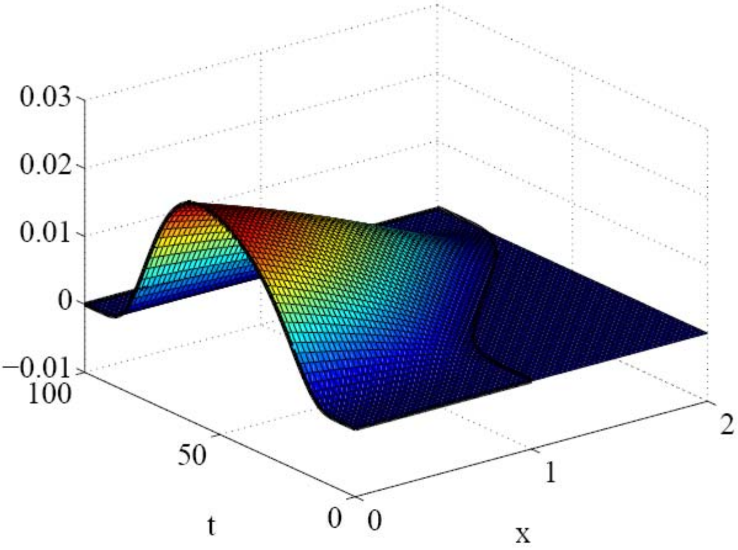
  Then $\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2}$ yields

  $$a_{i+2} = \tfrac{d}{dt}a_i - a_{i-1}\tfrac{d}{dt}y + \nu a_{i+1} + \rho\sum_{k=0}^{i}\begin{pmatrix} i \\ k \end{pmatrix} a_{i-k}a_k$$

  and the boundary conditions: $a_0 = 0$ and $a_1 = -\frac{d}{dt}y$.
- The series defining θ admits a strictly positive radius of convergence as soon as y is of Gevrey-order α strictly less than 2.

Growth of the liquide zone with $\theta \geq 0$

$\nu = 0.5$, $\rho = 1.5$, y goes from 1 to 2.



Conclusion for PDE

- For other 1D PDE of engineering interest with motion planning see the book of J. Rudolph: Flatness Based Control of Distributed Parameter Systems (Shaker-Germany, 2003)
- For tracking and feedback stabilization on linear 1D diffusion and wave equations, see the book of M. Krstić and A. Smyshlyaev : Boundary Control of PDEs: a Course on Backstepping Designs (SIAM, 2008).
- Open questions:
  - Combine divergent series and smallest-term summation (see the PhD of Th. Meurer: Feedforward and Feedback Tracking Control of Diffusion-Convection-Reaction Systems using Summability Methods (Stuttgart, 2005)).
  - 2D heat equation with a scalar control $u(t)$: with modal decomposition and symbolic computations, we get $u(s) = P(s)y(s)$ with $P(s)$ an entire function (coding the spectrum) of order 1 but infinite type $|P(s)| \leq M \exp(K|s|\log(|s|))$. It yields divergence series for any C^∞ function $y \neq 0$ with compact support.

$u(s) = P(s)y(s)$ for 1D and 2D heat equations

- 1D heat equation: eigenvalue asymptotics $\lambda_n \sim -n^2$:

$$\text{Prototype:} \quad P(s) = \prod_{n=1}^{+\infty} \left(1 - \frac{s}{n^2}\right) = \frac{\sinh(\pi\sqrt{s})}{\pi\sqrt{s}}$$

  entire function of order $1/2$.

- 2D heat equation in a domain Ω with a single scalar control $u(t)$ on the boundary $\partial\Omega_1$ ($\partial\Omega = \partial\Omega_1 \bigcup \partial\Omega_2$):

$$\frac{\partial\theta}{\partial t} = \Delta\theta \text{ on } \Omega, \quad \theta = u(t) \text{ on } \partial\Omega_1, \quad \frac{\partial\theta}{\partial n} = 0 \text{ on } \partial\Omega_2$$

