Modeling and Control of Quantum Systems

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Outline

Bilinear Schrödinger equation

Un-measured quantum system $\rightarrow$ **Bilinear Schrödinger equation**

$$i\frac{d}{dt}|\psi\rangle = (H_0 + u(t)H_1)\,|\psi\rangle\,,$$

- $|\psi\rangle \in \mathcal{H}$ the system's wavefunction with $\big\| \, |\psi\rangle \, \big\|_{\mathcal{H}} = 1$;
- the free Hamiltonian, H_0, is a Hermitian operator defined on $\mathcal{H}$;
- the control Hamiltonian, H_1, is a Hermitian operator defined on $\mathcal{H}$;
- the control $u(t) : \mathbb{R}^+ \mapsto \mathbb{R}$ is a scalar control.

Here we consider the case of finite dimensional $\mathcal{H}$

Almost periodic control

We consider the controls of the form

$$u(t) = \epsilon \left(\sum_{j=1}^{r} \mathbf{u}_j e^{i\omega_j t} + \mathbf{u}_j^* e^{-i\omega_j t} \right)$$

- $\epsilon > 0$ is a small parameter;
- $\epsilon \mathbf{u}_j$ is the constant complex amplitude associated to the pulsation $\omega_j \geq 0$;
- r stands for the number of independent pulsations ($\omega_j \neq \omega_k$ for $j \neq k$).

We are interested in approximations, for ϵ tending to 0^+, of trajectories $t \mapsto |\psi_\epsilon\rangle_t$ of

$$\frac{d}{dt} |\psi_\epsilon\rangle = \left(A_0 + \epsilon \left(\sum_{j=1}^{r} \mathbf{u}_j e^{i\omega_j t} + \mathbf{u}_j^* e^{-i\omega_j t} \right) A_1 \right) |\psi_\epsilon\rangle$$

where $A_0 = -iH_0$ and $A_1 = -iH_1$ are skew-Hermitian.

Rotating frame

Consider the following change of variables

$$|\psi_\epsilon\rangle_t = e^{A_0 t} |\phi_\epsilon\rangle_t .$$

The resulting system is said to be in the "interaction frame"

$$\frac{d}{dt} |\phi_\epsilon\rangle = \epsilon B(t) |\phi_\epsilon\rangle$$

where $B(t)$ is a skew-Hermitian operator whose time-dependence is almost periodic:

$$B(t) = \sum_{j=1}^{r} \mathbf{u}_j e^{i\omega_j t} e^{-A_0 t} A_1 e^{A_0 t} + \mathbf{u}_j^* e^{-i\omega_j t} e^{-A_0 t} A_1 e^{A_0 t}.$$

Main idea

We can write

$$B(t) = \bar{B} + \frac{d}{dt} \widetilde{B}(t),$$

where $\bar{B}$ is a constant skew-Hermitian matrix and $\widetilde{B}(t)$ is a bounded almost periodic skew-Hermitian matrix.

Multi-frequency averaging: first order

Consider the two systems

$$\frac{d}{dt}\left|\phi_\epsilon\right\rangle = \epsilon\left(\bar{B} + \frac{d}{dt}\widetilde{B}(t)\right)\left|\phi_\epsilon\right\rangle,$$

and

$$\frac{d}{dt}\left|\phi_\epsilon^{1^{\text{st}}}\right\rangle = \epsilon\bar{B}\left|\phi_\epsilon^{1^{\text{st}}}\right\rangle,$$

initialized at the same state $\left|\phi_\epsilon^{1^{\text{st}}}\right\rangle_0 = \left|\phi_\epsilon\right\rangle_0$.

Theorem: first order approximation (Rotating Wave Approximation)

Consider the functions $\left|\phi_\epsilon\right\rangle$ and $\left|\phi_\epsilon^{1^{\text{st}}}\right\rangle$ initialized at the same state and following the above dynamics. Then, there exist $M > 0$ and $\eta > 0$ such that for all $\epsilon \in]0, \eta[$ we have

$$\max_{t\in\left[0,\frac{1}{\epsilon}\right]} \left\| \left|\phi_\epsilon\right\rangle_t - \left|\phi_\epsilon^{1^{\text{st}}}\right\rangle_t \right\| \leq M\epsilon$$

Multi-frequency averaging: first order

Proof's idea

Almost periodic change of variables:

$$|\chi_\epsilon\rangle = (1 - \epsilon\widetilde{B}(t))\,|\phi_\epsilon\rangle$$

well-defined for $\epsilon > 0$ sufficiently small.
The dynamics can be written as

$$\frac{d}{dt}\,|\chi_\epsilon\rangle = (\epsilon\bar{B} + \epsilon^2 F(\epsilon, t))\,|\chi_\epsilon\rangle$$

where $F(\epsilon, t)$ is uniformly bounded in time.

