# Modeling and Control of the LKB Photon-Box: [1] Quantum Non-Demolition (QND) Measurement

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Several slides have been used during the IHP course (fall 2010) given with Mazyar Mirrahimi (INRIA) see:
http://cas.ensmp.fr/~rouchon/QuantumSyst/index.html

# Outline

1. Quantum measurement
   - Projective measurement
   - Positive Operator Valued Measurement (POVM)
   - Quantum Non-Demolition (QND) measurement
   - Stochastic process attached to a POVM
2. A discrete-time open system: the LKB photon box
   - General case
   - Dispersive case
   - The open-loop Markov chain

For the system defined on Hilbert space $\mathcal{H}$, take

- an observable $\mathcal{O}$ (Hermitian operator) defined on $\mathcal{H}$:

$$\mathcal{O} = \sum_{\nu} \lambda_\nu P_\nu,$$

  where $\lambda_\nu$'s are the eigenvalues of $\mathcal{O}$ and $P_\nu$ is the projection operator over the associated eigenspace; $\mathcal{O}$ can be degenerate and therefore the projection operator $P_\nu$ is not necessarily a rank-1 operator.

- a quantum state (a priori mixed) given by the density operator $\rho$ on $\mathcal{H}$, Hermitian, positive and of trace 1; $\mathrm{Tr}\left(\rho^2\right) \leq 1$ with equality only when $\rho$ is an orthogonal projector on some pure quantum state $|\psi\rangle$, i.e., $\rho = |\psi\rangle \langle\psi|$.

Projective measurement of the physical observable $\mathcal{O} = \sum_\nu \lambda_\nu P_\nu$ for the quantum state $\rho$:

1. The probability of obtaining the value $\lambda_\nu$ is given by $p_\nu = \mathrm{Tr}\,(\rho P_\nu)$; note that $\sum_\nu p_\nu = 1$ as $\sum_\nu P_\nu = \mathbf{1}_\mathcal{H}$ ($\mathbf{1}_\mathcal{H}$ represents the identity operator of $\mathcal{H}$).
2. After the measurement, the conditional (a posteriori) state $\rho_+$ of the system, given the outcome $\lambda_\nu$, is

$$\rho_+ = \frac{P_\nu\ \rho\ P_\nu}{p_\nu} \quad \text{(collapse of the wave packet)}$$

3. When $\rho = |\psi\rangle\langle\psi|$, $p_\nu = \langle\psi|P_\nu|\psi\rangle$, $\rho_+ = |\psi_+\rangle\langle\psi_+|$ with $|\psi_+\rangle = \frac{P_\nu\psi}{\sqrt{p_\nu}}$.

$\mathcal{O}$ non degenerate: von Neumann measurement.

**Example**: $\mathcal{H} = \mathbb{C}^2$, $|\psi\rangle = (|g\rangle + |e\rangle)/\sqrt{2}$, $\mathcal{O} = \sigma_z$; measuring consists in turning on, for a small time, a laser resonant between $|g\rangle$ and a highly unstable third state $|f\rangle$; fluorescence means $|\psi_+\rangle = |g\rangle$, no fluorescence means $|\psi_+\rangle = |e\rangle$.

# Positive Operator Valued Measurement (POVM) (1)

System $S$ of interest (a quantized electromagnetic field) interacts with the meter $M$ (a probe atom), and the experimenter measures projectively the meter $M$ (the probe atom). Need for a **Composite system**: $\mathcal{H}_S \otimes \mathcal{H}_M$ where $\mathcal{H}_S$ and $\mathcal{H}_M$ are the Hilbert space of $S$ and $M$.

Measurement process in three successive steps:

1. Initially the quantum state is separable

$$\mathcal{H}_S \otimes \mathcal{H}_M \ni |\Psi\rangle = |\psi_S\rangle \otimes |\theta_M\rangle$$

   with a well defined and known state $|\theta_M\rangle$ for $M$.

2. Then a Schrödinger evolution during a small time (unitary operator $U_{S,M}$) of the composite system from $|\psi_S\rangle \otimes |\theta_M\rangle$ and producing $U_{S,M}\left( |\psi_S\rangle \otimes |\theta_M\rangle \right)$, entangled in general.
3. Finally a projective measurement of the meter $M$: $\mathcal{O}_M = \mathbf{1}_S \otimes \left( \sum_\nu \lambda_\nu P_\nu \right)$ the measured observable for the meter. Projection operator $P_\nu$ is a rank-1 projection in $\mathcal{H}_M$ over the eigenstate $|\lambda_\nu\rangle \in \mathcal{H}_M$: $P_\nu = |\lambda_\nu\rangle \langle\lambda_\nu|$.

