Modeling and control of quantum systems

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Chapter 1

Introduction

1.1 Control of a classical harmonic oscillator: the PID controller

Consider a classical harmonic oscillator of pulsation ω where the position y is measured and the control is given by a force u. Moreover, we assume that the oscillator is also subject to an additional unknown force w, the perturbation. Thus, the model is given by

$$\frac{d^2}{dt^2}y = -\omega^2 y + u + w.$$

In the Laplace domain, it reads $y = \frac{u+w}{s^2+\omega^2}$ with $s = \frac{d}{dt}$ and is associated to the bloc diagram of Figure 1.1. It corresponds to the following system of first order ordinary differential equations:

$$\frac{d}{dt}x_1 = x_2, \quad \frac{d}{dt}x_2 = -\omega^2 x_1 + u + w$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ is the state-space (phase-space) and the measured output is just $y = x_1$.

The control goal is to maintain the output y close the reference constant value y^r and to reduce the influence of the unknown perturbation w. As illustrated by bloc diagram of

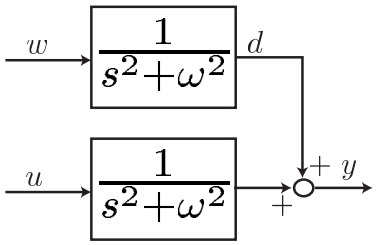

Figure 1.1: Bloc diagram of a classical harmonic oscillator; the Laplace variable s corresponds to the operator $\frac{d}{dt}$.

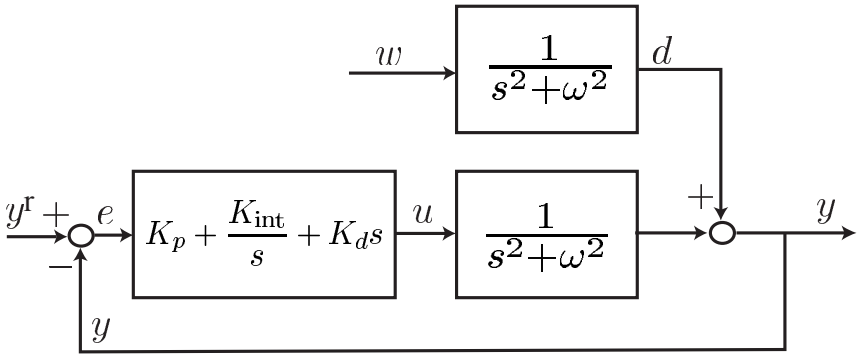

Figure 1.2: The PID control of a classical harmonic oscillator

Figure 1.2, this goal is achieved by a feedback loop with a Proportional Integral Derivative (PID) controller,

$$u = -K_p(y - y^r) - K_d\frac{d}{dt}(y - y^r) - K_{\text{int}}\int (y - y^r), \tag{1.1}$$

where the positive gains $(K_p, K_{\text{int}}, K_d)$ are tuned as follows

$$K_p = \Omega^2, \quad K_d = 2\xi\Omega, \quad , K_{\text{int}} = \epsilon\Omega^3$$

with positive parameters $\Omega \sim \omega$, $\xi \sim 1$ and $\epsilon \ll 1$.

Let us analyze this tuning by the behavior of the closed-loop system. Denote by x_3 the integral term. The closed-loop system corresponds to the following three ordinary differential equations (y^r is constant here):

$$\frac{d}{dt}x_1 = x_2, \quad \frac{d}{dt}x_2 = -\omega^2 x_1 - \Omega^2(x_1 - y^r) - 2\xi\Omega x_2 - x_3 + w, \quad \frac{d}{dt}x_3 = \epsilon\Omega^3(x_1 - y^r).$$

This system admits two time-scales, the fast one attached to the fast variable (x_1, x_2) and the slow one attached to x_3 (see appendix D). The fast dynamics in (x_1, x_2) corresponds to a forced damped harmonic oscillator:

$$\frac{d^2}{dt^2}x_1 = -(\omega^2 + \Omega^2)x_1 - 2\xi\frac{d}{dt}x_1 + w + \Omega^2 y^r - x_3.$$

For w constant and since x_3 is almost constant, x_1 converges exponentially towards the quasi steady-state $\frac{w+\Omega^2 y^r - x_3}{\omega^2+\Omega^2}$. Thus the slow dynamics of x_3 is approximatively given by

$$\frac{d}{dt}x_3 = \epsilon\frac{\Omega^3}{\omega^2 + \Omega^2}(w - \omega^2 y^r - x_3)$$

where we have replaced x_1 by $\frac{w+\Omega^2 y^r - x_3}{\omega^2+\Omega^2}$. The slow state x_3 converges exponentially towards $w - \omega^2 y^r$. We have proved that for ϵ small enough and $\Omega, \xi > 0$, the closed loop system is exponentially stable and that $y = x_1$ converges always towards y^r whatever the value of the unknown constant w is.

Let us notice three important points:

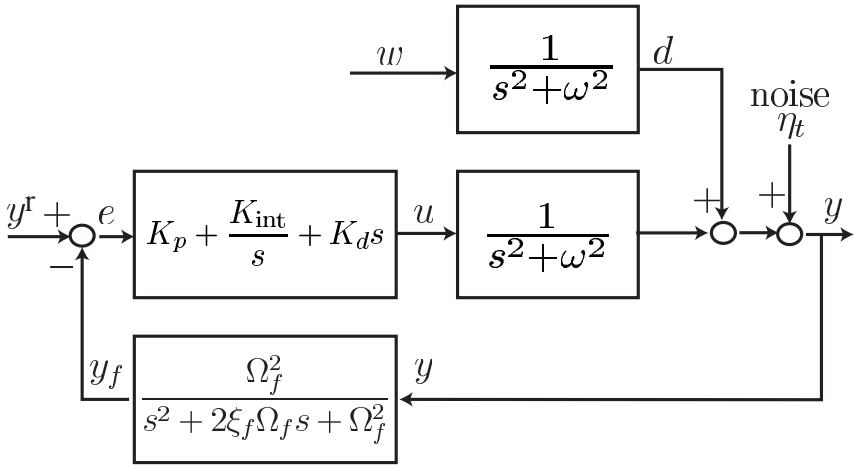

Figure 1.3: The PID control of a classical harmonic oscillator including a filter of the noisy signal y

- The closed-loop stability is exponential: we can tune separately the convergence speed of the fast sub-system in (x_1, x_2) with Ω and ξ and of the slow one in x_3 with ϵ.

- The closed-loop stability does not depend on the precise value of the parameter ω nor on w: only the order of magnitude for ω is necessary to tune the PID gains K_p, K_{int} and K_d.

- For w constant, x_1 converges always towards the reference y^r.

The two first points illustrate the notion of stability and robustness, the third one the notion of precision and performance.

Let us complete this analysis by adding noise to the measurement of x_1: now the output map becomes $y = x_1 + \eta_t$ where η_t is a white noise of standard deviation σ (δ is the Dirac distribution at zero): $\mathbb{E}(\eta_{t_1}\eta_{t_2}) = \sigma^2\delta(t_1 - t_2)$. With such noisy measurement, the PID controller is not realistic since u includes $-K_d\frac{d}{dt}\eta_t$ and becomes unbounded with a derivative gain $K_d > 0$. As illustrated on Figure 1.3, pre-filtering y is needed before computing u. Consider the second order low-pass filter (cut-off pulsation $\Omega_f > 0$ and damping coefficient $\xi_f > 1$ to avoid the resonance pick at Ω_f)

$$y_f = \frac{\Omega_f^2}{s^2 + 2\xi_f\Omega_f s + \Omega_f^2} y$$

and replace y by y_f in the PID formula (1.1). This PID controller with the pre-filtered y_f yields to a stable closed loop system for previous settings of K_p, K_{int} and K_d and if we take $\Omega_f \gg \Omega$. The state space description of the closed-loop system includes now the two

additional states (z_1, z_2) of the filter:

$$\begin{aligned}
&\frac{d}{dt}z_1 = z_2, \quad \frac{d}{dt}z_2 = -2\xi_f\Omega_f z_2 - \Omega_f^2(z_1 - x_1 - \eta_t)\\
&\frac{d}{dt}x_1 = x_2, \quad \frac{d}{dt}x_2 = -\omega^2 x_1 - \Omega^2(z_1 - y^r) - 2\xi\Omega z_2 - x_3 + w\\
&\frac{d}{dt}x_3 = \epsilon\Omega^3(z_1 - y^r).
\end{aligned}$$

With the above tuning, it admits now 3 time scales, a very fast one with (z_1, z_2), a fast one with (x_1, x_2) and a slow one with x_3. The quasi-static approximation for the very fast scale yields to $z_1 \approx x_1$ and $z_2 \approx x_2$. We recover the original two time-scales of the closed-loop system without the pre-filter.

We have seen here the key role of the pre-filter to overcome the presence of noise in the measurement process of y. The additional state variables (z_1, z_2, x_3) introduced by the pre-filter and controller are directly related to the state of an observer that reconstructs in real-time and in a causal manner the variable (x_1, x_2) and the constant parameter w from the knowledge of the past values of u and y. An asymptotic observer[1] for

$$\frac{d}{dt}x_1 = x_2, \quad \frac{d}{dt}x_2 = -\omega^2 x_1 + u + w, \quad \frac{d}{dt}w = 0, \quad y = x_1 + \eta_t$$

admits the following form

$$\frac{d}{dt}\hat{x}_1 = \hat{x}_2 - L_1(\hat{x}_1 - y), \quad \frac{d}{dt}\hat{x}_2 = -\omega^2\hat{x}_1 + u + \hat{w} - L_2(\hat{x}_1 - y), \quad \frac{d}{dt}\hat{w} = -L_w(\hat{x}_1 - y)$$

where, if we take the observers gains L_1, L_2 and L_w as

$$L_1 = 2\xi_f\Omega_f, \quad L_2 = \Omega_f^2, \quad L_w = \epsilon\Omega^3$$

we recover, up to some slight modifications, the filter state variables (z_1, z_2) as $(\hat{x}_1, \hat{x}_2)$ and the integral term x_3 as $\hat{w}$. With this interpretation the PID control with the pre-filtered y is in fact closely related to the state feedback

$$u = -K_1(x_1 - y^r) - K_2 x_2 - w, \quad K_1 = \Omega^2, \quad K_2 = 2\xi\Omega$$

where x_1, x_2 and w have been replaced by their estimations $\hat{x}_1$, $\hat{x}_2$ and $\hat{w}$. We recover here the general form of an observer-controller combining state estimation and state feedback.

Let us complete this essentially feedback scheme by a feed-forward part. The goal now is to follow a time varying references $y^r(t)$. We consider the tracking problem and not only the regulation problem. From $\frac{d^2}{dt^2}y = -\omega^2 y + u + w$, it is clear that y^r should be at least C^2 and that the reference control u^r, whenever $w = 0$, is given by

$$u^r = \frac{d^2}{dt^2}y^r + \omega^2 y^r$$

[1] A Kalman filter is an asymptotic observer where the correction gains L are parameterized by the noise level of the measurement output and of the system's dynamics.