  Eigenvalue asymptotics $\lambda_n \sim -n$

$$\text{Prototype:} \quad P(s) = \prod_{n=1}^{+\infty} \left(1 + \frac{s}{n}\right) \exp(-s/n) = \frac{\exp(-\gamma s)}{s\Gamma(s)}$$

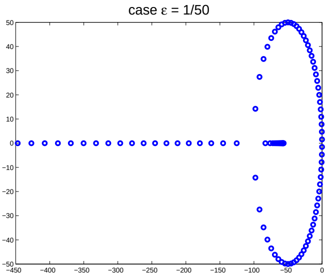
  entire function of order 1 but of infinite type[15]

[15] For the links between the distributions of the zeros and the order at infinity of entire functions see the book of B.Ja Levin: Distribution of Zeros of Entire Functions; AMS, 1972.

Symbolic computations with Laplace variable $s = \frac{d}{dt}$

- Wave 1D: $u = \cosh(s)y$. General case is similar: $u = P(s)y$ where the zeros of P are the eigen-values $\pm i\omega_n$ with asymptotic $\omega_n \sim n$; $P(s)$ entire function of order 1 and finite type (in time domain: advance/delay operator with compact support).
- Diffusion 1D: $u = \cosh\left(\sqrt{s}\right) u$. General case is similar: $u = P(s)y$ where the zeros of P are the eigen-values $-\lambda_n$ with asymptotic $\lambda_n \sim n^2$; $P(s)$ entire function of order $1/2$ (in time domain: ultra-distribution made of an infinite sum of Dirac derivatives applied on Gevrey functions with compact support of order < 2).
- Wave 2D: since $\omega_n \sim \sqrt{n}$, P entire with order 2 but infinite type; prototype $P(s) = \prod_{n=1}^{+\infty}\left(1 - \frac{s^2}{n}\right)\exp(s^2/n) = \frac{-\exp(\gamma s^2)}{s^2\Gamma(-s^2)}$.
  Diffusion 2D: since $\lambda_n \sim -n$, P entire with order 1 but infinite type; prototype $P(s) = \prod_{n=1}^{+\infty}\left(1 + \frac{s}{n}\right)\exp(-s/n) = \frac{\exp(-\gamma s)}{s\Gamma(s)}$.
  Open Question: interpretation of $P(s)$ in time domain as operator on a set of time functions $y(t)$...

Wave 1D with internal damping



$$\frac{\partial^2 H}{\partial t^2} = \frac{\partial^2 H}{\partial x^2} + \epsilon \frac{\partial^3 H}{\partial x^2 \partial t}$$

$$H(0,t) = 0, \quad H(1,t) = u(t)$$

where the eigenvalues are the zeros of

$$P(s) = \cosh\left(\frac{s}{\sqrt{\epsilon s + 1}}\right).$$

Approximate controllability depends on the functional space chosen to have a well-posed Cauchy problem[16]

[16] Rosier-R, CAO'06. 13th IFAC Workshop on Control Applications of Optimisation. 2006.

Propagation of electro-magnetic wave in a partially transparent medium:

$$\frac{\partial^2}{\partial t^2}(E + D) = c^2 \frac{\partial^2}{\partial x^2} E, \quad \frac{\partial^2 D}{\partial t^2} = \omega_0^2(\epsilon E - D)$$

where ω_0 is associated to an adsorption ray and ϵ is the coupling constant between medium of polarization P and travelling field E

- The eigenvalues rely on the analytic function ($s = d/dt$ Laplace variable, L length)

$$Q^{\pm}(s, L) = \exp\left(\pm \frac{Ls}{c}\sqrt{\left(1 + \frac{\epsilon s^2}{\omega_0^2 + s^2}\right)}\right)$$

  The essential singularity in $s = \pm\imath\omega_0$ yields an accumulation of eigenvalues around $\pm\omega_0$.
- Few works on this kind of PDE with spectrum that accumulates at finite distance.