Define the measurement operators $\mathcal{M}_\nu$ via

$$\forall\,|\psi_S\rangle \in \mathcal{H}_S, \quad U_{S,M}\big(\,|\psi_S\rangle \otimes |\theta_M\rangle\,\big) = \sum_\nu \big(\mathcal{M}_\nu\,|\psi_S\rangle\,\big) \otimes |\lambda_\nu\rangle\,.$$

Then $\sum_\nu \mathcal{M}_\nu^\dagger \mathcal{M}_\nu = \mathbf{1}_S$. The set $\{\mathcal{M}_\nu\}$ defines a Positive Operator Valued Measurement (POVM).

In $\mathcal{H}_S \otimes \mathcal{H}_M$, projective measurement of $\mathcal{O}_M = \mathbf{1}_S \otimes \big(\sum_\nu \lambda_\nu P_\nu\big)$ with quantum state $U_{S,M}\big(\,|\psi_S\rangle \otimes |\theta_M\rangle\,\big)$:

1. The probability of obtaining the value $\lambda_\nu$ is given by $p_\nu = \langle\psi_S|\,\mathcal{M}_\nu^\dagger \mathcal{M}_\nu\,|\psi_S\rangle$
2. After the measurement, the conditional (a posteriori) state of the system, given the outcome $\lambda_\nu$, is

$$|\psi_S\rangle_+ = \frac{\mathcal{M}_\nu\,|\psi_S\rangle}{\sqrt{p_\nu}}.$$

For mixed state $\rho$ (instead of pure state $|\psi_S\rangle$):

$p_\nu = \mathrm{Tr}\left(\mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger\right)$ and $\rho_+ = \frac{\mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger}{\mathrm{Tr}\left(\mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger\right)}$,

# Quantum Non-Demolition (QND) measurement (1)

$U_{S,M}$ is the propagator generated by $H = H_S + H_M + H_{SM}$ where $H_S$ (resp. $H_M$, $H_{SM}$) describes the system (resp. the meter , system-meter interaction). For time-invariant $H$: $U_{S,M} = e^{-i\tau H}$ where $\tau$ is the interaction time.

A necessary condition for meter measurement to encode some information on the system $S$ itself: $[H, \mathcal{O}_M] \neq 0$. When $H_M = 0$, this necessary condition reads $[H_{SM}, \mathcal{O}_M] \neq 0$.

Proof: otherwise $\mathcal{O}_M U_{S,M} = U_{S,M} \mathcal{O}_M$. With $\mathcal{O}_M = \sum_\nu \lambda_\nu \mathbf{1}_S \otimes |\lambda_\nu\rangle\langle\lambda_\nu|$ we have

$$\forall \nu, \quad \mathcal{O}_M U_{S,M}\big(|\psi_S\rangle \otimes |\lambda_\nu\rangle\big) = U_{S,M}\mathcal{O}_M\big(|\psi_S\rangle \otimes |\lambda_\nu\rangle\big) = \lambda_\nu U_{S,M}\big(|\psi_S\rangle \otimes |\lambda_\nu\rangle\big).$$

Thus, necessarily $U_{S,M}\big(|\psi_S\rangle \otimes |\lambda_\nu\rangle\big) = \big(U_\nu |\psi_S\rangle\big) \otimes |\lambda_\nu\rangle$ where $U_\nu$ is a unitary transformation on $\mathcal{H}_S$ only. With $|\theta_M\rangle = \sum_\nu \theta_\nu |\lambda_\nu\rangle$, we get:

$$\forall |\psi_S\rangle \in \mathcal{H}_S U_{S,M}\big(|\psi_S\rangle \otimes |\theta_M\rangle\big) = \sum_\nu \theta_\nu \big(U_\nu |\psi_S\rangle\big) \otimes |\lambda_\nu\rangle$$

Then measurement operators $\mathcal{M}_\nu$ are equal to $\theta_\nu U_\nu$. The probability to get measurement outcome $\nu$, $\langle\psi_S|\mathcal{M}_\nu^\dagger \mathcal{M}_\nu|\psi_S\rangle = |\theta_\nu|^2$, is completely independent of systems state $|\psi_S\rangle$.