Multi-frequency averaging: second order

More precisely, the dynamics of $|\chi_\epsilon\rangle$ is given by

$$\frac{d}{dt}|\chi_\epsilon\rangle = \left(\epsilon\bar{B} + \epsilon^2[\bar{B}, \widetilde{B}(t)] - \epsilon^2\widetilde{B}(t)\frac{d}{dt}\widetilde{B}(t) + \epsilon^3 E(\epsilon, t)\right)|\chi_\epsilon\rangle$$

- $E(\epsilon, t)$ is still almost periodic but its entries are no more linear combinations of time-exponentials;
- $\widetilde{B}(t)\frac{d}{dt}\widetilde{B}(t)$ is an almost periodic operator whose entries are linear combinations of oscillating time-exponentials.

We can write

$$\widetilde{B}(t)\frac{d}{dt}\widetilde{B}(t) = \bar{D} + \frac{d}{dt}\widetilde{D}(t)$$

where $\widetilde{D}(t)$ is almost periodic. We have

$$\frac{d}{dt}|\chi_\epsilon\rangle = \left(\epsilon\bar{B} - \epsilon^2\bar{D} + \epsilon^2\frac{d}{dt}\left([\bar{B}, \widetilde{C}(t)] - \widetilde{D}(t)\right) + \epsilon^3 E(\epsilon, t)\right)|\chi_\epsilon\rangle$$

where the skew-Hermitian operators $\bar{B}$ and $\bar{D}$ are constants and the other ones $\widetilde{C}$, $\widetilde{D}$, and E are almost periodic.

Multi-frequency averaging: second order

Consider the two systems

$$\frac{d}{dt}|\phi_\epsilon\rangle = \epsilon\left(\bar{B} + \frac{d}{dt}\widetilde{B}(t)\right)|\phi_\epsilon\rangle\,,$$

and

$$\frac{d}{dt}\left|\phi_\epsilon^{2^{\text{nd}}}\right\rangle = (\epsilon\bar{B} - \epsilon^2\bar{D})\left|\phi_\epsilon^{2^{\text{nd}}}\right\rangle\,,$$

initialized at the same state $\left|\phi_\epsilon^{2^{\text{nd}}}\right\rangle_0 = |\phi_\epsilon\rangle_0$.

Theorem: second order approximation

Consider the functions $|\phi_\epsilon\rangle$ and $\left|\phi_\epsilon^{2^{\text{nd}}}\right\rangle$ initialized at the same state and following the above dynamics. Then, there exist $M > 0$ and $\eta > 0$ such that for all $\epsilon \in]0, \eta[$ we have

$$\max_{t\in\left[0,\frac{1}{\epsilon^2}\right]} \left\| |\phi_\epsilon\rangle_t - \left|\phi_\epsilon^{2^{\text{nd}}}\right\rangle_t \right\| \leq M\epsilon$$

Multi-frequency averaging: second order

Proof's idea

Another almost periodic change of variables

$$|\xi_\epsilon\rangle = \left(\mathbf{1} - \epsilon^2\left([\bar{B}, \widetilde{C}(t)] - \widetilde{D}(t)\right)\right)|\chi_\epsilon\rangle.$$

The dynamics can be written as

$$\frac{d}{dt}|\xi_\epsilon\rangle = \left(\epsilon\bar{B} - \epsilon^2\bar{D} + \epsilon^3 G(\epsilon, t)\right)|\xi_\epsilon\rangle$$

where G is almost periodic and therefore uniformly bounded in time.

