A tutorial introduction to quantum feedback

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Underlying issues

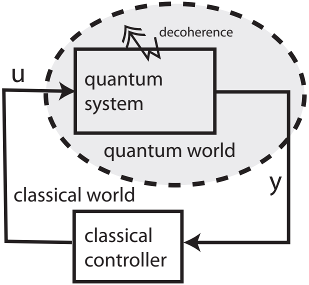
Quantum Error Correction (QEC) is based on a discrete-time feedback loop

- Current experiments: 10^{-3} is the typical error probability during elementary gates (manipulations) involving few physical qubits.
- High-order error-correcting codes with an important overhead; **more than 1000 physical qubits to encode a controllable logical qubit**[1].
- Today, no such controllable logical qubit has been built.
- Key issue: reduction by several magnitude orders of such error rates, far below the threshold required by actual QEC, to build a controllable logical qubit encoded in a reasonable number of physical qubits and protected by QEC.

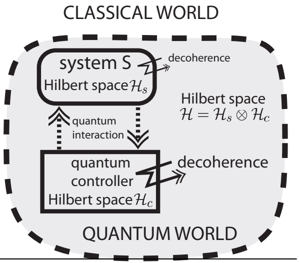
Control engineering can play a crucial role to build a controllable logical qubit protected by **adapted feedback schemes increasing precision and stability.**

[1] A.G. Fowler, M. Mariantoni, J.M. Martinis, A.N. Cleland: Surface codes: Towards practical large-scale quantum computation. Phys. Rev. A, 2012.

Two kinds of quantum feedback[2]



Measurement-based feedback: controller is classical; measurement back-action on the quantum system of Hilbert space $\mathcal{H}$ is stochastic (collapse of the wave-packet); the measured output y is a classical signal; the control input u is a classical variable appearing in some controlled Schrödinger equation; $u(t)$ depends on the past measurements $y(\tau)$, $\tau \leq t$.



Coherent/autonomous feedback and reservoir/dissipation engineering: the **system of Hilbert space $\mathcal{H}_s$** is coupled to **the controller, another quantum system**; the composite system of Hilbert space $\mathcal{H}_s \otimes \mathcal{H}_c$, is an open-quantum system relaxing to some target (separable) state. Relaxation behaviors in open quantum systems can be exploited: optical pumping of Alfred Kastler, physics Nobel prize 1966.

[2] Wiseman/Milburn: Quantum Measurement and Control, 2009, Cambridge University Press.

Outline

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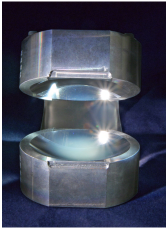
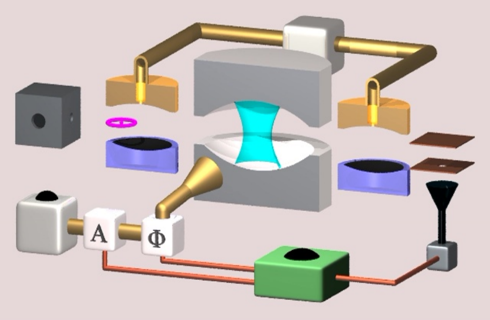
The first experimental realization of a quantum-state feedback

microwave photons
(10 GHz)





Experiment: C. Sayrin, I. Dotsenko, X. Zhou, B. Peaudecerf, T. Rybarczyk, S. Gleyzes, P. Rouchon, M. Mirrahimi, H. Amini, M. Brune, J.M. Raimond, **S. Haroche**: Real-time quantum feedback prepares and stabilizes photon number states. Nature, **2011**, 477, 73-77.

Theory: I. Dotsenko, M. Mirrahimi, M. Brune, **S. Haroche**, J.M. Raimond, P. Rouchon: Quantum feedback by discrete quantum non-demolition measurements: towards on-demand generation of photon-number states. Physical Review A, **2009**, 80: 013805-013813.
M. Mirrahimi et al. CDC 2009, 1451-1456, **2009**.
H. Amini et al. IEEE Trans. Automatic Control, 57 (8): 1918–1930, **2012**.
R. Somaraju et al., Rev. Math. Phys., 25, 1350001, **2013**.
H. Amini et. al., Automatica, 49 (9): 2683-2692, **2013**.

Experimental closed-loop data

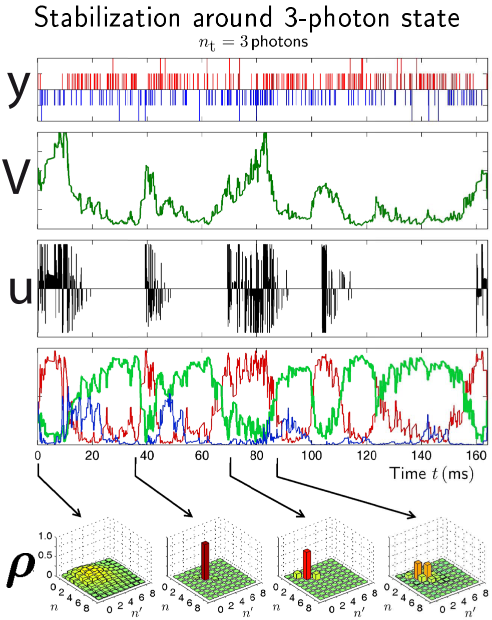
C. Sayrin et. al., Nature 477, 73-77, Sept. 2011.

Decoherence due to finite photon life-time (70 ms)

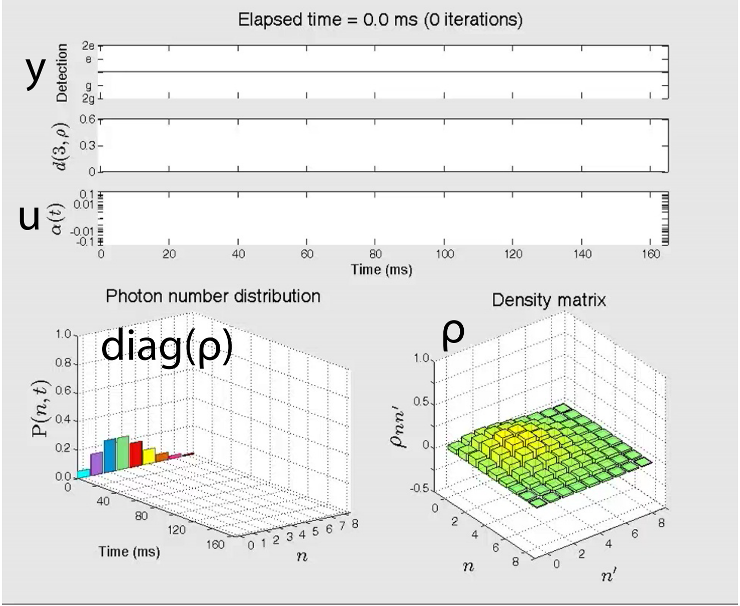
Detection efficiency 40%
Detection error rate 10%
Delay $d = 4$ sampling periods

The quantum filter includes cavity decoherence, detector imperfections and delays (Bayes law).

Truncation to 9 photons



Feedback stabilization around 3-photon state: experimental data



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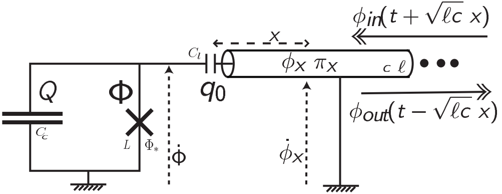
Transmon regime [3]



$$\frac{d}{dt}\Phi = \tfrac{1}{C}Q + 2\epsilon u - \epsilon^2\sqrt{\tfrac{\ell}{c}}\tfrac{\Phi_*}{L}\sin\left(\tfrac{1}{\Phi_*}\Phi\right)$$

$$\frac{d}{dt}Q = -\tfrac{\Phi_*}{L}\sin\left(\tfrac{1}{\Phi_*}\Phi\right)$$

with $y = u - \epsilon\sqrt{\tfrac{\ell}{c}}\tfrac{\Phi_*}{L}\sin\left(\tfrac{1}{\Phi_*}\Phi\right)$.

