Modeling and Control of Quantum Systems

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http://cas.ensmp.fr/~rouchon/QuantumSyst/index.html

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Outline

Projective measurement (1)

For the system defined on Hilbert space $\mathcal{H}$, take

- an observable $\mathcal{O}$ (Hermitian operator) defined on $\mathcal{H}$:

  $$\mathcal{O} = \sum_{\nu} \lambda_\nu P_\nu,$$

  where λ_ν's are the eigenvalues of $\mathcal{O}$ and P_ν is the projection operator over the associated eigenspace; $\mathcal{O}$ can be degenerate and therefore the projection operator P_ν is not necessarily a rank-1 operator.
- a quantum state (a priori mixed) given by the density operator ρ on $\mathcal{H}$, Hermitian, positive and of trace 1; $\mathrm{Tr}\left(\rho^2\right) \leq 1$ with equality only when ρ is an orthogonal projector on some pure quantum state $|\psi\rangle$, i.e., $\rho = |\psi\rangle \langle\psi|$.

Projective measurement of the physical observable $\mathcal{O} = \sum_\nu \lambda_\nu P_\nu$ for the quantum state ρ:

1. The probability of obtaining the value λ_ν is given by $p_\nu = \mathrm{Tr}\,(\rho P_\nu)$; note that $\sum_\nu p_\nu = 1$ as $\sum_\nu P_\nu = \mathbf{1}_\mathcal{H}$ ($\mathbf{1}_\mathcal{H}$ represents the identity operator of $\mathcal{H}$).
2. After the measurement, the conditional (a posteriori) state ρ_+ of the system, given the outcome λ_ν, is

$$\rho_+ = \frac{P_\nu \, \rho \, P_\nu}{p_\nu} \quad \text{(collapse of the wave packet)}$$

3. When $\rho = |\psi\rangle\langle\psi|$, $p_\nu = \langle\psi|P_\nu|\psi\rangle$, $\rho_+ = |\psi_+\rangle\langle\psi_+|$ with $|\psi_+\rangle = \frac{P_\nu\psi}{\sqrt{p_\nu}}$.

$\mathcal{O}$ non degenerate: von Neumann measurement.

Example: $\mathcal{H} = \mathbb{C}^2$, $|\psi\rangle = (|g\rangle + |e\rangle)/\sqrt{2}$, $\mathcal{O} = \sigma_z$; measuring consists in turning on, for a small time, a laser resonant between $|g\rangle$ and a highly unstable third state $|f\rangle$; fluorescence means $|\psi_+\rangle = |g\rangle$, no fluorescence means $|\psi_+\rangle = |e\rangle$.

System S of interest (a quantized electromagnetic field) interacts with the meter M (a probe atom), and the experimenter measures projectively the meter M (the probe atom). Need for a **Composite system**: $\mathcal{H}_S \otimes \mathcal{H}_M$ where $\mathcal{H}_S$ and $\mathcal{H}_M$ are the Hilbert space of S and M.

Measurement process in three successive steps:

1. Initially the quantum state is separable

$$\mathcal{H}_S \otimes \mathcal{H}_M \ni |\Psi\rangle = |\psi_S\rangle \otimes |\theta_M\rangle$$

   with a well defined and known state $|\theta_M\rangle$ for M.

2. Then a Schrödinger evolution during a small time (unitary operator $U_{S,M}$) of the composite system from $|\psi_S\rangle \otimes |\theta_M\rangle$ and producing $U_{S,M}\big(|\psi_S\rangle \otimes |\theta_M\rangle\big)$, entangled in general.
3. Finally a projective measurement of the meter M: $\mathcal{O}_M = \mathbf{1}_S \otimes \left(\sum_\nu \lambda_\nu P_\nu\right)$ the measured observable for the meter. Projection operator P_ν is a rank-1 projection in $\mathcal{H}_M$ over the eigenstate $|\lambda_\nu\rangle \in \mathcal{H}_M$: $P_\nu = |\lambda_\nu\rangle\langle\lambda_\nu|$.

Define the measurement operators $\mathcal{M}_\nu$ via

$$\forall \left|\psi_S\right\rangle \in \mathcal{H}_S, \quad U_{S,M}\big(\left|\psi_S\right\rangle \otimes \left|\theta_M\right\rangle\big) = \sum_\nu \big(\mathcal{M}_\nu \left|\psi_S\right\rangle\big) \otimes \left|\lambda_\nu\right\rangle.$$

Then $\sum_\nu \mathcal{M}_\nu^\dagger \mathcal{M}_\nu = \mathbf{1}_S$. The set $\{\mathcal{M}_\nu\}$ defines a Positive Operator Valued Measurement (POVM).