Approximation recipes

We consider the Hamiltonian

$$H = H_0 + \sum_{k=1}^{m} u_k H_k, \qquad u_k(t) = \sum_{j=1}^{r} \mathbf{u}_{k,j} e^{\omega_j t} + \mathbf{u}^*_{k,j} e^{-\omega_j t}.$$

The Hamiltonian in interaction frame

$$H_{\text{int}}(t) = \sum_{k,j} \left(\mathbf{u}_{k,j} e^{\omega_j t} + \mathbf{u}^*_{k,j} e^{-\omega_j t} \right) e^{iH_0 t} H_k e^{-iH_0 t}$$

We define the first order Hamiltonian

$$H^{1^{\text{st}}}_{\text{rwa}} = \overline{H_{\text{int}}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T H_{\text{int}}(t) dt,$$

and the second order Hamiltonian

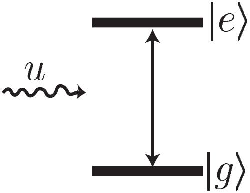
$$H^{2^{\text{nd}}}_{\text{rwa}} = H^{1^{\text{st}}}_{\text{rwa}} - i \overline{(H_{\text{int}} - \overline{H_{\text{int}}}) \left(\int_t (H_{\text{int}} - \overline{H_{\text{int}}}) \right)}$$

Slowly varying amplitudes

Remark

In the above analysis we have assumed the complex amplitudes $\mathbf{u}_{k,j}$ to be constant. However, the whole analysis holds for the case where each one $\mathbf{u}_{k,j}$'s is of a small magnitude, admits a finite number of discontinuities and, between two successive discontinuities, is a slowly time varying function that is continuously differentiable.

2-level system (1/2 spin)



The simplest quantum system: a ground state $|g\rangle$ of energy ω_g; an excited state $|e\rangle$ of energy ω_e. The quantum state $|\psi\rangle \in \mathbb{C}^2$ is a linear superposition $|\psi\rangle = \psi_g |g\rangle + \psi_e |e\rangle$ and obey to the Schrödinger equation (ψ_g and ψ_e depend on t).

Schrödinger equation for the uncontrolled 2-level system ($\hbar = 1$) :

$$\imath \frac{d}{dt} |\psi\rangle = H_0 |\psi\rangle = \big(\omega_e |e\rangle \langle e| + \omega_g |g\rangle \langle g|\big) |\psi\rangle$$

where H_0 is the Hamiltonian, a Hermitian operator $H_0^\dagger = H_0$. Energy is defined up to a constant: H_0 and $H_0 + \varpi(t)\mathbf{1}$ ($\varpi(t) \in \mathbb{R}$ arbitrary) are attached to the same physical system. If $|\psi\rangle$ satisfies $i\hbar \frac{d}{dt} |\psi\rangle = H_0 |\psi\rangle$ then $|\chi\rangle = e^{-i\vartheta(t)} |\psi\rangle$ with $\frac{d}{dt}\vartheta = \varpi$ obeys to $i\hbar \frac{d}{dt} |\chi\rangle = (H_0 + \varpi I) |\chi\rangle$. Thus for any ϑ, $|\psi\rangle$ and $e^{-i\vartheta} |\psi\rangle$ represent the same physical system: The global phase of a quantum system $|\psi\rangle$ can be chosen arbitrarily at any time.

Take origin of energy such that ω_g (resp. ω_e) becomes $-\frac{\omega_e-\omega_g}{2}$ (resp. $\frac{\omega_e-\omega_g}{2}$) and set $\omega_{eg} = \omega_e - \omega_g$
The solution of $i\frac{d}{dt}|\psi\rangle = H_0|\psi\rangle = \frac{\omega_{eg}}{2}(|e\rangle\langle e| - |g\rangle\langle g|)|\psi\rangle$ is

$$|\psi\rangle_t = \psi_{g0} e^{\frac{i\omega_{eg}t}{2}} |g\rangle + \psi_{e0} e^{\frac{-i\omega_{eg}t}{2}} |e\rangle .$$

With a classical electromagnetic field described by $u(t) \in \mathbb{R}$, the coherent evolution the controlled Hamiltonian

$$H(t) = \frac{\omega_{eg}}{2}\sigma_z + \frac{u(t)}{2}\sigma_x = \frac{\omega_{eg}}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + \frac{u(t)}{2}(|e\rangle\langle g| + |g\rangle\langle e|)$$

The controlled Schrödinger equation $i\hbar\frac{d}{dt}|\psi\rangle = (H_0 + uH_1)|\psi\rangle$ reads:

$$i\frac{d}{dt}\begin{pmatrix}\psi_e\\ \psi_g\end{pmatrix} = \frac{\omega_{eg}}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_e\\ \psi_g\end{pmatrix} + \frac{u(t)}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_e\\ \psi_g\end{pmatrix}.$$

The 3 Pauli Matrices[1]

$$\sigma_x = |e\rangle\langle g| + |g\rangle\langle e|, \quad \sigma_y = -i|e\rangle\langle g| + i|g\rangle\langle e|, \quad \sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$$

[1] They correspond, up to multiplication by i, to the 3 imaginary quaternions.