The POVM $(\mathcal{M}_\nu)$ (system $S$, interaction with the meter $M$ via $H = H_S + H_M + H_{SM}$, von Neumann measurements on the meter via $\mathcal{O}_M$) is a QND measurement of the system observable $\mathcal{O}_S$ if the eigenspaces of $\mathcal{O}_S$ are invariant with respect to the measurement operators $\mathcal{M}_\nu$. A sufficient but not necessary condition for this is $[H, \mathcal{O}_S] = 0$.
Under this condition $\mathcal{O}_S$ and $U_{S,M}$ commute. Assume $\mathcal{O}_S$ non degenerate and take the eigenstate $|\mu\rangle$ to the eigenvalue $\mu \in \mathbb{R}$:

$$\mathcal{O}_S U_{S,M}\big( |\mu\rangle \otimes |\theta_M\rangle \big) = U_{S,M}\mathcal{O}_S\big( |\mu\rangle \otimes |\theta_M\rangle \big) = \mu U_{S,M}\big( |\mu\rangle \otimes |\theta_M\rangle \big).$$

Thus $U_{S,M}\big( |\mu\rangle \otimes |\theta_M\rangle \big) = |\mu\rangle \otimes \big(U_\mu |\theta_M\rangle \big)$ with $U_\mu$ unitary on $\mathcal{H}_M$. We also have

$$U_{S,M}\big( |\mu\rangle \otimes |\theta_M\rangle \big) = \sum_\nu \mathcal{M}_\nu |\mu\rangle \otimes |\lambda_\nu\rangle \,.$$

Thus necessarily,each $\mathcal{M}_\nu |\mu\rangle$ is colinear to $|\mu\rangle$.
When $\rho = |\mu\rangle \langle\mu|$, the conditional state remains unchanged $\rho_+ = \mathbb{M}_\nu(\rho)$ whatever the meter measure outcome $\nu$ is.
When the spectrum of $\mathcal{O}_S$ is degenerate: for all $\nu$, $\mathcal{M}_\nu P_\mu = P_\mu \mathcal{M}_\nu$ where $P_\mu$ is the projector on the eigenspace associated to $\mu$:

- To the POVM $(\mathcal{M}_\nu)$ on $\mathcal{H}_S$ is attached a stochastic process of quantum state $\rho$

$$\rho_+ = \frac{\mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger}{\mathrm{Tr}\left(\mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger\right)} \text{ with probability } p_\nu = \mathrm{Tr}\left(\mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger\right).$$

- For any observable $A$ on $\mathcal{H}_S$, its conditional expectation value after the transition knowing the state $\rho$

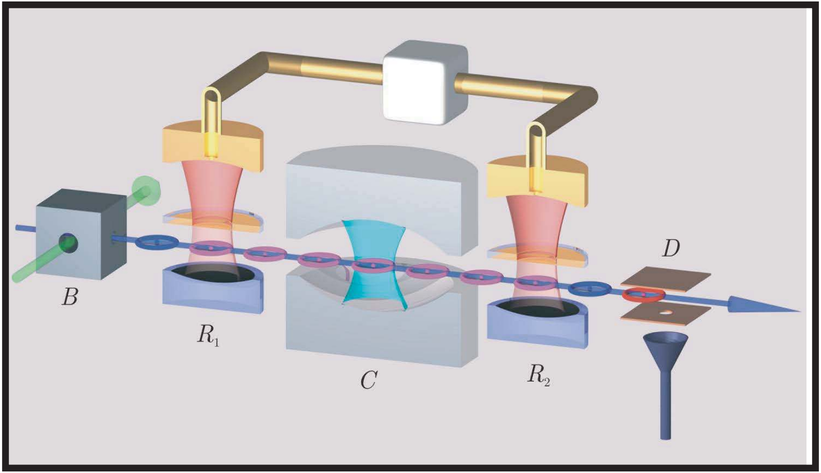
$$\mathbb{E}\left(\mathrm{Tr}\left(A\,\rho_+\right)|\rho\right) = \mathrm{Tr}\left(A\,\mathbb{K}\rho\right)$$

where the linear map $\rho \mapsto \mathbb{K}\rho = \sum_\nu \mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger$ is a Kraus map.