In $\mathcal{H}_S \otimes \mathcal{H}_M$, projective measurement of $\mathcal{O}_M = \mathbf{1}_S \otimes \big(\sum_\nu \lambda_\nu P_\nu\big)$ with quantum state $U_{S,M}\big(\left|\psi_S\right\rangle \otimes \left|\theta_M\right\rangle\big)$:

1. The probability of obtaining the value λ_ν is given by $p_\nu = \left\langle\psi_S\right| \mathcal{M}_\nu^\dagger \mathcal{M}_\nu \left|\psi_S\right\rangle$
2. After the measurement, the conditional (a posteriori) state of the system, given the outcome λ_ν, is

$$\left|\psi_S\right\rangle_+ = \frac{\mathcal{M}_\nu \left|\psi_S\right\rangle}{\sqrt{p_\nu}}.$$

For mixed state ρ (instead of pure state $\left|\psi_S\right\rangle$):
$p_\nu = \operatorname{Tr}\left(\mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger\right)$ and $\rho_+ = \frac{\mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger}{\operatorname{Tr}\left(\mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger\right)}$,

Quantum Non-Demolition (QND) measurement (1)

$U_{S,M}$ is the propagator generated by $H = H_S + H_M + H_{SM}$ where H_S (resp. H_M, H_{SM}) describes the system (resp. the meter , system-meter interaction). For time-invariant H: $U_{S,M} = e^{-i\tau H}$ where τ is the interaction time.

A necessary condition for meter measurement to encode some information on the system S itself: $[H, \mathcal{O}_M] \neq 0$. When $H_M = 0$, this necessary condition reads $[H_{SM}, \mathcal{O}_M] \neq 0$.

Proof: otherwise $\mathcal{O}_M U_{S,M} = U_{S,M} \mathcal{O}_M$. With $\mathcal{O}_M = \sum_\nu \lambda_\nu \mathbf{1}_S \otimes |\lambda_\nu\rangle$ we have

$$\forall \nu, \quad \mathcal{O}_M U_{S,M}\big(|\psi_S\rangle \otimes |\lambda_\nu\rangle\big) = U_{S,M}\mathcal{O}_M\big(|\psi_S\rangle \otimes |\lambda_\nu\rangle\big) = \lambda_\nu U_{S,M}\big(|\psi_S\rangle \otimes |\lambda_\nu\rangle\big).$$

Thus, necessarily $U_{S,M}\big(|\psi_S\rangle \otimes |\lambda_\nu\rangle\big) = \big(U_\nu |\psi_S\rangle\big) \otimes |\lambda_\nu\rangle$ where U_ν is a unitary transformation on $\mathcal{H}_S$ only. With $|\theta_M\rangle = \sum_\nu \theta_\nu |\lambda_\nu\rangle$, we get:

$$\forall |\psi_S\rangle \in \mathcal{H}_S U_{S,M}\big(|\psi_S\rangle \otimes |\theta_M\rangle\big) = \sum_\nu \theta_\nu \big(U_\nu |\psi_S\rangle\big) \otimes |\lambda_\nu\rangle$$

Then measurement operators $\mathcal{M}_\nu$ are equal to $\theta_\nu U_\nu$. The probability to get measurement outcome ν, $\langle \psi_S | \mathcal{M}_\nu^\dagger \mathcal{M}_\nu | \psi_S \rangle = |\theta_\nu|^2$, is completely independent of systems state $|\psi_S\rangle$.

Quantum Non-Demolition (QND) measurement (2)

The POVM $(\mathcal{M}_\nu)$ (system S, interaction with the meter M via $H = H_S + H_M + H_{SM}$, von Neumann measurements on the meter via $\mathcal{O}_M$) is a QND measurement of the system observable $\mathcal{O}_S$ if the eigenspaces of $\mathcal{O}_S$ are invariant with respect to the measurement operators $\mathcal{M}_\nu$. A sufficient but not necessary condition for this is $[H, \mathcal{O}_S] = 0$.
Under this condition $\mathcal{O}_S$ and $U_{S,M}$ commute. Assume $\mathcal{O}_S$ non degenerate and take the eigenstate $|\mu\rangle$ to the eigenvalue $\mu \in \mathbb{R}$:

$$\mathcal{O}_S U_{S,M}\big(\,|\mu\rangle \otimes |\theta_M\rangle\,\big) = U_{S,M}\mathcal{O}_S\big(\,|\mu\rangle \otimes |\theta_M\rangle\,\big) = \mu U_{S,M}\big(\,|\mu\rangle \otimes |\theta_M\rangle\,\big).$$

Thus $U_{S,M}\big(\,|\mu\rangle \otimes |\theta_M\rangle\,\big) = |\mu\rangle \otimes \big(U_\mu\,|\theta_M\rangle\,\big)$ with U_μ unitary on $\mathcal{H}_M$. We also have

$$U_{S,M}\big(\,|\mu\rangle \otimes |\theta_M\rangle\,\big) = \sum_\nu \mathcal{M}_\nu\,|\mu\rangle \otimes |\lambda_\nu\rangle\,.$$

Thus necessarily,each $\mathcal{M}_\nu\,|\mu\rangle$ is colinear to $|\mu\rangle$.
When $\rho = |\mu\rangle\,\langle\mu|$, the conditional state remains unchanged $\rho_+ = \mathbb{M}_\nu(\rho)$ whatever the meter measure outcome ν is.
When the spectrum of $\mathcal{O}_S$ is degenerate: for all ν, $\mathcal{M}_\nu P_\mu = P_\mu \mathcal{M}_\nu$ where P_μ is the projector on the eigenspace associated to μ:

Stochastic process attached to a POVM

- To the POVM $(\mathcal{M}_\nu)$ on $\mathcal{H}_S$ is attached a stochastic process of quantum state ρ

$$\rho_+ = \frac{\mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger}{\text{Tr}\left(\mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger\right)} \text{ with probability } p_\nu = \text{Tr}\left(\mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger\right).$$

- For any observable A on $\mathcal{H}_S$, its conditional expectation value after the transition knowing the state ρ

