Modeling and Control of Quantum Systems

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Outline

1. Measurement uncertainties, Bayesian filter and decoherence
2. Markov chains, martingales and convergence theorems
3. Asymptotic behavior of LKB-Photon box (dispersive case)
4. Quantum separation principle
5. Lyapunov feedback for LKB-photon box
6. Realistic closed-loop simulations

Why density matrices (1)

Measurement in $|g\rangle$

$$|g\rangle \otimes \mathcal{M}_g |\psi\rangle + |e\rangle \otimes \mathcal{M}_e |\psi\rangle \longrightarrow \frac{|g\rangle \otimes \mathcal{M}_g |\psi\rangle}{\left\| \mathcal{M}_g |\psi\rangle \right\|_{\mathcal{H}}},$$

Measurement in $|e\rangle$

$$|g\rangle \otimes \mathcal{M}_g |\psi\rangle + |e\rangle \otimes \mathcal{M}_e |\psi\rangle \longrightarrow \frac{|e\rangle \otimes \mathcal{M}_e |\psi\rangle}{\left\| \mathcal{M}_e |\psi\rangle \right\|_{\mathcal{H}}},$$

Why density matrices (2)

The atom-detector does not always detect the atoms.
Therefore 3 outcomes:
Atom in $|g\rangle$, Atom in $|e\rangle$, No detection

Best estimate for the no-detection case

$$\mathbb{E}\left(|\psi\rangle_+ \mid |\psi\rangle\right) = \left\|\mathcal{M}_g |\psi\rangle\right\|_{\mathcal{H}} \mathcal{M}_g |\psi\rangle + \left\|\mathcal{M}_e |\psi\rangle\right\|_{\mathcal{H}} \mathcal{M}_e |\psi\rangle$$

This is not a well-defined wavefunction

Barycenter in the sense of geodesics of $\mathbb{S}(\mathcal{H})$
not invariant with respect to a change of global phase

We need a barycenter in the sense of the projective space
$\mathbb{CP}(\mathcal{H}) \equiv \mathbb{S}(\mathcal{H})/\mathbb{S}^1$

Why density matrices (3)

Projector over the state $|\psi\rangle$: $P_{|\psi\rangle} = |\psi\rangle\langle\psi|$

Detection in $|g\rangle$: the projector is given by

$$P_{|\psi_+\rangle} = \frac{\mathcal{M}_g|\psi\rangle\langle\psi|\mathcal{M}_g^\dagger}{\left\|\mathcal{M}_g|\psi\rangle\right\|_{\mathcal{H}}^2} = \frac{\mathcal{M}_g|\psi\rangle\langle\psi|\mathcal{M}_g^\dagger}{\left|\left\langle\psi \mid \mathcal{M}_g^\dagger\mathcal{M}_g \mid \psi\right\rangle\right|^2} = \frac{\mathcal{M}_g|\psi\rangle\langle\psi|\mathcal{M}_g^\dagger}{\mathrm{Tr}\left(\mathcal{M}_g|\psi\rangle\langle\psi|\mathcal{M}_g^\dagger\right)}$$

Detection in $|e\rangle$: the projector is given by

$$P_{|\psi_+\rangle} = \frac{\mathcal{M}_e|\psi\rangle\langle\psi|\mathcal{M}_e^\dagger}{\mathrm{Tr}\left(\mathcal{M}_e|\psi\rangle\langle\psi|\mathcal{M}_e^\dagger\right)}$$

Probabilities:

$$p_g = \mathrm{Tr}\left(\mathcal{M}_g|\psi\rangle\langle\psi|\mathcal{M}_g^\dagger\right) \quad \text{and} \quad p_e = \mathrm{Tr}\left(\mathcal{M}_e|\psi\rangle\langle\psi|\mathcal{M}_e^\dagger\right)$$

Why density matrices (4)

Imperfect detection: barycenter

$$|\psi\rangle\langle\psi| \longrightarrow p_g \frac{\mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger}{\mathrm{Tr}\left(\mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger\right)} + p_e \frac{\mathcal{M}_e |\psi\rangle\langle\psi| \mathcal{M}_e^\dagger}{\mathrm{Tr}\left(\mathcal{M}_e |\psi\rangle\langle\psi| \mathcal{M}_e^\dagger\right)}$$
$$= \mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger + \mathcal{M}_e |\psi\rangle\langle\psi| \mathcal{M}_e^\dagger.$$

This is not anymore a projector: no well-defined wave function

New state space of quantum states ρ:

$$\mathcal{X} = \{\rho \in \mathcal{L}(\mathcal{H}) \mid \rho^\dagger = \rho, \rho \geq 0, \mathrm{Tr}(\rho) = 1\}$$

Pure quantum states ρ correspond to rank 1 projectors and thus to wave functions $|\psi\rangle$ with $\rho = |\psi\rangle\langle\psi|$.

Kraus map

What if we do not detect the atoms after they exit R_2?

The “best estimate” of the cavity state is given by its expectation value

$$\rho_+ = p_{g,k}\mathbb{M}_g(\rho) + p_{e,k}\mathbb{M}_e(\rho) = \mathcal{M}_g \rho \mathcal{M}_g^\dagger + \mathcal{M}_e \rho \mathcal{M}_e^\dagger =: \mathbb{K}(\rho).$$

This linear map is called the Kraus map associated to the Kraus operators $\mathcal{M}_g$ and $\mathcal{M}_e$.

In the same way and through a Bayesian filter we can take into account various uncertainties.

Some uncertainties

Pulse occupation The probability that a pulse is occupied by an atom is given by η_a ($\eta_a \in (0, 1]$ is called the pulse occupancy rate);

Detector efficiency The detector can miss an atom with a probability of $1 - \eta_d$ ($\eta_d \in (0, 1]$ is called the detector's efficiency rate);

Detector faults The detector can make a mistake by detecting an atom in $|g\rangle$ while it is in the state $|e\rangle$ or vice-versa; this happens with a probability of η_f ($\eta_f \in [0, 1/2]$ is called the detector's fault rate);

We basically have three possibilities for the detection output:

Atom detected in $|g\rangle$ either the atom is really in the state $|g\rangle$ or the detector has made a mistake and it is actually in the state $|e\rangle$;

Atom detected in $|e\rangle$ either the atom is really in the state $|e\rangle$ or the detector has made a mistake and it is actually in the state $|g\rangle$;

No atom detected either the pulse has been empty or the detector has missed the atom.

