Modeling and Control of the LKB Photon-Box: [1] Introduction

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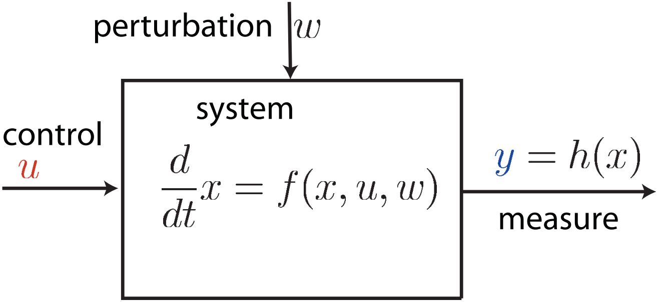
[1] LKB: Laboratoire Kastler Brossel, ENS, Paris.
Several slides have been used during the IHP course (fall 2010) given with Mazyar Mirrahimi (INRIA) see:
http://cas.ensmp.fr/~rouchon/QuantumSyst/index.html

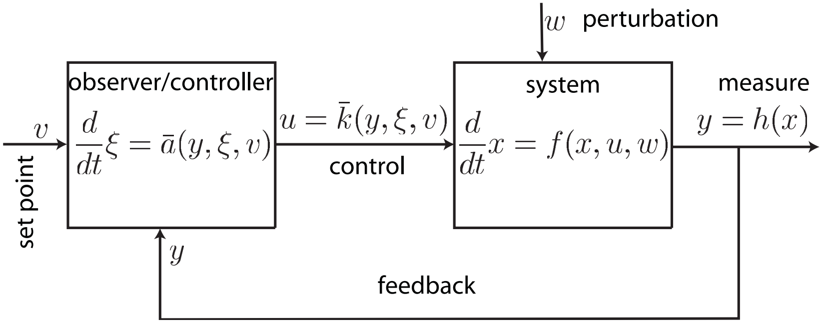
Outline



For the harmonic oscillator of pulsation ω with measured position y, controlled by the force u and subject to an additional unknown force w.

$$x = (x_1, x_2) \in \mathbb{R}^2, \qquad y = x_1$$

$$\frac{d}{dt}x_1 = x_2, \quad \frac{d}{dt}x_2 = -\omega^2 x_1 + u + w$$



Proportional Integral Derivative (PID) for $\frac{d^2}{dt^2}y = -\omega^2 y + u + w$ with the set point $v = y^{\text{set point}}$

$$u = -K_p\left(y - y^{\text{set point}}\right) - K_d \frac{d}{dt}\left(y - y^{\text{set point}}\right) - K_{\text{int}} \int \left(y - y^{\text{set point}}\right)$$

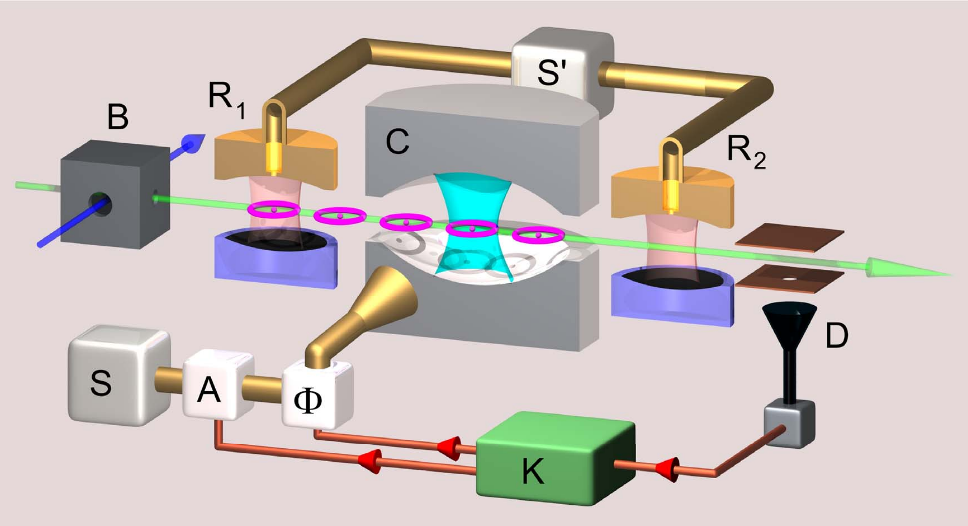
with the positive gains $(K_p, K_d, K_{\text{int}})$ tuned as follows $(0 < \Omega_0 \sim \omega$, $0 < \xi \sim 1$, $0 < \epsilon \ll 1$:

$$K_p = \Omega_0^2, \quad K_d = 2\xi\Omega_0, \quad , K_{\text{int}} = \epsilon\Omega_0^3.$$