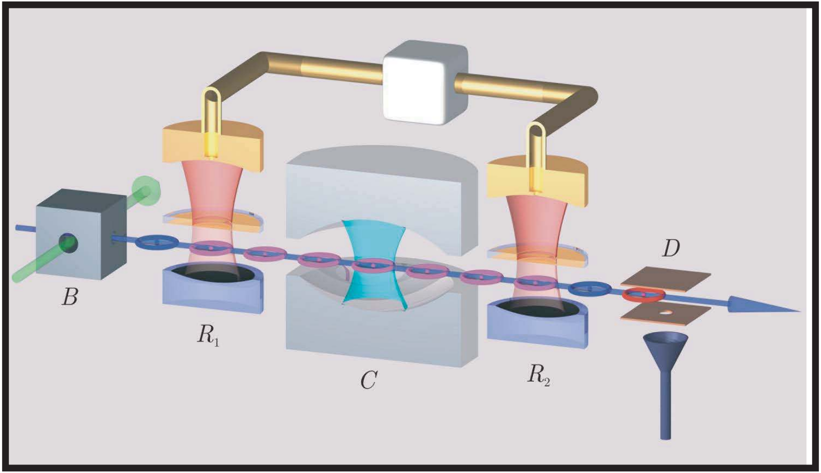
$$\mathbb{E}\left(\text{Tr}\left(A\,\rho_+\right)|\rho\right) = \text{Tr}\left(A\,\mathbb{K}\rho\right)$$

where the linear map $\rho \mapsto \mathbb{K}\rho = \sum_\nu \mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger$ is a Kraus map.

- If $\bar{A}$ is a stationary point of the adjoint Kraus map $\mathbb{K}^*$, $\mathbb{K}^*\bar{A} = \sum_\nu \mathcal{M}_\nu^\dagger \bar{A} \mathcal{M}_\nu$, then $\text{Tr}\left(\bar{A}\rho\right)$ is a martingale:

$$\mathbb{E}\left(\text{Tr}\left(\bar{A}\,\rho_+\right) \mid \rho\right) = \text{Tr}\left(\bar{A}\,\mathbb{K}\rho\right) = \text{Tr}\left(\rho\,\mathbb{K}^*\bar{A}\right) = \text{Tr}\left(\rho\bar{A}\right).$$

- QND measurement of $\mathcal{O}_S = \sum_\mu \sigma_\mu P_\mu$: $\mathbb{K}^* P_\mu = P_\mu$ and each $\bar{\rho} = P_\mu / \text{Tr}\left(P_\mu\right)$ is a fixed point of the above stochastic process ($\rho_+ \equiv \bar{\rho}$ if $\rho = \bar{\rho}$)



Atoms get out of box B one by one, undergo then a first Rabi pulse in Ramsey zone R_1, become entangled with electromagnetic field trapped in C, undergo a second Rabi pulse in Ramsey zone R_2 and finally are measured in the detector D.

- System S corresponds to a quantized mode in C:

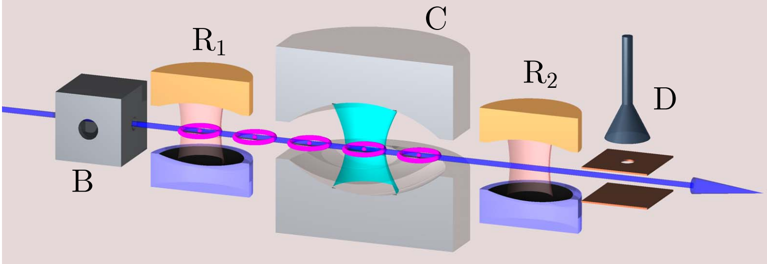
$$\mathcal{H}_S = \left\{ \sum_{n=0}^{\infty} \psi_n |n\rangle \;|\; (\psi_n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

  where $|n\rangle$ represents the Fock state associated to exactly n photons inside the cavity

- Meter M is associated to atoms : $\mathcal{H}_M = \mathbb{C}^2$, each atom admits two-level and is described by a wave function $c_g |g\rangle + c_e |e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$; atoms leaving B are all in state $|g\rangle$

- When atom comes out B, the state $|\Psi\rangle_B \in \mathcal{H}_M \otimes \mathcal{H}_S$ of the composite system atom/field is separable

$$|\Psi\rangle_B = |g\rangle \otimes |\psi\rangle .$$



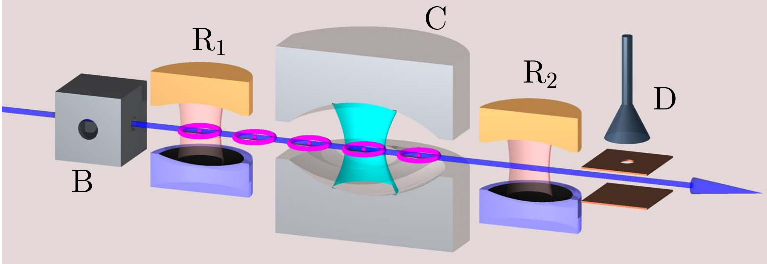
- When atom comes out B: $|\Psi\rangle_B = |g\rangle \otimes |\psi\rangle$.
- When atom comes out the first Ramsey zone R_1 the state remains separable but has changed to

$$|\Psi\rangle_{R_1} = (U_{R_1} \otimes \mathbf{1})\, |\Psi\rangle_B = (U_{R_1}\, |g\rangle) \otimes |\psi\rangle$$

where the unitary transformation performed in R_1 only affects the atom:

$$U_{R_1} = e^{-i\frac{\theta_1}{2}(x_1\sigma_x + y_1\sigma_y + z_1\sigma_z)} = \cos(\tfrac{\theta_1}{2}) - i\sin(\tfrac{\theta_1}{2})(x_1\sigma_x + y_1\sigma_y + z_1\sigma_z)$$

corresponds, in the Bloch sphere representation, to a rotation of angle θ_1 around $x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k}$ $(x_1^2 + y_1^2 + z_1^2 = 1)$