Atom detected in $|g\rangle$

Either the atom is actually in the state $|e\rangle$ and the detector has made a mistake by detecting it in $|g\rangle$ (this happens with a probability p_g^f) or the atom is really in the state $|g\rangle$ (this happens with probability $1 - p_g^f$).

Conditional probablity p_g^f: We apply the Bayesian formula

$$p_g^f = \frac{\eta_f p_e}{\eta_f p_e + (1 - \eta_f) p_g},$$

where $p_g = \mathrm{Tr}\left(\mathcal{M}_g \rho \mathcal{M}_g^\dagger\right)$ and $p_e = \mathrm{Tr}\left(\mathcal{M}_e \rho \mathcal{M}_e^\dagger\right)$.

Conditional evolution of density matrix:

$$\begin{aligned}\rho_+ &= p_g^f \mathbb{M}_e(\rho) + (1 - p_g^f)\mathbb{M}_g(\rho) \\ &= \frac{\eta_f}{\eta_f p_e + (1-\eta_f) p_g} \mathcal{M}_e \rho \mathcal{M}_e^\dagger + \frac{1-\eta_f}{\eta_f p_e + (1-\eta_f) p_g} \mathcal{M}_g \rho \mathcal{M}_g^\dagger.\end{aligned}$$

Atom detected in $|e\rangle$

In the same way

$$\rho_+ = \frac{\eta_f}{\eta_f p_g + (1-\eta_f)p_e} \mathcal{M}_g \rho \mathcal{M}_g^\dagger + \frac{1-\eta_f}{\eta_f p_g + (1-\eta_f)p_e} \mathcal{M}_e \rho \mathcal{M}_e^\dagger.$$

No atom detected

Either the pulse has been empty (this happens with a probability p_{na}) or there has been an atom which has not been detected by the detector (this happens with the probability $1 - p_{\text{na}}$).

Conditional probability p_{na}:

$$p_{\text{na}} = \frac{1 - \eta_a}{\eta_a(1 - \eta_d) + (1 - \eta_a)} = \frac{1 - \eta_a}{1 - \eta_a \eta_d}.$$

In such case the density matrix remains untouched.
The undetected atom case leads to an evolution of the density matrix through the Kraus representation.

Conditional evolution:

$$\begin{aligned} \rho_+ &= p_{\text{na}}\ \rho + (1 - p_{\text{na}})(\mathcal{M}_g \rho \mathcal{M}_g^\dagger + \mathcal{M}_e \rho \mathcal{M}_e^\dagger) \\ &= \frac{1 - \eta_a}{1 - \eta_a \eta_d} \rho + \frac{\eta_a (1 - \eta_d)}{1 - \eta_a \eta_d} (\mathcal{M}_g \rho \mathcal{M}_g^\dagger + \mathcal{M}_e \rho \mathcal{M}_e^\dagger). \end{aligned}$$

Cavity decay

Absorption of photon by cavity mirrors characterized by photon life-time inside the cavity $T_{\text{cav}} = 1/\kappa_{\text{loss}}$.
When $T_{\text{cav}} \gg \tau_a$ (τ_a sampling time, time interval between two atoms)[1]:

$$\rho_+ = \begin{cases} \frac{\mathcal{M}_{\text{loss}}\rho\mathcal{M}_{\text{loss}}^\dagger}{\text{Tr}\left(\mathcal{M}_{\text{loss}}\rho\mathcal{M}_{\text{loss}}^\dagger\right)} = \frac{a\rho a^\dagger}{\text{Tr}(\mathbf{N}\rho)} & \text{prob. } \kappa_{\text{loss}}\tau_a \text{Tr}\left(\mathbf{N}\rho\right); \\ \frac{\mathcal{M}_{\text{no-loss}}\rho\mathcal{M}_{\text{no-loss}}^\dagger}{\text{Tr}\left(\mathcal{M}_{\text{no-loss}}\rho\mathcal{M}_{\text{no-loss}}^\dagger\right)} & \text{prob. } 1 - \kappa_{\text{loss}}\tau_a \text{Tr}\left(\mathbf{N}\rho\right); \end{cases}$$

where, up to second order terms in $\kappa_{\text{loss}}\tau_a$,

$$\mathcal{M}_{\text{loss}} = \sqrt{\kappa_{\text{loss}}\tau_a}\, a, \qquad \mathcal{M}_{\text{no-loss}} = \mathbf{1} - \frac{\kappa_{\text{loss}}\tau_a}{2} a^\dagger a.$$

Associated Kraus map:

$$\rho \mapsto \mathcal{M}_{\text{loss}}\rho\mathcal{M}_{\text{loss}}^\dagger + \mathcal{M}_{\text{no-loss}}\rho\mathcal{M}_{\text{no-loss}}^\dagger$$
$$= \rho + \kappa_{\text{loss}}\tau_a \left(a\rho a^\dagger - \tfrac{1}{2}a^\dagger a\rho - \tfrac{1}{2}\rho a^\dagger a\right),$$

[1] LKB Experimental setup: $\tau_a \sim 1O^{-4}$ s and $T_{\text{cav}} \sim 10^{-1}$ s.

Cavity decay and thermal photons (1)

The thermal photon gain can be treated through the measurement operator $\mathcal{M}_{\text{gain}} = \sqrt{\kappa_{\text{gain}}\tau_a}\,a^\dagger$ instead of $\mathcal{M}_{\text{loss}} = \sqrt{\kappa_{\text{loss}}\tau_a}\,a$ where κ_{loss} and κ_{gain} are expressed in term of cavity decay time T_{cav} and n_{th} thermal photon number[2]

$$\kappa_{\text{loss}} = \frac{1+n_{\text{th}}}{T_{\text{cav}}}, \qquad \kappa_{\text{gain}} = \frac{n_{\text{th}}}{T_{\text{cav}}}.$$

