Modeling and Control of Quantum Systems

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Outline

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Controllability of bilinear Schrödinger equations[1]

Schrödinger equation

$$i\frac{d}{dt}|\psi\rangle = \left(H_0 + \sum_{k=1}^{m} u_k H_k\right)|\psi\rangle$$

State controllability

For any $|\psi_a\rangle$ and $|\psi_b\rangle$ on the unit sphere of $\mathcal{H}$, there exist a time $T > 0$, a global phase $\theta \in [0, 2\pi[$ and a piecewise continuous control $[0, T] \ni t \mapsto u(t)$ such that the solution with initial condition $|\psi\rangle_0 = |\psi_a\rangle$ satisfies $|\psi\rangle_T = e^{i\theta}|\psi_b\rangle$.

[1] See, e.g., *Introduction to Quantum Control and Dynamics* by D. D'Alessandro. Chapman & Hall/CRC, 2008.

Controllability of bilinear Schrödinger equations

Propagator equation:

$$i\frac{d}{dt}U = \left(H_0 + \sum_{k=1}^{m} u_k H_k\right) U, \quad U(0) = \mathbf{1}$$

We have $|\psi\rangle_t = U(t)\,|\psi\rangle_0$.

Operator controllability

For any unitary operator V on $\mathcal{H}$, there exist a time $T > 0$, a global phase θ and a piecewise continuous control $[0, T] \ni t \mapsto u(t)$ such that the solution of propagator equation satisfies $U_T = e^{i\theta} V$.

Operator controllability implies state controllability

Lie-algebra rank condition

$$\frac{d}{dt}U = \left(A_0 + \sum_{k=1}^{m} u_k A_k\right) U$$

with $A_k = H_k/i$ are skew-Hermitian. We define

$$\begin{aligned}
\mathcal{L}_0 &= \text{span}\{A_0, A_1, \ldots, A_m\} \\
\mathcal{L}_1 &= \text{span}(\mathcal{L}_0, [\mathcal{L}_0, \mathcal{L}_0]) \\
\mathcal{L}_2 &= \text{span}(\mathcal{L}_1, [\mathcal{L}_1, \mathcal{L}_1]) \\
&\vdots \\
\mathcal{L} = \mathcal{L}_\nu &= \text{span}(\mathcal{L}_{\nu-1}, [\mathcal{L}_{\nu-1}, \mathcal{L}_{\nu-1}])
\end{aligned}$$

Lie Algebra Rank Condition

Operator controllable if, and only if, the Lie algebra generated by the $m+1$ skew-Hermitian matrices $\{-iH_0, -iH_1, \ldots, -iH_m\}$ is either $su(n)$ or $u(n)$.

Exercice

Show that $i\frac{d}{dt}|\psi\rangle = \left(\frac{\omega_{eg}}{2}\sigma_z + \frac{u}{2}\sigma_x\right)|\psi\rangle$, $|\psi\rangle \in \mathbb{C}^2$ *is controllable.*

A simple sufficient condition

We consider $H = H_0 + uH_1$, $(|j\rangle)_{j=1,\dots,n}$ the eigenbasis of H_0.
We assume $H_0 |j\rangle = \omega_j |j\rangle$ where $\omega_j \in \mathbb{R}$, we consider a graph G:

$$V = \{|1\rangle, \dots, |n\rangle\}, \quad E = \{(|j_1\rangle, |j_2\rangle) \mid 1 \leq j_1 < j_2 \leq n, \ \langle j_1|H_1|j_2\rangle \neq 0\}.$$

G amits a degenerate transition if there exist $(|j_1\rangle, |j_2\rangle) \in E$ and $(|l_1\rangle, |l_2\rangle) \in E$, admitting the same transition frequencies,

$$|\omega_{j_1} - \omega_{j_2}| = |\omega_{l_1} - \omega_{l_2}|.$$

A sufficient controllability condition

Remove from E, all the edges with identical transition frequencies. Denote by $\bar{E} \subset E$ the reduced set of edges without degenerate transitions and by $\bar{G} = (V, \bar{E})$. If $\bar{G}$ is connected, then the system is operator controllable.

The dynamics of the 2-qubit system (state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$) obey

$$i\frac{d}{dt}|\psi\rangle = (H_0 + uH_1)|\psi\rangle = (Z_1Z_2 + u(X_1 + X_2))|\psi\rangle \qquad (1)$$

with $u \in \mathbb{R}$ as control.