Pauli matrices and some formula

$$\sigma_x = |e\rangle\langle g| + |g\rangle\langle e|\,,\ \sigma_y = -i\,|e\rangle\langle g| + i\,|g\rangle\langle e|\,,\ \sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$$

$$\sigma_x^2 = \mathbf{1}, \quad \sigma_x\sigma_y = i\sigma_z, \quad [\sigma_x, \sigma_y] = 2i\sigma_z, \ \text{circular permutation} \ \ldots$$

- Since for any $\theta \in \mathbb{R}$, $e^{i\theta\sigma_x} = \cos\theta + i\sin\theta\sigma_x$ (idem for σ_y and σ_z), the solution of $i\frac{d}{dt}|\psi\rangle = \frac{\omega_{eg}}{2}\sigma_z|\psi\rangle$ is

$$|\psi\rangle_t = e^{\frac{-i\omega_{eg}t}{2}\sigma_z}|\psi\rangle_0 = \left(\cos\left(\frac{\omega_{eg}t}{2}\right)\mathbf{1} - i\sin\left(\frac{\omega_{eg}t}{2}\right)\sigma_z\right)|\psi\rangle_0$$

- For $\alpha, \beta = x, y, z$, $\alpha \neq \beta$ we have

$$\sigma_\alpha e^{i\theta\sigma_\beta} = e^{-i\theta\sigma_\beta}\sigma_\alpha, \qquad \left(e^{i\theta\sigma_\alpha}\right)^{-1} = \left(e^{i\theta\sigma_\alpha}\right)^\dagger = e^{-i\theta\sigma_\alpha}.$$

and also

$$e^{-\frac{i\theta}{2}\sigma_\alpha}\sigma_\beta e^{\frac{i\theta}{2}\sigma_\alpha} = e^{-i\theta\sigma_\alpha}\sigma_\beta = \sigma_\beta e^{i\theta\sigma_\alpha}$$

RWA and resonant control

In $i\frac{d}{dt}|\psi\rangle = \left(\frac{\omega_{eg}}{2}\sigma_z + \frac{u}{2}\sigma_x\right)|\psi\rangle$, take a resonant control $u = \mathbf{u}e^{i\omega_{eg}t} + \mathbf{u}^* e^{-i\omega_{eg}t}$ with $\mathbf{u}$ slowly varying complex amplitude $\left|\frac{d}{dt}\mathbf{u}\right| \ll \omega_{eg}|\mathbf{u}|$. Set $H_0 = \frac{\omega_{eg}}{2}\sigma_z$ and $\epsilon H_1 = \frac{u}{2}\sigma_x$ and consider $|\psi\rangle = e^{-\frac{i\omega_{eg}t}{2}\sigma_z}|\phi\rangle$ to eliminate the drift H_0 and to get the Hamiltonian in the interaction frame:

$$i\frac{d}{dt}|\phi\rangle = \frac{u}{2}e^{\frac{i\omega_{eg}t}{2}\sigma_z}\sigma_x e^{-\frac{i\omega_{eg}t}{2}\sigma_z}|\phi\rangle = H_{\text{int}}|\phi\rangle$$

with $H_{\text{int}} = \frac{u}{2}e^{i\omega_{eg}t}\overbrace{\frac{\sigma_x + i\sigma_y}{2}}^{\sigma^+=|e\rangle\langle g|} + \frac{u}{2}e^{-i\omega_{eg}t}\overbrace{\frac{\sigma_x - i\sigma_y}{2}}^{\sigma^-=|g\rangle\langle e|}$

The RWA consists in neglecting the oscillating terms at frequency $2\omega_{eg}$ when $|\mathbf{u}| \ll \Omega$:

$$H_{int} = \left(\frac{\mathbf{u}e^{2i\omega_{eg}t} + \mathbf{u}^*}{2}\right)\sigma^+ + \left(\frac{\mathbf{u} + \mathbf{u}^* e^{-2i\omega_{eg}t}}{2}\right)\sigma^-.$$