$\mathcal{H}_{sys}(\Phi, Q) = \frac{1}{2C}Q^2 - \frac{\Phi_*^2}{L}\cos\left(\frac{1}{\Phi_*}\Phi\right)$ with nonlinearity:

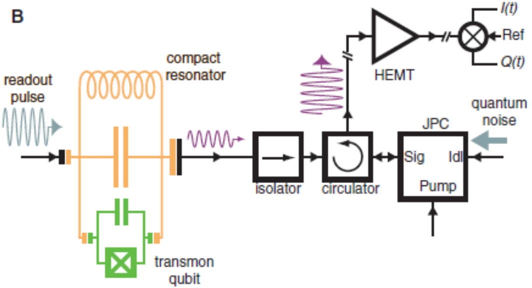
- anharmonic spectrum with frequency transition between the ground and first excited states larger than frequency transition between first and second excited states.
- qubit model based on restriction to these two slowest energy levels, $|g\rangle$ and $|e\rangle$, with pulsation $\omega_q \sim 1/\sqrt{LC}$.

Two weak coupling regimes:

- resonant, in/out wave pulsation ω_q;
- off-resonant , in/out wave pulsation $\omega_q + \Delta$ with $|\Delta| \ll \omega_q$.

[3] J. Koch et al.: Charge-insensitive qubit design derived from the Cooper pair box. Phys. Rev. A, 76:042319, 2007.

A key physical example in circuit quantum electrodynamics [4]



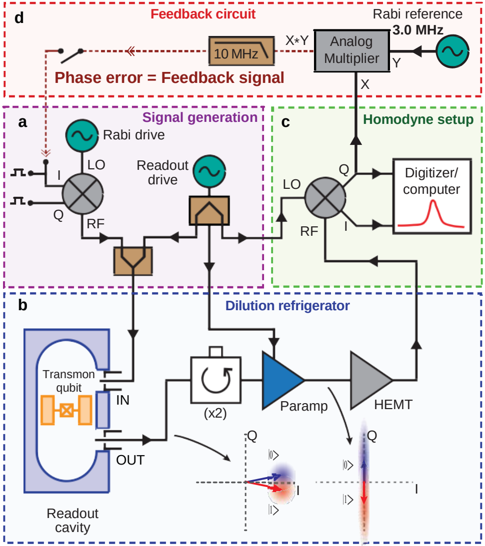
Superconducting qubit dispersively coupled to a cavity traversed by a microwave signal (input/output theory). The back-action on the single qubit state of a single measurement of one output field quadrature y is described by a simple SME for the qubit density operator ρ.

$$d\rho_t = \Big(-\tfrac{i}{2}[\omega_q \widehat{\sigma}_z, \rho_t] + \gamma(\widehat{\sigma}_z \rho \widehat{\sigma}_z - \rho_t)\Big)dt + \sqrt{\eta\gamma}\Big(\widehat{\sigma}_z \rho_t + \rho_t \widehat{\sigma}_z - 2\,\mathrm{Tr}\left(\widehat{\sigma}_z \rho_t\right)\rho_t\Big)dW_t$$

with y_t given by $dy_t = 2\sqrt{\eta\gamma}\,\mathrm{Tr}\left(\widehat{\sigma}_z \rho_t\right)dt + dW_t$ where $\gamma \geq 0$ is related to the measurement strength and $\eta \in [0,1]$ is the detection efficiency.

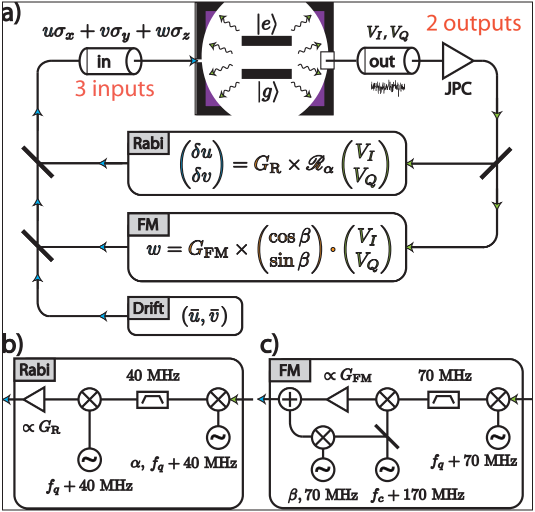
[4] M. Hatridge et al. Quantum Back-Action of an Individual Variable-Strength Measurement. Science, 2013, 339, 178-181.

First SISO measurement-based feedback for a superconducting qubit [5]



[5] R. Vijay, . . . , I. Siddiqi. Stabilizing Rabi oscillations in a superconducting qubit using quantum feedback. Nature 490, 77-80, October 2012.

First MIMO measurement-based feedback for a superconducting qubit [6]



[6] P. Campagne-Ibarcq, . . . , B. Huard: Using Spontaneous Emission of a Qubit as a Resource for Feedback Control. Phys. Rev. Lett. 117(6), 2016.

Outline

Feedback with classical controllers

The Haroche Photon-Box

Super-conducting qubit

Dynamics of open quantum systems

Feedback with quantum controllers

Quantum dissipation engineering

Cat-qubit and autonomous correction of bit-flips

GKP-qubit and autonomous correction of bit and phase flips

Quantum feedback engineering

Dynamics of open quantum systems based on three quantum features [7]

1. Schrödinger ($\hbar = 1$): wave funct. $|\psi\rangle \in \mathcal{H}$, density op. $\rho \sim |\psi\rangle\langle\psi|$

$$\frac{d}{dt}|\psi\rangle = -i\widehat{H}|\psi\rangle, \quad \widehat{H} = \widehat{H}_0 + u\widehat{H}_1 = \widehat{H}^\dagger, \quad \frac{d}{dt}\rho = -i[\widehat{H}, \rho].$$

2. Origin of dissipation: collapse of the wave packet induced by the measurement of $\widehat{O} = \widehat{O}^\dagger$ with spectral decomp. $\sum_y \lambda_y \widehat{P}_y$:
   - measurement outcome y with proba. $\mathbb{P}_y = \langle\psi|\widehat{P}_y|\psi\rangle = \text{Tr}\left(\rho\widehat{P}_y\right)$ depending on $|\psi\rangle$, ρ just before the measurement
   - measurement back-action if outcome y:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\widehat{P}_y|\psi\rangle}{\sqrt{\langle\psi|\widehat{P}_y|\psi\rangle}}, \quad \rho \mapsto \rho_+ = \frac{\widehat{P}_y\rho\widehat{P}_y}{\text{Tr}\left(\rho\widehat{P}_y\right)}$$

3. Tensor product for the description of composite systems (A, B):
   - Hilbert space $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$
   - Hamiltonian $\widehat{H} = \widehat{H}_a \otimes \widehat{I}_b + \widehat{H}_{ab} + \widehat{I}_a \otimes \widehat{H}_b$
   - observable on sub-system B only: $\widehat{O} = \widehat{I}_a \otimes \widehat{O}_a$.

[7] S. Haroche and J.M. Raimond (2006). *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts.

Structure of discrete-time dynamical models

Four modeling features[8]:

1. Schrödinger equations defining unitary transformations.
2. **Randomness**, irreversibility and dissipation induced by the measurement of observables with degenerate spectra.
3. Entanglement and tensor product for composite systems.
4. Classical probability (e.g. Bayes law) to include classical noises, measurement errors and uncertainties.

$\Rrightarrow$ **Hidden-state controlled Markov system**

Control input $\mathbf{u}$, state ρ (density op.), measured output $\mathbf{y}$:

$$\rho_{\mathbf{t}+1} = \frac{\mathcal{K}_{\mathbf{u_t},\mathbf{y_t}}(\rho_\mathbf{t})}{\mathrm{Tr}\left(\mathcal{K}_{\mathbf{u_t},\mathbf{y_t}}(\rho_\mathbf{t})\right)}, \text{ with proba. } \mathbb{P}\Big(\mathbf{y_t} \;/\rho_t, \mathbf{u_t}\Big) = \mathrm{Tr}\left(\mathcal{K}_{\mathbf{u_t},\mathbf{y_t}}(\rho_t)\right)$$

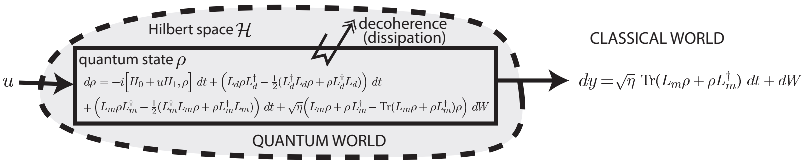
where $\mathcal{K}_{\mathbf{u},\mathbf{y}}(\rho) = \sum_{\mu=1}^{m} \eta_{\mathbf{y},\mu} \widehat{M}_{\mathbf{u},\mu} \rho \widehat{M}_{\mathbf{u},\mu}^\dagger$ with **left stochastic matrix** $(\eta_{\mathbf{y},\mu})$ and Kraus operators $\widehat{M}_{\mathbf{u},\mu}$ satisfying $\sum_\mu \widehat{M}_{\mathbf{u},\mu}^\dagger \widehat{M}_{\mathbf{u},\mu} = \widehat{I}$.