Up to second order term in $\frac{\tau_a}{T_{\text{cav}}}$ we have

$$\rho_+ = \begin{cases} \frac{\mathcal{M}_{\text{loss}}\rho\mathcal{M}_{\text{loss}}^\dagger}{\text{Tr}\left(\mathcal{M}_{\text{loss}}\rho\mathcal{M}_{\text{loss}}^\dagger\right)} = \frac{a\rho a^\dagger}{\text{Tr}(\mathbf{N}\rho)} & \text{prob. } p_{\text{loss}} = \kappa_{\text{loss}}\tau_a \text{Tr}\left(\mathbf{N}\rho\right); \\ \frac{\mathcal{M}_{\text{gain}}\rho\mathcal{M}_{\text{gain}}^\dagger}{\text{Tr}\left(\mathcal{M}_{\text{gain}}\rho\mathcal{M}_{\text{gain}}^\dagger\right)} = \frac{a^\dagger\rho a}{\text{Tr}((\mathbf{N}+\mathbf{1})\rho)} & \text{prob. } p_{\text{gain}} = \kappa_{\text{gain}}\tau_a \text{Tr}\left((\mathbf{N}+\mathbf{1})\rho\right); \\ \frac{\mathcal{M}_{\text{no}}\rho\mathcal{M}_{\text{no}}^\dagger}{\text{Tr}\left(\mathcal{M}_{\text{no}}\rho\mathcal{M}_{\text{no}}^\dagger\right)} & \text{prob. } 1 - p_{\text{loss}} - p_{\text{gain}}; \end{cases}$$

with

$$\mathcal{M}_{\text{no}} = \mathbf{1} - \frac{\kappa_{\text{loss}}\tau_a}{2}a^\dagger a - \frac{\kappa_{\text{gain}}\tau_a}{2}aa^\dagger = \left(1 - \frac{\kappa_{\text{gain}}\tau_a}{2}\right)\mathbf{1} - \frac{(\kappa_{\text{loss}}+\kappa_{\text{gain}})\tau_a}{2}\mathbf{N}.$$

[2] LKB Experimental setup: $n_{\text{th}} \sim \frac{1}{20}$.

Cavity decay and thermal photons (2)

The Kraus map reads:

$$
\begin{aligned}
\rho \mapsto \mathcal{M}_{\text{loss}}\rho\mathcal{M}_{\text{loss}}^{\dagger} + \mathcal{M}_{\text{gain}}\rho\mathcal{M}_{\text{gain}}^{\dagger} + \mathcal{M}_{\text{no}}\rho\mathcal{M}_{\text{no}}^{\dagger} \\
= \rho + \tfrac{(1+n_{\text{th}})\tau_a}{T_{\text{cav}}}\left(a\rho a^{\dagger} - \tfrac{1}{2}a^{\dagger}a\rho - \tfrac{1}{2}\rho a^{\dagger}a\right) \\
+ \tfrac{n_{\text{th}}\tau_a}{T_{\text{cav}}}\left(a^{\dagger}\rho a - \tfrac{1}{2}aa^{\dagger}\rho - \tfrac{1}{2}\rho aa^{\dagger}\right)
\end{aligned}
$$

Stability and convergence of stochastic processes (1)

Convergence of a random process

Consider (X_n) a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in a Banach space $\mathcal{X}$. The random process X_n is said to,

1. converge in probability towards the random variable X if for all $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(|X_n - X| > \epsilon\right) = \lim_{n\to\infty} \mathbb{P}\left(\omega \in \Omega \mid \|X_n(\omega) - X(\omega)\| > \epsilon\right) = 0;$$

2. converge almost surely towards the random variable X if

$$\mathbb{P}\left(\lim_{n\to\infty} X_n = X\right) = \mathbb{P}\left(\omega \in \Omega \mid \lim_{n\to\infty} X_n(\omega) = X(\omega)\right) = 1;$$

3. converge in mean towards the random variable X if

$$\lim_{n\to\infty} \mathbb{E}\left(\|X_n - X\|\right) = 0.$$

Mean convergence implies convergence in probability.
Almost sure convergence implies convergence in probability.

Stability and convergence of stochastic processes (2)

Markov process

The sequence $(X_n)_{n=1}^{\infty}$ is called a Markov process, if for $n' > n$ and any measurable real function $f(x)$ with $\sup_x |f(x)| < \infty$,

$$\mathbb{E}\left(f(X_{n'}) \mid X_1, \dots, X_n\right) = \mathbb{E}\left(f(X_{n'}) \mid X_n\right).$$

Martingales

The sequence $(X_n)_{n=1}^{\infty}$ is called respectively a *supermartingale*, a *submartingale* or a martingale, if $\mathbb{E}\left(\|X_n\|\right) < \infty$ for $n = 1, 2, \cdots$, and

$$\mathbb{E}\left(X_n \mid X_1, \dots, X_m\right) \le X_m \qquad (\mathbb{P} \text{ almost surely}), \qquad n \ge m,$$

or

$$\mathbb{E}\left(X_n \mid X_1, \dots, X_m\right) \ge X_m \qquad (\mathbb{P} \text{ almost surely}), \qquad n \ge m,$$

or finally,

$$\mathbb{E}\left(X_n \mid X_1, \dots, X_m\right) = X_m \qquad (\mathbb{P} \text{ almost surely}), \qquad n \ge m.$$

Stability and convergence of stochastic processes (3)

Doob's Inequality

Let $\{X_n\}$ be a Markov chain on state space $\mathcal{X}$. Suppose that there is a non-negative function $V(x)$ satisfying $\mathbb{E}\left(V(X_1) \mid X_0 = x\right) - V(x) = -k(x)$, where $k(x) \geq 0$ on the set $\{x : V(x) < \lambda\} \equiv Q_\lambda$. Then

$$\mathbb{P}\left(\sup_{\infty > n \geq 0} V(X_n) \geq \lambda \mid X_0 = x\right) \leq \frac{V(x)}{\lambda}.$$

Corollary: stability in probability

Consider the same assumptions as in the above theorem. Assume moreover that there exists $\bar{x} \in \mathcal{X}$ such that $V(\bar{x}) = 0$ and that $V(x) \neq 0$ for all x different from $\bar{x}$. Then the Doob's inequality implies that the Markov process X_n is **stable in probability** around $\bar{x}$, i.e.

$$\lim_{x \to \bar{x}} \mathbb{P}\left(\sup_n \|X_n - \bar{x}\| \geq \epsilon \mid X_0 = x\right) = 0, \qquad \forall \epsilon > 0.$$

Stability and convergence of stochastic processes (4)

Kushner's invariance Theorem

Consider the same assumptions as that of the Doob's inequality. Let $\mu_0 = \sigma$ be concentrated on a state $x_0 \in Q_\lambda$, i.e. $\sigma(x_0) = 1$. Assume that $0 \leq k(X_n) \to 0$ in Q_λ implies that $X_n \to \{x \mid k(x) = 0\} \cap Q_\lambda \equiv K_\lambda$. For the trajectories never leaving Q_λ, X_n converges to K_λ almost surely. Also, the associated conditioned probability measures $\tilde{\mu}_n$ tend to the largest invariant set of measures $M_\infty \subset M$ whose support set is in K_λ. Finally, for the trajectories never leaving Q_λ, X_n converges, in probability, to the support set of M_∞.