Thus

$$\overline{H_{int}} = \frac{\mathbf{u}^*\sigma^+ + \mathbf{u}\sigma^-}{2}.$$

The decomposition of H_{int},

$$H_{\text{int}} = \underbrace{\frac{\mathbf{u}^*}{2}\sigma_+ + \frac{\mathbf{u}}{2}\sigma_-}_{\overline{H_{\text{int}}}} + \underbrace{\frac{\mathbf{u}e^{2i\omega_{eg}t}}{2}\sigma_+ + \frac{\mathbf{u}^* e^{-2i\omega_{eg}t}}{2}\sigma_-}_{H_{\text{int}}-\overline{H_{\text{int}}}},$$

provides the first order approximation (RWA) $H_{\text{rwa}}^{1^{\text{st}}} = \overline{H_{\text{int}}} = \lim_{T\to\infty} \frac{1}{T}\int_0^T H_{\text{int}}(t)dt$, and also the second order approximation $H_{\text{rwa}}^{2^{\text{nd}}} = H_{\text{rwa}}^{1^{\text{st}}} - i\overline{\left(H_{\text{int}} - \overline{H_{\text{int}}}\right)\left(\int_t (H_{\text{int}} - \overline{H_{\text{int}}})\right)}$. Since $\int_t H_{\text{int}} - \overline{H_{\text{int}}} = \frac{\mathbf{u}e^{2i\omega_{eg}t}}{4i\omega_{eg}}\sigma_+ - \frac{\mathbf{u}^* e^{-2i\omega_{eg}t}}{4i\omega_{eg}}\sigma_-$, we have

$$\overline{\left(H_{\text{int}} - \overline{H_{\text{int}}}\right)\left(\int_t (H_{\text{int}} - \overline{H_{\text{int}}})\right)} = -\frac{|\mathbf{u}|^2}{8i\omega_{eg}}\sigma_z$$

(use $\sigma_+^2 = \sigma_-^2 = 0$ and $\sigma_z = \sigma_+\sigma_- - \sigma_-\sigma_+$).
The second order approximation reads:

$$H_{\text{rwa}}^{2^{\text{nd}}} = H_{\text{rwa}}^{1^{\text{st}}} + \left(\frac{|\mathbf{u}|^2}{8\omega_{eg}}\right)\sigma_z = \frac{\mathbf{u}^*}{2}\sigma_+ + \frac{\mathbf{u}}{2}\sigma_- + \left(\frac{|\mathbf{u}|^2}{8\omega_{eg}}\right)\sigma_z.$$

The 2nd order correction $\frac{|\mathbf{u}|^2}{4\omega_r}\sigma_z$ is called the Bloch-Siegert shift.

Exercise: controllability of the 2-level systems and Rabi oscillation

Take the first order approximation

$$(\Sigma) \quad i\frac{d}{dt}|\phi\rangle = \frac{(\mathbf{u}^*\sigma^+ + \mathbf{u}\sigma^-)}{2}|\phi\rangle = \frac{(\mathbf{u}^*|e\rangle\langle g| + \mathbf{u}|g\rangle\langle e|)}{2}|\phi\rangle$$

with control $\mathbf{u} \in \mathbb{C}$.

1. Take constant control $\mathbf{u}(t) = \Omega_r e^{i\theta}$ for $t \in [0, T]$, $T > 0$. Show that $i\frac{d}{dt}|\phi\rangle = \frac{\Omega_r(\cos\theta\sigma_x + \sin\theta\sigma_y)}{2}|\phi\rangle$.
2. Set $\Theta_r = \frac{\Omega_r}{2}T$. Show that the solution at T of the propagator $U_t \in SU(2)$, $i\frac{d}{dt}U = \frac{\Omega_r(\cos\theta\sigma_x + \sin\theta\sigma_y)}{2}U$, $U_0 = \mathbf{1}$ is given by

$$U_T = \cos\Theta_r \mathbf{1} - i\sin\Theta_r(\cos\theta\sigma_x + \sin\theta\sigma_y),$$

3. Take a wave function $|\bar{\phi}\rangle$. Show that exist Ω_r and θ such that $U_T|g\rangle = e^{i\alpha}|\bar{\phi}\rangle$, where α is some global phase.
4. Prove that for any given two wave functions $|\phi_a\rangle$ and $|\phi_b\rangle$ exists a piece-wise constant control $[0, 2T] \ni t \mapsto \mathbf{u}(t) \in \mathbb{C}$ such that the solution of (Σ) with $|\phi\rangle_0 = |\phi_a\rangle$ satisfies $|\phi\rangle_T = e^{i\beta}|\phi_b\rangle$ for some global phase β.