The flatness characterization problem

$\frac{d}{dt}x = f(x,u)$ is said r-flat if exists a flat output y only function of $(x, u, \dot{u}, \ldots, u^{(r-1)})$; 0-flat means $y = h(x)$.
Example:

$$x_1^{(\alpha_1)} = u_1, \quad x_2^{(\alpha_2)} = u_2, \quad \frac{d}{dt}x_3 = u_1 u_2$$

is $[r := \min(\alpha_1, \alpha_2) - 1]$-flat with

$$y_1 = x_3 + \sum_{i=1}^{\alpha_1} (-1)^i x_1^{(\alpha_1 - i)} u_2^{(i-1)}, \quad y_2 = x_2,$$

Conjecture: there is no flat output depending on derivatives of u of order less than $r - 1$.
The main difficulty: for $\frac{d}{dt}x = f(x,u)$ with $y = h(x, u, \ldots, u^{(p)})$ as flat output, we do not know an upper-bound on p with respect to $n = \dim(x)$, $m = \dim(u)$, ...

Systems linearizable by static feedback

- A system which is linearizable by static feedback and coordinate change is flat: geometric necessary and sufficient conditions by Jakubczyk and Respondek (1980) (see also Hunt et al. (1983)).
- When there is only one control input, flatness reduces to static feedback linearizability (Charlet et al. (1989))

Affine control systems of small co-dimension

- Affine systems of codimension 1.

$$\frac{d}{dt}x = f_0(x) + \sum_{j=1}^{n-1} u_j g_j(x), \qquad x \in \mathbb{R}^n,$$

  is 0-flat as soon as it is controllable, Charlet et al. (1989)
- Affine systems with 2 inputs and 4 states. Necessary and sufficient conditions for 1-flatness (Pomet (1997)) give a good idea of the complexity of checking r-flatness even for r small.

Driftless systems with two controls.

- $$\frac{d}{dt}x = f_1(x)u_1 + f_2(x)u_2$$

  is flat if and only if the generic rank of E_k is equal to $k+2$ for $k = 0, \ldots, n-2$ where

  $$\begin{aligned} E_0 &:= \text{span}\{f_1, f_2\} \\ E_{k+1} &:= \text{span}\{E_k, [E_k, E_k]\}, \quad k \geq 0. \end{aligned}$$

  Proof: Martin and R. (1994) with a theorem of Cartan (1916) on Pfaffian systems.

- A flat two-input driftless system satisfying some additional regularity conditions (Murray (1994)) can be put into the *chained system*

  $$\frac{d}{dt}x_1 = u_1, \quad \frac{d}{dt}x_2 = u_2$$
  $$\frac{d}{dt}x_3 = x_2 u_1, \quad \ldots, \quad \frac{d}{dt}x_n = x_{n-1}u_1.$$

Codimension 2 driftless systems

- $$\frac{d}{dt}x = \sum_{i=1}^{n-2} u_i f_i(x), \quad x \in \mathbb{R}^n$$
  is flat as soon as it is controllable (Martin and R. (1995))
- Tools: exterior differential systems.
- Many nonholonomic control systems are flat.

The ruled-manifold criterion (R. (1995))

- Assume $\dot{x} = f(x, u)$ is flat. The projection on the p-space of the submanifold $p = f(x, u)$, where x is considered as a parameter, is a ruled submanifold for all x.
- Otherwise stated: eliminating u from $\dot{x} = f(x, u)$ yields a set of equations $F(x, \dot{x}) = 0$: for all (x, p) such that $F(x, p) = 0$, there exists $a \in \mathbb{R}^n$, $a \neq 0$ such that

$$\forall \lambda \in \mathbb{R}, \quad F(x, p + \lambda a) = 0.$$

- Proof elementary and derived from Hilbert (1912).
- Restricted version proposed by Sluis (1993).

Why static linearization coincides with flatness for single input systems ? Because a ruled-manifold of dimension 1 is just a straight line.

Proving that a multi-input system is not flat

$$\tfrac{d}{dt}x_1 = u_1, \quad \tfrac{d}{dt}x_2 = u_2, \quad \tfrac{d}{dt}x_3 = (u_1)^2 + (u_2)^3$$

is not flat The submanifold $p_3 = p_1^2 + p_2^3$ is not ruled: there is no $a \in \mathbb{R}^3$, $a \neq 0$, such that

$$\forall \lambda \in \mathbb{R}, p_3 + \lambda a_3 = (p_1 + \lambda a_1)^2 + (p_2 + \lambda a_2)^3.$$

Indeed, the cubic term in λ implies $a_2 = 0$, the quadratic term $a_1 = 0$ hence $a_3 = 0$.