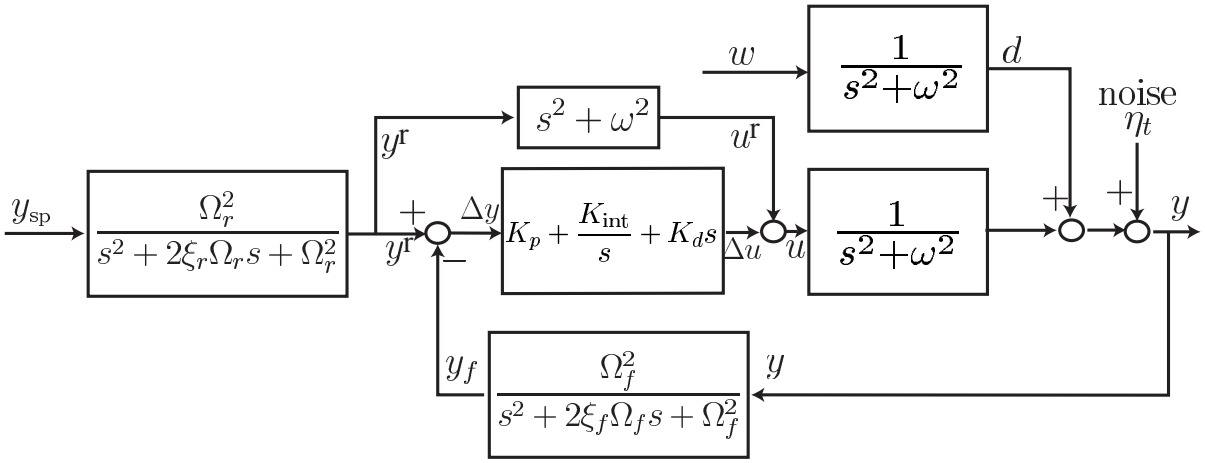

Figure 1.4: The PID control of a classical harmonic oscillator including a filter of the noisy signal y, a filter of the piece-wise continuous set-point $y_{\rm sp}$ and the feed-forward part u^r.

Exploiting the linearity of the system, we have

$$\frac{d^2}{dt^2}\Delta y = -\omega^2\Delta y + \Delta u + w$$

for $\Delta y = y - y^r$ and $\Delta u = u - u^r$. We recover previous settings with Δy, Δu and 0 instead of y, u and the constant reference y^r. Thus the complete feedback scheme with the feed-forward part brought by y^r, $\frac{d}{dt}y^r$ and $\frac{d^2}{dt^2}y^r$ is given by

$$u = \frac{d^2}{dt^2}y^r + \omega^2 y^r - K_p(y_f - y^r) - K_{\rm int}\int (y_f - y^r) - K_d\left(\tfrac{d}{dt}y_f - \tfrac{d}{dt}y^r\right).$$

This scheme requires a smooth reference signal y^r. In general it is not the case and we have to introduce some smoothing process. The simplest one consists in filtering a piece-wise continuous set-point signal $y_{\rm sp}$ to produce a smooth reference y^r. Here a second order filter is sufficient

$$\frac{d^2}{dt^2}y^r + 2\xi_r\Omega_r\frac{d}{dt}y^r + \Omega_r^2(y^r - y_{\rm sp}) = 0$$

with parameters $\Omega_r > 0$ and $\xi_r \sim 1$. The complete tracking scheme is illustrated on Figure (1.4): starting with the piece-wise continuous set-point $y_{\rm sp}$, an admissible reference trajectory (y^r, u^r) is generated from the output of a second order filter (model reference). Then the stabilizing PID controller is computed form the tracking error $e = y_f - y^r$. It provides the feedback correction Δu that is added to the feed-forward control u^r.

Modeling of a system with control input u, perturbation input w and measured output y yields in many cases to a state-space description $\frac{d}{dt}x = f(x,u,w)$ and an output map $y = h(x)$ as illustrated on Figure 1.5. We can gather in the same bloc, called controller bloc, the tracking feedback, the state observer and the feed-forward control based on the new input $y_{\rm sp}$. The inputs of the controller bloc are then y and $y_{\rm sp}$ (see Figure (1.6)). Its state

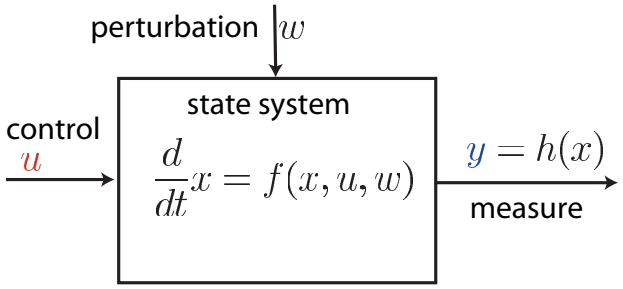

Figure 1.5: open-loop bloc diagram of a non-linear classical system

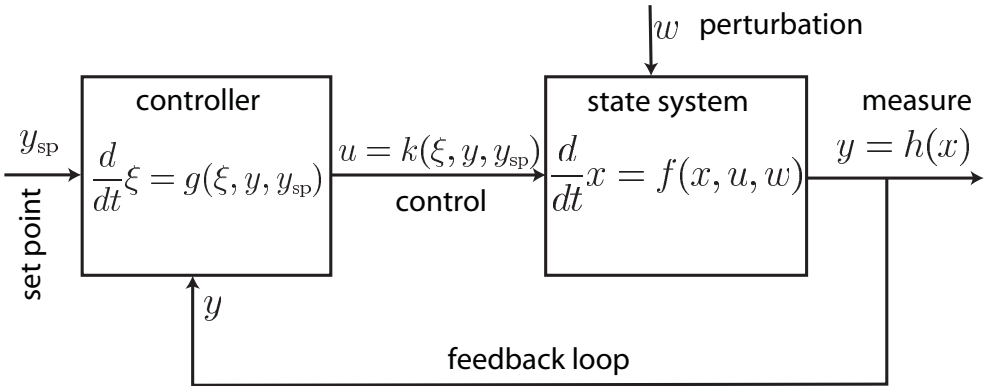

Figure 1.6: closed-loop bloc diagram of a non-linear classical system

is denoted by ξ and obeys $\frac{d}{dt}\xi = g(\xi, y, y_{sp})$. Its output is then $u = k(\xi, y, y_{sp})$. The control goal is then to construct, from the knowledge of the modeling equations, $\frac{d}{dt}x = f(x,u,w)$ and $y = h(x)$, these controller equations, $\frac{d}{dt}\xi = g(\xi, y, y_{sp})$ and $u = k(\xi, y, y_{sp})$, such that, for the closed-loop system, y follows the time-varying set-point y_{sp}. The controller has to be robust and to compensate modelling errors, measurement noise, unknown perturbations w. The practical implementation just consists in a numerical integration of the differential equations of the controller. For a small enough sampling time-step, the explicit first order Euler scheme is usually sufficient.

The elaboration of the controller equations from the modeling ones relies on several key notions that have been implicitly used for the harmonic oscillator controller displayed on Figure 1.4:

- Controllability: for any states x^a and x^b, the possibility to find a transition time $T > 0$ and an open-loop control $[0,T] \ni t \mapsto u(t)$ steering the system from x^a to x^b, i.e. such that solution of the initial value problem $\frac{d}{dt}x = f(x,u(t))$ with $x(0) = x^a$ satisfies $x(T) = x^b$; for the harmonic oscillator such open-loop controls are given by $u(t) = \frac{d^2}{dt^2}\gamma(t) + \omega^2\gamma(t)$ where $\gamma(t)$ are smooth functions (at least piece-wise C^2) such that $(\gamma(0), \frac{d}{dt}\gamma(0)) = (x_1^a, x_2^a)$ and $(\gamma(T), \frac{d}{dt}\gamma(T)) = (x_1^b, x_2^b)$;

- Observability: from the knowledge of u and y one can recover without ambiguity the state x; for the harmonic oscillator $x_1 = y$ and $x_2 = \frac{d}{dt}y$.

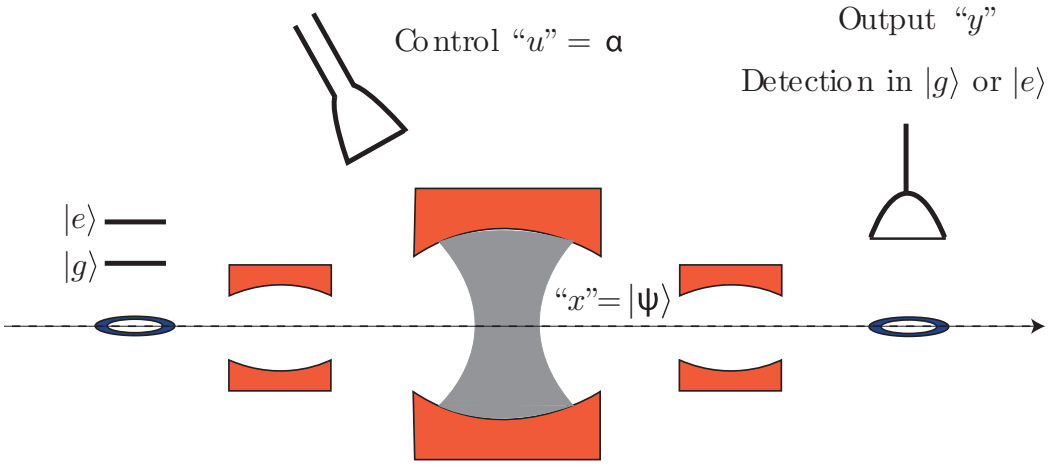

Figure 1.7: Simple schematic of LKB experiment for control of cavity field

- Feed-forward: $u = u^r(t)$ associated to reference trajectory $t \mapsto (x^r(t), u^r(t), y^r(t))$ (performance); for the harmonic oscillator, we have parameterized the reference trajectory via C^2 curves $t \mapsto \gamma(t)$ and $(x_1^r, x_2^r, u^r, y^r) = (\gamma, \frac{d}{dt}\gamma, \frac{d^2}{dt^2}\gamma + \omega^2\gamma, \gamma)$. Such motion planing solution has been used to get the feed-forward open-loop control u_r with $\gamma = y_r$.

- Feedback stabilization and tracking : $u = u^r(t) + \Delta u$ where Δu depends on the tracking state error $\Delta x = x - x^r$ to ensure that Δx tends to 0 in closed-loop; for the harmonic oscillator, $\Delta u = -K_p\Delta x_1 - K_d\Delta x_2$ is exponentially stabilizing as soon as the gains $K_p, K_d > 0$.

- Asymptotic observers and real-time state estimation: design a filter of the form $\frac{d}{dt}\hat{x} = f(\hat{x}, u) + L(h(\hat{x}) - y)$ that forgets its initial condition and converges towards x as t tends to infinity: for the harmonic oscillator

$$\frac{d}{dt}\hat{x}_1 = \hat{x}_2 - L_1(\hat{x}_1 - y), \quad \frac{d}{dt}\hat{x}_2 = -\omega^2\hat{x}_1 + u - L_2(\hat{x}_1 - y)$$

  converges exponentially toward (x_1, x_2) as soon as the observer gains $L_1, L_2 > 0$.

- Stability and robustness: for t large, Δx and $\hat{x} - x$ remain small even in presence of small modelling errors and noises; for the harmonic oscillator, the exponential convergence ensure robustness to errors.