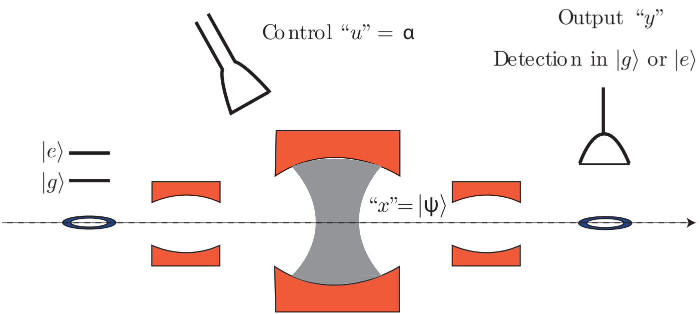
Control of a classical harmonic oscillator

- Controllability: the control u can steer the state x to any location ($\frac{d}{dt}x_1 = x_2$, $\frac{d}{dt}x_2 = -\omega^2 x_1 + u$).
- Observability: from the knowledge of u and y one can recover without ambiguity the state x. ($y = x_1$ and $x_2 = \frac{d}{dt}y$).
- Feed-forward $u = u^r(t)$ associated to reference trajectory $t \mapsto (x^r(t), u^r(t), y^r(t))$ (performance).
- Feed-back $u = u^r(t) + \Delta u$ where Δu depends on the measured output error $\Delta y = y - y^r(t)$ (stability).
- Stability and robustness : asymptotic regime for t large of Δx and Δy, sensitivity to perturbations and errors.