- If $\bar{A}$ is a stationary point of the adjoint Kraus map $\mathbb{K}^*$, $\mathbb{K}^*\bar{A} = \sum_\nu \mathcal{M}_\nu^\dagger \bar{A} \mathcal{M}_\nu$, then $\mathrm{Tr}\left(\bar{A}\rho\right)$ is a martingale:

$$\mathbb{E}\left(\mathrm{Tr}\left(\bar{A}\,\rho_+\right) \mid \rho\right) = \mathrm{Tr}\left(\bar{A}\,\mathbb{K}\rho\right) = \mathrm{Tr}\left(\rho\,\mathbb{K}^*\bar{A}\right) = \mathrm{Tr}\left(\rho\bar{A}\right).$$

- QND measurement of $\mathcal{O}_S = \sum_\mu \sigma_\mu P_\mu$: $\mathbb{K}^* P_\mu = P_\mu$ and each $\bar{\rho} = P_\mu / \mathrm{Tr}\left(P_\mu\right)$ is a fixed point of the above stochastic process ($\rho_+ \equiv \bar{\rho}$ if $\rho = \bar{\rho}$)




Atoms get out of box $B$ one by one, undergo then a first Rabi pulse in Ramsey zone $R_1$, become entangled with electromagnetic field trapped in $C$, undergo a second Rabi pulse in Ramsey zone $R_2$ and finally are measured in the detector $D$.

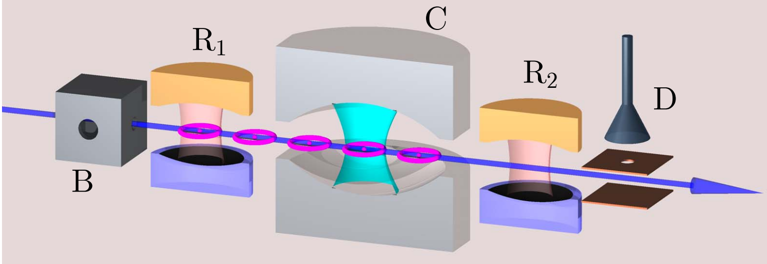
# The Markov chain model (1)

- System $S$ corresponds to a quantized mode in $C$:

$$\mathcal{H}_S = \left\{ \sum_{n=0}^{\infty} \psi_n |n\rangle \mid (\psi_n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

  where $|n\rangle$ represents the Fock state associated to exactly $n$ photons inside the cavity
- Meter $M$ is associated to atoms : $\mathcal{H}_M = \mathbb{C}^2$, each atom admits two-level and is described by a wave function $c_g |g\rangle + c_e |e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$; atoms leaving $B$ are all in state $|g\rangle$
- When atom comes out $B$, the state $|\Psi\rangle_B \in \mathcal{H}_M \otimes \mathcal{H}_S$ of the composite system atom/field is separable

$$|\Psi\rangle_B = |g\rangle \otimes |\psi\rangle .$$




- When atom comes out $B$: $|\Psi\rangle_B = |g\rangle \otimes |\psi\rangle$ .
- When atom comes out the first Ramsey zone $R_1$ the state remains separable but has changed to

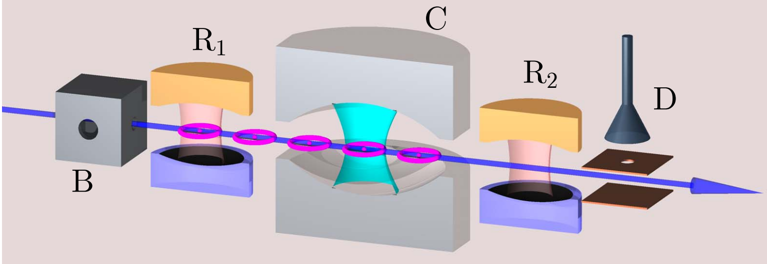
$$|\Psi\rangle_{R_1} = (U_{R_1} \otimes \mathbf{1}) |\Psi\rangle_B = (U_{R_1} |g\rangle) \otimes |\psi\rangle$$

where the unitary transformation performed in $R_1$ only affects the atom:

$$U_{R_1} = e^{-i\frac{\theta_1}{2}(x_1\sigma_x + y_1\sigma_y + z_1\sigma_z)} = \cos(\tfrac{\theta_1}{2}) - i\sin(\tfrac{\theta_1}{2})(x_1\sigma_x + y_1\sigma_y + z_1\sigma_z)$$