1.2 Control of a quantum harmonic oscillator: the LKB photon-box

Controlling a quantum harmonic oscillator is not as simple as controlling a classical one. Many problems and questions remain to be set and answered. The main difference between

classical and quantum systems is not the relation between the control input and the state: controllability extends directly to quantum systems and many feed-forward control strategy are used experimentally (they are addressed in Chapter 3). The major difference is due to the relation between the state and the measurement: this relation is drastically different from the classical case, as we have to take into account a non-deterministic back-action of the measurement process on the state (see Chapter 4 for models of open quantum systems). To have a first idea of this essential difference, let us consider the photon-box experiment conducted at the "Laboratoire Kasler-Brossel" (LKB) of Ecole Normale Supérieure in Paris (see [33, 28]) where a quantum harmonic oscillator is controlled and measured. As sketched on Figure 1.7, the state is described by a complex probability amplitude wave function $|\psi\rangle$[2]. It belongs to a Hilbert space $\mathcal{H}$ that we assume to be finite dimensional here. Thus $|\psi\rangle$ is just a vector with complex entries and of length one (finite dimensional truncation to $n^{\max}$ photons)

$$|\psi\rangle = \begin{pmatrix} \psi_0 \\ \vdots \\ \psi_{n^{\max}} \end{pmatrix}, \quad \| |\psi\rangle \|_{\mathcal{H}}^2 = \sum_n |\psi_n|^2 = 1.$$

This system admits a nice discrete-time formulation: the differential equations are then replaced by recurrence ones. If k is the time-index, $|\psi\rangle_k$ denotes the state at time or step k. The control input at time k is then denoted by $u_k \in \mathbb{C}$. The measure output at time k is denoted by y_k. It takes only two discrete values denoted by e and g: $y_k \in \{e, g\}$. Quantum physics provides a stochastic model, i.e., a controlled Markov chain of state $|\psi\rangle_k$ with state-dependent jump probabilities and where y_k is just the type of jump that has occurred at step k:

$$|\psi\rangle_{k+1} = \begin{cases} \dfrac{D_{u_k}\, \mathcal{M}_g\, |\psi\rangle_k}{\left\| \mathcal{M}_g\, |\psi\rangle_k \right\|_{\mathcal{H}}} & \text{if } y_k = g \ \left(\text{probability } p_{g,k} = \left\| \mathcal{M}_g\, |\psi\rangle_k \right\|_{\mathcal{H}}^2\right) \\ \dfrac{D_{u_k}\, \mathcal{M}_e\, |\psi\rangle_k}{\left\| \mathcal{M}_e\, |\psi\rangle_k \right\|_{\mathcal{H}}} & \text{if } y_k = e \ \left(\text{probability } p_{e,k} = \left\| \mathcal{M}_e\, |\psi\rangle_k \right\|_{\mathcal{H}}^2\right). \end{cases}$$

The measurement operators $\mathcal{M}_g$ and $\mathcal{M}_e$ are such that $\mathcal{M}_g^\dagger \mathcal{M}_g + \mathcal{M}_e^\dagger \mathcal{M}_e = \mathbf{1}$ since $p_{e,k} + p_{g,k} \equiv 1$. The control operator D_u is unitary, i.e. $D_u^\dagger D_u = \mathbf{1}$. It corresponds to the propagator of a u-dependent Schrödinger equation satisfied by $|\psi\rangle$.

Physically, $|\psi\rangle$ is the quantum state of an eigen-mode trapped between the two mirrors of a super-conducting electromagnetic cavity, u is the complex amplitude of a classical electromagnetic pulse and y_k corresponds to the energy measurement for the k'th two-level atom after its passage though the cavity and thus after its interaction with the cavity state $|\psi\rangle_k$. This explains why the measurement outcomes, g for ground state and e for excited state, have probabilities depending on $|\psi\rangle_k$.

[2]See appendix A for summary of Dirac notations of "Bra" and "Ket"

Matrices $\mathcal{M}_g$ and $\mathcal{M}_e$ admit simple forms when the atom/field interaction is dispersive (see [33]):

$$\mathcal{M}_g = \mathrm{diag}\left(\cos(\vartheta n + \varphi_0)_{n=0,\ldots,n^{\max}}\right., \quad \mathcal{M}_e = \mathrm{diag}\left(\sin(\vartheta n + \varphi_0)_{n=0,\ldots,n^{\max}}\right.$$

where ϑ and φ_0 are constant parameters. The unitary operator $D_u = e^{ua^\dagger - u^* a}$ is given via a truncation to $n^{\max}$ photons of the annihilation operator a (see Section 2.2):

$$a\,|\psi\rangle = a \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{n^{\max}-1} \\ \psi_{n^{\max}} \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \sqrt{2}\psi_2 \\ \sqrt{3}\psi_3 \\ \vdots \\ \sqrt{n^{\max}}\psi_{n^{\max}} \\ 0 \end{pmatrix}.$$

Notice that $a^\dagger a = \mathbf{N} = \mathrm{diag}(0, 1, \ldots, n^{\max})$ is the truncated photon-number operator.

For $u = 0$, $D_u = \mathbf{1}$ and any Fock state $|m\rangle = (\delta_{nm})_{n\in\{0,\ldots,n^{\max}\}}$ ($m \in \{0, n^{\max}\}$) is a fixed point of the open-loop dynamics: if $|\psi\rangle_0 = |m\rangle$ then for all $k > 0$, $|\psi\rangle_k = |m\rangle$ when $u = 0$. For ϑ/π irrational, $|\psi\rangle_k$ ($u = 0$) converges to one of these $|m\rangle$'s. Whatever the initial condition $|\psi\rangle_0$ is and for almost all realizations of the Markov chain starting form $|\psi\rangle_0$, $|\psi\rangle_k$ converges to one of the states $|m\rangle$ (see [6]); this limit state could change from one realization to another one; the probability to have $|m\rangle$ as limit state is given by $|\psi_{0,m}|^2$. Even if we start from the same initial state $|\psi\rangle_0$ the limit state is not deterministic. A natural goal of the control u will be to ensure a deterministic value $|n_{sp}\rangle$ of the limit state and to ensure that for almost all trajectories and initial states $|\psi\rangle_k$ tends to $|n_{sp}\rangle$ as $k \mapsto \infty$. We will see in Section 5.1 how to ensure such global stabilization towards $|n_{sp}\rangle$ via an observer/controller structure (quantum filter and Lyapunov feedback law).

The quantum analogue of unknown perturbations w is essentially played by the environment and the decoherence. For the photon box, the major perturbing events are photon destruction by the cavity mirrors: they produce a drastic jump of the state $|\psi\rangle_k$ since it is instantaneously replaced by $|\psi\rangle_{k^+} = \frac{a|\psi\rangle_k}{\|a|\psi\rangle_k\|}$. This jump is not recorded and there is no possibility to instantaneously know that a photon has been destroyed by the cavity. Consequently it is not possible to compensate exactly such large unknown jumps. However, they can be compensated asymptotically via a feedback loop as shown in [28]. Let us briefly explain how: assume, for simplicity, that $|\psi\rangle_k$ has reached $|n_{sp}\rangle$ and that a photon has just been adsorbed by the cavity. Then $|\psi\rangle_{k^+} = |n_{sp} - 1\rangle$. The only way to detect this jump relies in the statistics of future measures y_{k+d}. If we maintain $u = 0$, $|\psi\rangle_{k+d}$ remains constant $|n_{sp} - 1\rangle$ and the probability to detect g has changed to $\cos^2(\vartheta(n_{sp} - 1) + \varphi_0)$ instead of being equal to $\cos^2(\vartheta n_{sp} + \varphi_0)$ for $|\psi\rangle = |n_{sp}\rangle$. It takes a certain time to see that the measurement statics have changed. Consequently, a feedback scheme can compensate only asymptotically such perturbations: in closed-loop $|\psi\rangle_{k+d}$ will return to the goal state $|n_{sp}\rangle$ after a certain number of steps d if no new photon destruction occurs. The performance of the controller and its efficiency to compensate these unknown jumps depend directly on

the ratio between the convergence time from $|n_{sp}-1\rangle$ to $|n_{sp}\rangle$ and the photon's life time inside the cavity.

To conclude we recommend the following books dealing with control on one side and quantum systems on the other side:

- Mathematical system theory and control:
  - H.K. Khalil. *Nonlinear Systems*. MacMillan, 1992.
  - J.M. Coron. *Control and Nonlinearity*. American Mathematical Society, 2007.
  - D. D'Alessandro. *Introduction to Quantum Control and Dynamics*. Chapman & Hall/CRC, 2008.
- Quantum physics and quantum information
  - S. Haroche and J.M. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford University Press, 2006.
  - H.M. Wiseman and G.J. Milburn. *Quantum Measurement and Control*. Cambridge University Press, 2009.
  - M.A. Nielsen and I.L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.
  - D. Steck. Quantum and atom optics (course notes). http://atomoptics.uoregon.edu/ dsteck/teaching/quantum-optics/, 2010.

Chapter 2

Schrödinger models

The models considered here rely on the Schrödinger equation (throughout this document $\hbar$ is set to be 1 and therefore the physical units are the atomic ones).

$$i\frac{d}{dt}\left|\psi\right\rangle = H\left|\psi\right\rangle,$$

where the wave function $|\psi\rangle$ belongs to a Hilbert space $\mathcal{H}$ of finite or infinite dimension, is of unit length ($\langle\psi|\psi\rangle = 1$) and where the Hamiltonan H is a Hermitian operator on $\mathcal{H}$ ($H^\dagger = H$). H is time varying though an affine dependence on m scalar controls $u_k \in \mathbb{R}$, $k \in \{1, \ldots, m\}$, $H = H_0 + \sum_{k=1}^{m} u H_k$.

We start with the simple but important case of 2-level systems (spin-half system) where $\mathcal{H}$ is of dimension 2. Then we consider the quantized harmonic oscillator (spring system) the simplest infinite dimensional case where we will define, through the Hamiltonian H, the creation and annihilation operators. We continue with the composite system (spin-spring system) made of two kinds of sub-systems: 2-level systems and quantized harmonic oscillators. We consider then n-qubit systems appearing in quantum information and 3-level systems. We end with controlled Shrödinger partial differential equations (1-D and n-D cases).

We have recalled in appendix A, Bra and Ket notations (Dirac notations), usual computations with operators, their spectral decompositions, tensor product related to composite systems and the passage from the wave function $|\psi\rangle$ to the density operator ρ.

The first 3 sections of this chapter are directly inspired of [33][chapter 3].

2.1 2-level systems

2.1.1 Schrödinger equation and Pauli matrices

Take the system of figure 2.1. Typically, it corresponds to an electron around an atom. This electron is either in the ground state $|g\rangle$ of energy E_g, or in the excited state $|e\rangle$ of energy E_e ($E_g < E_e$). We discard the other energy levels. We proceed here similarly to flexible mechanical systems where one usually considers only few vibration modes: instead

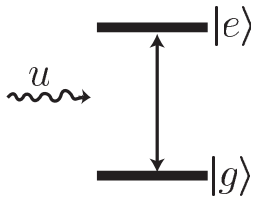

Figure 2.1: a 2-level system

of looking at the partial differential form of the Schrödinger equation describing the time evolution of the electron wave function, we consider only its components along two eigenmodes, one corresponds to the fundamental state and the other to the excited state. We will see below that controls are close to resonance and thus such an approximation is very natural.

The quantum state, described by $|\psi\rangle \in \mathbb{C}^2$ of length 1, $\langle\psi|\psi\rangle = 1$, is a linear superposition of $|g\rangle \in \mathbb{C}^2$, the ground state, and $|e\rangle \in \mathbb{C}^2$, the excited state, two orthogonal states, $\langle g|e\rangle = 0$, of length 1, $\langle g|g\rangle = \langle e|e\rangle = 1$:

$$|\psi\rangle = \psi_g |g\rangle + \psi_e |e\rangle$$

with $\psi_g, \psi_e \in \mathbb{C}$ the complex probability amplitudes[1]. This state $|\psi\rangle$ depends on time t. For this simple 2-level system, the Schrödinger equation is just an ordinary differential equation

$$i\frac{d}{dt}|\psi\rangle = H|\psi\rangle = \Big(E_g |g\rangle\langle g| + E_e |e\rangle\langle e|\Big)|\psi\rangle$$

completely characterized by H, the Hamiltonian operator ($H^\dagger = H$) corresponding to the system's energy (H is measured in frequency unit since we have assumed $\hbar = 1$) [2].

Since energies are defined up to a scalar, the Hamiltonians H and $H + u_0(t)I$ (with an arbitrary $u_0(t) \in \mathbb{R}$) describe the same physical system. If $|\psi\rangle$ obeys $i\frac{d}{dt}|\psi\rangle = H|\psi\rangle$ then $|\chi\rangle = e^{-i\theta_0(t)}|\psi\rangle$ with $\frac{d}{dt}\theta_0 = u_0$ satisfies $i\frac{d}{dt}|\chi\rangle = (H + u_0\mathbf{1})|\chi\rangle$ where $\mathbf{1} = |g\rangle\langle g| + |e\rangle\langle e|$ stands for the identity operator. Thus for all θ_0, $|\psi\rangle$ and $e^{-i\theta_0}|\psi\rangle$ are attached to the same physical system. The global phase of the quantum state $|\psi\rangle$ can be arbitrarily chosen. It is as if we can add a control u_0 of the global phase, this control input u_0 being arbitrary (gauge degree of freedom relative to the origin of the energy scale). Thus the one parameter family of Hamiltonians

$$\Big((E_g + u_0)|g\rangle\langle g| + (E_e + u_0)|e\rangle\langle e|\Big)_{u_0 \in \mathbb{R}}$$

[1] In a more standard formulation, $|g\rangle$ stands for $\begin{pmatrix}0\\1\end{pmatrix}$, $|e\rangle$ for $\begin{pmatrix}1\\0\end{pmatrix}$ and $|\psi\rangle$ for $\begin{pmatrix}\psi_e\\\psi_g\end{pmatrix}$.