The quantum harmonic oscillator lives on $L^2(\mathbb{R},\mathbb{C}) \sim l^2(\mathbb{C})$ with controlled Hamiltonian

$$-\frac{\omega_c}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega_c}{2}x^2 + \sqrt{2}ux \sim \omega_c\left(a^\dagger a + \frac{1}{2}\right) + u(a + a^\dagger)$$

(remember that $a = X + iP = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$).

The 2-level system lives on $\mathbb{C}^2$ with Hamiltonian $H_a = \frac{\omega_{eg}}{2}\sigma_z$.

The composite system lives on the tensor product $\mathbb{C}^2 \otimes L^2(\mathbb{R},\mathbb{C}) \sim \mathbb{C}^2 \otimes l^2(\mathbb{C})$ with controlled Hamiltonian

$$\frac{\omega_{eg}}{2}\,\sigma_z \otimes \mathbf{1}_{L^2(\mathbb{R},\mathbb{C})} + \omega_c\,\mathbf{1}_{\mathbb{C}^2} \otimes \left(a^\dagger a + \frac{1}{2}\right) + u\,\mathbf{1}_{\mathbb{C}^2} \otimes (a + a^\dagger)$$
$$-i\frac{\Omega}{2}\sigma_x \otimes (a^\dagger - a)$$

Shortcut notations for the Jaynes-Cummings Hamiltonian:

$$H_{JC} = \frac{\omega_{eg}}{2}\sigma_z + \omega_c\left(a^\dagger a + \frac{1}{2}\right) + u(a + a^\dagger) - i\frac{\Omega}{2}\sigma_x(a^\dagger - a)$$

with the usual scales $\Omega \ll \omega_c, \omega_{eg}$, $|\omega_c - \omega_{eg}| \ll \omega_c, \omega_{eg}$ and $|u| \ll \omega_c, \omega_{eg}$.

The Jaynes-Cummings PDE

The Schrödinger system

$$i\frac{d}{dt}|\psi\rangle = \left(\frac{\omega_{eg}}{2}\sigma_z + \omega_c\left(a^\dagger a + \frac{1}{2}\right) + u(a + a^\dagger) - i\frac{\Omega}{2}\sigma_x(a^\dagger - a)\right)|\psi\rangle$$

corresponds to two coupled scalar PDE's:

$$i\frac{\partial\psi_g}{\partial t} = \frac{\omega_c}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_g + \left(\sqrt{2}ux - \frac{\omega_{eg}}{2}\right)\psi_g + i\frac{\Omega}{\sqrt{2}}\frac{\partial}{\partial x}\psi_e$$

$$i\frac{\partial\psi_e}{\partial t} = \frac{\omega_c}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_e + \left(\sqrt{2}ux + \frac{\omega_{eg}}{2}\right)\psi_e + i\frac{\Omega}{\sqrt{2}}\frac{\partial}{\partial x}\psi_g$$

since $a = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$ and $|\psi\rangle$ corresponds to $(\psi_g(x,t), \psi_e(x,t))$ where $\psi_g(.,t), \psi_e(.,t) \in L^2(\mathbb{R},\mathbb{C})$ and $\|\psi_g\|^2 + \|\psi_e\|^2 = 1$.

Resonant control and passage to the interaction frame

In $H_{JC} = \frac{\omega_{eg}}{2}\sigma_z + \omega_c\left(a^\dagger a + \frac{1}{2}\right) + u(a + a^\dagger) - i\frac{\Omega}{2}\sigma_x(a^\dagger - a)$, $\omega_{eg} = \omega_c = \omega_r$ and $u(t) = \mathbf{u}e^{i\omega_r t} + \mathbf{u}^* e^{-i\omega_r t}$ with slowly varying complex amplitude $\mathbf{u}$ and $|\Omega|, |\mathbf{u}| \ll \omega_r$. Then $H_{JC} = H_0 + \epsilon H_1$ where ϵ is a small parameter and

$$
\begin{aligned}
H_0 &= \frac{\omega_r}{2}\sigma_z + \omega_r\left(a^\dagger a + \frac{1}{2}\right)\\
\epsilon H_1 &= (\mathbf{u}e^{i\omega_r t} + \mathbf{u}^* e^{-i\omega_r t})(a + a^\dagger) - i\frac{\Omega}{2}\sigma_x(a^\dagger - a).
\end{aligned}
$$