Kraus map $\mathcal{K}_\mathbf{u}$ (ensemble average, quantum channel)

$$\mathbb{E}\left(\rho_{t+1}|\rho_t\right) = \mathcal{K}_\mathbf{u}(\rho_t) = \sum_{\mathbf{y}} \mathcal{K}_{\mathbf{u},\mathbf{y}}(\rho_t) = \sum_\mu \widehat{M}_{\mathbf{u},\mu} \rho_t \widehat{M}_{\mathbf{u},\mu}^\dagger.$$

[8] See, e.g., books: E.B Davies in 1976; S. Haroche with J.M. Raimond in 2006; C. Gardiner with P. Zoller in 2014/2015.

Continuous dynamical models relying on Stochastic Master Equation (SME) [9]



Continuous-time models: stochastic differential systems (Itō formulation)
Control input u, state ρ (density op.), measured output y:

$$d\rho_t = \Big(-i[\widehat{H}_0 + \mathsf{u_t}\widehat{H}_1, \rho_t] + \sum_{\nu=d,m} \widehat{L}_\nu \rho_t \widehat{L}_\nu^\dagger - \frac{1}{2}(\widehat{L}_\nu^\dagger \widehat{L}_\nu \rho_t + \rho_t \widehat{L}_\nu^\dagger \widehat{L}_\nu)\Big) dt$$
$$+ \sqrt{\eta_m}\Big(\widehat{L}_m \rho_t + \rho_t \widehat{L}_m^\dagger - \ \mathrm{Tr}\left((\widehat{L}_m + \widehat{L}_m^\dagger)\rho_t\right) \rho_t\Big) d\mathsf{W_t}$$

driven by the Wiener process $\mathsf{W_t}$, with measurement $\mathsf{y_t}$,

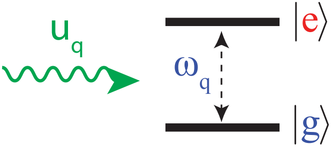
$$d\mathsf{y_t} = \sqrt{\eta_m}\ \mathrm{Tr}\left((\widehat{L}_m + \widehat{L}_m^\dagger)\, \rho_t\right) dt + d\mathsf{W_t} \quad \text{detection efficiencies } \eta_m \in [0,1].$$

Measurement backaction: $d\rho_t$ and $d\mathsf{y_t}$ share the same noises $d\mathsf{W_t}$. Very different from Kalman I/O state-space description used in control engineering.

[9] A. Barchielli, M. Gregoratti (2009): Quantum Trajectories and Measurements in Continuous Time: the Diffusive Case. Springer Verlag.

Qubit (2-level system, half-spin) [10]

- Hilbert space:
  $\mathcal{H} = \mathbb{C}^2 = \Big\{ c_g|g\rangle + c_e|e\rangle, \ c_g, c_e \in \mathbb{C} \Big\}.$
- Quantum state space:
  $\mathcal{D} = \{\rho \in \mathcal{L}(\mathbb{C}^2), \rho^\dagger = \rho, \ \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}\ .$
- Operators and commutations:
  $\widehat{\sigma}_- = |g\rangle\langle e|, \ \widehat{\sigma}_+ = \widehat{\sigma}_-^\dagger = |e\rangle\langle g|$
  $\widehat{X} \equiv \widehat{\sigma}_x = \widehat{\sigma}_- + \widehat{\sigma}_+ = |g\rangle\langle e| + |e\rangle\langle g|;$
  $\widehat{Y} \equiv \widehat{\sigma}_y = i\widehat{\sigma}_- - i\widehat{\sigma}_+ = i|g\rangle\langle e| - i|e\rangle\langle g|;$
  $\widehat{Z} \equiv \widehat{\sigma}_z = \widehat{\sigma}_+\widehat{\sigma}_- - \widehat{\sigma}_-\widehat{\sigma}_+ = |e\rangle\langle e| - |g\rangle\langle g|;$
  $\widehat{\sigma}_x^2 = \widehat{I}, \ \widehat{\sigma}_x\widehat{\sigma}_y = i\widehat{\sigma}_z, \ [\widehat{\sigma}_x, \widehat{\sigma}_y] = 2i\widehat{\sigma}_z, \ \ldots$

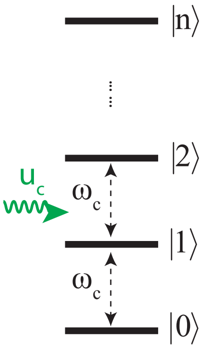


- Hamiltonian: $\widehat{H} = \omega_q\widehat{\sigma}_z/2 + u_q\widehat{\sigma}_x.$
- Bloch sphere representation:
  $\mathcal{D} = \Big\{ \frac{1}{2}\big(\widehat{I} + x\widehat{\sigma}_x + y\widehat{\sigma}_y + z\widehat{\sigma}_z\big) \ \Big| \ (x, y, z) \in \mathbb{R}^3, \ x^2 + y^2 + z^2 \leq 1 \Big\}$

[10] See S. M. Barnett, P.M. Radmore (2003): Methods in Theoretical Quantum Optics. Oxford University Press.

Quantum harmonic oscillator (spring system) [10]

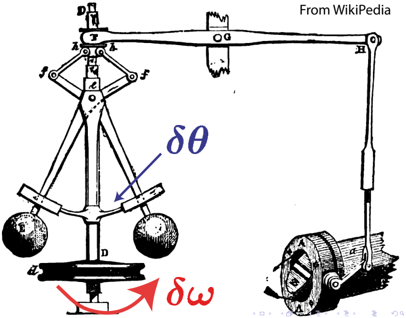
- Hilbert space:
  $\mathcal{H} = \left\{ \sum_{n\geq 0} \psi_n |n\rangle, \ (\psi_n)_{n\geq 0} \in \ell^2(\mathbb{C}) \right\} \equiv L^2(\mathbb{R}, \mathbb{C})$
- Quantum state space:
  $\mathcal{D} = \{\rho \in \mathcal{K}^1(\mathcal{H}), \rho^\dagger = \rho, \ \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}$.
- Operators and commutations:
  $\widehat{a}|n\rangle = \sqrt{n}\ |\text{n-1}\rangle$, $\widehat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$;
  $\widehat{N} = \widehat{a}^\dagger \widehat{a}$, $\widehat{N}|n\rangle = n|n\rangle$;
  $[\widehat{a}, \widehat{a}^\dagger] = \widehat{I}$, $\widehat{a} f(\widehat{N}) = f(\widehat{N} + \widehat{I})\widehat{a}$;
  $\widehat{D}_\alpha = e^{\alpha \widehat{a}^\dagger - \alpha^\dagger \widehat{a}} = e^{i\Re\alpha\Im\alpha} e^{i\sqrt{2}\Im\alpha x} e^{i\sqrt{2}\Re\alpha \frac{\partial}{\partial x}}$.
  $\widehat{a} = \widehat{X} + i\widehat{P} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$, $[\widehat{X}, \widehat{P}] = \imath \widehat{I}/2$.
- Hamiltonian: $\widehat{H} = \omega_c \widehat{a}^\dagger \widehat{a} + \mathrm{u_c}(\widehat{a} + \widehat{a}^\dagger)$.
  (associated classical dynamics:
  $\frac{dx}{dt} = \omega_c p$, $\frac{dp}{dt} = -\omega_c x - \sqrt{2} u_c$).



- Quasi-classical pure state ≡ coherent state $|\alpha\rangle$
  $\alpha \in \mathbb{C}: \ |\alpha\rangle = \sum_{n\geq 0} \left(e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \right) |n\rangle; \ |\alpha\rangle \equiv \frac{1}{\pi^{1/4}} e^{\imath\sqrt{2}\Im\alpha x} e^{-\frac{(x-\sqrt{2}\Re\alpha)^2}{2}}$
  $\widehat{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\widehat{D}_\alpha|0\rangle = |\alpha\rangle$.