1. Prove that X_1X_2 commutes with H_0 and with H_1.
2. Is the system controllable ?
3. Use the spectral basis of X_1X_2 and the decomposition $\text{span}\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} = \text{span}\{|{++}\rangle, |{--}\rangle\} \oplus \text{span}\{|{+-}\rangle, |{-+}\rangle\}$ with $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$, to deduce a splitting of this system into two separated systems on $\text{span}\{|{++}\rangle, |{--}\rangle\}$ and on $\text{span}\{|{+-}\rangle, |{-+}\rangle\}$.
4. Prove that one of these sub-systems is controllable and that the other one is not controllable.

Lyapunov control

Bilinear Schrödinger equation:

$$i\frac{d}{dt}|\psi\rangle = (H_0 + u(t)H_1)|\psi\rangle$$

Control task: to prepare $|\bar{\psi}\rangle$ such that

$$H_0|\bar{\psi}\rangle = \bar{\omega}|\bar{\psi}\rangle.$$

The states $|\psi\rangle$ and $e^{i\phi}|\psi\rangle$ represent the same physical states

We add a fictitious control:

$$i\frac{d}{dt}|\psi\rangle = (H_0 + u(t)H_1)|\psi\rangle + \omega(t)|\psi\rangle$$

$|\bar{\psi}\rangle$ is a stationary solution for $u(t) \equiv 0$ and $\omega(t) \equiv -\bar{\omega}$.

Lyapunov control

We look for feedback laws $u(t) = f(|\psi\rangle)$ and $\omega(t) = g(|\psi\rangle)$ such that the solution of

$$i\frac{d}{dt}|\psi\rangle = (H_0 + f(|\psi\rangle)H_1 + g(|\psi\rangle))|\psi\rangle$$

converges asymptotically towards $|\bar{\psi}\rangle$.

Remark

These feedback laws are calculated off-line and by simulating the closed-loop system and are then applied in open-loop on the real system.

A Lyapunov function

We consider

$$\mathcal{V}(|\psi\rangle) = \frac{1}{2}\left\| |\psi\rangle - |\bar{\psi}\rangle \right\|^2 = 1 - \Re(\langle \bar{\psi} \mid \psi \rangle).$$

We have

$$\frac{d}{dt}\mathcal{V} = -u(t)\Im\langle \bar{\psi} \mid H_1 \mid \psi \rangle - (\omega(t) + \bar{\omega})\Im(\langle \bar{\psi} \mid \psi \rangle)$$

Choice of feedback laws

$$u(t) \equiv a\Im(\langle \bar{\psi} \mid H_1 \mid \psi \rangle) \qquad \text{and} \qquad \omega(t) \equiv -\bar{\omega} + b\Im(\langle \bar{\psi} \mid \psi \rangle),$$

where $a, b > 0$.

Theorem (Lyapunov function and Lasalle invariance principle)

Take $\Omega \subset \mathbb{R}^n$ an open and non-empty subset of $\mathbb{R}^n$ and $\Omega \ni x \mapsto v(x) \in \mathbb{R}^n$ continuously differentiable function of x. Consider $\Omega \ni x \mapsto V(x) \in \mathbb{R}$ a continuously differentiable function of x and assume that

1. *there exits $c \in \mathbb{R}$ such that the subset $V_c = \{x \in \Omega \mid V(x) \leq c\}$ of $\mathbb{R}^n$ is compact (bounded and closed) and non-empty.*
2. *V is a decreasing time function for solutions of $\frac{d}{dt}x = v(x)$ inside V_c:*

$$\forall x \in V_c, \quad \tfrac{d}{dt}V(x) = \nabla V(x) \cdot v(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x)\, v_i(x) \leq 0$$