LKB Photon-box: feedback stabilization of photon-number states



Control of quantum harmonic oscillator: LKB photon-box

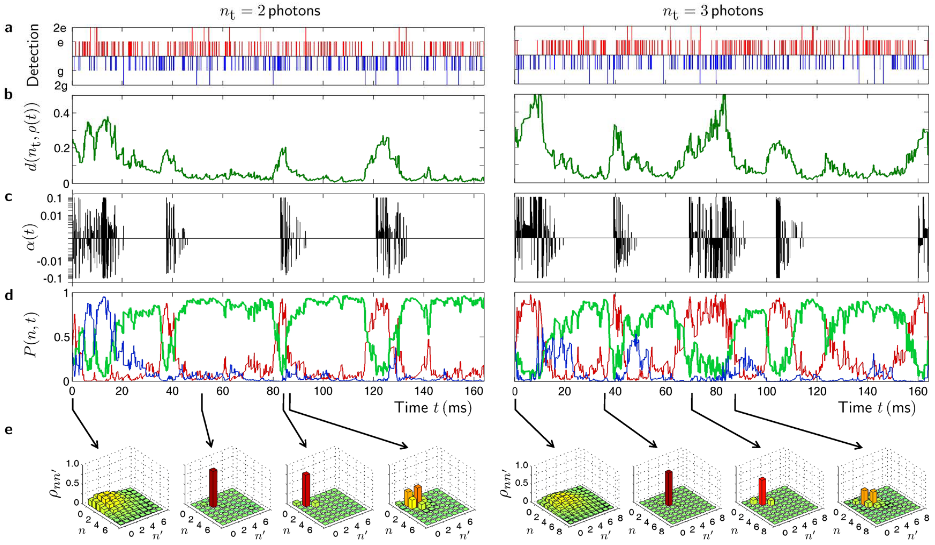


Simple schematic of LKB experiment for control of cavity field

A discrete-time system: non-linear Markov chain of state $|\psi\rangle$

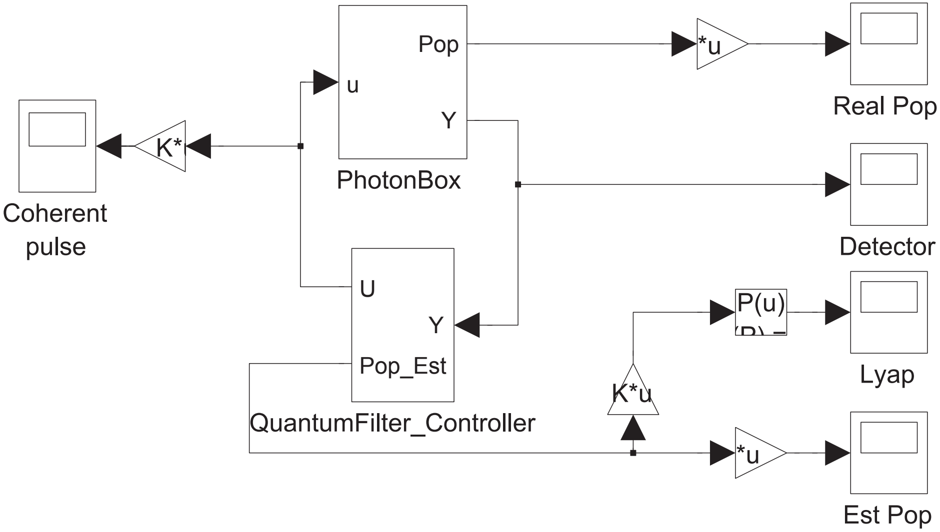
$$
|\psi\rangle_{k+1} = \begin{cases} \dfrac{D_\alpha \, \mathcal{M}_g \, |\psi\rangle_k}{\left\| \mathcal{M}_g \, |\psi\rangle_k \right\|_{\mathcal{H}}} & \text{Detect. in } |g\rangle \left(\text{proba. } \left\| \mathcal{M}_g \, |\psi\rangle_k \right\|^2_{\mathcal{H}}\right) \\[2ex] \dfrac{D_\alpha \, \mathcal{M}_e \, |\psi\rangle_k}{\left\| \mathcal{M}_e \, |\psi\rangle_k \right\|_{\mathcal{H}}} & \text{Detect. in } |e\rangle \left(\text{proba. } \left\| \mathcal{M}_e \, |\psi\rangle_k \right\|^2_{\mathcal{H}}\right) \end{cases}
$$

Photon-box: experimental data in closed-loop[2]

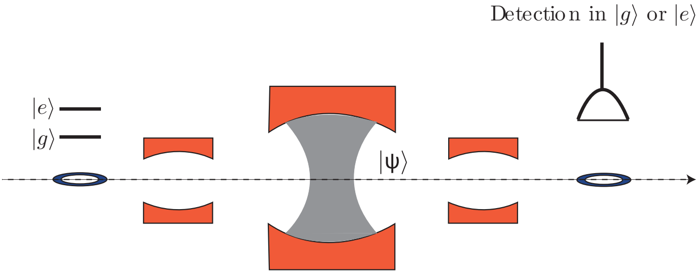


[2]C. Sayrin et al.: Real-time quantum feedback prepares and stabilizes photon number states. To appear in Nature. (http://arxiv.org/abs/1107.4027);

Photon-box: simulations in closed-loop



Photon-box (1): measurement process



Simple schematic of LKB experiment for measurement of cavity field

Photon-box (2) : atom-field entanglement

Initial state Atom in $|g\rangle$ and cavity in $|\psi\rangle \in \mathcal{H}$ where

$$\mathcal{H} = \left\{ \sum_{k=n}^{\infty} c_n |n\rangle \mid (c_n) \in l^2(\mathbb{C}) \right\}.$$

We can write the initial state as

$$|g\rangle \otimes |\psi\rangle \in \mathbb{C}^2 \otimes \mathcal{H}.$$

State before detection a joint unitary evolution implies an entangled state

$$|g\rangle \otimes \mathcal{M}_g |\psi\rangle + |e\rangle \otimes \mathcal{M}_e |\psi\rangle$$

where $\mathcal{M}_g$ and $\mathcal{M}_e$ are operators acting on $\mathcal{H}$.
The unitarity condition implies:

$$\mathcal{M}_g^\dagger \mathcal{M}_g + \mathcal{M}_e^\dagger \mathcal{M}_e = \mathbf{1}$$