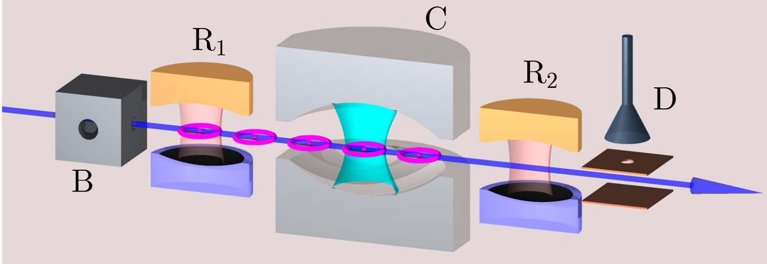
- When atom comes out the first Ramsey zone R_1:
  $|\Psi\rangle_{R_1} = (U_{R_1}|g\rangle) \otimes |\psi\rangle$.
- When atom comes out cavity C, the state does not remain separable: atom and field becomes entangled and the state is described by

$$|\Psi\rangle_C = U_C |\Psi\rangle_{R_1}$$

where the unitary transformation U_C on $\mathcal{H}_M \otimes \mathcal{H}_S$ is associated to a Jaynes-Cumming Hamiltonian:

$$H_C = \frac{\Delta}{2}\sigma_z + i\frac{\Omega}{2}(\sigma_- a^\dagger - \sigma_+ a)$$

Parameters: $\Delta = \omega_{eg} - \omega_c$, Ω.



- When atom comes out cavity C: $|\Psi\rangle_C = U_C\big((U_{R_1}|g\rangle)\otimes|\psi\rangle\big)$.
- When atom comes out second Ramsey zone R_2, the state becomes

$$|\Psi\rangle_{R_2} = (U_{R_2}\otimes \mathbf{1})\,|\Psi\rangle_C \text{ with } U_{R_2} = e^{-i\frac{\theta_2}{2}(x_2\sigma_x + y_2\sigma_y + z_2\sigma_z)}$$

- Just before the measurement in D, the state is given by

$$|\Psi\rangle_{R_2} = U_{SM}\big(|g\rangle\otimes|\psi\rangle\big) = |g\rangle\otimes\mathcal{M}_g|\psi\rangle + |e\rangle\otimes\mathcal{M}_e|\psi\rangle$$

where $U_{SM} = U_{R_2}U_CU_{R_1}$ is the total unitary transformation defining the linear measurement operators $\mathcal{M}_g$ and $\mathcal{M}_e$ on $\mathcal{H}_S$.

Just before the measurement in D, the atom/field state is:

$$|g\rangle \otimes \mathcal{M}_g |\psi\rangle + |e\rangle \otimes \mathcal{M}_e |\psi\rangle$$

Denote by $s \in \{g, e\}$ the measurement outcome in detector D: with probability $p_s = \left\langle \psi | \mathcal{M}_s^\dagger \mathcal{M}_s | \psi \right\rangle$ we get s. Just after the measurement outcome s, the state becomes separable:

$$|\Psi\rangle_D = \frac{1}{\sqrt{p_s}} \, |s\rangle \otimes (\mathcal{M}_s |\psi\rangle) = \frac{|s\rangle \otimes (\mathcal{M}_s |\psi\rangle)}{\sqrt{\left\langle \psi | \mathcal{M}_s^\dagger \mathcal{M}_s | \psi \right\rangle}}.$$

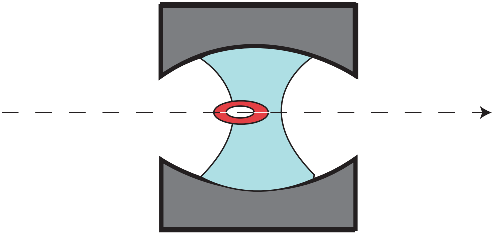
Markov process (density matrix formulation)

$$\rho_+ = \begin{cases} \mathbb{M}_g(\rho) = \frac{\mathcal{M}_g \rho \mathcal{M}_g^\dagger}{\mathrm{Tr}\left(\mathcal{M}_g \rho \mathcal{M}_g^\dagger\right)}, & \text{with probability } p_g = \mathrm{Tr}\left(\mathcal{M}_g \rho \mathcal{M}_g^\dagger\right); \\ \mathbb{M}_e(\rho) = \frac{\mathcal{M}_e \rho \mathcal{M}_e^\dagger}{\mathrm{Tr}\left(\mathcal{M}_e \rho \mathcal{M}_e^\dagger\right)}, & \text{with probability } p_e = \mathrm{Tr}\left(\mathcal{M}_e \rho \mathcal{M}_e^\dagger\right). \end{cases}$$

Exercice

Show that, for any density matrix ρ, $\mathcal{M}_g \rho \mathcal{M}_g^\dagger + \mathcal{M}_e \rho \mathcal{M}_e^\dagger$ does not depend on $(\theta_2, x_2, y_2, z_2)$, the parameters of the second Ramsey pulse in R_2.