[2] In a more standard formulation, $|g\rangle\langle g|$ stands for $\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$, $|e\rangle\langle e|$ for $\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$ and H for $\begin{pmatrix}E_e & 0\\0 & E_g\end{pmatrix}$.

describes the same system. It is then natural to take $u_0 = -\frac{E_e - E_g}{2}$ and to set $\omega_{eg} = E_e - E_g$, the pulsation of the photon emitted or absorbed during the transition between the ground and excited states. This frequency is associated to the light emitted by the electron during the jump from $|e\rangle$ to $|g\rangle$. This light is observed experimentally in spectroscopy: its frequency is a signature of the atom.

For the isolated system, the dynamics of $|\psi\rangle$ reads:

$$i\frac{d}{dt}|\psi\rangle = \frac{\omega_{eg}}{2}(|e\rangle\langle e| - |g\rangle\langle g|)|\psi\rangle\,.$$

Thus

$$|\psi\rangle_t = \psi_{g0}\, e^{\frac{i\omega_{eg}t}{2}}|g\rangle + \psi_{e0}\, e^{\frac{-i\omega_{eg}t}{2}}|e\rangle$$

where $|\psi\rangle_0 = \psi_{g0}|g\rangle + \psi_{e0}|e\rangle$. Usually, we denote by

$$\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$$

this Pauli matrix (see section A.6). Since $\sigma_z^2 = \mathbf{1}$, we have $e^{i\theta\sigma_z} = \cos\theta\mathbf{1} + i\sin\theta\sigma_z$ ($\theta \in \mathbb{R}$) and another expression of the time evolution of $|\psi\rangle$ is:

$$|\psi\rangle_t = e^{-\frac{i\omega_{eg}t}{2}\sigma_z}|\psi\rangle_0 = \cos\left(\frac{\omega_{eg}t}{2}\right)|\psi\rangle_0 - i\sin\left(\frac{\omega_{eg}t}{2}\right)\sigma_z|\psi\rangle_0\,.$$

Assume now that the system is in interaction with a classical electromagnetic field described by the control input $u(t) \in \mathbb{R}$. Then the evolution of $|\psi\rangle$ still results from a Schrödinger equation with a Hamiltonian depending on $u(t)$. In many cases, this controlled Hamiltonian admits the following form (dipolar and long wave-length approximations):

$$H(t) = \frac{\omega_{eg}}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + \frac{u(t)}{2}(|e\rangle\langle g| + |g\rangle\langle e|)$$

where u is homogenous to a frequency[3]. At this point, it is very convenient to introduced the two other Pauli matrices (see section A.6):

$$\sigma_x = |e\rangle\langle g| + |g\rangle\langle e|\,,\ \ \sigma_y = -i|e\rangle\langle g| + i|g\rangle\langle e|\,.$$

The controlled Hamiltonian is then $H = \frac{\omega_{eg}}{2}\sigma_z + \frac{u(t)}{2}\sigma_x$ and the dynamics of the wave function $|\psi\rangle$ reads:

$$i\frac{d}{dt}|\psi\rangle = \left(\tfrac{\omega_{eg}}{2}\sigma_z + \tfrac{u(t)}{2}\sigma_x\right)|\psi\rangle \tag{2.1}$$

Since σ_z and σ_x do not commute, there is no simple expression for the solution of the associated Cauchy problem when u depends on t (in general the system is not integrable).

[3] The Schrödinger equation $i\frac{d}{dt}|\psi\rangle = H|\psi\rangle$ reads with standard notations

$$i\frac{d}{dt}\begin{pmatrix}\psi_e\\ \psi_g\end{pmatrix} = \frac{\omega_{eg}}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_e\\ \psi_g\end{pmatrix} + \frac{u(t)}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_e\\ \psi_g\end{pmatrix}.$$

If we add the phase control $\varpi \in \mathbb{R}$, we have a 2-input control system

$$i\frac{d}{dt}|\phi\rangle = \left(\tfrac{\omega_{eg}}{2}\sigma_z + \tfrac{u(t)}{2}\sigma_x + \varpi(t)\mathbf{1}\right)|\phi\rangle \tag{2.2}$$

where $|\psi\rangle = e^{i\int_0^t \varpi(s)ds}|\phi\rangle$ is solution of (2.1).

2.1.2 Density operator and Bloch sphere

We start with $|\psi\rangle$ satisfying $i\frac{d}{dt}|\psi\rangle = H|\psi\rangle$. We consider the orthogonal projector $\rho = |\psi\rangle\langle\psi|$, corresponding to the density operator. Then ρ is Hermitian and semi-positive definite, satisfies $\mathrm{Tr}(\rho) = 1$, $\rho^2 = \rho$ and obeys the following equation:

$$\frac{d}{dt}\rho = -i[H, \rho]$$

where $[,]$ is the commutator: $[H, \rho] = H\rho - \rho H$. During the passage from $|\psi\rangle$ to the projector ρ we lose the global phase: for any angle θ, $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ yield to the same ρ. For a 2-level system $|\psi\rangle = \psi_g|g\rangle + \psi_e|e\rangle$ we have

$$|\psi\rangle\langle\psi| = |\psi_g|^2|g\rangle\langle g| + \psi_g\psi_e^*|g\rangle\langle e| + \psi_g^*\psi_e|e\rangle\langle g| + |\psi_e|^2|e\rangle\langle e|.$$

With

$$x = 2\Re(\psi_g\psi_e^*), \quad y = 2\Im(\psi_g\psi_e^*), \quad z = |\psi_e|^2 - |\psi_g|^2$$

we get the following expression

$$\rho = \frac{\mathbf{1} + x\sigma_x + y\sigma_y + z\sigma_z}{2}.$$

Thus $(x, y, z) \in \mathbb{R}^3$ can be seen as the coordinates in the orthogonal frame $(\vec{i}, \vec{j}, \vec{k})$ of a vector $\vec{M}$ in $\mathbb{R}^3$, called the Bloch vector:

$$\vec{M} = x\vec{i} + y\vec{j} + z\vec{k}.$$

Since $\mathrm{Tr}(\rho^2) = x^2 + y^2 + z^2 = 1$, $\vec{M}$ is of length one. For $|\psi\rangle$ solution of (2.1) or of (2.2), $\vec{M}$ evolves on the unit sphere of $\mathbb{R}^3$, called the Bloch sphere, according to

$$\frac{d}{dt}\vec{M} = (u\vec{i} + \omega_{eg}\vec{k}) \times \vec{M},$$

another equivalent writing for $\frac{d}{dt}\rho = -i\left[\frac{\omega_{eg}}{2}\sigma_z + \frac{u}{2}\sigma_x,\ \rho\right]$. Thus $u\vec{i} + \omega_{eg}\vec{k}$ is the instantaneous rotation velocity. Such geometric interpretation of the $|\psi\rangle$ dynamics on the Bloch sphere is very popular in magnetic resonance where the 2-level system corresponds to a spin-$\frac{1}{2}$ one. The knowledge of $\vec{M}$ is equivalent to the knowledge of $|\psi\rangle$, up to a global phase.

2.2 Harmonic oscillator and coherent states

A much more tutorial exposure is available in [22] and more advanced materials can be found in [8]. We just recall here some basic facts.

The Hamiltonian formulation of a classical harmonic oscillator of pulsation ω, $\frac{d^2}{dt^2}x = -\omega^2 x$, reads:

$$\frac{d}{dt}x = \omega p = \frac{\partial \mathcal{H}}{\partial p}, \quad \frac{d}{dt}p = -\omega x = -\frac{\partial \mathcal{H}}{\partial x}$$

where the classical Hamiltonian $\mathcal{H} = \frac{\omega}{2}(p^2 + x^2)$. The correspondence principle gives directly, from the classical Hamiltonian formulation, its quantization. The classical Hamiltonian becomes then an operator, H, operating on complex-value functions of one real variable $x \in \mathbb{R}$. The quantum state $|\psi\rangle$ is thus a function of x and t. It is also denoted here by $\psi(x,t)$. This function admits complex values and, for each time t, is square-integrable over $x \in \mathbb{R}$ with $\int |\psi(x,t)|^2 dx = 1$: at each time t, $|\psi\rangle_t \in L^2(\mathbb{R}, \mathbb{C})$.

The Hamiltonian operator H is obtained by replacing, in the classical Hamiltonian $\mathcal{H}$, x by the operator X, the multiplication by $\frac{x}{\sqrt{2}}$, p by the derivation $P = -\frac{i}{\sqrt{2}}\frac{\partial}{\partial x}$. Thus we have

$$H = \omega(P^2 + X^2) = -\frac{\omega}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega}{2}x^2.$$

The Schrödinger equation

$$i\frac{d}{dt}|\psi\rangle = H|\psi\rangle$$

is then a partial differential equation that determines the evolution of the probability amplitude wave function $\psi(x,t)$:

$$i\frac{\partial \psi}{\partial t}(x,t) = -\frac{\omega}{2}\frac{\partial^2 \psi}{\partial x^2}(x,t) + \frac{\omega}{2}x^2\psi(x,t), \quad x \in \mathbb{R}.$$

The average position is

$$\langle X\rangle_t = \langle\psi|X|\psi\rangle = \tfrac{1}{\sqrt{2}}\int_{-\infty}^{+\infty} x|\psi|^2 dx,$$

and average impulsion reads

$$\langle P\rangle_t = \langle\psi|P|\psi\rangle = -\tfrac{i}{\sqrt{2}}\int_{-\infty}^{+\infty} \psi^*\frac{\partial \psi}{\partial x}dx.$$

Exercice 2.2.1. *Verify via integration by part that $\langle P\rangle_t$ is real.*

With the annihilation and creation operators, a and $a^\dagger$,

$$a = X + iP = \tfrac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right), \quad a^\dagger = X - iP = \tfrac{1}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right) \tag{2.3}$$

we have

$$[X,P]=\tfrac{i}{2}\mathbf{1},\quad [a,a^\dagger]=\mathbf{1},\quad H=\omega(P^2+X^2)=\omega\left(a^\dagger a+\tfrac{1}{2}\mathbf{1}\right)$$

where $\mathbf{1}$ stands for the identity operator. Usually the above relations are expressed by replacing the identity operator $\mathbf{1}$ with the scalar 1. This yields to the compact expressions:

$$[X,P]=\tfrac{i}{2},\quad [a,a^\dagger]=1,\quad H=\omega(P^2+X^2)=\omega\left(a^\dagger a+\tfrac{1}{2}\right)$$

With $[a,a^\dagger]=1$, the spectral decomposition of $a^\dagger a$ is very simple and justifies the denomination of annihilation and creation operators for a and $a^\dagger$. The Hermitian operator $a^\dagger a$ admits $\mathbb{N}$ as non degenerate spectrum. The unitary eigen-state associated to the eigenvalue $n\in\mathbb{N}$ is denoted by $|n\rangle$: it is also called a Fock state and n is the number of quanta of vibration (phonon or photon). Moreover for any $n>0$,

$$a|n\rangle=\sqrt{n}\;|n-1\rangle,\quad a^\dagger|n\rangle=\sqrt{n+1}\;|n+1\rangle.$$

The ground state $|0\rangle$ satisfies $a|0\rangle=0$ and corresponds to the Gaussian function:

$$\psi_0(x)=\frac{1}{\pi^{1/4}}\exp(-x^2/2).$$

The operator a (resp. $a^\dagger$) is the annihilation (resp. creation) operator since it transfers $|n\rangle$ to $|n-1\rangle$ (resp. $|n+1\rangle$) and thus decreases (resp. increases) the quantum number by one unit.