Corollary: global stability

Consider the same assumptions as in the above theorem and assume moreover that $\bar{x} \in \mathcal{X}$ is the only point in Q_λ such that $V(\bar{x}) = 0$ and furthermore that the set K_λ is reduced to $\{\bar{x}\}$ (strict Lyapunov function). Then the equilibrium $\bar{x}$ is **globally stable in probability** in the set Q_λ, i.e. $\bar{x}$ is stable in probability and moreover

$$\mathbb{P}\left(\lim_{n\to\infty} X_n = \bar{x} \mid X_n \text{ never leaves } Q_\lambda\right) = 1.$$

Restriction to finite dimensional subspace spanned by the $n^{\max}+1$ first modes $\{|0\rangle, |1\rangle, \dots, |n^{\max}\rangle\}$.

$$\mathbf{N} = \mathrm{diag}(0, 1, \dots, n^{\max}), \qquad a\,|0\rangle = 0, \quad a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,.$$

The truncated creation operator $a^\dagger$ is the Hermitian conjugate of a. We still have $\mathbf{N} = a^\dagger a$, but truncation does not preserve the usual commutation $[a, a^\dagger] = 1$ (this is only valid when $n^{\max} = \infty$).

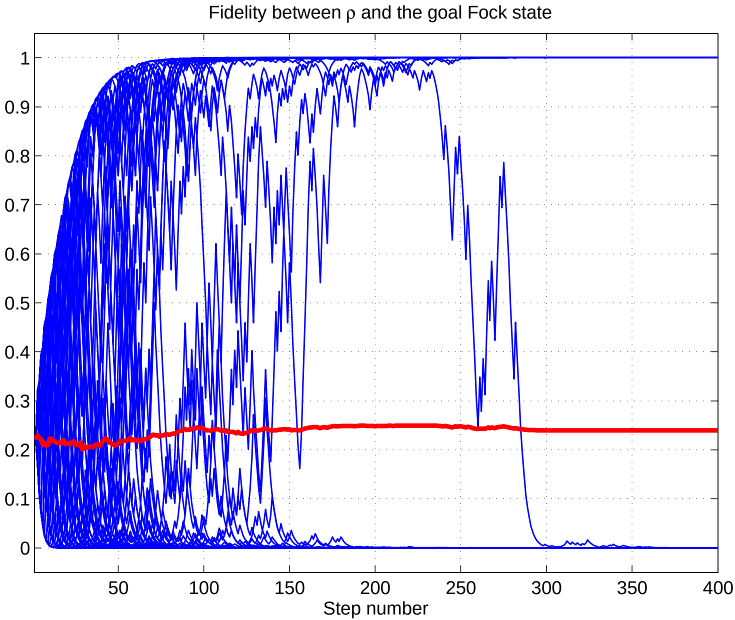
The Markov chain of state ρ ($\rho^\dagger = \rho$, $\rho \geq 0$ and $\mathrm{Tr}\,(\rho) = 1$):

$$\rho_{k+1} = \begin{cases} \mathbb{M}_g(\rho_k) = \dfrac{\mathcal{M}_g \rho_k \mathcal{M}_g^\dagger}{\mathrm{Tr}\left(\mathcal{M}_g \rho_k \mathcal{M}_g^\dagger\right)}, & \text{prob. } p_{g,k} = \mathrm{Tr}\left(\mathcal{M}_g \rho_k \mathcal{M}_g^\dagger\right); \\ \mathbb{M}_e(\rho_k) = \dfrac{\mathcal{M}_e \rho_k \mathcal{M}_e^\dagger}{\mathrm{Tr}\left(\mathcal{M}_e \rho_k \mathcal{M}_e^\dagger\right)}, & \text{prob. } p_{e,k} = \mathrm{Tr}\left(\mathcal{M}_e \rho_k \mathcal{M}_e^\dagger\right). \end{cases}$$

with $\mathcal{M}_g$ and $\mathcal{M}_e$ diagonal operators (dispersive atom/cavity interaction)

$$\mathcal{M}_g = \cos(\varphi_0 + N\vartheta), \quad \mathcal{M}_e = \sin(\varphi_0 + N\vartheta)$$

100 Monte-Carlo simulations ($\langle 3|\rho_k|3\rangle$ versus k)



Open-loop convergence of LKB-photon box (2)

Theorem

Consider the Markov process defined above with an initial density matrix ρ_0. Assume that the parameters φ_0, ϑ are chosen in order to have $\mathcal{M}_g = \cos(\varphi_0 + N\vartheta)$, $\mathcal{M}_e = \sin(\varphi_0 + N\vartheta)$ invertible and such that the spectrum of $\mathcal{M}_g^\dagger \mathcal{M}_g = \mathcal{M}_g^2$ and $\mathcal{M}_e^\dagger \mathcal{M}_e = \mathcal{M}_e^2$ are not degenerate. Then

1. *for any $n \in \{0, \ldots, n^{\max}\}$, $Tr(\rho_k |n\rangle \langle n|) = \langle n| \rho_k |n\rangle$ is a martingale*
2. *ρ_k converges with probability 1 to one of the $n^{\max} + 1$ Fock state $|n\rangle \langle n|$ with $n \in \{0, \ldots, n^{\max}\}$.*
3. *the probability to converge towards the Fock state $|n\rangle \langle n|$ is given by $Tr(\rho_0 |n\rangle \langle n|) = \langle n| \rho_0 |n\rangle$.*

The proof of point 2 is based on the Lyapunov functions

$$V_n(\rho) = f(\langle n|\rho|n\rangle) = \frac{\langle n|\rho|n\rangle + (\langle n|\rho|n\rangle)^2}{2}$$

where $f(x) = \frac{x+x^2}{2}$.

Since $f(x) = \frac{x+x^2}{2}$ obeys to the following convexity identity

$$\forall (x, y, \theta) \in [0, 1], \quad \theta f(x) + (1-\theta) f(y) = \frac{\theta(1-\theta)}{2}(x-y)^2 + f(\theta x + (1-\theta) y)$$

we have for any n, ($\varphi_n = \varphi_0 + n\vartheta$)

$$\mathbb{E}\left(V_n(\rho_{k+1}) \mid \rho_k \right) - V_n(\rho_k) =$$
$$\frac{\mathrm{Tr}\left(\mathcal{M}_g \rho_k \mathcal{M}_g^\dagger\right) \mathrm{Tr}\left(\mathcal{M}_e \rho_k \mathcal{M}_e^\dagger\right) (\langle n|\rho_k|n\rangle)^2}{2} \left(\frac{\cos^2\varphi_n}{\mathrm{Tr}\left(\mathcal{M}_g \rho_k \mathcal{M}_g^\dagger\right)} - \frac{\sin^2\varphi_n}{\mathrm{Tr}\left(\mathcal{M}_e \rho_k \mathcal{M}_e^\dagger\right)} \right)^2 .$$

Thus $V_n(\rho_k) = f(\langle n|\rho_k|n\rangle)$ is also a sub-martingale, $\mathbb{E}\left(V_n(\rho_{k+1}) \mid \rho_k \right) \geq V_n(\rho_k)$.
Moreover, $\mathbb{E}\left(V_n(\rho_{k+1}) \mid \rho_k \right) = V_n(\rho_k)$ implies that either $\langle n|\rho_k|n\rangle = 0$ or $\mathrm{Tr}\left(\mathcal{M}_g \rho_k \mathcal{M}_g^\dagger\right) = \cos^2\varphi_n$.