Atom-cavity coupling



The composite system lives on the Hilbert space $\mathbb{C}^2 \otimes L^2(\mathbb{R};\mathbb{C}) \sim \mathbb{C}^2 \otimes l^2(\mathbb{C})$ with the Jaynes-Cummings Hamiltonian

$$\frac{\omega_{eg}}{2}\sigma_z + \omega_c(a^\dagger a + \frac{1}{2}) + i\frac{\Omega(t)}{2}\sigma_x(a^\dagger - a),$$

with the usual scales $\Omega \ll \omega_c, \omega_{eg}$, $|\omega_c - \omega_{eg}| \ll \omega_c, \omega_{eg}$ and $|d\Omega/dt| \ll \omega_c\Omega, \omega_{eg}\Omega$.

Jaynes-Cumming model: RWA

We consider the change of frame: $|\psi\rangle = e^{-i\omega_c t(a^\dagger a + \frac{1}{2})} e^{-i\omega_c t \sigma_z} |\phi\rangle$.
The system becomes $i\frac{d}{dt}|\phi\rangle = H_{\text{int}}|\phi\rangle$ with

$$H_{\text{int}} = \frac{\Delta}{2}\sigma_z + i\frac{\Omega(t)}{2}(e^{-i\omega_c t}|g\rangle\langle e| + e^{i\omega_c t}|e\rangle\langle g|)(e^{i\omega_c t}a^\dagger - e^{-i\omega_c t}a),$$

where $\Delta = \omega_{eg} - \omega_c$.
The secular terms of H_{int} are given by (RWA, first order approximation):

$$H_{\text{rwa}} = \frac{\Delta}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + i\frac{\Omega(t)}{2}(|g\rangle\langle e|a^\dagger - |e\rangle\langle g|a).$$

We compute the propagator for the simple case where $\Omega(t)$ is constant.

Jaynes-Cumming propagator

Exercice: Let us assume that the Jaynes-Cumming propagator U_C admits the following form

$$U_C = e^{-i\tau\left(\frac{\Delta\left(|e\rangle\langle e| - |g\rangle\langle g|\right)}{2} + i\frac{\Omega\left(|g\rangle\langle e| a^\dagger - |e\rangle\langle g| a\right)}{2}\right)}$$

where τ is an interaction time.

- Show by recurrence on integer k that

$$\Big(\Delta\big(\,|e\rangle\langle e| - |g\rangle\langle g|\,\big) + i\Omega\big(\,|g\rangle\langle e|\, a^\dagger - |e\rangle\langle g|\, a\big)\Big)^{2k} = \\ |e\rangle\langle e| \left(\Delta^2 + (N+1)\Omega^2\right)^k + |g\rangle\langle g| \left(\Delta^2 + N\Omega^2\right)^k$$

and that

$$\Big(\Delta\big(\,|e\rangle\langle e| - |g\rangle\langle g|\,\big) + i\Omega\big(\,|g\rangle\langle e|\, a^\dagger - |e\rangle\langle g|\, a\big)\Big)^{2k+1} = \\ |e\rangle\langle e|\, \Delta \left(\Delta^2 + (N+1)\Omega^2\right)^k - |g\rangle\langle g|\, \Delta \left(\Delta^2 + N\Omega^2\right)^k \\ + i\Omega\Big(\,|g\rangle\langle e| \left(\Delta^2 + N\Omega^2\right)^k a^\dagger - |e\rangle\langle g|\, a \left(\Delta^2 + N\Omega^2\right)^k\Big).$$

- Deduce that

$$U_C = |g\rangle\langle g| \left(\cos\left(\frac{\tau\sqrt{\Delta^2+N\Omega^2}}{2}\right) + i\frac{\Delta\sin\left(\frac{\tau\sqrt{\Delta^2+N\Omega^2}}{2}\right)}{\sqrt{\Delta^2+N\Omega^2}} \right)$$

$$+ |e\rangle\langle e| \left(\cos\left(\frac{\tau\sqrt{\Delta^2+(N+1)\Omega^2}}{2}\right) - i\frac{\Delta\sin\left(\frac{\tau\sqrt{\Delta^2+(N+1)\Omega^2}}{2}\right)}{\sqrt{\Delta^2+(N+1)\Omega^2}} \right)$$

$$+ |g\rangle\langle e| \left(\frac{\Omega\sin\left(\frac{\tau\sqrt{\Delta^2+N\Omega^2}}{2}\right)}{\sqrt{\Delta^2+N\Omega^2}} \right) a^\dagger - |e\rangle\langle g| \, a \left(\frac{\Omega\sin\left(\frac{\tau\sqrt{\Delta^2+N\Omega^2}}{2}\right)}{\sqrt{\Delta^2+N\Omega^2}} \right)$$

where $N = a^\dagger a$ the photon-number operator (a is the photon annihilator operator).