Example of non-resonant interaction

$$\mathcal{M}_g = \cos(\vartheta \mathbf{N} + \varphi), \quad \mathcal{M}_e = \sin(\vartheta \mathbf{N} + \varphi), \quad \mathbf{N} = \text{diag}(n)$$

Photon-box (3): entanglement

Final state is inseparable: we can not write

$$|g\rangle \otimes \mathcal{M}_g |\psi\rangle + |e\rangle \otimes \mathcal{M}_e |\psi\rangle \neq \left(\tilde{\alpha} |g\rangle + \tilde{\beta} |e\rangle\right) \otimes \left(\sum_n \tilde{c}_n |n\rangle\right).$$

We can not associate to the cavity (nor to the atom) a well-defined wavefunction just before the measurement.

However, we can still compute the probability of having the atom in $|g\rangle$ or in $|e\rangle$:

$$P_g = \left\| \mathcal{M}_g |\psi\rangle \right\|^2_{\mathcal{H}}, \qquad P_e = \left\| \mathcal{M}_e |\psi\rangle \right\|^2_{\mathcal{H}}.$$

Photon-box (4): measurement and collapse

Measurement in $|g\rangle$

$$|g\rangle \otimes \mathcal{M}_g |\psi\rangle + |e\rangle \otimes \mathcal{M}_e |\psi\rangle \longrightarrow \frac{|g\rangle \otimes \mathcal{M}_g |\psi\rangle}{\left\| \mathcal{M}_g |\psi\rangle \right\|_{\mathcal{H}}},$$

Measurement in $|e\rangle$

$$|g\rangle \otimes \mathcal{M}_g |\psi\rangle + |e\rangle \otimes \mathcal{M}_e |\psi\rangle \longrightarrow \frac{|e\rangle \otimes \mathcal{M}_e |\psi\rangle}{\left\| \mathcal{M}_e |\psi\rangle \right\|_{\mathcal{H}}},$$

Photon-box (5): quantum Monte-Carlo trajectories

Stochastic evolution: ψ_k the wave function after the measurement of atom number $k-1$.

$$
|\psi\rangle_{k+1} = \begin{cases} \dfrac{D_\alpha \, \mathcal{M}_g \, |\psi\rangle_k}{\left\| \mathcal{M}_g \, |\psi\rangle_k \right\|_{\mathcal{H}}} & \text{Detect. in } |g\rangle \; \left(\text{proba. } \left\| \mathcal{M}_g \, |\psi\rangle_k \right\|^2_{\mathcal{H}}\right) \\[2ex] \dfrac{D_\alpha \, \mathcal{M}_e \, |\psi\rangle_k}{\left\| \mathcal{M}_e \, |\psi\rangle_k \right\|_{\mathcal{H}}} & \text{Detect. in } |e\rangle \; \left(\text{proba. } \left\| \mathcal{M}_e \, |\psi\rangle_k \right\|^2_{\mathcal{H}}\right) \end{cases}
$$

We have a Markov chain

Photon-box (6): imperfect measurement

The atom-detector does not always detect the atoms.
Therefore 3 outcomes:
Atom in $|g\rangle$, Atom in $|e\rangle$, No detection

Best estimate for the no-detection case

$$\mathbb{E}\left(|\psi\rangle_+ \mid |\psi\rangle\right) = \left\|\mathcal{M}_g |\psi\rangle\right\|_{\mathcal{H}} \mathcal{M}_g |\psi\rangle + \left\|\mathcal{M}_e |\psi\rangle\right\|_{\mathcal{H}} \mathcal{M}_e |\psi\rangle$$

This is not a well-defined wavefunction

Barycenter in the sense of geodesics of $\mathbb{S}(\mathcal{H})$
not invariant with respect to a change of global phase

We need a barycenter in the sense of the projective space
$\mathbb{CP}(\mathcal{H}) \equiv \mathbb{S}(\mathcal{H})/\mathbb{S}^1$

Photon-box (7): density matrix language

Projector over the state $|\psi\rangle$: $P_{|\psi\rangle} = |\psi\rangle\langle\psi|$