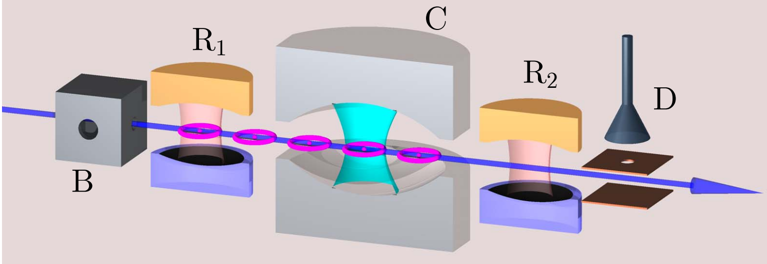
corresponds, in the Bloch sphere representation, to a rotation of angle $\theta_1$ around $x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k}$ $(x_1^2 + y_1^2 + z_1^2 = 1)$

# The Markov chain model (3)




- When atom comes out the first Ramsey zone $R_1$:
  $|\Psi\rangle_{R_1} = (U_{R_1}|g\rangle) \otimes |\psi\rangle$.
- When atom comes out cavity $C$, the state does not remain separable: atom and field becomes entangled and the state is described by
  $$|\Psi\rangle_C = U_C |\Psi\rangle_{R_1}$$
  where the unitary transformation $U_C$ on $\mathcal{H}_M \otimes \mathcal{H}_S$ is associated to a Jaynes-Cumming Hamiltonian:
  $$H_C = \frac{\Delta(t)}{2}\sigma_z + i\frac{\Omega(t)}{2}(\sigma_- a^\dagger - \sigma_+ a)$$
  Parameters: $\Delta(t) = \omega_{eg} - \omega_c$, $\Omega(t)$ depend on time $t$.

# The Markov chain model (4)




- When atom comes out cavity $C$: $|\Psi\rangle_C = U_C\big((U_{R_1}|g\rangle) \otimes |\psi\rangle\big)$.
- When atom comes out second Ramsey zone $R_2$, the state becomes

$$|\Psi\rangle_{R_2} = (U_{R_2} \otimes \mathbf{1})\,|\Psi\rangle_C \text{ with } U_{R_2} = e^{-i\frac{\theta_2}{2}(x_2\sigma_x + y_2\sigma_y + z_2\sigma_z)}$$

- Just before the measurement in $D$, the state is given by

$$|\Psi\rangle_{R_2} = U_{SM}\big(|g\rangle \otimes |\psi\rangle\big) = |g\rangle \otimes \mathcal{M}_g\,|\psi\rangle + |e\rangle \otimes \mathcal{M}_e\,|\psi\rangle$$

where $U_{SM} = U_{R_2}U_C U_{R_1}$ is the total unitary transformation defining the linear measurement operators $\mathcal{M}_g$ and $\mathcal{M}_e$ on $\mathcal{H}_S$.

# The Markov chain model (5)

Just before the measurement in $D$, the atom/field state is:

$$|g\rangle \otimes \mathcal{M}_g |\psi\rangle + |e\rangle \otimes \mathcal{M}_e |\psi\rangle$$

Denote by $s \in \{g, e\}$ the measurement outcome in detector $D$: with probability $p_s = \left\langle \psi | \mathcal{M}_s^\dagger \mathcal{M}_s | \psi \right\rangle$ we get $s$. Just after the measurement outcome $s$, the state becomes separable:

$$|\Psi\rangle_D = \tfrac{1}{\sqrt{p_s}} \; |s\rangle \otimes (\mathcal{M}_s |\psi\rangle) = \frac{|s\rangle \otimes (\mathcal{M}_s |\psi\rangle)}{\sqrt{\left\langle \psi | \mathcal{M}_s^\dagger \mathcal{M}_s | \psi \right\rangle}}.$$

Markov process (density matrix formulation)

$$\rho_+ = \begin{cases} \mathbb{M}_g(\rho) = \frac{\mathcal{M}_g \rho \mathcal{M}_g^\dagger}{\mathrm{Tr}\left(\mathcal{M}_g \rho \mathcal{M}_g^\dagger\right)}, & \text{with probability } p_g = \mathrm{Tr}\left(\mathcal{M}_g \rho \mathcal{M}_g^\dagger\right); \\ \mathbb{M}_e(\rho) = \frac{\mathcal{M}_e \rho \mathcal{M}_e^\dagger}{\mathrm{Tr}\left(\mathcal{M}_e \rho \mathcal{M}_e^\dagger\right)}, & \text{with probability } p_e = \mathrm{Tr}\left(\mathcal{M}_e \rho \mathcal{M}_e^\dagger\right). \end{cases}$$

**Exercice**

*Show that, for any density matrix $\rho$, $\mathcal{M}_g \rho \mathcal{M}_g^\dagger + \mathcal{M}_e \rho \mathcal{M}_e^\dagger$ does not depend on $(\theta_2, x_2, y_2, z_2)$, the parameters of the second Ramsey pulse in $R_2$.*

# Dispersive case with adiabatic coupling

We start from $|\Psi\rangle_B = |g\rangle\,|\psi\rangle$ and apply the transformations:

$U_{R_1} = e^{-i\frac{\pi}{4}\sigma_y}$, $U_C = |g\rangle\langle g|\, e^{i\phi(N)} + |e\rangle\langle e|\, e^{-i\phi(N+I)}$, $U_{R_2} = e^{-i\frac{\pi}{4}(-\sin\eta\sigma_x + \cos\eta\sigma_y)}$.