- In the resonant case, $\Delta = 0$, prove that:

$$U_C = |g\rangle\langle g| \cos\left(\frac{\Theta}{2}\sqrt{N}\right) + |e\rangle\langle e| \cos\left(\frac{\Theta}{2}\sqrt{N+1}\right) + |g\rangle\langle e| \left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{N}\right)}{\sqrt{N}}\right) a^\dagger - |e\rangle\langle g| \, a \left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{N}\right)}{\sqrt{N}}\right)$$

where $N = a^\dagger a$ is the photon number operator, the adjustable parameter Θ being the Rabi angle with zero photon. What is its value?

- In the dispersive case, $|\Delta| \gg |\Omega|$, and when the interaction time τ is large, $\Delta\tau \sim \left(\frac{\Delta}{\Omega}\right)^2$, show that, up to first order terms in Ω/Δ, we get

$$e^{-i\tau\left(\frac{\Delta\left(|e\rangle\langle e| - |g\rangle\langle g|\right)}{2} + i\frac{\Omega\left(|g\rangle\langle e| a^\dagger - |e\rangle\langle g| a\right)}{2}\right)} = |g\rangle\langle g| \, e^{i\left(\frac{\Delta\tau}{2} + \frac{\Omega^2\tau}{4\Delta}N\right)} + |e\rangle\langle e| \, e^{-i\left(\frac{\Delta\tau}{2} + \frac{\Omega^2\tau}{4\Delta}(N+1)\right)}.$$

Resonant case ($\Delta = 0$)

We take

$$U_{R_1} = e^{-i\frac{\theta_1}{2}\sigma_y} = \cos\left(\frac{\theta_1}{2}\right) + \sin\left(\frac{\theta_1}{2}\right)\left(\,|g\rangle\langle e| - |e\rangle\langle g|\,\right) \quad \text{and} \quad U_{R_2} = \mathbf{1}.$$

We were looking for $\mathcal{M}_g$ and $\mathcal{M}_e$ such that

$$U_{SM}\,|g\rangle \otimes |\psi\rangle = U_{R_2}U_C U_{R_1}\,|g\rangle \otimes |\psi\rangle = |g\rangle \otimes \mathcal{M}_g\,|\psi\rangle + |e\rangle \otimes \mathcal{M}_e\,|\psi\rangle\,.$$

We have

$$|\Psi\rangle_{R_1} = \left(\cos\left(\frac{\theta_1}{2}\right)|g\rangle - \sin\left(\frac{\theta_1}{2}\right)|e\rangle\right) \otimes |\psi\rangle\,.$$

and then

$$|\Psi\rangle_{R_2} = |\Psi\rangle_C =$$

$$|g\rangle \otimes \left(\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\Theta}{2}\sqrt{N}\right) - \sin\left(\frac{\theta_1}{2}\right)\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{N}\right)}{\sqrt{N}}\right)a^\dagger\right)|\psi\rangle$$

$$-\,|e\rangle \otimes \left(\sin\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\Theta}{2}\sqrt{N+1}\right) + \cos\left(\frac{\theta_1}{2}\right)a\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{N}\right)}{\sqrt{N}}\right)\right)|\psi\rangle\,.$$

Resonant case: measurement operators

$$\mathcal{M}_g = \cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\Theta}{2}\sqrt{N}\right) - \sin\left(\frac{\theta_1}{2}\right)\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{N}\right)}{\sqrt{N}}\right) a^\dagger$$

$$\mathcal{M}_e = -\sin\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\Theta}{2}\sqrt{N+1}\right) - \cos\left(\frac{\theta_1}{2}\right) a \left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{N}\right)}{\sqrt{N}}\right)$$

Exercice

Verify that these Kraus operators satisfy $\mathcal{M}_g^\dagger \mathcal{M}_g + \mathcal{M}_e^\dagger \mathcal{M}_e = \mathbf{1}$ *(hint: use,* $N = a^\dagger a$, $a\, f(N) = f(N+1)\, a$ *and* $a^\dagger f(N) = f(N-1)\, a^\dagger$*).*

Dispersive case ($|\Delta| \gg |\Omega|$)

We take

$$U_{R_1} = e^{-i\frac{\pi}{4}\sigma_y} \quad \text{and} \quad U_{R_2} = e^{-i\frac{\pi}{4}(-\sin\eta\sigma_x + \cos\eta\sigma_y)}$$

Therefore

$$|\Psi\rangle_{R_1} = \frac{|g\rangle - |e\rangle}{\sqrt{2}} \otimes |\psi\rangle\,.$$

Then

$$|\Psi\rangle_C = \tfrac{1}{\sqrt{2}}\,|g\rangle \otimes e^{-i\phi(N)}\,|\psi\rangle - \tfrac{1}{\sqrt{2}}\,|e\rangle \otimes e^{i\phi(N+1)}\,|\psi\rangle\,.$$

Finally

$$2\,|\Psi\rangle_{R_2} = \left(\,|g\rangle - e^{-i\eta}\,|e\rangle\,\right) \otimes e^{-i\phi(N)}\,|\psi\rangle - \left(e^{i\eta}\,|g\rangle + |e\rangle\,\right) \otimes e^{i\phi(N+1)}\,|\psi\rangle$$
$$= |g\rangle \otimes \left(e^{-i\phi(N)} - e^{i(\eta+\phi(N+1))}\right)|\psi\rangle - |e\rangle \otimes \left(e^{-i(\eta+\phi(N))} + e^{i\phi(N+1)}\right)|\psi\rangle$$

where $\phi(N) = \vartheta_0 + N\vartheta$ with $\vartheta_0 = -\frac{\Delta\tau}{2}$ and $\vartheta = -\frac{\Omega^2\tau}{4\Delta}$.