Add a control u and consider the controlled harmonic oscillator $\frac{d^2}{dt^2}x=-\omega^2x-\frac{1}{\sqrt{2}}u$. Its quantization yields the following controlled Hamiltonian[4]

$$H=\omega\left(a^\dagger a+\tfrac{1}{2}\right)+u(a+a^\dagger). \tag{2.4}$$

that corresponds to the following controlled partial differential equation

$$i\frac{\partial\psi}{\partial t}(x,t)=-\frac{\omega}{2}\frac{\partial^2\psi}{\partial x^2}(x,t)+\left(\tfrac{\omega}{2}x^2+\sqrt{2}ux\right)\psi(x,t) \tag{2.5}$$

with $\psi(.,t)\in L^2(\mathbb{R},\mathbb{C})$ such that $\|\psi\|_{L^2}=1$.

Exercice 2.2.2. *Set $X_\lambda=\frac{1}{2}\left(e^{-i\lambda}a+e^{i\lambda}a^\dagger\right)$ for any angle λ. Prove that $\left[X_\lambda,X_{\lambda+\frac{\pi}{2}}\right]=\frac{i}{2}$.*

Exercice 2.2.3. *Denote by $\mathbf{N}=a^\dagger a$ the photon number operator. Show that for any analytic function f, we have the following identities:*

$$af(\mathbf{N})=f(\mathbf{N}+\mathbf{1})a,\quad a^\dagger f(\mathbf{N})=f(\mathbf{N}-\mathbf{1})a^\dagger.$$

Deduce that $e^{i\theta\mathbf{N}}ae^{-i\theta\mathbf{N}}=e^{-i\theta}a$ and $e^{i\theta\mathbf{N}}a^\dagger e^{-i\theta\mathbf{N}}=e^{i\theta}a^\dagger$.

[4]Notice the similarity with the controlled Hamiltonian of a 2-level system where the annihilation operator a is replaced by $\sigma_-=|g\rangle\langle e|$, the jump operator from the excited state $|e\rangle$ to the ground state $|g\rangle$.

For any complex number, denote by D_α the unitary operator

$$D_\alpha = e^{\alpha a^\dagger - \alpha^* a} \tag{2.6}$$

also called the (Glauber) displacement operator. Notice that $D_\alpha^{-1} = D_\alpha^\dagger = D_{-\alpha}$. When the operators A and B commute with their commutator, i.e., when $[A,[A,B]] = [B,[A,B]] = 0$ we have the identity

$$e^{A+B} = e^A\ e^B\ e^{-\frac{1}{2}[A,B]} \tag{2.7}$$

known as the Glauber formula. Using this formula with $A = \alpha a^\dagger$ and $B = -\alpha^* a$, we get another expression of D_α

$$D_\alpha = e^{-\frac{|\alpha|^2}{2}}\ e^{\alpha a^\dagger} e^{-\alpha^* a} = e^{+\frac{|\alpha|^2}{2}}\ e^{-\alpha^* a} e^{\alpha a^\dagger}. \tag{2.8}$$

The term "displacement" used for D_α comes from the fact that

$$D_{-\alpha} a D_\alpha = a + \alpha \quad \text{and} \quad D_{-\alpha} a^\dagger D_\alpha = a^\dagger + \alpha^*. \tag{2.9}$$

This can be proved applying the Cambell-Baker-Hausdorff formula:

$$e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \dots. \tag{2.10}$$

Exercice 2.2.4.

1. *Using (2.7), prove the relationships (2.8) and (2.9). Prove also that* $D_{-\alpha} X D_\alpha = X + \Re\alpha$ *and* $D_{-\alpha} P D_\alpha = P + \Im\alpha$.

2. *Prove also that, for any* $\alpha, \beta \in \mathbb{C}$, *we have* $D_{\alpha+\beta} = e^{\frac{\alpha^*\beta - \alpha\beta^*}{2}} D_\alpha D_\beta$.

3. *Deduce, for* $\beta = \epsilon$ *small, the following approximation*

$$D_{\alpha+\epsilon} D_{-\alpha} = \left(1 + \tfrac{\alpha\epsilon^* - \alpha^*\epsilon}{2}\right) \mathbf{1} + \epsilon a^\dagger - \epsilon^* a + O(|\epsilon|^2).$$

4. *Show that, when* α *is a smooth function of time,*

$$\left(\frac{d}{dt} D_\alpha\right) D_{-\alpha} = \left(\tfrac{\alpha \frac{d}{dt}\alpha^* - \alpha^* \frac{d}{dt}\alpha}{2}\right) \mathbf{1} + \left(\frac{d}{dt}\alpha\right) a^\dagger - \left(\frac{d}{dt}\alpha^*\right) a.$$

Coherent state of amplitude $\alpha \in \mathbb{C}$, usually denoted by $|\alpha\rangle$, is defined via D_α:

$$|\alpha\rangle = D_\alpha |0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{+\infty} \tfrac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{2.11}$$

where, for each integer n, $|n\rangle$ is the Fock state with n photon(s). One has to be careful with this notation: for $\alpha = n$ positive integer, the coherent state of amplitude α and the Fock state with n photon(s) are denoted with a similar symbol $|\alpha\rangle$ or $|n\rangle$ but these states coincide only when $\alpha = n = 0$ (vacuum state). Coherent states are eigenstates of the annihilation operator a:

$$a |\alpha\rangle = \alpha |\alpha\rangle .$$

Exercice 2.2.5. *Prove the above equality and also the second equality of (2.11) using (2.8).*

2.3 The Jaynes-Cummings model

This model describes a composite system, made of a 2-level system of states $|g\rangle$ and $|e\rangle$ and a quantized harmonic oscillator with a control u. It is widely used in physics literature and often models an atom with two electronic levels that is quasi-resonantly in interaction with a quantized mode of an electromagnetic cavity (cavity QED –Quantum Electrodynamics– with a Rydberg atom [33]). It can also represent a circuit QED. The quantum state $|\psi\rangle$ lives thus in the tensor product of $\mathbb{C}^2$ and $L^2(\mathbb{R},\mathbb{C})$[5]. Thus $|\psi\rangle$ admits two components $(\psi_g(x,t),\psi_e(x,t))$ where, for each t, the complex value functions ψ_g and ψ_e belong to $L^2(\mathbb{R},\mathbb{C})$. The Hamiltonian of this composite system is the sum of three Hamiltonians: the Hamiltonian H_a of the 2-level system alone (a for atom), the Hamiltonian of the controlled harmonic oscillator alone H_c (c for cavity) and finally the interaction Hamiltonian H_{int} (*int* for interaction). We have

$$H_a = \tfrac{\omega_{eg}}{2}(|e\rangle\langle e| - |g\rangle\langle g|) = \tfrac{\omega_{eg}}{2}\sigma_z, \quad H_c = \omega_c\left(a^\dagger a + \tfrac{1}{2}\right) + u(a+a^\dagger)$$

where atomic and cavity pulsations, ω_{eg} and ω_c, are close. Since $|\psi\rangle \in \mathbb{C}^2 \otimes L^2(\mathbb{R},\mathbb{C})$, we should write (to be rigorous):

$$H_a = \tfrac{\omega_{eg}}{2}\ \sigma_z \otimes \mathbf{1}_{L^2(\mathbb{R},\mathbb{C})}, \quad H_c = \omega_c\ \mathbf{1}_{\mathbb{C}^2} \otimes \left(a^\dagger a + \tfrac{1}{2}\right) + u\ \mathbf{1}_{\mathbb{C}^2} \otimes (a+a^\dagger)$$

where $\mathbf{1}_{L^2(\mathbb{R},\mathbb{C})}$ and $\mathbf{1}_{\mathbb{C}^2}$ are identity operators on $L^2(\mathbb{R},\mathbb{C})$ and $\mathbb{C}^2$. Since these rigorous notations are quite inefficient and here unnecessary, we abandon the tensor products sign and identity operators, as done previously. Thus H_a and H_c commute since they act on different spaces. However, the interaction Hamiltonian H_{ac} is based on a true tensor product of two non trivial operators. It admits the following form (dipolar and long wavelength approximations):

$$H_{ac} = i\tfrac{\Omega}{2}(|e\rangle\langle g| + |g\rangle\langle e|)(a^\dagger - a) = i\tfrac{\Omega}{2}\sigma_x(a^\dagger - a)$$

where the tensor product is noted as a simple product [6]. The pulsation Ω is called the vacuum Rabi pulsation. Thus the complete Hamiltonian, called Jaynes and Cummings Hamiltonian [34], reads with these compact notations:

$$H_{JC} = \tfrac{\omega_{eg}}{2}\sigma_z + \omega_c\left(a^\dagger a + \tfrac{1}{2}\right) + u(a+a^\dagger) + i\tfrac{\Omega}{2}\sigma_x(a^\dagger - a). \tag{2.12}$$

The usual scale assumptions are:

$$\Omega \ll \omega_c, \omega_{eg}, \qquad |\omega_c - \omega_{eg}| \ll \omega_c, \omega_{eg} \qquad \text{and} \qquad |u| \ll \omega_c, \omega_{eg}.$$

[5] See appendix A for a rapid introdocution on composite systems and tensor product.

[6] The rigorous expression is $H_{\text{int}} = \frac{\Omega_0}{2}\sigma_x \otimes (a+a^\dagger)$.

The wave function $|\psi\rangle$ obeys to a Schrödinger equation $i\frac{d}{dt}|\psi\rangle = H_{JC}|\psi\rangle$ that can be seen as a set of two real partial differential equations for $\psi_g(x,t)$ and $\psi_e(x,t)$

$$\begin{aligned} i\frac{\partial\psi_g}{\partial t} &= \tfrac{\omega_c}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_g + \left(\sqrt{2}ux - \tfrac{\omega_{eg}}{2}\right)\psi_g - i\frac{\Omega}{\sqrt{2}}\frac{\partial}{\partial x}\psi_e \\ i\frac{\partial\psi_e}{\partial t} &= \tfrac{\omega_c}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_e + \left(\sqrt{2}ux + \tfrac{\omega_{eg}}{2}\right)\psi_e - i\frac{\Omega}{\sqrt{2}}\frac{\partial}{\partial x}\psi_g \end{aligned} \tag{2.13}$$

with $\|\psi_g\|^2_{L^2} + \|\psi_e\|^2_{L^2} = 1$.

Exercice 2.3.1. *Consider the canonical basis* $\{|g,n\rangle, |e,n\rangle\}_{n\in\mathbb{N}}$ *and set*

$$|\psi\rangle = \sum_{n^\in\mathbb{N}} \psi_{g,n}|g,n\rangle + \psi_{e,n}|e,n\rangle \quad \textit{with} \quad \psi_{g,n},\psi_{e,n}\in\mathbb{C}.$$

Write down the differential equations satisfied by $\psi_{g,n}$ *and* $\phi_{e,n}$ *when* $|\psi\rangle$ *obeys* (2.13).

2.4 n-qubit system

The term qubit is just another denomination for 2-level systems (see 2.1). An n-qubit system is a composite system made of n qubits, each one living in the Hilbert space $\mathbb{C}^2$ with Hilbert basis $|g\rangle$ (g for ground state) and $|e\rangle$ (e for excited state). Its state belongs to $\overbrace{\mathbb{C}^2\otimes\mathbb{C}^2\ldots\otimes\mathbb{C}^2}^{n \text{ times}}$ that is isomorphic to $\mathbb{C}^{2^n}$. This is very different from a Cartesian product that will produce $\mathbb{C}^{2n}$ (see section A.3). It is usual in quantum information to denote by $|0\rangle$ the excited state $|e\rangle$ and by $|1\rangle$ the ground state $|g\rangle$. It is also usual to denote by X_k (resp. Y_k, Z_k) the Pauli operator σ_x (resp. σ_y, σ_z) acting only on the qubit number k.