Then for any initial condition $x^0 \in V_c$, the solution of $\frac{d}{dt}x = v(x)$ remains in V_c, is defined for all $t > 0$ (no explosion in finite time) and converges towards the largest invariant set included in

$$\left\{x \in V_c \mid \tfrac{d}{dt}V(x) = 0\right\}.$$

$d\mathcal{V}/dt = 0$ and invariance

$$\begin{aligned}
\Im(\langle \bar\psi \mid \psi \rangle) &= 0,\\
\Im(\langle \bar\psi \mid H_1 \mid \psi \rangle) &= 0,\\
\Re(\langle \bar\psi \mid [H_0, H_1] \mid \psi \rangle) &= 0,\\
&\vdots\\
\Im\left(\left\langle \bar\psi \mid \mathrm{ad}_{H_1}^{2k} H_0 \mid \psi \right\rangle\right) &= 0,\\
\Re\left(\left\langle \bar\psi \mid \mathrm{ad}_{H_1}^{2k+1} H_0 \mid \psi \right\rangle\right) &= 0.
\end{aligned}$$

Assume that the spectrum of H_0 is not $\bar\omega$-degenerate: i.e. H_0 is not degenerate and for any two eigenvalues $\omega_\alpha \neq \omega_\beta$, $|\omega_\alpha - \bar\omega| \neq |\omega_\beta - \bar\omega|$;

Ω-limit set

Intersection of $\mathbb{S}^{2n-1}$ with $\mathbb{R}\left|\bar\psi\right\rangle \bigcup_\alpha \mathbb{C}\left|\psi_\alpha\right\rangle$, where $\left|\psi_\alpha\right\rangle$ is any eigenvector of H_0 non co-linear with $\left|\bar\psi\right\rangle$ and satisfying $\langle \bar\psi \mid H_1 \mid \psi_\alpha \rangle = 0$.

Convergence Analysis

Theorem

Under the assumption of H_0 not $\bar{\omega}$-degenerate and mono-photonic transitions to $|\bar{\psi}\rangle$ ($\langle\bar{\psi} \mid H_1 \mid \psi_\alpha\rangle \neq 0$ for all eigenvector $|\psi_\alpha\rangle$ of H_0), the Ω-limit set reduces to $\{|\bar{\psi}\rangle, -|\bar{\psi}\rangle\}$. The equilibrium $-|\bar{\psi}\rangle$ is unstable and the attraction region for the equilibrium $|\psi\rangle$ is exactly $\mathbb{S}^{2n-1}/\{-|\bar{\psi}\rangle\}$.

Remark

Assumptions of H_0 not $\bar{\omega}$-degenerate and mono-photonic transitions to $|\bar{\psi}\rangle$

$$\leftrightarrow$$

Controllability of linearized system around $(|\psi\rangle, u, \omega) = (|\bar{\psi}\rangle, 0, -\bar{\omega})$

Relaxing the assumptions: tracking

Main idea: stabilizing around another reference trajectory, around which the linearized system is controllable.

Reference trajectory:

$$i\frac{d}{dt}|\psi_r\rangle = (H_0 + u_r(t)H_1 + \omega_r(t))|\psi_r\rangle$$

Same Lyapunov function: $\mathcal{V}(t, |\psi\rangle) = 1 - \Re(\langle\psi_r(t) \mid \psi\rangle)$.

Feedback laws:

$$u(t, |\psi\rangle) = u_r(t) + a\Im(\langle\psi_r(t) \mid H_1 \mid \psi\rangle),$$
$$\omega(t, |\psi\rangle) = \omega_r(t) + b\Im(\langle\psi_r(t) \mid \psi\rangle)$$

Tracking and quantum gate design

We consider a drift-less propagator dynamics:

$$i\frac{d}{dt}U = \left(\omega \mathbf{1} + \sum_{k=1}^{m} u_k H_k\right) U, \qquad U\Big|_{t=0} = \mathbf{1}.$$

Periodic reference trajectory: u_k^r and ω_r periodic and odd.

Main idea

By a Coron's result, as soon as $\text{Lie}(H_1, \ldots, H_m) = su(n)$, one can find reference controls ω^r and u_k^r around which the linearized system is controllable.

Lyapunov function: $\mathcal{V}(U, U^r) = n - \Re(\text{Tr}\left(U^\dagger U^r\right))$.
Feedback laws:

$$u_k = u_k^r - a_k \Im(\text{Tr}\left(U^\dagger H_k U^r\right)),$$

$$\omega = \omega^r - b\Im(\text{Tr}\left(U^\dagger U^r\right)).$$

Remark

The LaSalle's invariance principle also works for time-periodic systems; only one needs to be be careful about the notion of invariance:

A set S is said to be invariant for the time-periodic system $\frac{d}{dt}x = v(x, t)$ if, for all $x_0 \in S$ there exists a time $t_0 > 0$ such that the solution starting from x_0 at time t_0 remains in the set S for all $t \geq t_0$.