Kraus operators

Taking φ_0 an arbitrary phase and $\eta = 2(\varphi_0 - \vartheta_0) - \vartheta - \pi$, we find

$$\mathcal{M}_g = \cos(\varphi_0 + N\vartheta), \quad \mathcal{M}_e = \sin(\varphi_0 + N\vartheta)$$

Markov chain model: summary

Therefore the Markov chain model is given by

$$\rho_{k+1} = \mathbb{M}_{s_k}(\rho_k) = \frac{\mathcal{M}_{s_k}\rho_k\mathcal{M}_{s_k}^\dagger}{\mathrm{Tr}\left(\mathcal{M}_{s_k}\rho_k\mathcal{M}_{s_k}^\dagger\right)},$$

where $s_k = g$ or e with associated probabilities $p_{g,k}$ and $p_{e,k}$ given by

$$p_{g,k} = \mathrm{Tr}\left(\mathcal{M}_g\rho_k\mathcal{M}_g^\dagger\right) \quad \text{and} \quad p_{e,k} = \mathrm{Tr}\left(\mathcal{M}_e\rho_k\mathcal{M}_e^\dagger\right).$$

Here $\mathcal{M}_g$ and $\mathcal{M}_e$ are given by

$$\mathcal{M}_g = \cos(\varphi_0 + N\vartheta), \quad \mathcal{M}_e = \sin(\varphi_0 + N\vartheta)$$

This is a QND measurement for the observable N of photon number. Indeed, as the Kraus operators $\mathcal{M}_g$ and $\mathcal{M}_e$ commute with N, the mean value of N does not change through the measurement procedure:

$$\mathbb{E}\left(\mathrm{Tr}\left(N\rho_{k+1}\right) \mid \rho_k\right) = \mathrm{Tr}\left(N\rho_k\right).$$

Also, the eigenstates of the observable N (the Fock states) are invariant with respect to the measurement procedure:

$$\mathbb{M}_g(|n\rangle\langle n|) = |n\rangle\langle n| \quad \text{and} \quad \mathbb{M}_e(|n\rangle\langle n|) = |n\rangle\langle n| \qquad \text{for all } n.$$

Why density matrices (1)

Measurement in $|g\rangle$

$$|g\rangle \otimes \mathcal{M}_g |\psi\rangle + |e\rangle \otimes \mathcal{M}_e |\psi\rangle \longrightarrow \frac{|g\rangle \otimes \mathcal{M}_g |\psi\rangle}{\left\| \mathcal{M}_g |\psi\rangle \right\|_{\mathcal{H}}},$$

Measurement in $|e\rangle$

$$|g\rangle \otimes \mathcal{M}_g |\psi\rangle + |e\rangle \otimes \mathcal{M}_e |\psi\rangle \longrightarrow \frac{|e\rangle \otimes \mathcal{M}_e |\psi\rangle}{\left\| \mathcal{M}_e |\psi\rangle \right\|_{\mathcal{H}}},$$

Why density matrices (2)

The atom-detector does not always detect the atoms.
Therefore 3 outcomes:
Atom in $|g\rangle$, Atom in $|e\rangle$, No detection

Best estimate for the no-detection case

$$\mathbb{E}\left(|\psi\rangle_+ \mid |\psi\rangle\right) = \left\|\mathcal{M}_g |\psi\rangle\right\|_{\mathcal{H}} \mathcal{M}_g |\psi\rangle + \left\|\mathcal{M}_e |\psi\rangle\right\|_{\mathcal{H}} \mathcal{M}_e |\psi\rangle$$

This is not a well-defined wavefunction

Barycenter in the sense of geodesics of $\mathbb{S}(\mathcal{H})$
not invariant with respect to a change of global phase

We need a barycenter in the sense of the projective space
$\mathbb{CP}(\mathcal{H}) \equiv \mathbb{S}(\mathcal{H})/\mathbb{S}^1$

Why density matrices (3)

Projector over the state $|\psi\rangle$: $P_{|\psi\rangle} = |\psi\rangle\langle\psi|$

Detection in $|g\rangle$: the projector is given by

$$P_{|\psi_+\rangle} = \frac{\mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger}{\left\| \mathcal{M}_g |\psi\rangle \right\|_{\mathcal{H}}^2} = \frac{\mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger}{\left| \left\langle \psi \mid \mathcal{M}_g^\dagger \mathcal{M}_g \mid \psi \right\rangle \right|^2} = \frac{\mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger}{\mathrm{Tr}\left(\mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger \right)}$$

Detection in $|e\rangle$: the projector is given by

$$P_{|\psi_+\rangle} = \frac{\mathcal{M}_e |\psi\rangle\langle\psi| \mathcal{M}_e^\dagger}{\mathrm{Tr}\left(\mathcal{M}_e |\psi\rangle\langle\psi| \mathcal{M}_e^\dagger \right)}$$

Probabilities:

$$p_g = \mathrm{Tr}\left(\mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger \right) \quad \text{and} \quad p_e = \mathrm{Tr}\left(\mathcal{M}_e |\psi\rangle\langle\psi| \mathcal{M}_e^\dagger \right)$$

Why density matrices (4)