With this notation the canonical basis of a 2-qubit system is

$$|0\rangle\otimes|0\rangle = |00\rangle, \quad |0\rangle\otimes|1\rangle = |01\rangle, \quad |1\rangle\otimes|0\rangle = |10\rangle, \quad |1\rangle\otimes|1\rangle = |11\rangle$$

and, for example,

$$X_1|00\rangle = |10\rangle, \quad X_1|01\rangle = |11\rangle, \quad X_1|10\rangle = |00\rangle, \quad X_1|11\rangle = |01\rangle$$

whereas

$$X_2|00\rangle = |01\rangle, \quad X_2|01\rangle = |00\rangle, \quad X_2|10\rangle = |11\rangle, \quad X_2|11\rangle = |10\rangle$$

Similarly, the canonical basis of 3-qubit system reads

$$|000\rangle, \quad |001\rangle, \quad |010\rangle, \quad |100\rangle, \quad |011\rangle, \quad |101\rangle, \quad |110\rangle, \quad |111\rangle.$$

and, for any $q_1, q_2 \in \{0,1\}$,

$$X_3|q_1q_20\rangle = |q_1q_21\rangle, \quad X_3|q_1q_21\rangle = |q_1q_20\rangle$$

Take the case $n = 2$ and assume an Ising interaction with $H_0 = JZ_1Z_2$ and that the scalar control u addresses collectively the 2-qubit system with $H_1 = J(X_1 + X_2)$. Then the dynamics of the 2-qubit system (state $|\psi\rangle \in \mathbb{C}^2$) obey

$$i\frac{d}{dt}|\psi\rangle = (H_0 + uH_1)|\psi\rangle = J(Z_1Z_2 + u(X_1 + X_2))|\psi\rangle \tag{2.14}$$

with $u \in \mathbb{R}$ as control and J as parameter.

Generalization to a linear chain of n spins yields to an n-qubit state $|\psi\rangle \in \mathbb{C}^{2^n}$ obeying to

$$i\frac{d}{dt}|\psi\rangle = J\left(\sum_{k=1}^{n-1} Z_kZ_{k+1} + u\sum_{k=1}^{n} X_k\right)|\psi\rangle\,. \tag{2.15}$$

2.5 3-level systems

The Hilbert space is $\mathbb{C}^3$ with an orthonormal frame $(|g\rangle, |e\rangle, |f\rangle)$. The controlled Hamiltonian is of the form

$$\begin{aligned} H &= \omega_g|g\rangle\langle g| + \omega_e|e\rangle\langle e| + \omega_f|f\rangle\langle f| \\ &+ u\mu_{ge}\Big(|g\rangle\langle e| + |e\rangle\langle g|\Big) + u\mu_{ef}\Big(|e\rangle\langle f| + |f\rangle\langle e|\Big) + u\mu_{gf}\Big(|g\rangle\langle f| + |f\rangle\langle g|\Big) \end{aligned} \tag{2.16}$$

where $u \in \mathbb{R}$ is the control and $(\omega_g, \omega_e, \omega_f, \mu_{ge}, \mu_{ef}, \mu_{gf})$ are physical constants with $\omega_g \leq \omega_e < \omega_f$. For $u = 0$, $|g\rangle$ is always a ground state with lowest energy. When $\omega_g = \omega_e$, the ground level is degenerate and is spanned by $|g\rangle$ and $|e\rangle$. Depending on the values of the coupling constants μ_{ge}, μ_{ef} and μ_{gf} we have illustrated on figure 2.2 the three important configurations:

ladder systems when $\mu_{gf} = 0$ and $\mu_{ge}, \mu_{ef} > 0$.

V-systems when $\mu_{ef} = 0$ and $\mu_{ge}, \mu_{gf} > 0$.

Λ-systems when $\mu_{ge} = 0$ and $\mu_{ef}, \mu_{gf} > 0$.

2.6 Partial differential systems

2.6.1 1D-particle in a moving potential well

We consider the control of a quantum particle represented by a complex probability amplitude $\mathbb{R} \ni x \mapsto \psi(x,t)$ solution of

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + (V(x) + \ddot{\zeta}(t)x)\psi \tag{2.17}$$

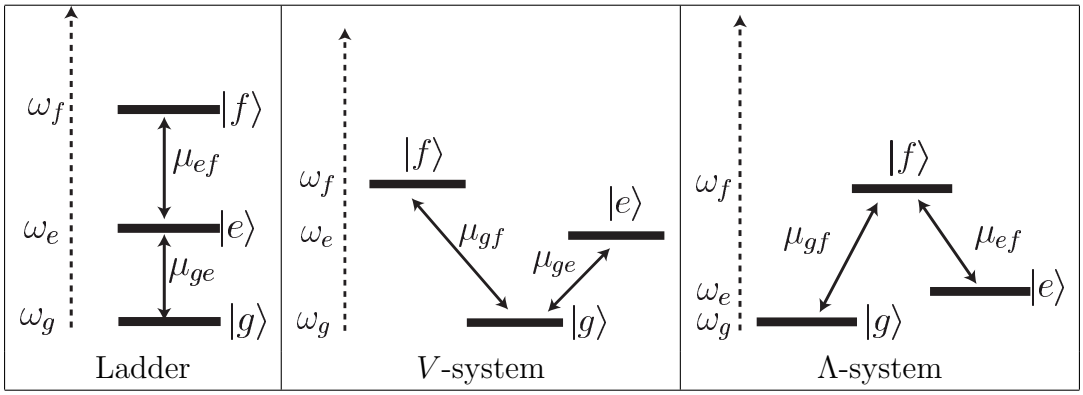

Figure 2.2: Three important configurations of 3-level systems with controlled Hamiltonian (2.16).

This 1-D Schrödinger equation describes the non relativistic motion of a single charged particle (mass $m = 1$, $\hbar = 1$) with a potential V in a non Galilean frame x of absolute position $z = x + \zeta$ where $\zeta(t)$ corresponding to the position of the well. Changes of independent variables $(t, x) \mapsto (t, z)$ and dependent variable $\psi \mapsto \phi$ defined by

$$\psi(t, z - \zeta) = \exp\left(i\left(-z\dot{\zeta} - \zeta\dot{\zeta} + \frac{1}{2}\int_0^t \dot{\zeta}^2\right)\right)\phi(t, z)$$

transform (2.17)

$$i\frac{\partial\phi}{\partial t} = -\frac{1}{2}\frac{\partial^2\phi}{\partial z^2} + V(z - \zeta)\phi \tag{2.18}$$

corresponding to the Schrödinger equation in a Galilean frame z. In (2.17) and (2.18), the control is $u = \ddot{\zeta} \in \mathbb{R}$ the acceleration. The position ζ and it velocity $\dot{\zeta}$ have to be included into the state: in the Galilean frame the state is $(\phi, \zeta, \dot{\zeta})$ and the dynamics is given by (2.18) with $\frac{d}{dt}\zeta = \dot{\zeta}$ and $\frac{d}{dt}\dot{\zeta} = u$; in the moving frame the state is $(\psi, \zeta, \dot{\zeta})$ and, for the dynamics, (2.18) is replaced by (2.17).

Controllability depends strongly on the shape of the potential V. We just highlight two types of potential shape.

- The periodic potential: $V(x) \equiv V(x + a)$, of the period $a > 0$.
- The box potential: $V(x) = 0$ for $x \in [-a/2, a/2]$ and $V(x) = +\infty$ for x outside $[-a/2, a/2]$.

2.6.2 Schrödinger equation with d-dimensional spatial domain

Typically, the spatial dimension $d = 3$ and the wave function $\psi(x, t)$ depends on $x \in \mathbb{R}^d$, where $\mathbb{R}^d$ is considered as a Euclidian space. The controlled Schrödinger equation reads then

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + \left(V(x) + \sum_{k=1}^{m} u(t)\mu_k(x)\right)\psi \tag{2.19}$$

where $\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial {x_j}^2}$ stands for the Laplacian operator, $x \mapsto V(x) \in \mathbb{R}$ is the free potential, $(u_k)_{k=1,\dots,m}$ are m independent scalar controls depending on t only, and, for each k, $x \mapsto \mu_k(x) \in \mathbb{R}$ is the coupling potential with the control u_k. With $d = 1$, $m = 1$, $\mu_1(x) = x$ and $\ddot{\zeta} = u_1$ we re-discover the 1D-particle in a moving potential (2.17). Notice that formally with the operators $H_0 = -\Delta + V(x)$ and $H_k = \mu_k(x)$ for $k = 1, \dots, m$, the dynamics (2.19) admits the usual bilinear form (versus the state ψ and the controls u_k):

$$i\frac{\partial \psi}{\partial t} = \left(H_0 + \sum_{k=1}^{m} u_k H_k \right) \psi \tag{2.20}$$

where H_0 and H_k are Hermitian operator acting on an Hilbert space $\mathcal{H}$, u_k are scalar controls and ψ belongs to the unit sphere of $\mathcal{H}$.

A common procedure to derive ordinary differential approximations (finite dimensional reduction) of (2.19) is based on the spectral decomposition of H_0. More precisely, assume that H_0 admits $n > 0$ orthonormal eigenstates $\phi_l(x)$ with eigenvalues ω_l, $l = 1, \dots, n$. Then ψ is approximated by $\sum_{l=1}^{n} \psi_l(t)\phi_l(x)$ where $\psi_l(t)$ obeys to the truncated equations:

$$i\frac{d}{dt}\psi_l = \omega_l \psi_l + \sum_{k=1}^{m} \sum_{j=1}^{n} u_k \mu_{k,lj} \psi_j$$

where the coupling coefficients are given by

$$\mu_{k,lj} = \langle \phi_j | H_k | \phi_l \rangle = \int_{\mathbb{R}^d} \phi_j^*(x) \mu_k(x) \phi_l(x) \, dx.$$

With Dirac notations (see appendix A) these ordinary differential equations read

$$i\frac{d}{dt} |\psi\rangle = \left(\sum_{l=1}^{n} \omega_l \, |l\rangle \langle l| + \sum_{k=1}^{m} u_k \left(\sum_{l_1,l_2=1}^{n} \mu_{k,l_1 l_2} \, |l_1\rangle \langle l_2| \right) \right) |\psi\rangle$$

where $|l\rangle$ stands for ϕ_l and $|\psi\rangle = \sum_{l=1}^{n} \psi_l \, |l\rangle$ for the finite approximation of the infinite dimension state $\psi(x,t)$.

Chapter 3

Open-loop control of Schrödinger-type models

This chapter investigates the following question: for $|\psi\rangle$ obeying a controlled Schrödinger equation $i\frac{d}{dt}|\psi\rangle = (H_0 + \sum_{k=1}^{m} uH_k)|\psi\rangle$ with a given initial condition, find an open-loop control $[0,T] \ni\mapsto u(t)$ such that at the final time T, $|\psi\rangle$ has reached a pre-specified target state. This question is directly linked to controllability, a fundamental notion of system theory that is considered in section 3.1. In the other sections of this chapter, emphasis is put on different methods to construct efficiency open-loop steering controls from one state to another one: resonant control and the rotation wave approximation are treated in section 3.3; quasi-static controls exploiting adiabatic invariance are presented in section 3.4; optimal control techniques minimizing $\int u^2$ are investigated in section 3.5; finally, section 3.6 is devoted to feedback stabilization relying on control Lyapunov functions provided by fidelity to the target state.

3.1 Controllability

3.1.1 Some definitions

This subsection is directly inspired from chapter 3 of [26] where the proof of results recalled here are given and where finer controllability characterizations can be found.

Assume that the Hilbert space $\mathcal{H}$ is of finite dimension n and consider a quantum system with wave function $|\psi\rangle$ on the unit sphere of $\mathcal{H}$ and satisfying the following controlled Schrödinger equation

$$i\frac{d}{dt}|\psi\rangle = \left(H_0 + \sum_{k=1}^{m} u_k H_k\right)|\psi\rangle \tag{3.1}$$

where $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ is formed by m independent controls and H_0, H_1, $\ldots$, H_m are $m+1$ Hermitian operators on $\mathcal{H}$.

Associated to (3.1) we have the Liouville equation for the density operator ρ, a positive

(not necessarily definite positive) Hermitian operator on $\mathcal{H}$ with unit trace ($\mathrm{Tr}\,(\rho) = 1$):

$$i\frac{d}{dt}\rho = \left[\left(H_0 + \sum_{k=1}^{m} u_k H_k\right)\ ,\ \rho\right] \tag{3.2}$$

where [. , .] denotes the commutator. If $|\psi\rangle$ satisfies (3.1), then the projector $\rho = |\psi\rangle\langle\psi|$ satisfies (3.2).