For given T, $|\psi_a\rangle$ and $|\psi_b\rangle$, find the open-loop control $[0,T] \ni t \mapsto u(t)$ such that

$$\min_{\substack{u_k \in L^2([0,T],\mathbb{R}) \\ i\frac{d}{dt}|\psi\rangle = (H_0 + \sum_{k=1}^m u_k H_k)|\psi\rangle \\ |\psi\rangle_{t=0} = |\psi_a\rangle \,,\ |\langle\psi_b|\psi\rangle|^2_{t=T} = 1}} \frac{1}{2}\int_0^T \left(\sum_{k=1}^m u_k^2\right)$$

Since the initial and final constraints are difficult to satisfy simultaneously from a numerical point of view, consider the second problem where the final constraint is penalized with $\alpha > 0$:

$$\min_{\substack{u_k \in L^2([0,T],\mathbb{R}) \\ i\frac{d}{dt}|\psi\rangle = (H_0 + \sum_{k=1}^m u_k H_k)|\psi\rangle \\ |\psi\rangle_{t=0} = |\psi_a\rangle}} \frac{1}{2}\int_0^T \left(\sum_{k=1}^m u_k^2\right) + \frac{\alpha}{2}\left(1 - |\langle\psi_b|\psi\rangle|^2_T\right)$$

First order stationary conditions

For two-points problem, the first order stationary conditions read:

$$\begin{cases} i\frac{d}{dt}\left|\psi\right\rangle = (H_0 + \sum_{k=1}^m u_k H_k)\left|\psi\right\rangle, \ t \in (0,T) \\ i\frac{d}{dt}\left|p\right\rangle = (H_0 + \sum_{k=1}^m u_k H_k)\left|p\right\rangle, \ t \in (0,T) \\ u_k = -\Im\Big(\left\langle p|H_k|\psi\right\rangle\Big), k = 1,\dots,m, \ \ t \in (0,T) \\ \left|\psi\right\rangle_{t=0} = \left|\psi_a\right\rangle, \ |\left\langle\psi_b|\psi\right\rangle|^2_{t=T} = 1 \end{cases}$$

For the relaxed problem, the first order stationary conditions read:

$$\begin{cases} i\frac{d}{dt}\left|\psi\right\rangle = (H_0 + \sum_{k=1}^m u_k H_k)\left|\psi\right\rangle, \ t \in (0,T) \\ i\frac{d}{dt}\left|p\right\rangle = (H_0 + \sum_{k=1}^m u_k H_k)\left|p\right\rangle, \ t \in (0,T) \\ u_k = -\Im\Big(\left\langle p|H_k|\psi\right\rangle\Big), k = 1,\dots,m, \ \ t \in (0,T) \\ \left|\psi\right\rangle_{t=0} = \left|\psi_a\right\rangle, \ \left|p\right\rangle_{t=T} = -\alpha\left\langle\psi_b|\psi\right\rangle_{t=T} \ \left|\psi_b\right\rangle. \end{cases}$$

The dynamical system

$$(\Sigma)\left\{\begin{array}{c} i\frac{d}{dt}\left|\psi\right\rangle = (H_0 + \sum_{k=1}^m u_k H_k)\left|\psi\right\rangle, \; t\in(0,T) \\ i\frac{d}{dt}\left|p\right\rangle = (H_0 + \sum_{k=1}^m u_k H_k)\left|p\right\rangle, \; t\in(0,T) \\ u_k = -\Im\left(\left\langle p|H_k|\psi\right\rangle\right), k=1,\ldots,m, \;\; t\in(0,T) \end{array}\right.$$

is Hamiltonian with $\left|\psi\right\rangle$ and $\left|p\right\rangle$ being the conjugate variables. The underlying Hamiltonian function is given by (Pontryaguin Maximum Principle): $\overline{\mathbb{H}}(\left|\psi\right\rangle, \left|p\right\rangle) = \min_{u\in\mathbb{R}^m} \mathbb{H}(\left|\psi\right\rangle, \left|p\right\rangle, u)$ where

$$\mathbb{H}(\left|\psi\right\rangle, \left|p\right\rangle, u) = \frac{1}{2}\left(\sum_{k=1}^m u_k^2\right) + \Im\left(\left\langle p \middle| H_0 + \sum_{k=1}^m u_k H_k \middle| \psi \right\rangle\right).$$