Outline

Watt regulator: classical analogue of a quantum controller. [11]



Third order system

The first variations of speed $\delta\omega$ and governor angle $\delta\theta$ obey to

$$\frac{\mathrm{d}}{\mathrm{dt}}\delta\omega = -\mathrm{a}\delta\theta$$
$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}\delta\theta = -\tilde{\ }\frac{\mathrm{d}}{\mathrm{dt}}\delta\theta - \dot{\ }^2(\delta\theta - \mathrm{b}\delta\omega)$$

with (a, b, Λ, Ω) positive parameters.

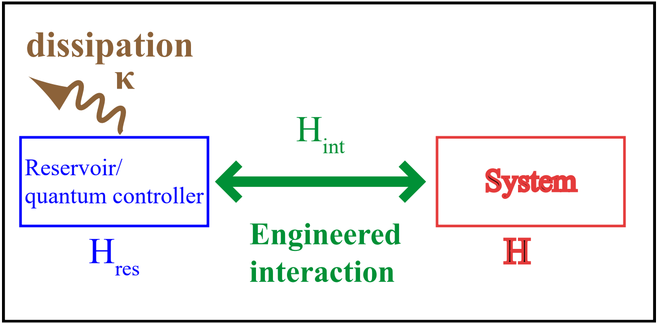
$$\frac{d^3}{dt^3}\delta\omega + \Lambda\frac{d^2}{dt^2}\delta\omega + \Omega^2\frac{d}{dt}\delta\omega + ab\Omega^2\delta\omega = 0.$$

Characteristic polynomial $P(s) = s^3 + \Lambda s^2 + \Omega^2 s + ab\Omega^2$ with roots having negative real parts iff $\Lambda > ab$: governor damping must be strong enough to ensure asymptotic stability.

Key issues: asymptotic stability and convergence rates.

[11] J.C. Maxwell: On governors. Proc. of the Royal Society, No.100, 1868.

Reservoir/dissipation engineering and quantum controller (1) [12]

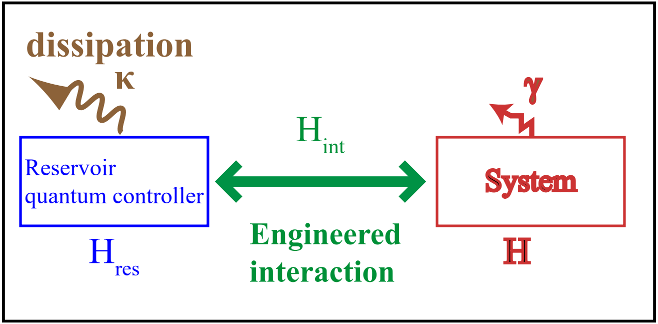


$$\widehat{H} = \widehat{H}_{res} + \widehat{H}_{int} + \widehat{H}$$

If $\rho \underset{t\to\infty}{\rightarrow} \rho_{res} \otimes \rho_{\text{target}}$ exponentially with rate $\kappa > 0$ large enough then ......

[12] See, e.g., the lectures of H. Mabuchi delivered at the "Ecole de physique des Houches", July 2011.

Reservoir/dissipation engineering and quantum controller (2)



$$\widehat{H} = \widehat{H}_{\text{res}} + \widehat{H}_{\text{int}} + \widehat{H}$$

. . . . . . $\rho \underset{t\to\infty}{\longrightarrow} \rho_{res} \otimes \rho_{\text{target}} + \delta\rho$, with $\|\delta\rho\|$ remaining small for $\gamma \ll \kappa$.

Quantum dynamics with dissipation (decoherence)

Gorini–Kossakowski –Sudarshan–Lindblad (GKSL) master equation:

$$\frac{d}{dt}\rho = -i[\widehat{H}_0 + u\widehat{H}_1, \rho] + \sum_\nu \left(\widehat{L}_\nu \rho \widehat{L}_\nu^\dagger - \frac{1}{2}(\widehat{L}_\nu^\dagger \widehat{L}_\nu \rho + \rho \widehat{L}_\nu^\dagger \widehat{L}_\nu) \right)$$

- Preservation of trace, hermiticity and positivity: ρ lies in the set of Hermitian and trace-class operators that are non-negative and of trace one.
- Invariance under unitary transformations.
  A time-varying change of frame $\rho \mapsto \widehat{U}_t^\dagger \rho \widehat{U}_t$ with $\widehat{U}_t$ unitary.
  The new density operator obeys to a similar master equation where $\widehat{H}_0 + u\widehat{H}_1 \mapsto \widehat{U}_t^\dagger(\widehat{H}_0 + u\widehat{H}_1)\widehat{U}_t + i\widehat{U}_t^\dagger \left(\frac{d}{dt}\widehat{U}_t\right)$ and $\widehat{L}_\nu \mapsto \widehat{U}_t^\dagger \widehat{L}_\nu \widehat{U}_t$.
- "L^1-contraction" properties. Such master equations generate contraction semi-groups for many distances (nuclear distance[13], Hilbert metric on the cone of non negative operators[14]).
- If the Hermitian operator $\widehat{A}$ satisfies the operator inequality

$$i[\widehat{H}_0 + u\widehat{H}_1, \widehat{A}] + \sum_\nu \left(\widehat{L}_\nu^\dagger \widehat{A} \widehat{L}_\nu - \frac{1}{2}(\widehat{L}_\nu^\dagger \widehat{L}_\nu \widehat{A} + \widehat{A} \widehat{L}_\nu^\dagger \widehat{L}_\nu) \right) \leq 0$$

  then $V(\rho) = \operatorname{Tr}\left(\widehat{A}\rho\right)$ is a Lyapunov function when $\widehat{A} \geq 0$.

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[13] D.Petz (1996). Monotone metrics on matrix spaces. Linear Algebra and its Applications

[14] R. Sepulchre, A. Sarlette, PR (2010). Consensus in non-commutative spaces. IEEE-CDC.

Outline

Feedback with classical controllers

The Haroche Photon-Box

Super-conducting qubit

Dynamics of open quantum systems

Feedback with quantum controllers

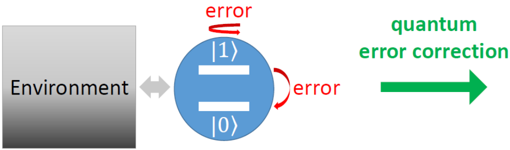
Quantum dissipation engineering

Cat-qubit and autonomous correction of bit-flips

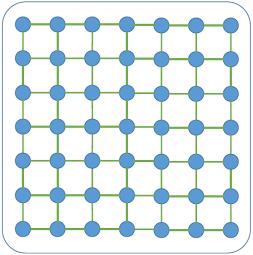
GKP-qubit and autonomous correction of bit and phase flips

Quantum feedback engineering

QEC: 2D redundancy to correct bit-flip and phase-flip errors



Physical qubit
electron spin,
trapped ion, ...



Protected qubit
Overhead " $\times$ 10^3"

Bosonic code with cat-qubits

- Quantum error corrrection requires redundancy.
- Bosonic code: instead of encoding a logical qubit in N physical qubits living in $\mathbb{C}^{2^N}$, encode a logical qubit in an harmonic oscillator living in Fock space $\text{span}\{|0\rangle, |1\rangle, \ldots, |n\rangle, \ldots\} \sim L^2(\mathbb{R}, \mathbb{C})$ of infinite dimension.
- Cat-qubit [15]: $|\psi_L\rangle \in \text{span}\{|\alpha\rangle, |\text{-}\alpha\rangle\}$ where $|\alpha\rangle$ is the coherent state of real amplitude α: $\widehat{a}|\alpha\rangle = \alpha|\alpha\rangle$ with $\widehat{a} = (\widehat{q} + i\widehat{p})/\sqrt{2}$ and $[\widehat{q}, \widehat{p}] = i$:

$$|\psi\rangle \sim \psi(q) \in L^2(\mathbb{R}, \mathbb{C}),\ \widehat{q}|\psi\rangle \sim q\psi(q),\ \widehat{p}|\psi\rangle \sim -i\frac{d\psi}{dq}(q),\ |\alpha\rangle \sim \frac{\exp\left(-\frac{(q-\alpha\sqrt{2})^2}{2}\right)}{\sqrt{2\pi}}.$$

- Stabilisation of cat-qubit via a single Lindblad dissipator $\widehat{L} = \widehat{a}^2 - \alpha^2$. For any initial density operator $\rho(0)$, the solution $\rho(t)$ of

$$\frac{d}{dt}\rho = \widehat{L}\rho\widehat{L}^\dagger - \frac{1}{2}(\widehat{L}^\dagger\widehat{L}\rho + \rho\widehat{L}^\dagger\widehat{L})$$

converges exponentially towards a steady-state density operator since

$$\frac{d}{dt}\ \text{Tr}\left(\widehat{L}^\dagger\widehat{L}\rho\right) \leq -2\ \text{Tr}\left(\widehat{L}^\dagger\widehat{L}\rho\right), \quad \ker\widehat{L} = \text{span}\{|\alpha\rangle, |\text{-}\alpha\rangle\}.$$

Any density operator with support in $\text{span}\{|\alpha\rangle, |\text{-}\alpha\rangle\}$ is a steady-state.