Open-loop convergence of LKB-photon box (4)

For each n, we apply now the Kushner's invariance theorem to the Markov process ρ_k and the sub-martingale $V_n(\rho_k)$. This theorem implies that the Markov process ρ_k converges in probability to the largest invariant subset of

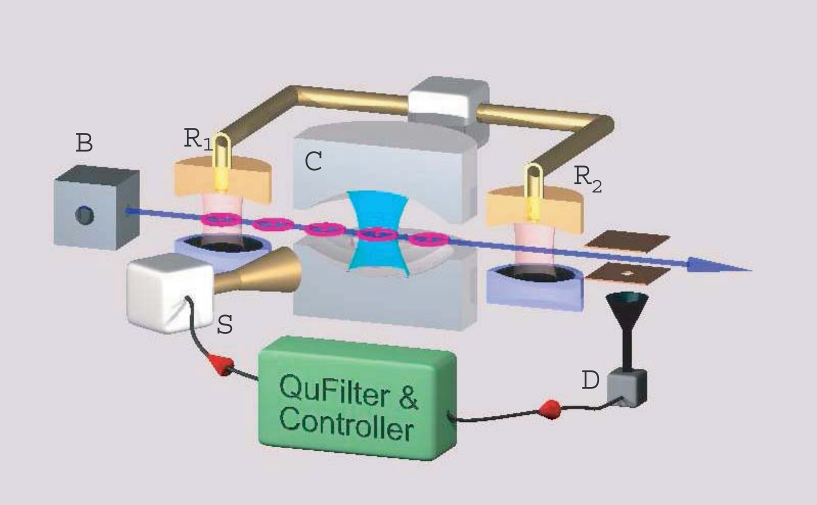
$$\left\{\rho \mid \operatorname{Tr}\left(\mathcal{M}_g \rho \mathcal{M}_g^\dagger\right) = \cos^2\varphi_n \text{ or } \langle n|\rho|n\rangle = 0\right\}.$$

We have

- the set $\{\rho \mid \langle n|\rho|n\rangle = 0\}$ is invariant.
- The largest invariant subset included in $\left\{\rho \mid \operatorname{Tr}\left(\mathcal{M}_g \rho \mathcal{M}_g^\dagger\right) = \cos^2\varphi_n\right\}$ is reduced to $\{|n\rangle\langle n|\}$

This yields convergence in probability.
Additional technical arguments (dominate convergence and Doob's first martingale convergence theorem, see the notes) ensure almost-sure convergence.

LKB-photon box: feedback control



Controlled coherent field injection inside the cavity between two atom passages.

LKB-photon box: model with control

Coherent field injection:

$$\rho_+ = \mathbb{D}_\alpha(\rho) := D_\alpha \rho D_\alpha^\dagger,$$

where $D_\alpha = \exp(\alpha a^\dagger - \alpha^* a)$ is a unitary operator called the displacement operator.
Remember that $D_\alpha^\dagger = D_{-\alpha}$ and $D_0 = \mathbf{1}$ and

$$|\alpha\rangle = D_\alpha |0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle .$$

Controlled Markov chain:

$$\rho_{k+1} = \mathbb{M}_{s_k}(\rho_{k+\frac{1}{2}}), \qquad \rho_{k+\frac{1}{2}} = \mathbb{D}_{\alpha_k}(\rho_k).$$

Quantum filter for feedback control

$$\rho_{k+1} = \mathbb{M}_{s_k}(\rho_{k+\frac{1}{2}}), \qquad \rho_{k+\frac{1}{2}} = \mathbb{D}_{\alpha_k}(\rho_k).$$

We wish to find the control α_k as a function of the k first measured jumps. In this aim we need to estimate the state of the system.

We start with the ideal case (no measurement uncertainties nor decoherence): Best estimate is given by the system dynamics itself.

Quantum filter

$$\rho^{\text{est}}_{k+1} = \mathbb{M}_{s_k}(\rho^{\text{est}}_{k+\frac{1}{2}}), \qquad \rho^{\text{est}}_{k+\frac{1}{2}} = \mathbb{D}_{\alpha_k}(\rho^{\text{est}}_k),$$

where the values for $s_k \in \{g, e\}$ are given by the measurement results and α_k is a function of ρ^{est}_k: $\alpha_k = \alpha(\rho^{\text{est}}_k)$.

A quantum separation principle

System+Filter dynamics:

$$\rho_{k+1} = \mathbb{M}_{s_k}(\rho_{k+\frac{1}{2}}), \qquad \rho_{k+\frac{1}{2}} = \mathbb{D}_{\alpha_k}(\rho_k),$$
$$\rho^{\text{est}}_{k+1} = \mathbb{M}_{s_k}(\rho^{\text{est}}_{k+\frac{1}{2}}), \qquad \rho^{\text{est}}_{k+\frac{1}{2}} = \mathbb{D}_{\alpha_k}(\rho^{\text{est}}_k),$$

where s_k takes the values g or e with probabilities $p_{g,k}$ and $p_{e,k}$ given by

$$p_{g,k} = \text{Tr}\left(\mathcal{M}_g \rho_{k+\frac{1}{2}} \mathcal{M}_g^\dagger\right), \qquad p_{e,k} = \text{Tr}\left(\mathcal{M}_e \rho_{k+\frac{1}{2}} \mathcal{M}_e^\dagger\right)$$

and where $\alpha_k = \alpha(\rho^{\text{est}}_k)$.

Theorem: a quantum separation principle

Consider a closed-loop system of the above form. Assume moreover that, whenever $\rho^{\text{est}}_0 = \rho_0$ (so that the quantum filter coincides with the closed-loop dynamics, $\rho^{\text{est}} \equiv \rho$), the closed-loop system converges almost surely towards a fixed pure state $\bar{\rho}$. Then, for any choice of the initial state ρ^{est}_0, such that ker$\rho^{\text{est}}_0 \subset$ kerρ_0, the trajectories of the system-filter converge almost surely towards the same pure state: $\rho_k, \rho^{\text{est}}_k \rightarrow \bar{\rho}$.