Thus for any solutions $(\left|\psi\right\rangle, \left|p\right\rangle)$ of (Σ) ,

$$\overline{\mathbb{H}}(\left|\psi\right\rangle, \left|p\right\rangle) = \Im\left(\left\langle p\left|H_0\right|\psi\right\rangle\right) - \frac{1}{2}\left(\sum_{k=1}^m \Im\Big(\left\langle p|H_k|\psi\right\rangle\Big)^2\right).$$

is independent of t.

Main difficulty: such systems are not, in general, integrable in the Arnold-Liouville sense.

Monotone numerical scheme for the relaxed problem (1)[2]

Take an L^2 control $[0,T] \ni t \mapsto u(t)$ ($\dim(u) = 1$ here) and denote by

- $|\psi_u\rangle$ the solution of forward system $i\frac{d}{dt}|\psi\rangle = (H_0 + uH_1)|\psi\rangle$ starting from $|\psi_a\rangle$.
- $|p_u\rangle$ the adjoint associated to u, i.e. the solution of the backward system $i\frac{d}{dt}|p_u\rangle = (H_0 + uH_1)|p_u\rangle$ with $|p_u\rangle_T = -\alpha P|\psi_u\rangle_T$, P projector on $|\psi_b\rangle$, $P|\phi\rangle \equiv \langle\psi_b|\phi\rangle\ |\psi_b\rangle$.
- $J(u) = \frac{1}{2}\int_0^T u^2 + \frac{\alpha}{2}(1 - |\langle\psi_b|\psi_u\rangle|_T^2)$.

Starting from an initial guess $u^0 \in L^2([0,T],\mathbb{R})$, the monotone scheme generates a sequence of controls $u^\nu \in L^2([0,T],\mathbb{R})$, $\nu = 1, 2, \ldots$, such that the cost $J(u^\nu)$ is decreasing, $J(u^{\nu+1}) \le J(u^\nu)$.

[2] D. Tannor, V. Kazakov, and V. Orlov. *Time Dependent Quantum Molecular Dynamics*, chapter Control of photochemical branching: Novel procedures for finding optimal pulses and global upper bounds, pages 347–360. Plenum, 1992.

Assume that, at step ν, we have computed the control u^ν, the associated quantum state $|\psi^\nu\rangle = |\psi_{u^\nu}\rangle$ and its adjoint $|p^\nu\rangle = |p_{u^\nu}\rangle$. We get their new time values $u^{\nu+1}$, $\left|\psi^{\nu+1}\right\rangle$ and $\left|p^{\nu+1}\right\rangle$ in two steps:

1. Imposing $u^{\nu+1} = -\Im\left(\left\langle p^\nu \left|H_1\right| \psi^{\nu+1}\right\rangle\right)$ is just a feedback; one get $u^{\nu+1}$ just by a forward integration of the nonlinear Schrödinger equation,

$$i\frac{d}{dt}|\psi\rangle = \left(H_0 - \Im\left(\left\langle p^\nu \left|H_1\right| \psi\right\rangle\right) H_1\right)|\psi\rangle , \quad |\psi\rangle_0 = |\psi_a\rangle ,$$

   that provides $[0, T] \ni t \mapsto \left|\psi^{\nu+1}\right\rangle$ and the new control $u^{\nu+1}$.

2. Backward integration from $t = T$ to $t = 0$ of

$$i\frac{d}{dt}|p\rangle = \left(H_0 + u^{\nu+1}(t)H_1\right)|p\rangle , \quad |p\rangle_T = -\alpha\left\langle\psi_b|\psi^{\nu+1}\right\rangle_T |\psi_b\rangle$$

   yields to the new adjoint trajectory $[0, T] \ni t \mapsto \left|p^{\nu+1}\right\rangle$.

Monotone numerical scheme for the relaxed problem (3)

Why $J(u^{\nu+1}) \leq J(u^{\nu})$?