Associated to (3.1) we have also the propagator equation

$$i\frac{d}{dt}U = \left(H_0 + \sum_{k=1}^{m} u_k H_k\right) U \tag{3.3}$$

where U is a unitary operator on $\mathcal{H}$ (U belongs to the compact Lie group $U(n)$, the set of $n \times n$ unitary matrices).

It is clear that we can express the solutions of (3.1) and (3.2) as soon as we have the solution of (3.3) $t \mapsto U_t$ starting from identity, $U_0 = \mathbf{1}$:

$$|\psi\rangle_t = U_t\,|\psi\rangle_0\,, \quad \rho_t = U_t \rho_0 U_t^\dagger.$$

When u is time-dependent, (3.3) is not integrable in general and we do not have explicit expression for U_t, $|\psi\rangle_t$ and ρ_t.

Since $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ for any phase $\theta \in [0, 2\pi[$ represent the same physical state, we have the following controllability definition underlying quantum state preparation.

Definition 3.1.1 (State Controllability)**.** *The controlled Schrödinger system* (3.1) *is said state controllable if, and only if, for any* $|a\rangle$ *and* $|b\rangle$ *on the unit sphere of* $\mathcal{H}$*, exist a time* $T > 0$*, a global phase* $\theta \in [0, 2\pi[$ *and a piecewise continuous control* $[0, T] \ni t \mapsto u(t)$ *such that the solution of* (3.1) *with initial condition* $|\psi\rangle_0 = |a\rangle$ *satisfies* $|\psi\rangle_T = e^{i\theta}|b\rangle$*.*

Remark 3.1.2. *In the above definition, the controllability time* T *cannot be chosen arbitrarily short, even for the finite dimensional case (there exists a non-zero minimal control time).*

Exercice 3.1.3. *Consider the example of a qubit following the dynamics*

$$i\frac{d}{dt}|\psi\rangle = \sigma_z|\psi\rangle + u(t)\sigma_x|\psi\rangle\,.$$

While we will assume the controllability of the above system (see the next subsection for a proof), show through a geometrical argument that, by considering an initial state $|g\rangle$ *and a target state* $\frac{1}{\sqrt{2}}(|g\rangle + |e\rangle)$*, the control time cannot be made arbitrarily small.*

For the Liouville equation we have a similar definition without an arbitrary global phase θ:

Definition 3.1.4 (Density Matrix Controllability). *The controlled Liouville equation (3.2) is said density matrix controllable if, and only if, for any density operator ϱ and any unitary operator V on $\mathcal{H}$, exist a time $T > 0$ and a piecewise continuous control $[0, T] \ni t \mapsto u(t)$ such that the solution of (3.2) with initial condition $\rho_0 = \varrho$ satisfies $\rho_T = V\varrho V^\dagger$.*

Exercice 3.1.5. *Why cannot we replace $V\varrho V^\dagger$ in definition 3.1.4 by an arbitrary density operator?*

For the propagator we have the following controllability definition underlying quantum gate design:

Definition 3.1.6 (Operator Controllability). *The controlled operator equation (3.3) is said operator controllable if, and only if, for unitary operators V and W on $\mathcal{H}$, exist a time $T > 0$, a global phase θ and a piecewise continuous control $[0, T] \ni t \mapsto u(t)$ such that the solution of (3.3) with initial condition $U_0 = V$ satisfies $U_T = e^{i\theta}W$.*

Exercice 3.1.7. *Show that operator controllability implies state and density-matrix controllability.*

It is more difficult to prove that density-matrix controllability is equivalent to operator controllability (see [26]) and to find an example showing that state controllability does not imply operator controllability.

3.1.2 The Lie algebra rank condition

Set $A_k = -iH_k$ for $k = 0, \ldots, m$. Then (3.3) reads

$$\frac{d}{dt}U = \left(A_0 + \sum_{k=1}^{m} u_k A_k\right) U$$

where the A_k's are skew-Hermitian operators ($A_k^\dagger = -A_k$). The set of all skew-Hermitian operators is usually denoted by $u(n)$ and forms a Lie algebra:

- It is a real vector space of dimension n^2.
- It is closed for the commutator: if A and B are skew-Hermitian, then $[A, B]$ is also skew-Hermitian

The sub-set of $u(n)$ with zero trace is denoted by $su(n)$. It is also a Lie algebra (a sub Lie algebra of $u(n)$) of dimension $n^2 - 1$. The exponential maps $u(n)$ to $U(n)$ since for any $A \in u(n)$, e^A is in $U(n)$ since $(e^A)^\dagger = e^{A^\dagger} = e^{-A} = (e^A)^{-1}$. If, moreover $A \in su(n)$, then $\det(e^A) = e^{\mathrm{Tr}(A)} = e^0 = 1$ and thus $e^A \in SU(n)$, where $SU(n)$ denotes the set of $n \times n$ unitary matrices with unit determinant (a sub-group of the group $U(n)$).

The Lie algebra spanned by the A_k's, denoted by $\mathcal{L} = \mathrm{Lie}(A_0, \ldots, A_m)$ is the real vector space formed by any real linear combination of finite-length Lie brackets (i.e. commutators)

of A_k's. The Lie algebra $\mathcal{L}$ is obtained after a finite number of steps ν (in any case $\nu \leq n^2-2$) of the following process starting from the real vector space $\mathcal{L}_0 = \text{span}(A_0, \ldots, A_m)$ spanned by the A_k's:

$$\begin{aligned} \mathcal{L}_1 &= \text{span}(\mathcal{L}_0, [\mathcal{L}_0, \mathcal{L}_0]) \\ \mathcal{L}_2 &= \text{span}(\mathcal{L}_1, [\mathcal{L}_1, \mathcal{L}_1]) \\ &\vdots \\ \mathcal{L} = \mathcal{L}_\nu &= \text{span}(\mathcal{L}_{\nu-1}, [\mathcal{L}_{\nu-1}, \mathcal{L}_{\nu-1}]) \end{aligned}$$

where, for example, $\text{span}(\mathcal{L}_0, [\mathcal{L}_0, \mathcal{L}_0])$ is a shortcut notation for the vector space spanned by $\mathcal{L}_0$ and all the skew-Hermitian matrices $[A, B]$ with $A, B \in \mathcal{L}_0$. An explicit computation of this increasing sequence of vector spaces $\mathcal{L}_\sigma \subset \mathcal{L}_{\sigma+1}$ consists in starting from a basis of the vector space $\mathcal{L}_\sigma$, to compute all the commutators between two elements of this basis, to complete this basis by commutators adding independent direction to form the basis of $\mathcal{L}_{\sigma+1}$.

Exercice 3.1.8. *Compute the sequence $(\mathcal{L}_\sigma)_{0\leq\sigma\leq\nu}$ for $n = 2$, $m = 1$, $A_0 = i(\sigma_z + \mathbf{1})$ and $A_1 = i\sigma_x$ and show that $\nu = 2$ and $\mathcal{L} = u(2)$. What are ν and $\mathcal{L}$ when $A_0 = i\sigma_z$ and $A_1 = i\sigma_x$?*

The following controllability theorem is recalled in [26] and goes back to [37]:

Theorem 3.1.9 (Lie Algebra Rank Condition)**.** *The system (3.3) is operator controllable in the sense of definition 3.1.6 if, and only if, the Lie algebra generated by the $m+1$ skew-Hermitian matrices $\{-iH_0, -iH_1, \ldots, -iH_m\}$ is either $su(n)$ or $u(n)$.*

This result solves a fundamental issue. But the practical issues remain to be addressed, in particular, the explicit construction of the open-loop steering control. More precisely, assume that (3.3) is operator controllable . Then,

- once the initial and final values V and W of definition 3.1.6 are given, the above theorem does not gives a transition time $T > 0$ nor an explicit open-loop control $[0, T] \ni t \mapsto u(t)$ that steers the propagator from $U_0 = V$ to $U_T = e^{i\theta}W$.
- once the initial and final values $|\psi_a\rangle$ and $|\psi_b\rangle$ of definition 3.1.1 are given, the above theorem does not gives a transition time $T > 0$ nor an explicit open-loop control $[0, T] \ni t \mapsto u(t)$ that steers the state from $|\psi\rangle_0 = |\psi_a\rangle$ to $|\psi\rangle_T = e^{i\theta}|\psi_b\rangle$.
- once the initial and final values ψ_a and ψ_b of definition 3.1.4 are given, the above theorem does not gives a transition time $T > 0$ nor an associated explicit open-loop control $[0, T] \ni t \mapsto u(t)$ that steers the density operator from $\rho_0 = |\psi_a\rangle\langle\psi_a|$ to $\rho_T = |\psi_b\rangle\langle\psi_b|$.

Exercice 3.1.10. *Deduce from Theorem 3.1.9 the necessary and sufficient conditions on ω_{eg} for the operator controllability of the 2-level system (2.1).*

Exercice 3.1.11. *Take the 2-qubit Ising system (2.14) with $J \neq 0$.*

1. *Prove that X_1X_2 commutes with Z_1Z_2 and $X_1 + X_2$.*
2. *Is the system controllable ?*
3. *Use the spectral basis of X_1X_2 and the decomposition $span\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} = span\{|++\rangle, |--\rangle\} \oplus span\{|+-\rangle, |-+\rangle\}$ with $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$, to deduce a splitting of this system into two separated systems on $span\{|++\rangle, |--\rangle\}$ and on $span\{|+-\rangle, |-+\rangle\}$.*
4. *Prove that one of these sub-systems is controllable and that the other one is not controllable.*

3.1.3 A simple sufficient controllability condition

Consider the single control Hamiltonian $H = H_0 + uH_1$ and take an orthonormal basis $(|j\rangle)_{j=1,\dots,n}$ where H_0 is diagonal. For each j, $H_0|j\rangle = \omega_j|j\rangle$ where $\omega_j \in \mathbb{R}$ is the eigenvalue associated to $|j\rangle$. To (H_0, H_1) is associated the following non oriented graph $G = (V, E)$ with vertices V formed by the n eigenstates $|j\rangle$ and the edges E connecting the pair of eigenstates $(|j_1\rangle, |j_2\rangle)$ such that $\langle j_1|H_1|j_2\rangle \neq 0$:

$$V = \{|1\rangle, \dots, |n\rangle\}, \quad E = \{(|j_1\rangle, |j_2\rangle) \mid 1 \leq j_1 < j_2 \leq n,\ \langle j_1|H_1|j_2\rangle \neq 0\}.$$

This connectivity graph is physically related to single-photon transitions. When the spectrum of H_0 is degenerate, this graph depends in general on the choice of the arthonormal basis $\{|j\rangle\}$. The graph G is said to have a degenerate transition if exist two distinct edges, $(|j_1\rangle, |j_2\rangle) \in E$ and $(|l_1\rangle, |l_2\rangle) \in E$, admitting the same transition frequencie, i.e., $|\omega_{j_1} - \omega_{j_2}| = |\omega_{l_1} - \omega_{l_2}|$.

The following theorem is proved in [68] for state controllability and in [5] for operator controllability.

Theorem 3.1.12. *Consider a finite dimensional controlled Hamiltonian $H_0 + uH_1$ with a single scalar control u. Consider for a spectral decomposition of H_0, the graph G defined here above. Remove from E, all the edges with identical transition frequencies. Denote by $\bar{E} \subset E$ the reduced set of edges without degenerate transitions and by $\bar{G} = (V, \bar{E})$ the resulting sub-graph of G. If $\bar{G}$ is connected, then $i\frac{d}{dt}|\psi\rangle = (H_0 + uH_1)|\psi\rangle$ is state and also operator controllable (definitions 3.1.1 and 3.1.6).*

Notice that the transition degeneracies can be changed if instead of considering the graph associated to (H_0, H_1) we consider the graph associated to $(H_0 + \bar{u}H_1, H_1)$ where H_0 is replaced by $\bar{H}_0 = H_0 + \bar{u}H_1$ with some real constant $\bar{u}$. Since controllabilities of $H_0 + uH_1$ and $\bar{H}_0 + uH_1$ are identical (these two systems are equivalent up to a shift on u), it is often possible to deduce controllability from Theorem 3.1.12 by choosing carefully the shift $\bar{u}$ in order to remove eventual degeneracies (Stark shift usually removes degeneracies).