Proof (1)

$\mathbb{E}\left(\mathrm{Tr}\left(\rho_k \bar{\rho}\right) \mid \rho_0, \rho_0^{\mathrm{est}}\right)$ depends linearly on ρ_0 even though we are applying a feedback control.

Indeed, we can write

$$\alpha_k = \alpha(\rho_0^{\mathrm{est}}, s_0, \ldots, s_{k-1}),$$

and simple computations imply

$$\mathbb{E}\left(\mathrm{Tr}\left(\bar{\rho}\rho_k\right) \mid \rho_0, \rho_0^{\mathrm{est}}\right) = \sum_{s_0,\ldots,s_{k-1}} \mathrm{Tr}\left(\bar{\rho}\, \widetilde{\mathbb{M}}_{s_{k-1}} \circ \mathbb{D}_{\alpha_{k-1}} \circ \ldots \circ \widetilde{\mathbb{M}}_{s_0} \circ \mathbb{D}_{\alpha_0}(\rho_0)\right)$$

where

$$\widetilde{\mathbb{M}}_s \rho = \mathcal{M}_s \rho \mathcal{M}_s^\dagger.$$

So, we easily have the linearity of $\mathbb{E}\left(\mathrm{Tr}\left(\rho_k \bar{\rho}\right) \mid \rho_0, \rho_0^{\mathrm{est}}\right)$ with respect to ρ_0.

The rest of the proof follows from the assumption $\ker \rho_0^{\mathrm{est}} \subset \ker \rho_0$ which implies the existence of a constant $\gamma > 0$ and a density matrix ρ_0^c, such that

$$\rho_0^{\mathrm{est}} = \gamma \rho_0 + (1-\gamma)\rho_0^c.$$

Proof (2)

We know that if both the system and filter start at ρ_0^{est}, we have the almost sure convergence. This, together with dominated convergence theorem implies

$$\lim_{k\to\infty} \mathbb{E}\left(\text{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0^{\text{est}}, \rho_0^{\text{est}}\right) = 1.$$

By the linearity of $\mathbb{E}\left(\text{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0, \rho_0^{\text{est}}\right)$ with respect to ρ_0, we have

$$\mathbb{E}\left(\text{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0^{\text{est}}, \rho_0^{\text{est}}\right) = \gamma\mathbb{E}\left(\text{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0, \rho_0^{\text{est}}\right) + (1-\gamma)\mathbb{E}\left(\text{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0^c, \rho_0^{\text{est}}\right),$$

and as both $\mathbb{E}\left(\text{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0, \rho_0^{\text{est}}\right)$ and $\mathbb{E}\left(\text{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0^c, \rho_0^{\text{est}}\right)$ are less than or equal to one, we necessarily have that both of them converge to 1:

$$\lim_{k\to\infty} \mathbb{E}\left(\text{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0, \rho_0^{\text{est}}\right) = 1.$$

This implies the almost sure convergence of the physical system towards the pure state $\bar{\rho}$.

Controlled Markov chain

Hilbert space after a Galerkin approximation:

$$\mathcal{H} = \left\{ \sum_{n=0}^{n^{\max}} c_n \left|n\right\rangle \ \middle|\ (c_n)_{n=0}^{n^{\max}} \in \mathbb{C} \right\}$$

Dynamics:

$$\rho_{k+1/2} = \mathbb{D}_{\alpha_k}(\rho_k) := D(\alpha_k)\rho_k D(\alpha_k)^\dagger$$

$$\rho_{k+1} = \mathbb{M}_{s_k}(\rho_{k+1/2}) = \frac{M_{s_k}\rho_{k+1/2}M_{s_k}^\dagger}{\operatorname{Tr}\left(M_{s_k}\rho_{k+1/2}M_{s_k}^\dagger\right)}, \qquad s_k = g, e.$$

where

- α_k is the feedback control (function of ρ_k) and $D(\alpha)$ is a unitary operator (coherent evolution semi-group),

$$D(\alpha) := \exp(\alpha a^\dagger - \alpha^* a), \qquad \text{for } \alpha \in \mathbb{C}.$$

Lyapunov control for stabilizing $\bar{\rho} = |\bar{n}\rangle\langle\bar{n}|$

Choosing α_k such that $\mathbb{E}\left(\mathrm{Tr}\left(\rho_k \bar{\rho}\right)\right)$ is increasing.

We have

$$\rho_{k+1} = \begin{cases} \dfrac{M_g \rho_{k+1/2} M_g^\dagger}{\mathrm{Tr}\left(M_g \rho_{k+1/2} M_g^\dagger\right)}, & \text{with probability} \quad \mathrm{Tr}\left(M_g \rho_{k+1/2} M_g^\dagger\right), \\ \dfrac{M_e \rho_{k+1/2} M_e^\dagger}{\mathrm{Tr}\left(M_e \rho_{k+1/2} M_e^\dagger\right)}, & \text{with probability} \quad \mathrm{Tr}\left(M_e \rho_{k+1/2} M_e^\dagger\right), \end{cases}$$

So

$$\begin{aligned} \mathbb{E}\left(\mathrm{Tr}\left(\rho_{k+1}\bar{\rho}\right) \mid \rho_{k+1/2}\right) &= \mathrm{Tr}\left(|\bar{n}\rangle\langle\bar{n}| M_g \rho_{k+1/2} M_g^\dagger\right) + \mathrm{Tr}\left(|\bar{n}\rangle\langle\bar{n}| M_e \rho_{k+1/2} M_e^\dagger\right) \\ &= \mathrm{Tr}\left(|\bar{n}\rangle\langle\bar{n}| \rho_{k+1/2}\right), \end{aligned}$$

as

$$M_g^\dagger |\bar{n}\rangle\langle\bar{n}| M_g + M_e^\dagger |\bar{n}\rangle\langle\bar{n}| M_e = \left(\cos^2 + \sin^2\right) |\bar{n}\rangle\langle\bar{n}| = |\bar{n}\rangle\langle\bar{n}| .$$

Lyapunov control: continued

Furthermore

$$\rho_{k+1/2} = D(\alpha_k)\rho_k D(-\alpha_k),$$

and we can show in $\mathcal{H}$, that

$$D_\alpha \rho D_\alpha^\dagger = e^{\alpha a^\dagger - \alpha^* a} \rho e^{-(\alpha a^\dagger - \alpha^* a)} = \rho + [\alpha a^\dagger - \alpha^* a, \rho] + O(|\alpha|^2).$$