- Because we have the identity for any open-loop controls u and v.

$$J(u) - J(v) = -\frac{\alpha}{2}\left(\langle \psi_u - \psi_v | P | \psi_u - \psi_v \rangle\right)_T + \frac{1}{2}\left(\int_0^T (u-v)\left(u + v + 2\Im\left(\langle p_v \left| H_1 \right| \psi_u \rangle\right)\right)\right).$$

- If $u = -\Im\left(\langle p_v \left| H_1 \right| \psi_u \rangle\right)$ for all $t \in [0, T)$, we have

$$J(u) - J(v) = -\frac{\alpha}{2}\left(\langle \psi_u - \psi_v | P | \psi_u - \psi_v \rangle\right)_T - \frac{1}{2}\left(\int_0^T (u-v)^2\right)$$

  and thus $J(u) \leq J(v)$.

- Take $v = u^{\nu}$, $u = u^{\nu+1}$: then $|p_v\rangle = |p^{\nu}\rangle$, $|\psi_v\rangle = |\psi^{\nu}\rangle$, $|p_u\rangle = \left|p^{\nu+1}\right\rangle$ and $|\psi_u\rangle = \left|\psi^{\nu+1}\right\rangle$.

Proof of

$$J(u) - J(v) = -\frac{\alpha}{2}\left(\langle \psi_u - \psi_v | P | \psi_u - \psi_v \rangle\right)_T + \frac{1}{2}\left(\int_0^T (u-v)\left(u+v+2\Im\left(\langle p_v | H_1 | \psi_u \rangle\right)\right)\right).$$

Start with

$$J(u)-J(v) = -\frac{\alpha\left(\langle \psi_u-\psi_v|P|\psi_u-\psi_v\rangle_T+\langle \psi_u-\psi_v|P|\psi_v\rangle_T+\langle \psi_v|P|\psi_u-\psi_v\rangle_T\right)}{2} + \int_0^T \frac{(u-v)(u+v)}{2}.$$

Hermitian product of $i\frac{d}{dt}(|\psi_u\rangle - |\psi_v\rangle) = (H_0 + vH_1)(|\psi_u\rangle - |\psi_v\rangle) + (u-v)H_1 |\psi_u\rangle$ **with** $|p_v\rangle$**:**

$$\left\langle p_v \left| \frac{d(\psi_u-\psi_v)}{dt} \right.\right\rangle = \left\langle p_v \left| \frac{H_0+vH_1}{i} \right| \psi_u - \psi_v \right\rangle + \left\langle p_v \left| \frac{(u-v)H_1}{i} \right| \psi_u \right\rangle.$$

Integration by parts (use $|\psi_v\rangle_0 = |\psi_u\rangle_0$, $|p_v\rangle_T = -\alpha P |\psi_v\rangle_T$ **and** $\frac{d}{dt}\langle p_v| = -\langle p_v|\left(\frac{H_0+vH_1}{i}\right)$**):**

$$\int_0^T \left\langle p_v \left| \frac{d(\psi_u-\psi_v)}{dt} \right.\right\rangle = \langle p_v|\psi_u - \psi_v\rangle_T - \langle p_v|\psi_u - \psi_v\rangle_0 - \int_0^T \left\langle \frac{dp_v}{dt} \right| \psi_u - \psi_v \Big\rangle$$
$$= -\alpha\langle \psi_v|P|\psi_u - \psi_v\rangle_T + \int_0^T \left\langle p_v \left| \frac{H_0+vH_1}{i} \right| \psi_u - \psi_v \right\rangle$$

Thus $-\alpha\langle \psi_v|P|\psi_u - \psi_v\rangle_T = \int_0^T \left\langle p_v \left| \frac{(u-v)H_1}{i} \right| \psi_u \right\rangle$ **and**
$\alpha\Re\left(\langle \psi_v|P|\psi_u - \psi_v\rangle_T\right) = -\int_0^T \Im\left(\langle p_v |(u-v)H_1| \psi_u\rangle\right)$. **Finally we have**

$$J(u) - J(v) = -\frac{\alpha}{2}\left(\langle \psi_u - \psi_v | P | \psi_u - \psi_v \rangle\right)_T + \frac{1}{2}\left(\int_0^T (u-v)\left(u+v+2\Im\left(\langle p_v | H_1 | \psi_u \rangle\right)\right)\right).$$

Optimality and resonance (1)[3]