[15] M. Mirrahimi, Z. Leghtas, ..., **M. Devoret**: Dynamically protected cat-qubits: a new paradigm for universal quantum computation. 2014, New Journal of Physics.

Asymmetrically Threaded SQUID (ATS) stabilizing a cat-qubit [16]

9

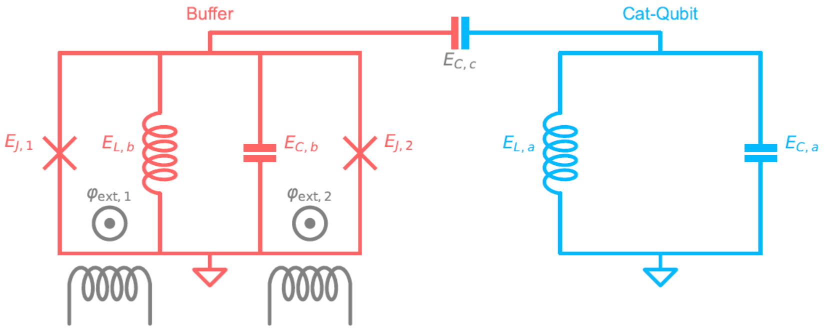
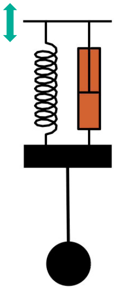


Figure S3. Equivalent circuit diagram. The cat-qubit (blue), a linear resonator, is capacitively coupled to the buffer (red). One recovers the circuit of Fig. 2 by replacing the buffer inductance with a 5-junction array and by setting $\varphi_\Sigma = (\varphi_{\text{ext},1} + \varphi_{\text{ext},2})/2$ and $\varphi_\Delta = (\varphi_{\text{ext},1} - \varphi_{\text{ext},2})/2$. Not shown here: the buffer is capacitively coupled to a transmission line, the cat-qubit resonator is coupled to a transmon qubit

[16] R. Lescanne, M. Villiers, Th. Peronnin, . . . , M. Mirrahimi and Z. Leghtas: Exponential suppression of bit-flips in a qubit encoded in an oscillator. 2020, Nature Physics

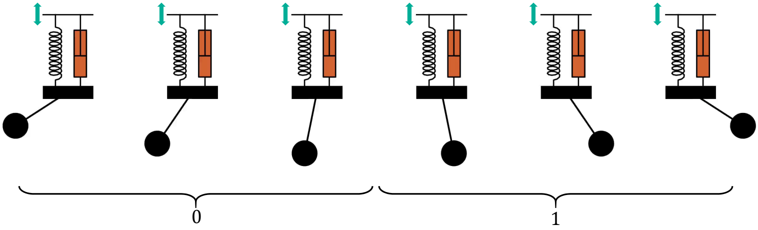
MAIN IDEA IN A CLASSICAL PICTURE



Driven damped oscillator
coupled to a pendulum.

Courtesy of Raphaël Lescanne

A BI-STABLE SYSTEM

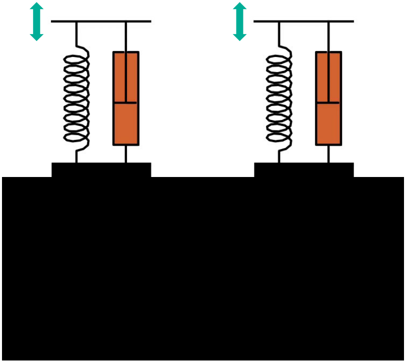


There are **2 steady states** in which we can encode information

Courtesy of Raphaël Lescanne

MAIN IDEA IN A CLASSICAL PICTURE

Stabilization regardless of the state



Neither the **drive** nor the **dissipation** can **distinguish** between 0 and 1

Important to preserve quantum coherence

Courtesy of Raphaël Lescanne

Master equations of the ATS super-conducting circuit

Oscillator $\widehat{a}$ with **quantum controller based on a damped oscillator** $\widehat{b}$:

$$\frac{d}{dt}\rho = g_2\left[(\widehat{a}^2 - \alpha^2)\widehat{b}^\dagger - ((\widehat{a}^\dagger)^2 - \alpha^2)\widehat{b}\,,\ \rho\right] + \kappa_b\left(\widehat{b}\rho\widehat{b}^\dagger - (\widehat{b}^\dagger\widehat{b}\rho + \rho\widehat{b}^\dagger\widehat{b})/2\right)$$

with $\alpha \in \mathbb{R}$ such that $\alpha^2 = u/g_2$, the drive amplitude $u \in \mathbb{R}$ applied to mode $\widehat{b}$ and $1/\kappa_b > 0$ the life-time of photon in mode $\widehat{b}$.
Any density operators $\bar{\rho} = \bar{\rho}_a \otimes |0\rangle\langle 0|_b$ **is a steady-state as soon as the support of $\bar{\rho}_a$ belongs to the two dimensional vector space spanned by the quasi-classical wave functions $|\alpha\rangle$ and $|\text{-}\alpha\rangle$** (range$(\bar{\rho}_a) \subset$ span$\{|\alpha\rangle, |\text{-}\alpha\rangle\}$)

Usually $\kappa_b \gg |g_2|$, mode $\widehat{b}$ relaxes rapidly to vaccuum $|0\rangle\langle 0|_b$, can be eliminated adiabatically (**singular perturbations**, second order corrections) to provides the slow evolution of mode $\widehat{a}$

$$\frac{d}{dt}\rho_a = \tfrac{4|g_2|^2}{\kappa_b}\left(\widehat{L}\rho\widehat{L}^\dagger - \frac{1}{2}(\widehat{L}^\dagger\widehat{L}\rho + \rho\widehat{L}^\dagger\widehat{L})\right) \text{ with } \widehat{L} = \widehat{a}^2 - \alpha^2.$$

Convergence via the exponential Lyapunov function $V(\rho) = \text{ Tr}\left(\widehat{L}^\dagger\widehat{L}\rho\right)$ [17]

[17] For a mathematical proof of convergence analysis in an adapted Banach space, see :R. Azouit, A. Sarlette, PR: Well-posedness and convergence of the Lindblad master equation for a quantum harmonic oscillator with multi-photon drive and damping. 2016, ESAIM: COCV.

Cat-qubit: exponential suppression of bit-flip for large α.

Since $\langle\alpha|\text{-}\alpha\rangle = e^{-2\alpha^2} \approx 0$:

$$|0_L\rangle \approx |\alpha\rangle,\ |1_L\rangle \approx |\text{-}\alpha\rangle,\ |+_L\rangle \propto \frac{|\alpha\rangle+|\text{-}\alpha\rangle}{\sqrt{2}},\ |-_L\rangle \propto \frac{|\alpha\rangle-|\text{-}\alpha\rangle}{\sqrt{2}}.$$

Photon loss as dominant error channel (dissipator $\widehat{a}$ with $0 < \kappa_1 \ll 1$):

$$\frac{d}{dt}\rho_a = \mathcal{D}_{\widehat{a}^2-\alpha^2}(\rho) + \kappa_1 \mathcal{D}_{\widehat{a}}(\rho)$$

with $\mathcal{D}_{\widehat{L}}(\rho) = \widehat{L}\rho\widehat{L}^\dagger - \frac{1}{2}(\widehat{L}^\dagger\widehat{L}\rho + \rho\widehat{L}^\dagger\widehat{L})$.