So

$$\mathrm{Tr}\left(\rho_{k+1/2}\bar{\rho}\right) = \mathrm{Tr}\left(\rho_k\bar{\rho}\right) + \alpha_k \mathrm{Tr}\left([|\bar{n}\rangle\langle\bar{n}|, a^\dagger]\rho_k\right) - \alpha_k^* \mathrm{Tr}\left([|\bar{n}\rangle\langle\bar{n}|, a]\rho_k\right) + O(|\alpha_k|^2).$$

Therefore, taking

$$\alpha_k = \epsilon \mathrm{Tr}\left(|\bar{n}\rangle\langle\bar{n}|[\rho_k, a]\right) = \epsilon\left(\mathrm{Tr}\left([|\bar{n}\rangle\langle\bar{n}|, a^\dagger]\rho_k\right)\right)^*,$$

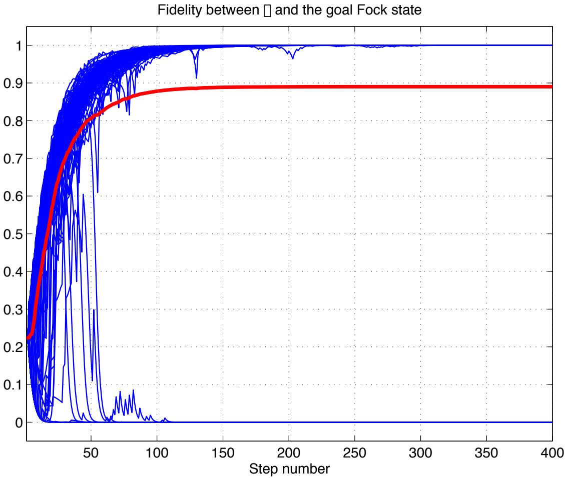
for sufficiently small $\epsilon > 0$, we have

$$\mathrm{Tr}\left(\rho_{k+1/2}\bar{\rho}\right) \geq \mathrm{Tr}\left(\rho_k\bar{\rho}\right) \quad \Longrightarrow \quad \mathbb{E}\left(\mathrm{Tr}\left(\rho_{k+1}\bar{\rho}\right) \mid \rho_k\right) \geq \mathrm{Tr}\left(\rho_k\bar{\rho}\right)$$

$\mathrm{Tr}\left(\rho_k\bar{\rho}\right)$ is a sub-martingale

Bad attractors

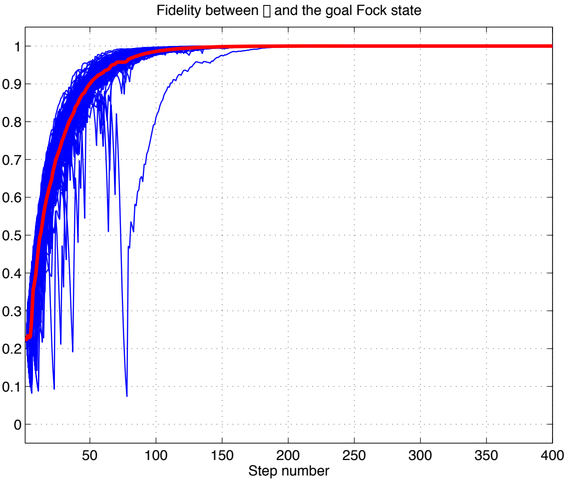
We do not have semi-global stabilization ...



$\mathrm{Tr}\left(\rho_k \bar{\rho}\right)$ converges almost surely towards a random variable with values 0 or 1

Modified feedback law

$$\alpha_k = \begin{cases} \epsilon \mathrm{Tr}\left(\bar{\rho}[\rho_k, a]\right) & \text{if } \mathrm{Tr}\left(\bar{\rho}\rho_k\right) \geq \eta \\ \underset{|\alpha| \leq \bar{\alpha}}{\mathrm{argmax}}\ \mathrm{Tr}\left(\bar{\rho}\mathbb{D}_\alpha(\rho_k)\right) & \text{if } \mathrm{Tr}\left(\bar{\rho}\rho_k\right) < \eta \end{cases}$$



Main result

Closed-loop Markov chain:

$$\rho_{k+1} = \mathbb{M}_{s_k}(\rho_{k+\frac{1}{2}}), \qquad \rho_{k+\frac{1}{2}} = \mathbb{D}_{\alpha_k}(\rho_k),$$

with

$$\alpha_k = \begin{cases} \epsilon \text{Tr}\left(\bar{\rho}[\rho_k, a]\right) & \text{if } \text{Tr}\left(\bar{\rho}\rho_k\right) \geq \eta \\ \underset{|\alpha| \leq \bar{\alpha}}{\text{argmax}}\ \text{Tr}\left(\bar{\rho}\mathbb{D}_\alpha(\rho_k)\right) & \text{if } \text{Tr}\left(\bar{\rho}\rho_k\right) < \eta \end{cases}$$

Theorem

Consider the above closed-loop quantum system. For small enough parameters $\epsilon, \eta > 0$ in the feedback scheme, the trajectories converge almost surely toward the target Fock state $\bar{\rho}$.

Proof's scheme

Four steps:

1. First, we show that for small enough η, the trajectories starting within the set $\mathcal{S}_{<\eta} = \{\rho \mid \mathrm{Tr}\,(\bar{\rho}\rho) < \eta\}$ always reach in one step the set $\mathcal{S}_{\geq 2\eta} = \{\rho \mid \mathrm{Tr}\,(\bar{\rho}\rho) \geq 2\eta\}$;
2. next, we show that the trajectories starting within the set $\mathcal{S}_{\geq 2\eta}$, will never hit the set $\mathcal{S}_{<\eta}$ with a uniformly non-zero probability $p_\eta > 0$ (Doob's inequality);
3. we prove an inequality showing that, for small enough ϵ, $\mathcal{V}(\rho_k) = f(\mathrm{Tr}\,(\bar{\rho}\rho_k))$ with $f(x) = \frac{x^2+x}{2}$ is a sub-martingale within $\mathcal{S}_{\geq\eta} = \{\rho \mid \mathrm{Tr}\,(\bar{\rho}\rho) \geq \eta\}$;
4. finally, we combine the previous step and the Kushner's invariance principle, to prove that almost all trajectories remaining inside $\mathcal{S}_{\geq\eta}$ converge towards $\bar{\rho}$.