For given T, $a_k \geq 0$ and $b_k \geq 0$ ($\sum_{k=1}^n a_k^2 = \sum_{k=1}^n b_k^2 = 1$),

$$\min_{\substack{\mathbf{u}_{k,l} \in L^2([0,T],\mathbb{C}),\ (k,l)\in I \\ i\frac{d}{dt}|\psi\rangle = \left(\sum_{(k,l)\in I}\mu_{kl}\mathbf{u}_{kl}|k\rangle\langle l|\right)|\psi\rangle, \\ |\langle k|\psi\rangle|^2_{t=0} = a_k^2,\ |\langle k|\psi\rangle|^2_{t=T} = b_k^2,\ k=1,\dots,n}} \frac{1}{2}\int_0^T \left(\sum_{(k,l)\in I} |\mathbf{u}_{kl}|^2\right)$$

admits the same minimal cost as the following reduced problem

$$\min_{\substack{v_{k,l} \in L^2([0,T],\mathbb{R}),\ v_{kl} = -v_{l,k},\ (k,l)\in I \\ \frac{d}{dt}|\phi\rangle = \left(\sum_{(k,l)\in I}\mu_{kl}v_{kl}|k\rangle\langle l|\right)|\phi\rangle \\ \langle k|\phi\rangle|_{t=0} = a_k,\ \langle k|\phi\rangle_{t=T} = b_k,\ k=1,\dots,n}} \frac{1}{2}\int_0^T \left(\sum_{(k,l)\in I} |v_{kl}|^2\right)$$

where the components of $|\psi\rangle = |\phi\rangle$ remain real, the $\mathbf{u}_{kl}$'s are purely imaginary, $\mathbf{u}_{kl} = iv_{kl}$ ($v_{kl} \in \mathbb{R}$ with $v_{kl} = -v_{lk}$).

[3] U. Boscain and G. Charlot. Resonance of minimizers for n-level quantum systems with an arbitrary cost. *ESAIM COCV*, 10:593–614, 2004.

Optimality and resonance (2)

- Go back to resulting optimal physical controls ($\mathbf{u}_{kl} = iv_{kl}$):

$$\mathbf{u}_{kl}(t)e^{i(\omega_k-\omega_l)t}+\mathbf{u}^*_{kl}(t)e^{-i(\omega_k-\omega_l)t} = -2v_{kl}(t)\sin\left((\omega_k-\omega_l)t\right).$$

- They are in resonance with the frequency transition between $|k\rangle$ and $|l\rangle$. They contain only amplitude modulations (up to a π phase-shift since v_{kl} can pass through zero).
- For drift-less quantum systems

$$i\frac{d}{dt}|\psi\rangle = \left(\sum_{(k,l)\in I} \mu_{kl}\mathbf{u}_{kl}|k\rangle\langle l|\right)|\psi\rangle$$

population transfer minimizing the L^2 control norm is achieved by resonant controls $\mathbf{u}_{kl} = iv_{kl}$ with $v_{kl} \in \mathbb{R}$ (the reduction of the problem to a real case of half dimension).

Optimality and resonance (3)

Associated to any $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ consider

$$|\psi\rangle \mapsto |\psi^\theta\rangle = \left(\sum_{k=1}^{n} e^{i\theta_k} |k\rangle\langle k|\right)|\psi\rangle, \qquad \mathbf{u}_{kl} \mapsto \mathbf{u}_{kl}^{\theta} = e^{i(\theta_k - \theta_l)}\mathbf{u}_{kl}.$$

These transformations leave unchanged cost and constraints of

$$\min_{\substack{\mathbf{u}_{k,l} \in L^2([0,T],\mathbb{C}),\ (k,l)\in I \\ i\frac{d}{dt}|\psi\rangle = \left(\sum_{(k,l)\in I} \mu_{kl}\mathbf{u}_{kl}|k\rangle\langle l|\right)|\psi\rangle, \\ |\langle k|\psi\rangle|^2_{t=0} = a_k^2,\ |\langle k|\psi\rangle|^2_{t=T} = b_k^2,\ k=1,\ldots,n}} \frac{1}{2}\int_0^T \left(\sum_{(k,l)\in I} |\mathbf{u}_{kl}|^2\right).$$

that coincides with

$$\min_{\substack{\mathbf{u}_{k,l} \in L^2([0,T],\mathbb{C}),\ (k,l)\in I \\ i\frac{d}{dt}|\psi\rangle = \left(\sum_{(k,l)\in I} \mu_{kl}\mathbf{u}_{kl}|k\rangle\langle l|\right)|\psi\rangle, \\ \langle k|\psi\rangle_{t=0} = a_k,\ |\langle k|\psi\rangle|^2_{t=T} = b_k^2,\ k=1,\ldots,n}} \frac{1}{2}\int_0^T \left(\sum_{(k,l)\in I} |\mathbf{u}_{kl}|^2\right).$$