H_{int} is obtained by setting $|\psi\rangle = e^{-i\omega_r t\left(a^\dagger a + \frac{1}{2}\right)} e^{\frac{-i\omega_r t}{2}\sigma_z}|\phi\rangle$ in $i\frac{d}{dt}|\psi\rangle = H_{JC}|\psi\rangle$ to get $i\frac{d}{dt}|\phi\rangle = H_{int}|\phi\rangle$ with

$$
\begin{aligned}
H_{\text{int}} = &\left(\mathbf{u}e^{i\omega_r t} + \mathbf{u}^* e^{-i\omega_r t}\right)\left(e^{-i\omega_r t}a + e^{i\omega_r t}a^\dagger\right)\\
&- i\frac{\Omega}{2}\left(e^{-i\omega_r t}|g\rangle\langle e| + e^{i\omega_r t}|e\rangle\langle g|\right)\left(e^{i\omega_r t}a^\dagger - e^{-i\omega_r t}a\right)
\end{aligned}
$$

where we used

$$
e^{\frac{i\theta}{2}\sigma_z}\,\sigma_x e^{-\frac{i\theta}{2}\sigma_z} = e^{-i\theta}\sigma_- + e^{i\theta}\sigma_+, \quad e^{i\theta\left(a^\dagger a + \frac{1}{2}\right)}\,a\,e^{-i\theta\left(a^\dagger a + \frac{1}{2}\right)} = e^{-i\theta}a
$$

The secular terms in H_{int} are given by (RWA, first order approximation)

$$H_{\text{rwa}}^{1^{\text{st}}} = \mathbf{u}a + \mathbf{u}^* a^\dagger - i\tfrac{\Omega}{2}\left(|g\rangle \langle e| a^\dagger - |e\rangle \langle g| a \right)$$

Set $H_{\text{rwa}}^{1^{\text{st}}} = H_0 + u_1 H_1 + u_2 H_2$ where $\mathbf{u} = \frac{1}{\sqrt{2}}(u_1 + i u_2)$, $u_1, u_2 \in \mathbb{R}$:

$$H_0 = -\frac{\Omega}{2}(X\sigma_y + P\sigma_x), \; H_1 = \tfrac{a+a^\dagger}{\sqrt{2}} = \sqrt{2}X, \; H_2 = \tfrac{a-a^\dagger}{i\sqrt{2}} = \sqrt{2}P.$$

The quantum state $|\phi\rangle$ is described by two elements of $L^2(\mathbb{R},\mathbb{C})$, ϕ_g and ϕ_e, whose time evolution is given by

$$i\frac{\partial \phi_g}{\partial t} = \left(u_1 x + i u_2 \frac{\partial}{\partial x}\right)\phi_g + i\tfrac{\Omega}{2\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)\phi_e$$
$$i\frac{\partial \phi_e}{\partial t} = \left(u_1 x + i u_2 \frac{\partial}{\partial x}\right)\phi_e + i\tfrac{\Omega}{2\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)\phi_g$$

since X stands for $\frac{x}{\sqrt{2}}$ and P for $-\frac{i}{\sqrt{2}}\frac{\partial}{\partial x}$.

Exercise: JC systems with impulse controls

Consider the average JC model (resonant case, $\mathbf{u} \in \mathbb{C}$ as control.).

$$i\frac{d}{dt}|\psi\rangle = \left(i\frac{\Omega}{2}(\sigma_+ a - \sigma_- a^\dagger) + \mathbf{u}a^\dagger + \mathbf{u}^* a\right)|\psi\rangle$$