Step 2: Doob's inequality

Doob's Inequality

Let $\{X_n\}$ be a Markov chain on state space $\mathcal{X}$. Suppose that there is a non-negative function $V(x)$ satisfying $\mathbb{E}\left(V(X_1) \mid X_0 = x\right) - V(x) = -k(x)$, where $k(x) \geq 0$ on the set $\{x : V(x) < \lambda\} \equiv Q_\lambda$. Then

$$\mathbb{P}\left(\sup_{\infty > n \geq 0} V(X_n) \geq \lambda \mid X_0 = x\right) \leq \frac{V(x)}{\lambda}.$$

Here we take $V(\rho_k) = 1 - \operatorname{Tr}\left(\bar{\rho}\rho_k\right)$ which is a super-martingale. We have:

$$\mathbb{P}(\sup_{k' \geq k}(1 - \operatorname{Tr}\left(\bar{\rho}\rho_{k'}\right))) \geq 1 - \eta \mid \rho_k \in \mathcal{S}_{\geq 2\eta}) \leq \frac{1 - \operatorname{Tr}\left(\bar{\rho}\rho_k\right)}{1 - \eta} \leq \frac{1 - 2\eta}{1 - \eta},$$

and thus

$$\begin{aligned}
\mathbb{P}\left(\inf_{k' \geq k} \operatorname{Tr}\left(\bar{\rho}\rho_{k'}\right) > \eta \mid \operatorname{Tr}\left(\bar{\rho}\rho_k\right) \geq 2\eta\right) &= 1 - \mathbb{P}(\sup_{k' \geq k}(1 - \operatorname{Tr}\left(\bar{\rho}\rho_{k'}\right))) \\
&\geq 1 - \eta \mid \operatorname{Tr}\left(\bar{\rho}\rho_k\right) \geq 2\eta) \\
&\geq 1 - \frac{1 - 2\eta}{1 - \eta} = \frac{\eta}{1 - \eta} = p_\eta.
\end{aligned}$$

Realistic simulations (Matlab script `PhotonBox.m`) (1)

We take into account the detector's efficiency (η_d), detection faults (η_f), pulse occupation (η_a), decoherence ($\frac{(1+n_{th})\tau_a}{T_{cav}}$), thermal photons ($\frac{n_{th}\tau_a}{T_{cav}}$).

System simulation:

$$\rho_{k+1} = \mathbb{M}_{r_k} \circ \mathbb{M}_{s_k} \circ \mathbb{D}_{\alpha_k}(\rho_k),$$

where $s_k \in \{g, e, u\}$, $r_k \in \{\text{loss}, \text{gain}, \text{no}\}$ are random variables admitting probability distributions depending of ρ_k and α_k:

$$\begin{aligned}
\mathbb{P}(s_k = g) &= \eta_a \mathrm{Tr}\left(\mathcal{M}_g^\dagger \mathcal{M}_g \mathbb{D}_{\alpha_k}(\rho_k)\right), \\
\mathbb{P}(s_k = e) &= \eta_a \mathrm{Tr}\left(\mathcal{M}_e^\dagger \mathcal{M}_e \mathbb{D}_{\alpha_k}(\rho_k)\right), \\
\mathbb{P}(s_k = u) &= 1 - \eta_a, \\
\mathbb{P}(r_k = \text{loss}) &= \tfrac{(1+n_{th})\tau_a}{T_{cav}} \mathrm{Tr}\left(a^\dagger a \, \mathbb{M}_{s_k} \circ \mathbb{D}_{\alpha_k}(\rho_k)\right), \\
\mathbb{P}(r_k = \text{gain}) &= \tfrac{n_{th}\tau_a}{T_{cav}} \mathrm{Tr}\left(a a^\dagger \, \mathbb{M}_{s_k} \circ \mathbb{D}_{\alpha_k}(\rho_k)\right), \\
\mathbb{P}(r_k = \text{no}) &= 1 - \mathbb{P}(r_k = \text{loss}) - \mathbb{P}(r_k = \text{gain}).
\end{aligned}$$

Realistic simulations (Matlab script `PhotonBox.m`) (2)

Filter simulation:

$$\rho_{k+1}^{\text{est}} = \mathbb{T} \circ \mathbb{B}_{s_k} \circ \mathbb{D}_{\alpha_k}(\rho_k^{\text{est}}),$$

where the $s_k \in \{g, e, u\}$ is the detection result (atom in $|g\rangle$, in $|e\rangle$ or undetected).
Furthermore $\mathbb{B}_s$ is the Bayesian filter given by:

$$\mathbb{B}_g(\rho) = \frac{1-\eta_f}{(1-\eta_f)p_g + \eta_f p_e} \mathcal{M}_g \rho \mathcal{M}_g^\dagger + \frac{\eta_f}{(1-\eta_f)p_g + \eta_f p_e} \mathcal{M}_e \rho \mathcal{M}_e^\dagger,$$

$$\mathbb{B}_e(\rho) = \frac{1-\eta_f}{(1-\eta_f)p_e + \eta_f p_g} \mathcal{M}_e \rho \mathcal{M}_e^\dagger + \frac{\eta_f}{(1-\eta_f)p_e + \eta_f p_g} \mathcal{M}_g \rho \mathcal{M}_g^\dagger,$$

$$\mathbb{B}_u(\rho) = \frac{1-\eta_a}{1-\eta_a\eta_d} \rho + \frac{\eta_a(1-\eta_d)}{1-\eta_a\eta_d} \left(\mathcal{M}_g \rho \mathcal{M}_g^\dagger + \mathcal{M}_e \rho \mathcal{M}_e^\dagger \right),$$

where $p_g = \text{Tr}\left(\mathcal{M}_g^\dagger \mathcal{M}_g \rho\right)$, $p_e = \text{Tr}\left(\mathcal{M}_e^\dagger \mathcal{M}_e \rho\right)$, η_f is the detection fault rate, η_a is the pulse occupation rate and η_d is the detection's efficiency rate.
The super-operator $\mathbb{T}$, modeling the decoherence, is given by:

$$\mathbb{T}(\rho) = \rho + \frac{(1+n_{\text{th}})\tau_a}{T_{\text{cav}}} \left(a\rho a^\dagger - \tfrac{1}{2} a^\dagger a \rho - \tfrac{1}{2} \rho a^\dagger a \right) + \frac{n_{\text{th}}\tau_a}{T_{\text{cav}}} \left(a^\dagger \rho a - \tfrac{1}{2} a a^\dagger \rho - \tfrac{1}{2} \rho a a^\dagger \right)$$