Detection in $|g\rangle$: the projector is given by

$$P_{|\psi_+\rangle} = \frac{\mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger}{\left\| \mathcal{M}_g |\psi\rangle \right\|_{\mathcal{H}}^2} = \frac{\mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger}{\left| \left\langle \psi \mid \mathcal{M}_g^\dagger \mathcal{M}_g \mid \psi \right\rangle \right|^2} = \frac{\mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger}{\mathrm{Tr}\left(\mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger \right)}$$

Detection in $|e\rangle$: the projector is given by

$$P_{|\psi_+\rangle} = \frac{\mathcal{M}_e |\psi\rangle\langle\psi| \mathcal{M}_e^\dagger}{\mathrm{Tr}\left(\mathcal{M}_e |\psi\rangle\langle\psi| \mathcal{M}_e^\dagger \right)}$$

Probabilities:

$$p_g = \mathrm{Tr}\left(\mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger \right) \quad \text{and} \quad p_e = \mathrm{Tr}\left(\mathcal{M}_e |\psi\rangle\langle\psi| \mathcal{M}_e^\dagger \right)$$

Photon-box (8): density matrix language

Imperfect detection: barycenter

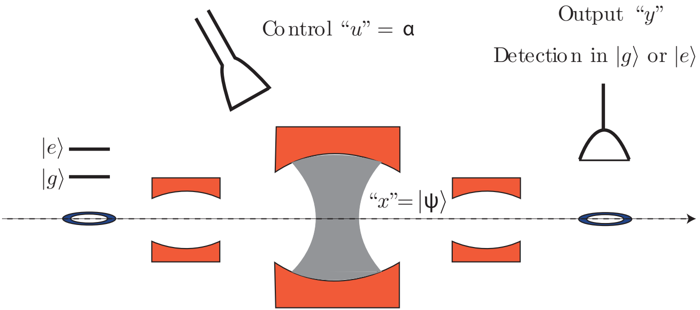
$$|\psi\rangle\langle\psi| \longrightarrow p_g \frac{\mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger}{\mathrm{Tr}\left(\mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger\right)} + p_e \frac{\mathcal{M}_e |\psi\rangle\langle\psi| \mathcal{M}_e^\dagger}{\mathrm{Tr}\left(\mathcal{M}_e |\psi\rangle\langle\psi| \mathcal{M}_e^\dagger\right)}$$
$$= \mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger + \mathcal{M}_e |\psi\rangle\langle\psi| \mathcal{M}_e^\dagger.$$

This is not anymore a projector: no well-defined wave function

Adapted state space

$$\mathcal{X} = \{\rho \in \mathcal{L}(\mathcal{H}) \mid \rho^\dagger = \rho, \rho \geq 0, \mathrm{Tr}(\rho) = 1\}$$

A classical control input



The control input $u = \alpha$ is classical and acts on the state $|\psi\rangle$ according to the unitary transformation D_α (displacement of amplitude α):

$$|\psi\rangle \mapsto D_\alpha |\psi\rangle = e^{\alpha \mathbf{a}^\dagger - \alpha^* \mathbf{a}} |\psi\rangle .$$

Outline of the lectures

Introduction: LKB Photon-Box, experimental data and simulations, non-linear state feedback stabilizing Fock states.

Spin systems: two-level systems, Dirac notations, Pauli matrices, density matrix as a Bloch vector, RWA, averaging, Rabi oscillation, adiabatic invariance and propagator.

Spin-Spring systems: harmonic oscillator, creation/annihilation operators, unitary displacement operator, coherent states, Jaynes-Cummings model, composite systems and tensor products, RWA and dressed states, dispersive and resonant propagators.

Quantum Non-Demolition (QND) measurement: LKB photon Box, QND photon counting, Positive Operator Valued Measurement (POVM), discrete-time quantum trajectories and Markov chains, Kraus maps.

Feedback stabilization with QND measures: martingales and Lyapunov functions, stochastic convergence, construction of strict control Lyapunov function, feedback stabilization.

State estimations: quantum filtering, ideal case, experimental case including detection errors, Bayes law.

Main references

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Specific references on the Photon-Box

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  R. Somaraju, M. Mirrahimmi, P. Rouchon: Approximate stabilization of an infinite dimensional quantum stochastic system. CDC/ECC 2011 (http://arxiv.org/abs/1103.1724).