Imperfect detection: barycenter

$$|\psi\rangle\langle\psi| \longrightarrow p_g \frac{\mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger}{\mathrm{Tr}\left(\mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger\right)} + p_e \frac{\mathcal{M}_e |\psi\rangle\langle\psi| \mathcal{M}_e^\dagger}{\mathrm{Tr}\left(\mathcal{M}_e |\psi\rangle\langle\psi| \mathcal{M}_e^\dagger\right)}$$
$$= \mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger + \mathcal{M}_e |\psi\rangle\langle\psi| \mathcal{M}_e^\dagger.$$

This is not anymore a projector: no well-defined wave function

New state space of quantum states ρ:

$$\mathcal{X} = \{\rho \in \mathcal{L}(\mathcal{H}) \mid \rho^\dagger = \rho, \rho \geq 0, \mathrm{Tr}(\rho) = 1\}$$

Pure quantum states ρ correspond to rank 1 projectors and thus to wave functions $|\psi\rangle$ with $\rho = |\psi\rangle\langle\psi|$.

Kraus map

What if we do not detect the atoms after they exit R_2?

The "best estimate" of the cavity state is given by its expectation value

$$\rho_+ = p_{g,k}\mathbb{M}_g(\rho) + p_{e,k}\mathbb{M}_e(\rho) = \mathcal{M}_g \rho \mathcal{M}_g^\dagger + \mathcal{M}_e \rho \mathcal{M}_e^\dagger =: \mathbb{K}(\rho).$$

This linear map is called the Kraus map associated to the Kraus operators $\mathcal{M}_g$ and $\mathcal{M}_e$.

In the same way and through a Bayesian filter we can take into account various uncertainties.

Some uncertainties

Pulse occupation The probability that a pulse is occupied by an atom is given by η_a ($\eta_a \in (0, 1]$ is called the pulse occupancy rate);

Detector efficiency The detector can miss an atom with a probability of $1 - \eta_d$ ($\eta_d \in (0, 1]$ is called the detector's efficiency rate);

Detector faults The detector can make a mistake by detecting an atom in $|g\rangle$ while it is in the state $|e\rangle$ or vice-versa; this happens with a probability of η_f ($\eta_f \in [0, 1/2]$ is called the detector's fault rate);

We basically have three possibilities for the detection output:

Atom detected in $|g\rangle$ either the atom is really in the state $|g\rangle$ or the detector has made a mistake and it is actually in the state $|e\rangle$;

Atom detected in $|e\rangle$ either the atom is really in the state $|e\rangle$ or the detector has made a mistake and it is actually in the state $|g\rangle$;

No atom detected either the pulse has been empty or the detector has missed the atom.

Atom detected in $|g\rangle$

Either the atom is actually in the state $|e\rangle$ and the detector has made a mistake by detecting it in $|g\rangle$ (this happens with a probability p_g^f) or the atom is really in the state $|g\rangle$ (this happens with probability $1 - p_g^f$).

Conditional probablity p_g^f: We apply the Bayesian formula

$$p_g^f = \frac{\eta_f p_e}{\eta_f p_e + (1-\eta_f)p_g},$$

where $p_g = \mathrm{Tr}\left(\mathcal{M}_g \rho \mathcal{M}_g^\dagger\right)$ and $p_e = \mathrm{Tr}\left(\mathcal{M}_e \rho \mathcal{M}_e^\dagger\right)$.

Conditional evolution of density matrix:

$$\begin{aligned}\rho_+ &= p_g^f \mathbb{M}_e(\rho) + (1-p_g^f)\mathbb{M}_g(\rho) \\ &= \frac{\eta_f}{\eta_f p_e + (1-\eta_f)p_g}\mathcal{M}_e \rho \mathcal{M}_e^\dagger + \frac{1-\eta_f}{\eta_f p_e + (1-\eta_f)p_g}\mathcal{M}_g \rho \mathcal{M}_g^\dagger.\end{aligned}$$

Atom detected in $|e\rangle$

In the same way

$$\rho_+ = \frac{\eta_f}{\eta_f p_g + (1-\eta_f)p_e}\mathcal{M}_g \rho \mathcal{M}_g^\dagger + \frac{1-\eta_f}{\eta_f p_g + (1-\eta_f)p_e}\mathcal{M}_e \rho \mathcal{M}_e^\dagger.$$

No atom detected

Either the pulse has been empty (this happens with a probability p_{na}) or there has been an atom which has not been detected by the detector (this happens with the probability $1 - p_{\text{na}}$).

Conditional probability p_{na}:

$$p_{\text{na}} = \frac{1-\eta_a}{\eta_a(1-\eta_d)+(1-\eta_a)} = \frac{1-\eta_a}{1-\eta_a\eta_d}.$$

In such case the density matrix remains untouched.
The undetected atom case leads to an evolution of the density matrix through the Kraus representation.

Conditional evolution:

$$\begin{aligned}\rho_+ &= p_{\text{na}}\ \rho + (1-p_{\text{na}})(\mathcal{M}_g\rho\mathcal{M}_g^\dagger + \mathcal{M}_e\rho\mathcal{M}_e^\dagger)\\ &= \frac{1-\eta_a}{1-\eta_a\eta_d}\rho + \frac{\eta_a(1-\eta_d)}{1-\eta_a\eta_d}(\mathcal{M}_g\rho\mathcal{M}_g^\dagger + \mathcal{M}_e\rho\mathcal{M}_e^\dagger).\end{aligned}$$