- if $\rho(0) = |0_L\rangle\langle 0_L|$ or $|1_L\rangle\langle 1_L|$, $\rho(t)$ converges to a statistical mixture of quasi-classical states close to $\frac{1}{2}|\alpha\rangle\langle\alpha| + \frac{1}{2}|\text{-}\alpha\rangle\langle\text{-}\alpha|$ in a time

  $$T_{bit-flip} \sim \frac{e^{2\alpha^2}}{\kappa_1}$$

  since $\widehat{a}|0_L\rangle \approx \alpha|0_L\rangle$ and $\widehat{a}|1_L\rangle \approx -\alpha|1_L\rangle$.
- if $\rho(0) = |+_L\rangle\langle +_L|$ or $|-_L\rangle\langle -_L|$, $\rho(t)$ converges also to the same statistical mixture in a time

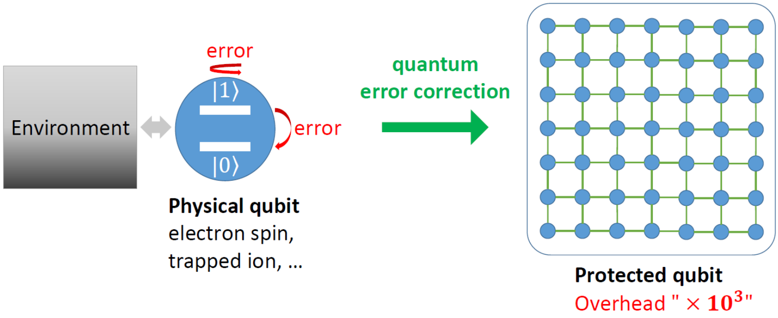
  $$T_{phase-flip} \sim \frac{1}{\kappa_1\alpha^2}$$

  since $\widehat{a}|+_L\rangle = \alpha| - L\rangle$ and $\widehat{a}|-_L\rangle = \alpha| + L\rangle$.

Take α large to ignore bit-flip and to correct only the phase-flip with $1D$ repetition code: important overhead reduction investigated by the startup **Alice&Bob** and also by **AWS**.

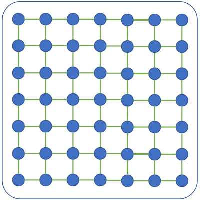
Outline

QEC: 2D redundancy to correct bit-flip and phase-flip errors



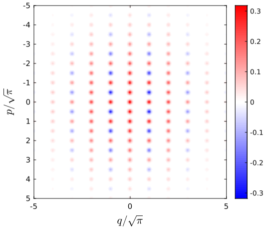
Local noise assumption (1)

DV-QEC



$$\mathcal{H} = \left(\mathbb{C}^2\right)^{\otimes N}$$
Dimension: 2^N

CV-QEC



$$\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})$$
Dimension: $+\infty$

Wave function $|\psi\rangle : \mathbb{R} \ni q \mapsto \psi(q) \in \mathbb{C}$, and Wigner function

$$\mathbb{R}^2 \ni (q, p) \mapsto W^{|\psi\rangle\langle\psi|}(q, p) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \psi^*\left(q - \tfrac{u}{2}\right)\psi\left(q + \tfrac{u}{2}\right) e^{-2ipu} du.$$

Local error operators $\hat{q}$ and $\hat{p}$ ($[\hat{q}, \hat{p}] = i$) on $|\psi\rangle$: small random shifts along q ($e^{i\pm\epsilon\hat{p}} \equiv e^{\pm\epsilon d/dq}$) and p ($e^{i\pm\epsilon\hat{q}} \equiv e^{\mp\epsilon d/dp}$) similar to diffusion along q and p axis for $W^{|\psi\rangle\langle\psi|}$.

Local noise assumption (2)

For a density operator ρ, its Wigner function

$$\mathbb{R}^2 \ni (q,p) \mapsto W^\rho(q,p) \in \mathbb{R}$$

reads ($\widehat{a} = \frac{\widehat{q}+i\widehat{p}}{\sqrt{2}}$)

$$W^\rho(q,p) = \tfrac{1}{\pi}\ \mathrm{Tr}\left(e^{i\pi\widehat{a}^\dagger\widehat{a}}\ e^{i(p\widehat{q}-q\widehat{p})}\ \rho\ e^{-i(p\widehat{q}-q\widehat{p})} \right)$$

Since

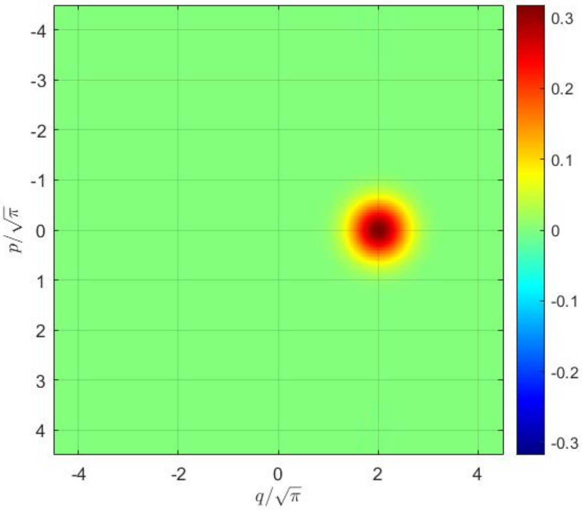
$$W^{\mathcal{D}_{\widehat{q}}(\rho)} = \frac{1}{2}\frac{\partial^2}{\partial p^2} W^\rho, \quad W^{\mathcal{D}_{\widehat{p}}(\rho)} = \frac{1}{2}\frac{\partial^2}{\partial q^2} W^\rho$$

and

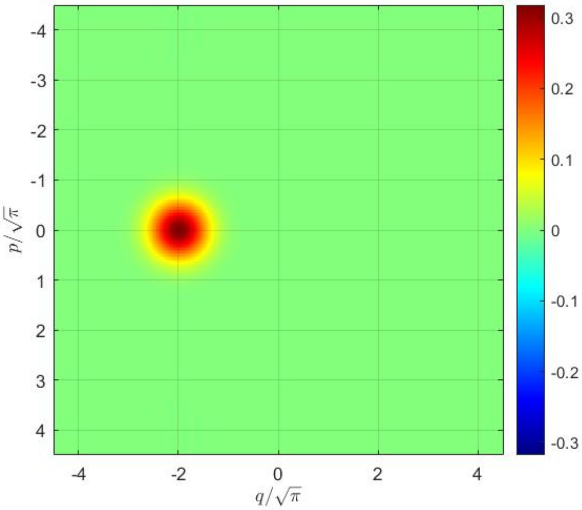
$$W^{\mathcal{D}_{\widehat{a}}(\rho)} = \frac{1}{2}\frac{\partial}{\partial q}(qW^\rho) + \frac{1}{2}\frac{\partial}{\partial p}(pW^\rho) + \frac{1}{2}\frac{\partial^2}{\partial q^2}W^\rho + \frac{1}{2}\frac{\partial^2}{\partial p^2}W^\rho$$

dominant errors on ρ correspond to local differential operators in the phase-space (q,p).

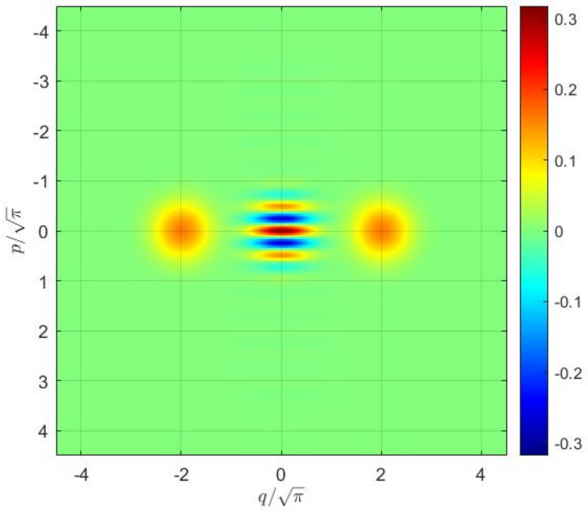
Wigner function of coherent state $|\sqrt{2\pi}\rangle \equiv \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(q-2\sqrt{\pi})^2}{2}\right) \approx |0_L\rangle$