1. Set $\mathbf{v} \in \mathbb{C}$ solution of $\frac{d}{dt}\mathbf{v} = -i\mathbf{u}$ and consider the following change of frame $|\phi\rangle = D_{-\mathbf{v}}|\psi\rangle$ with the displacement operator $D_{-\mathbf{v}} = e^{-\mathbf{v}a^\dagger + \mathbf{v}^* a}$. Show that, up to a global phase change, we have, with $\tilde{\mathbf{u}} = i\frac{\Omega}{2}\mathbf{v}$,

$$i\frac{d}{dt}|\phi\rangle = \left(\frac{i\Omega}{2}(\sigma_+ a - \sigma_- a^\dagger) + (\tilde{\mathbf{u}}\sigma_+ + \tilde{\mathbf{u}}^*\sigma_-)\right)|\phi\rangle$$

2. Take the orthonormal basis $\{|g,n\rangle, |e,n\rangle\}$ with $n \in \mathbb{N}$ being the photon number and where for instance $|g,n\rangle$ stands for the tensor product $|g\rangle \otimes |n\rangle$. Set $|\phi\rangle = \sum_n \phi_{g,n}|g,n\rangle + \phi_{e,n}|e,n\rangle$ with $\phi_{g,n}, \phi_{e,n} \in \mathbb{C}$ depending on t and $\sum_n |\phi_{g,n}|^2 + |\phi_{e,n}|^2 = 1$. Show that, for $n \geq 0$

$$i\frac{d}{dt}\phi_{g,n+1} = -i\frac{\Omega}{2}\sqrt{n+1}\phi_{e,n} + \tilde{\mathbf{u}}^*\phi_{e,n+1}, \quad i\frac{d}{dt}\phi_{e,n} = i\frac{\Omega}{2}\sqrt{n+1}\phi_{g,n+1} + \tilde{\mathbf{u}}\phi_{g,n}$$

and $i\frac{d}{dt}\phi_{g,0} = \tilde{\mathbf{u}}^*\phi_{e,0}$.

3. Assume that $|\phi\rangle_0 = |g,0\rangle$. Construct an open-loop control $[0,T] \ni t \mapsto \tilde{\mathbf{u}}(t)$ such that $|\phi\rangle_T = |g,1\rangle$.
4. Generalize the above open-loop control when the goal state $|\phi\rangle_T$ is $|g,n\rangle$ with any arbitrary photon number n.

With $A = \alpha a^\dagger$ and $B = -\alpha^* a$, Glauber formula gives:

$$D_\alpha = e^{-\frac{|\alpha|^2}{2}}\, e^{\alpha a^\dagger} e^{-\alpha^* a} = e^{+\frac{|\alpha|^2}{2}}\, e^{-\alpha^* a} e^{\alpha a^\dagger}$$

$$D_{-\alpha} a D_\alpha = a + \alpha \quad \text{and} \quad D_{-\alpha} a^\dagger D_\alpha = a^\dagger + \alpha^*.$$

With $A = 2i\Im\alpha X \sim i\sqrt{2}\Im\alpha x$ and $B = -2\imath\Re\alpha P \sim -\sqrt{2}\Re\alpha\frac{\partial}{\partial x}$, Glauber formula gives[2]:

$$D_\alpha = e^{-i\Re\alpha\Im\alpha}\, e^{i\sqrt{2}\Im\alpha x} e^{-\sqrt{2}\Re\alpha\frac{\partial}{\partial x}}$$

$$(D_\alpha\,|\psi\rangle)_{x,t} = e^{-i\Re\alpha\Im\alpha}\, e^{i\sqrt{2}\Im\alpha x}\psi(x - \sqrt{2}\Re\alpha, t)$$

For any $\alpha, \beta, \epsilon \in \mathbb{C}$, we have

$$D_{\alpha+\beta} = e^{\frac{\alpha^*\beta - \alpha\beta^*}{2}} D_\alpha D_\beta$$

$$D_{\alpha+\epsilon} D_{-\alpha} = \left(1 + \tfrac{\alpha\epsilon^* - \alpha^*\epsilon}{2}\right)\mathbf{1} + \epsilon a^\dagger - \epsilon^* a + O(|\epsilon|^2)$$

$$\left(\tfrac{d}{dt} D_\alpha\right) D_{-\alpha} = \left(\frac{\alpha\frac{d}{dt}\alpha^* - \alpha^*\frac{d}{dt}\alpha}{2}\right)\mathbf{1} + \left(\tfrac{d}{dt}\alpha\right) a^\dagger - \left(\tfrac{d}{dt}\alpha^*\right) a.$$

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[2] Remember that a time-delay of r corresponds to the operator $e^{-r\frac{d}{dt}}$.