Optimality and resonance (4)

- Set $\psi_k = \langle k|\psi\rangle$ and $\mathbf{z}_{kl} = \psi_k \psi_l^*$: $\frac{d}{dt}(|\psi_k|^2) = \sum_{l \mid (k,l)\in I} \mu_{kl} \frac{\mathbf{u}_{kl}\mathbf{z}_{kl}^* - \mathbf{u}_{kl}^*\mathbf{z}_{kl}}{i}$
Evolution of the direction of ψ_k in the complex plane is governed by

$$\psi_k^* \frac{d}{dt}\psi_k - \psi_k \frac{d}{dt}\psi_k^* = \sum_{l \mid (k,l)\in I} \mu_{kl} \frac{\mathbf{u}_{kl}\mathbf{z}_{kl}^* + \mathbf{u}_{kl}^*\mathbf{z}_{kl}}{i}.$$

- For $(k,l) \in I$ set $v_{kl}(t) = \begin{cases} 0, & \text{if } \mathbf{z}_{kl}(t) = 0; \\ \frac{\mathbf{u}_{kl}(t)\mathbf{z}_{kl}^*(t) - \mathbf{u}_{kl}^*(t)\mathbf{z}_{kl}(t)}{2i|\mathbf{z}_{kl}(t)|}, & \text{if } \mathbf{z}_{kl}(t) \neq 0;. \end{cases}$.
- We have $v_{kl} = -v_{lk}$ since $\mathbf{u}_{kl}^* = \mathbf{u}_{lk}$ and $z_{kl}^* = z_{lk}$. Moreover $|v_{kl}| \leq |\mathbf{u}_{kl}|$. Thus each v_{kl} belongs to $L^2([0,T],\mathbb{R})$ and the solution $|\phi\rangle$ of $\frac{d}{dt}\phi_k = \sum_{l \mid (k,l)\in I} \mu_{kl} v_{kl}\phi_l$, $\phi_k(0) = a_k, \quad k = 1,\ldots,n$ coincides with $\phi_k = |\psi_k|$.
- To summarize: starting from complex controls $\mathbf{u}_{kl} \in L^2([0,T],\mathbb{C})$ satisfying the constraints of the full problem, we have constructed real controls $v_{kl} \in L^2([0,T],\mathbb{C})$ satisfying the constraints of the reduced problem; the cost associated to $\mathbf{u}_{kl}$ is larger than the cost associated to v_{kl} since $|v_{kl}| \leq |u_{kl}|$.

Outline of the 8 lectures

Lect. 1 (Oct. 4) Introduction on LKB Photon-Box: control issues for classical and quantum oscillators (creation/annihilation operator, coherent state).

Part 1, open-loop control of Schrödinger systems:

- Lect. 2 (Oct. 11) RWA and multi-frequency averaging; 2-level system (half spin) and Jaynes-Cummings model (spin-spring)
- Lect. 3 (Oct. 25) Law-Eberly method for trapped ions; adiabatic invariance and control.
- Lect. 4 (Nov. 22) Controllability, Lyapounov control and optimal control

Part 2, closed-loop control of open quantum systems:

- Lect. 5 (Nov. 29) Measurement and quantum trajectories (discrete time, Kraus operators, LKB-photon box)
- Lect. 6 (Dec. 6) Feedback stabilization (Photon-box, quantum filter, Lyapunov, separation principle, delay compensation)
- Lect. 7 (Dec. 13) Quantum trajectories (continuous time with Poisson process, Lindblad operators, time/scale reduction, synchronization loop on a Λ-system)
- Lect. 8 (Dec. 14) Quantum trajectories (continuous time with Wiener process, homodyn detection, Lyapunov feedback stabilization of entangled states).