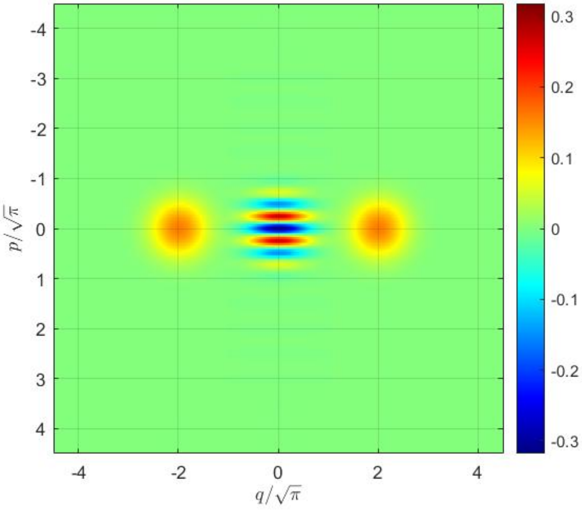
Wigner function of coherent state $|{-\sqrt{2\pi}}\rangle \equiv \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(q+2\sqrt{\pi})^2}{2}\right) \approx |1_L\rangle$



Wigner function of $|+_L\rangle \propto \frac{|\sqrt{2\pi}\rangle + |-\sqrt{2\pi}\rangle}{\sqrt{2}}$ ("Schrödinger phase cat")



Wigner function of $|-_L\rangle \propto \frac{|\sqrt{2\pi}\rangle - |-\sqrt{2\pi}\rangle}{\sqrt{2}}$ ("Schrödinger phase cat")



Grid-states and GKP-qubits

- Poisson summation formula: the Fourier transform of Dirac comb $f(q)$ of period T is a Dirac comb $g(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(q) e^{-iqp} dq$ of period $2\pi/T$.

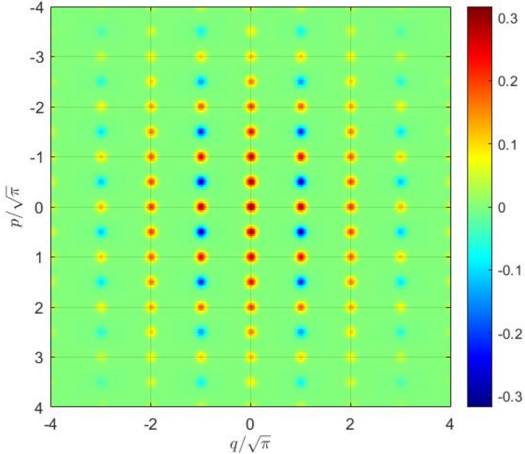
infinite energy grid-states	q representation	p representation
$\|0_L\rangle$	$\sum_k \delta(q - 2k\sqrt{\pi})$	$\sum_k \delta(p - k\sqrt{\pi})$
$\|1_L\rangle$	$\sum_k \delta(q - 2(k+1)\sqrt{\pi})$	$\sum_k (-1)^k \delta(p - k\sqrt{\pi})$
$\|+_L\rangle \sim \|0_L\rangle + \|1_L\rangle$	$\sum_k \delta(q - k)\sqrt{\pi})$	$\sum_k \delta(p - 2k\sqrt{\pi})$
$\|-_L\rangle \sim \|0_L\rangle - \|1_L\rangle$	$\sum_k (-1)^k \delta(q - k)\sqrt{\pi})$	$\sum_k \delta(p - 2(k+1)\sqrt{\pi})$

- Pauli operators of a GKP-qubit[18] with Bloch coordinates $(x, y, z) \in \mathbb{R}^3$:
$$\widehat{Z} = \text{sign}(\cos(\sqrt{\pi}\widehat{q})),\ \widehat{X} = \text{sign}(\cos(\sqrt{\pi}\widehat{p})) \text{ and } \widehat{Y} = -i\widehat{Z}\widehat{X}.$$
- 4 stabilizer operators $\widehat{S}$ relying on commuting modular operators in $\widehat{q}$ and $\widehat{p}$:
$$\forall \widehat{S} \in \{e^{i2\sqrt{\pi}\widehat{q}}, e^{i2\sqrt{\pi}\widehat{p}}, e^{-i2\sqrt{\pi}\widehat{q}}, e^{-i2\sqrt{\pi}\widehat{p}}\} \text{ and } \forall |\psi_L\rangle \in \text{span}\{|0_L\rangle, |1_L\rangle\}\text{: } \widehat{S}|\psi_L\rangle = |\psi_L\rangle$$
- Finite energy regularization with $0 < \epsilon \ll 1$,
$$|0_\epsilon\rangle \approx e^{-\epsilon \widehat{a}^\dagger \widehat{a}} |0_L\rangle, \quad |1_\epsilon\rangle \approx e^{-\epsilon \widehat{a}^\dagger \widehat{a}} |0_L\rangle,$$
where $\widehat{a}^\dagger \widehat{a} = \frac{1}{2}(\widehat{q}^2 + \widehat{p}^2) \sim \frac{1}{2}(q^2 + \partial^2/\partial q^2)$, provides a finite-energy code space where **any small local error can be corrected[19].**

[18] D. Gottesman, A. Kitaev and J. Preskill: Encoding a qubit in an oscillator. Physical Review A, 2001.

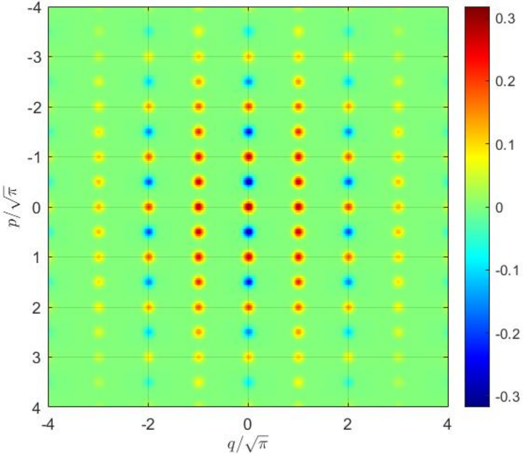
[19] 3 recent experiments stabilizing GKP-qubits via classical controllers: Ph. Campagne-Ibarcq et al. “Quantum error correction of a qubit encoded in grid states of an oscillator” Nature (2020); B. de Neeve et al. “Error correction of a logical grid state qubit by dissipative pumping” Nature (2022); V. Sivak et al. “Real-Time Quantum Error Correction beyond Break-Even” Nature (2023).

Wigner function of the GKP finite energy grid-state $|0_\epsilon\rangle$ [20]



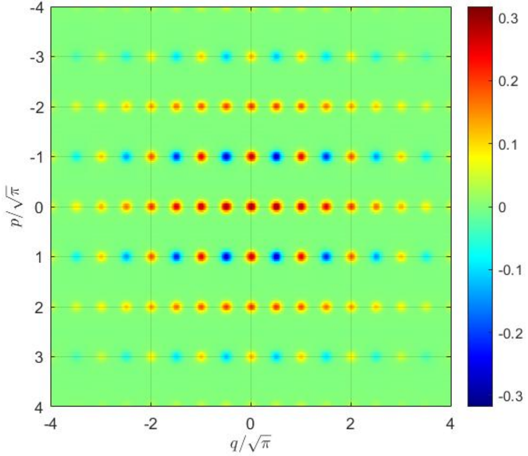
[20] $|0_\epsilon\rangle \approx e^{-\epsilon q^2} \sum_k \exp\left(-\frac{(q-2k\sqrt{\pi})^2}{\epsilon}\right)$ with $\epsilon = \frac{1}{30}$.

Wigner function of the GKP finite energy grid-state $|1_\epsilon\rangle$ [21]



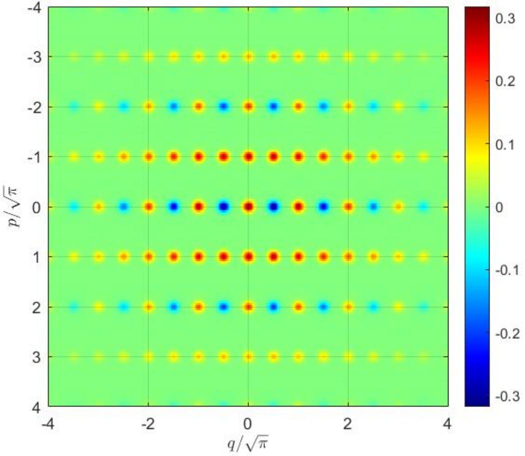
[21] $|1_\epsilon\rangle \approx e^{-\epsilon q^2} \sum_k \exp\left(-\frac{(q-(2k+1)\sqrt{\pi})^2}{\epsilon}\right)$ with $\epsilon = \frac{1}{30}$.