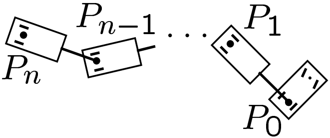
The system $\frac{d}{dt}x_3 = \left(\frac{d}{dt}x_1\right)^2 + \left(\frac{d}{dt}x_2\right)^2$ does not define a ruled submanifold of $\mathbb{R}^3$: it is not flat in $\mathbb{R}$. But it defines a ruled submanifold in $\mathbb{C}^3$: in fact it is flat in $\mathbb{C}$, with the flat output

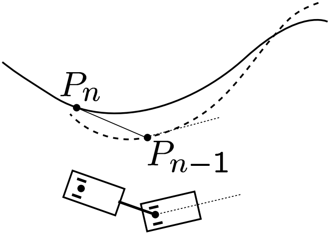
$$y_1 = x_3 - (\dot{x}_1 - \dot{x}_2\sqrt{-1})(x_1 + x_2\sqrt{-1})$$
$$y_2 = x_1 + x_2\sqrt{-1}.$$

JBP result on equivalent systems SIAM JCO (2010)

- Take two explicit analytic systems $\frac{d}{dt}x = f(x, u)$ and $\frac{d}{dt}z = g(z, v)$ with $\dim u = \dim v$ but not necessarily $\dim x$ equals to $\dim z$. Assume that they are equivalent via a possible dynamic state feedback. Then we have
  - if $\dim x < \dim z$ then $\frac{d}{dt}x = f(x, u)$ is ruled.
  - if $\dim z < \dim x$ then $\frac{d}{dt}z = g(z, v)$ is ruled.
  - if $\dim x = \dim z$ either they are equivalent by static feedback or they are both ruled.
- The system $\frac{d}{dt}x = f(x, u)$ (resp. $\frac{d}{dt}z = g(z, v)$ is said ruled when after elimination of u (resp. v), the implicit system $F(x\frac{d}{dt}x) = 0$ (resp. $G(x, \frac{d}{dt}x) = 0$) is ruled in the sense of the ruled manifold criterion explained here above.

Geometric construction: $SE(2)$ invariance





$$P_{n-1} = P_n + d_n \frac{dP_n}{ds_n}$$

- Invariance versus actions of the group $SE(2)$.
- Flat outputs are not unique: $(\xi = x_n, \zeta = y_n + \frac{d}{dt} x_n)$ is another flat output since $x_n = \xi$ and $y_n = \zeta - \frac{d}{dt}\xi$.
- The flat output (x_n, y_n) formed by the cartesian coordinates of P_n seems more adapted than (ξ, ζ): the output map h isequivariant.

Why the flat output $z := (x, y)$ is better than the flat output $\tilde{z} := (x, y + \dot{x})$?

Each symmetry of the system induces a transformation on the flat output z

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} z_1 \cos\alpha - z_2 \sin\alpha + a \\ z_1 \sin\alpha + z_2 \cos\alpha + b \end{pmatrix}$$

which does not involve derivatives of z
This point transformation, generates an endogenous transformation $(z, \dot{z}, \dots) \mapsto (Z, \dot{Z}, \dots)$ that is holonomic.

Why the flat output $z := (x, y)$ is better than the flat output $\tilde{z} := (x, y + \dot{x})$?