Therefore

$$|\Psi\rangle_{R_1} = \frac{|g\rangle - |e\rangle}{\sqrt{2}} \otimes |\psi\rangle\,.$$

Then

$$|\Psi\rangle_C = \tfrac{1}{\sqrt{2}}\,|g\rangle \otimes e^{i\phi(N)}\,|\psi\rangle - \tfrac{1}{\sqrt{2}}\,|e\rangle \otimes e^{-i\phi(N+1)}\,|\psi\rangle\,.$$

Finally

$$2\,|\Psi\rangle_{R_2} = \big(\,|g\rangle - e^{-i\eta}\,|e\rangle\,\big) \otimes e^{i\phi(N)}\,|\psi\rangle - \big(e^{i\eta}\,|g\rangle + |e\rangle\,\big) \otimes e^{-i\phi(N+1)}\,|\psi\rangle$$
$$= |g\rangle \otimes \left(e^{i\phi(N)} - e^{i(\eta-\phi(N+1))}\right)|\psi\rangle - |e\rangle \otimes \left(e^{-i(\eta-\phi(N))} + e^{-i\phi(N+1)}\right)|\psi\rangle\,.$$

With linear approximation of $\phi$ (valid when $\Delta \gg \Omega_0$), $\phi(N) = \vartheta_0 + N\vartheta$, we get

## Kraus operators

Taking $\varphi_0$ an arbitrary phase and $\eta = 2(\vartheta_0 - \varphi_0) + \vartheta - \pi$, we find

$$|\Psi\rangle_{R_2} = e^{i\theta_g}\,|g\rangle \otimes \mathcal{M}_g\,|\psi\rangle + e^{i\theta_e}\,|e\rangle \otimes \mathcal{M}_e\,|\psi\rangle$$

where $\theta_g$ and $\theta_e$ are constant phases and

$$\mathcal{M}_g = \cos(\varphi_0 + N\vartheta), \qquad \mathcal{M}_e = \sin(\varphi_0 + N\vartheta)$$

# Markov chain model: summary

Therefore the Markov chain model is given by

$$\rho_{k+1} = \mathbb{M}_{s_k}(\rho_k) = \frac{\mathcal{M}_{s_k}\rho_k\mathcal{M}_{s_k}^\dagger}{\text{Tr}\left(\mathcal{M}_{s_k}\rho_k\mathcal{M}_{s_k}^\dagger\right)},$$

where $s_k = g$ or $e$ with associated probabilities $p_{g,k}$ and $p_{e,k}$ given by

$$p_{g,k} = \text{Tr}\left(\mathcal{M}_g\rho_k\mathcal{M}_g^\dagger\right) \quad \text{and} \quad p_{e,k} = \text{Tr}\left(\mathcal{M}_e\rho_k\mathcal{M}_e^\dagger\right).$$

Here $\mathcal{M}_g$ and $\mathcal{M}_e$ are given by

$$\mathcal{M}_g = \cos(\varphi_0 + N\vartheta), \quad \mathcal{M}_e = \sin(\varphi_0 + N\vartheta)$$

This is a QND measurement for the observable $N$ of photon number. Indeed, as the Kraus operators $\mathcal{M}_g$ and $\mathcal{M}_e$ commute with $N$, the mean value of $N$ does not change through the measurement procedure:

$$\mathbb{E}\left(\text{Tr}\left(N\rho_{k+1}\right) \mid \rho_k\right) = \text{Tr}\left(N\rho_k\right).$$

Also, the eigenstates of the observable $N$ (the Fock states) are invariant with respect to the measurement procedure:

$$\mathbb{M}_g(|n\rangle\langle n|) = |n\rangle\langle n| \quad \text{and} \quad \mathbb{M}_e(|n\rangle\langle n|) = |n\rangle\langle n| \qquad \text{for all } n.$$