Wigner function of the GKPfinite energy grid-state $|+_\epsilon\rangle$ [22]



[22] $|+_\epsilon\rangle \approx e^{-\epsilon q^2} \sum_k \exp\left(\frac{(q-k\sqrt{\pi})^2}{\epsilon}\right) \equiv e^{-\epsilon p^2} \sum_k \exp\left(\frac{(p-2k\sqrt{\pi})^2}{\epsilon}\right)$.

Wigner function of the GKP finite energy grid-state $|-_\epsilon\rangle$ [23]



[23] $|-_\epsilon\rangle \approx e^{-\epsilon q^2} \sum_k (-1)^k \exp\left(-\frac{(q-k\sqrt{\pi})^2}{\epsilon}\right) \equiv e^{-\epsilon p^2} \sum_k \exp\left(-\frac{(p-(2k+1)\sqrt{\pi})^2}{\epsilon}\right).$

Exponential stabilisation of finite energy GKP-qubits[24]

- 4 regularized stabilizers:

$$\widehat{S}_{\epsilon,k} \triangleq e^{-(\epsilon - i\frac{k\pi}{2})\widehat{a}^\dagger \widehat{a}}\ e^{i2\sqrt{\pi}\widehat{q}}\ e^{(\epsilon - i\frac{k\pi}{2})\widehat{a}^\dagger \widehat{a}}, \quad k = 0, 1, 2, 3.$$

- Master equation with 4 dissipators $\widehat{M}_{\epsilon,k} = \widehat{S}_{\epsilon,k} - \widehat{I}$

$$\frac{d}{dt}\rho = \sum_{k=0}^{3} \mathcal{D}_{\widehat{M}_{\epsilon,k}}(\rho)$$

- Lyapunov function:

$$V(\rho) = \sum_k \text{Tr}\left(\widehat{M}_{\epsilon,k}^\dagger \widehat{M}_{\epsilon,k}\rho\right) \text{ with } \frac{d}{dt}V \leq -\left(32\pi^2\epsilon^2 + O(\epsilon^3)\right)V$$

ensuring exponential convergence towards the finite-energy code space

$$\text{span}\left\{e^{-\epsilon\widehat{a}^\dagger\widehat{a}}|0_L\rangle, e^{-\epsilon\widehat{a}^\dagger\widehat{a}}|1_L\rangle\right\}$$

[24] L.A. Sellem, Ph. Campagne-Ibarcq, M. Mirrahimi, A. Sarlette, PR: Exponential convergence of a dissipative quantum system towards finite-energy grid states of an oscillator: IEEE CDC 2022 (arXiv:2203.16836).

Approximated Lindblad dissipators with exponentially small decoherence rates [25]

Replace the ideal dissipators $\widehat{M}_{\epsilon,k}$ by more realistic dissipators $\widehat{L}_{\epsilon,k}$ derived from a first-order approximation in ϵ:

$$\widehat{L}_{\epsilon,k} \triangleq e^{i\frac{k\pi}{2}\widehat{a}^\dagger\widehat{a}} \left(e^{-2\pi\epsilon}e^{i2\sqrt{\pi}\widehat{q}}(\widehat{I} - 2\epsilon\sqrt{\pi}\widehat{p}) - \widehat{I}\right) e^{-i\frac{k\pi}{2}\widehat{a}^\dagger\widehat{a}}$$

For ρ governed by master equation $\frac{d}{dt}\rho = \sum_{k=0}^{3} \mathcal{D}_{\widehat{L}_{\epsilon,k}}(\rho)$:

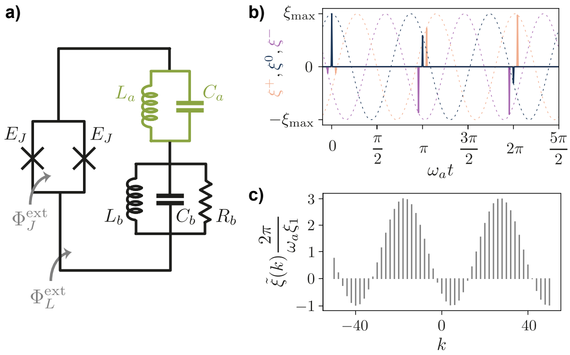
- Energy $\operatorname{Tr}\left(\widehat{a}^\dagger\widehat{a}\rho\right)$ remains finite and for t large is less than $\frac{1}{2\epsilon} + 0(1)$.
- For any 2π periodic function $f(\theta)$, one has

$$\frac{d}{dt}\operatorname{Tr}\left(f(\sqrt{\pi}\widehat{q})\rho\right) = -4\epsilon\pi e^{-2\pi\epsilon}\operatorname{Tr}\left(\left(\sin(2\sqrt{\pi}\widehat{q})f'(\sqrt{\pi}\hat{q}) - \epsilon\pi e^{-2\pi\epsilon}f''(\sqrt{\pi}\widehat{q})\right)\rho\right).$$

- Spect. $(\lambda_n)_{n\geq 0}$ of Witten Laplacian $\mathcal{L}_\sigma(f(\theta)) = \sin(2\theta)f'(\theta) - \sigma f''(\theta)$ with 2π-periodic function $f(\theta)$ and $0 < \sigma \ll 1$:
$$\lambda_0 = 0 < \lambda_1 \sim \frac{4}{\pi}e^{-1/\sigma} < 1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_n \leq \ldots$$
with eigenfunction $f_1(\theta) \approx \operatorname{sign}(\cos\theta)$ corresponding to λ_1. Thus $z \approx \operatorname{Tr}(f_1(\sqrt{\pi}\widehat{q})\rho)$ is almost constant: $\frac{d}{dt}z \approx -16\epsilon\exp\left(-\frac{1}{\epsilon\pi}\right) z$.
Similar exponentially small decays for (x, y, z) with quadrature noises,
i.e. when $\frac{d}{dt}\rho = \sum_{k=0}^{3}\mathcal{D}_{\widehat{L}_{\epsilon,k}}(\rho) + \kappa_q\mathcal{D}_{\widehat{q}}(\rho) + \kappa_p\mathcal{D}_{\widehat{p}}(\rho)$ $(\kappa_q, \kappa_q \ll 1)$

[25] L.A. Sellem, R. Robin, Ph. Campagne-Ibarcq, PR: Stability and decoherence rates of a GKP qubit protected by dissipation. IFAC WC 2023 (arXiv:2304.03806).

Engineering modular dissipation with super-conducting Josephson circuits [26]



High impedance $\sqrt{L_a/C_a}$ and low pulsation $1/\sqrt{L_a C_a}$ for storage mode $\widehat{a}$.

High pulsation $1/\sqrt{L_b C_b}$ of damped mode $\widehat{b}$ (quantum controller $R_b > 0$).

Josephson energy E_J between $\hbar/\sqrt{L_a C_a}$ and $\hbar/\sqrt{L_b C_b}$.

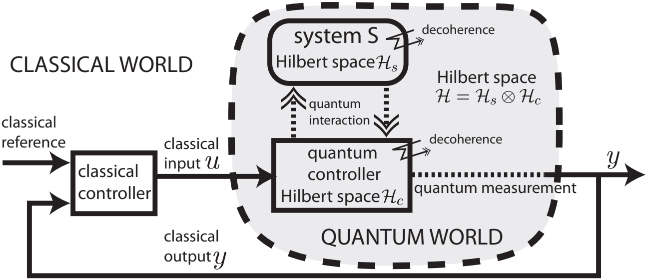
Classical open-loop control signals $\Phi_J^{ext}(t)$ and $\Phi_L^{ext}(t)$ made of short pulses.

Mathematical analysis to recover master equation with dissipators $\widehat{L}_k$.

Numerical simulations to test robustness versus experimental imperfections.

[26] L.A. Sellem, A. Salette, Z. Leghtas, M. Mirrahimi, PR, Ph. Campagne-Ibarcq: A GKP qubit protected by dissipation in a high-impedance superconducting circuit driven by a microwave frequency comb. Under review in PRX (arXiv:2304.01425).

Quantum feedback engineering for robust quantum information processing



To protect quantum information stored in system S:

- fast stabilization and protection mainly achieved by **quantum controllers** (autonomous feedback stabilizing decoherence-free sub-spaces);
- slow decoherence and perturbations, parameter estimation mainly tackled by **classical controllers and estimation algorithms** (measurement-based feedback and estimation "finishing the job")

Need of adapted mathematical and numerical methods for high-precision dynamical modeling and control with (stochastic) master equations.

Quantic research group ENS/Inria/Mines/CNRS, June 2023