On the contrary

$$\begin{pmatrix} x \\ y + \dot{x} \end{pmatrix} = \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \quad \longmapsto \begin{pmatrix} \tilde{Z}_1 \\ \tilde{Z}_2 \end{pmatrix} = \begin{pmatrix} X \\ Y + \dot{X} \end{pmatrix}$$
$$= \begin{pmatrix} \tilde{z}_1 \cos\alpha + (\dot{\tilde{z}}_1 - \tilde{z}_2)\sin\alpha + a \\ \tilde{z}_1 \sin\alpha + \tilde{z}_2 \cos\alpha + (\ddot{\tilde{z}}_1 - \dot{\tilde{z}}_2)\sin\alpha + b \end{pmatrix}$$

is not a point transformation and does not give to a holonomic transformation. It is endogenous since its inverse is

$$\begin{pmatrix} \tilde{Z}_1 \\ \tilde{Z}_2 \end{pmatrix} \quad \longmapsto \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} = \begin{pmatrix} (\tilde{Z}_1 - a)\cos\alpha - (\dot{\tilde{Z}}_1 - \tilde{Z}_2)\sin\alpha \\ (\tilde{Z}_1 - a)\sin\alpha + (\tilde{Z}_2 - b)\cos\alpha - (\ddot{\tilde{Z}}_1 - \dot{\tilde{Z}}_2)\sin\alpha \end{pmatrix}$$

Symmetry preserving flat output

- Take the implicit system $F(x, \ldots, x^{(r)}) = 0$ with flat output $y = h(x, \ldots, x^{(\alpha)}) \in \mathbb{R}^m$ (i.e. $x = \mathcal{A}(y, \ldots, y^{(\beta)})$)
- Assume that the group G acting on the x-space via the family of diffeomorphism $X = \phi_g(x)$ ($x = \phi_{g^{-1}}(X)$) leaves the ideal associated to the set of equation $F = 0$ invariante:

$$F(x, \ldots, x^{(r)}) = 0 \Longleftrightarrow F\left((\phi_g(x), \ldots, \phi_g^{(r)}(x, \ldots, x^{(r)})\right) = 0$$

- Question: we wonder if exists always an equivariante flat output $\bar{y} = \bar{h}(x, \ldots, \bar{x}^{(\bar{\alpha})})$, i.e. such that exists an action of G on the y-space via the family of diffeomorphisms $\bar{Y} = \rho_g(\bar{y})$ satisfying

$$\rho_g(y) \equiv h\left(\phi_g(x), \ldots \phi_g^{(\bar{\alpha})}(x, \ldots, x^{(\bar{r})})\right).$$

two different flat outputs correspond via a "non-linear uni-modular transformation ":

$$\bar{y} = \psi(y, \ldots, y^{(\mu)}) \quad \text{with inverse} \quad y = \bar{\psi}(\bar{y}, \ldots, \bar{y}^{(\bar{\mu})})$$

Flat outputs as potentials and gauge degree of freedom

Maxwell's equations in vacuum imply that the magnetic field H is divergent free:

$$\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} + \frac{\partial H_3}{\partial x_3} = 0$$

When $H = \nabla \times A$ the constraint $\nabla \cdot H = 0$ is automatically satisfied
The potential A is a priori not uniquely defined, but up to an arbitrary gradient field, the gauge degree of freedom. The symmetries indicate how to use this degree of freedom to fix a "natural" potential.
For flat systems: a flat output is a "potential" for the underdetermined differential equation $\dot{x} - f(x, u) = 0$; endogenous transformations on the flat output correspond to gauge degrees of freedom.

Open problems

- $\frac{d}{dt}x = f(x,u)$ with $y = h(x,u,...,u^{(r)})$, r-flatness: bounds on r with respect to $\dim(x)$ and $\dim(u)$.
- Symmetries and flat-output preserving symmetries: are time-invariant systems flat with a time invariant flat output map (a first step to prove that linearization via exogenous dynamics feedback, implies flatness).
- Are the intrinsic and extrinsic definitions of flat systems equivalent ?
- Flatness of JBP example

Jean-Baptiste Pomet example SIAM JCO (2010)

- The system

$$\frac{d}{dt}x_3 - x_2 - \left(\frac{d}{dt}x_1\right)\left(\frac{d}{dt}x_2 - x_3\frac{d}{dt}x_1\right)^2 = 0$$

  is ruled with a single linear direction
  $a(x,\dot{x}) = (1, x_3, (\dot{x}_2 - x_3\dot{x}_1)^2)^T$.
- There is no flat output y depending only on x and $\dot{x}$ (this system is not 1-flat)
- Conjecture: this system is not flat.