Exercice 3.1.13. *Deduce from Theorem 3.1.12 sufficient conditions on the parameters of the 3-level system (2.16) for its controllability.*

It is interesting to notice that similar sufficient controllability conditions are valid in the infinite dimensional cases. In [21] state and density matrix approximate controllabilities are proved if the spectrum $(\omega_n)_{n\in\mathbb{N}}$ of H_0 is discrete, and similar non-degeneracy and graph connectivity assumptions are satisfied.

3.1.4 Harmonic oscillator

The evolution of the quantized harmonic oscillator (2.4) is given by the bilinear controlled Schrödinger equation $i\frac{d}{dt}|\psi\rangle = (H_0 + u\ H_1)|\psi\rangle$, with

- $H_0 = \omega a^\dagger a$ the free evolution Hamiltonian (we have removed the $\omega/2$ off-set by a change of global phase);
- $H_1 = (a^\dagger + a)$ the interaction operator with the classical field described by the scalar control $u \in \mathbb{R}$.

The state $|\psi\rangle$ can be seen as an element of $L^2(\mathbb{R}, \mathbb{C})$. It is thus of infinite dimension and Theorem 3.1.9 cannot be used rigorously since it holds true only for finite dimensional systems. Nevertheless, we can compute, at least formally, the controllability Lie algebra using the usual commutation rule $[a, a^\dagger] = 1$. Since

$$[a^\dagger a, a^\dagger + a] = (a^\dagger - a), \quad [a^\dagger a, a^\dagger - a] = (a^\dagger + a), \quad [a^\dagger + a, a^\dagger - a] = 2$$

we see that

$$\text{Lie}\,\{iH_0, iH_1\} = \text{span}\{ia^\dagger a, i(a + a^\dagger), a - a^\dagger, i\mathbf{1}\}.$$

It is of dimension 4: the system cannot be controllable.

We can decompose the system into a controllable part (of dimension 2) and an uncontrollable part of infinite dimension. This decomposition is based on a time-dependent unitary transformation: in the new representation, we extract, from the dynamics, an autonomous Schrödinger equation modeling the quantum fluctuations around the average value of a. This unitary transformation corresponds to a displacement operator D_α, introduced in (2.6), where $\alpha \in \mathbb{C}$ is a well-chosen time-dependent complex amplitude. Such transformations are commonly used in quantum optics and underly the fact that classical currents and sources (generalizing the role played by u) only generate classical light (quasi-classical states of the field generalizing the coherent states introduced in (2.2)) (see, e.g., [23][complement B_{III}, page 217]). Here, we only propose a control theoretical interpretation (see [52] for a more detailed exposure).

Set $\langle a\rangle = \langle\psi|a\psi\rangle$ the average value of a. Since $|\psi\rangle$ depends on t, $\langle a\rangle$ depends also on t. It is complex since a is not Hermitian. Applying the dynamics of $|\psi\rangle$ and the commutation relation $[a, a^\dagger] = 1$, we have

$$\frac{d}{dt}\langle a\rangle = -i\omega\langle a\rangle - iu \tag{3.4}$$

From $\langle a\rangle = X + iP$, we have $\langle a\rangle = \langle X\rangle + i\langle P\rangle$ where $\langle X\rangle = \langle\psi|X|\psi\rangle \in \mathbb{R}$ and $\langle P\rangle = \langle\psi|P|\psi\rangle \in \mathbb{R}$. Consequently (3.4) reads:

$$\frac{d}{dt}\langle X\rangle = \omega\langle P\rangle, \quad \frac{d}{dt}\langle P\rangle = -\omega\langle X\rangle - u.$$

Consider the time-varying displacement operator

$$D_{\langle a\rangle} = e^{\langle a\rangle a^\dagger - \langle a\rangle^* a}.$$

The action of this unitary transformation on operator a is a translation by the quantity $\langle a\rangle$ (see section (2.2))):

$$D_{-\langle a\rangle}\ a\ D_{\langle a\rangle} = a + \langle a\rangle, \qquad D_{-\langle a\rangle}\ a^\dagger\ D_{\langle a\rangle} = a^\dagger + \langle a\rangle^*,$$

Set $|\phi\rangle = D_{-\langle a\rangle}\ |\psi\rangle$. Then

$$\begin{aligned} i\frac{d}{dt}|\phi\rangle &= \left(D_{-\langle a\rangle}(\omega a^\dagger a + u(a+a^\dagger))D_{\langle a\rangle}\right)|\phi\rangle + i\left(\frac{d}{dt}D_{-\langle a\rangle}\right)D_{\langle a\rangle}\ |\phi\rangle \\ &= \left(\omega(a^\dagger + \langle a\rangle^*)(a+\langle a\rangle) + u(a + \langle a\rangle + a^\dagger + \langle a\rangle^*)\right)|\phi\rangle \\ &\quad + i\left(\frac{\langle a\rangle\frac{d\langle a\rangle^*}{dt} - \frac{d\langle a\rangle}{dt}\langle a\rangle^*}{2} - \frac{d\langle a\rangle}{dt}a^\dagger + \frac{d\langle a\rangle^*}{dt}a\right)|\phi\rangle \\ &= \omega a^\dagger a\,|\phi\rangle + \left(\omega|\langle a\rangle|^2 + u\tfrac{\langle a\rangle+\langle a\rangle^*}{2}\right)|\phi\rangle \end{aligned}$$

since, with the last formula of exercise 2.2.4, we have

$$\left(\frac{d}{dt}D_{-\langle a\rangle}\right)D_{\langle a\rangle} = \left(\frac{\langle a\rangle\frac{d}{dt}\langle a\rangle^* - \langle a\rangle^*\frac{d}{dt}\langle a\rangle}{2}\right)\mathbf{1} - \left(\frac{d}{dt}\langle a\rangle\right)a^\dagger + \left(\frac{d}{dt}\langle a\rangle^*\right)a.$$

Set $|\chi\rangle = e^{i\theta_t}|\phi\rangle$ with the global phase $\theta_t = \int_0^t (|\langle a\rangle|^2 + u\Re(\langle a\rangle))$. It yields to the following autonomous Schrödinger equation:

$$i\ \frac{d}{dt}|\chi\rangle = \omega a^\dagger a\,|\chi\rangle\,. \tag{3.5}$$

The dynamics of $|\psi\rangle$ can be decomposed into two parts:

- a controllable part of dimension two (3.4) associated to the average of a, the classical phase of a harmonic oscillator.
- an uncontrollable part of infinite dimension (3.5) corresponding to the quantum fluctuations around the controllable phase-space dynamics.

These computations might be extended to n harmonic oscillators admitting the same control u but with different frequencies (see [51] for more details).

Exercice 3.1.14. *Consider the Jaynes-Cummings model presented in Section 2.3:*

$$i\frac{d}{dt}|\psi\rangle = \left(\tfrac{\omega_{eg}}{2}\sigma_z + \omega_c\left(a^\dagger a + \tfrac{1}{2}\right) + u(a+a^\dagger) - i\tfrac{\Omega}{2}\sigma_x(a^\dagger - a)\right)|\psi\rangle\,.$$

The Hamiltonians are therefore given by:

$$H_0 = \tfrac{\omega_{eg}}{2}\sigma_z + \omega_c\left(a^\dagger a + \tfrac{1}{2}\right) - i\tfrac{\Omega}{2}\sigma_x(a^\dagger - a), \qquad H_1 = a + a^\dagger.$$

By computing the commutators, show that the Lie algebra $Lie\{iH_0, iH_1\}$ contains all skew Hermitian operators of the form $i\sigma_\xi X^n P^m$ with integer $n, m \geq 0$ and $\xi = x, y, z$. This fact tends to indicate that the Jaynes-Cummings model may be controllable (at least approximatively in an appropriate functional space).

3.1.5 Ensemble controllability and robustness issues

Let us finish by an interesting robustness notion encountered in magnetic resonance: ensemble controllability as stated in [45] when we face a continuum of parameter values. We just consider here an example. For the system

$$i\frac{d}{dt}|\psi\rangle = \left(\frac{\Delta}{2}\sigma_z + \frac{\mathbf{u}}{2}\sigma_- + \frac{\mathbf{u}^*}{2}\sigma_+\right)|\psi\rangle$$

depending on the parameter Δ, the problem reads as follows: find a unique open-loop control $[0,T] \ni t \mapsto \mathbf{u}(t) \in \mathbb{C}$ ensuring the (approximated) transfer of $|\psi\rangle_0^\Delta = |g\rangle$ towards $|\psi\rangle_T^\Delta = |e\rangle$ where $|\psi\rangle_t^\Delta$ is the solution corresponding to the parameter Δ:

$$i\frac{d}{dt}|\psi\rangle^\Delta = \left(\frac{\Delta}{2}\sigma_z + \frac{\mathbf{u}}{2}\sigma_- + \frac{\mathbf{u}^*}{2}\sigma_+\right)|\psi\rangle^\Delta\,.$$

The difficulty comes from the fact that Δ takes any value in the interval $[\Delta_0, \Delta_1]$ ($\Delta_0 < \Delta_1$ are given) whereas $\mathbf{u}(t)$ is independent of Δ. The goal is to control via the same input an infinite number (a continuum) of similar systems differing only by the value of Δ. This is a special controllability problem of an infinite dimensional system with a continuous spectra: for $\mathbf{u} = 0$, the spectrum is on the imaginary axis, $\left[\frac{-i\Delta_1}{2}, \frac{-i\Delta_0}{2}\right] \cup \left[\frac{i\Delta_0}{2}, \frac{i\Delta_1}{2}\right]$. This infinite dimensional system is particularly interesting to better understand the controllability issue in the presence of a continuous part in the spectrum. For a first set of mathematical results on ensemble controllability of this system see [46, 11]. For approximate controllability results in the presence of a continuous part in the spectrum of H_0 see [49].

3.2 Linearized systems around stationary states

We sketch here basic results with a single control u when $i\frac{d}{dt}|\psi\rangle = (H_0 + uH_1)|\psi\rangle$ is linearized around an eigenstate $|\bar{\psi}\rangle$ of H_0. Up to a shift on H_0 and a global phase change

we can always assume that $H_0 \left|\bar{\psi}\right\rangle = 0$. Denote by $|1\rangle = \left|\bar{\psi}\right\rangle$ and by $|2\rangle$, ..., $|n\rangle$ the remaining eigen-states of H_0: $H_0 |k\rangle = \omega_k$ with $\omega_k \in \mathbb{R}$ ($\omega_1 = 0$). We can always assume that $(|1\rangle, |2\rangle, \ldots, |n\rangle)$ is an ortho-normal frame.

Set $|\psi\rangle = \sum_{k=1}^n \psi_k |k\rangle$ with $\psi_k \in \mathbb{C}$ and $\sum_{k=1}^n |\psi_k|^2 = 1$. The first variation of $|\psi\rangle$ around $|1\rangle$, $\delta|\psi\rangle$, reads:

$$\delta|\psi\rangle = \sum_{k=1}^n \delta\psi_k |k\rangle \quad \text{with} \quad \delta\psi_1 + \delta\psi_1^* = 0$$

where each $\delta\psi_k$ is a small complex number, $|\delta\psi_k| \ll 1$. Then, up to second order terms in $\delta|\psi\rangle$ and u, $i\frac{d}{dt}|\psi\rangle = (H_0 + uH_1)|\psi\rangle$, reads

$$i\frac{d}{dt}\delta\psi_k = \omega_k\delta\psi_k + b_k u, \quad k = 1, \ldots, n$$

with $b_k = \langle 1|H_1|k\rangle$.

Let us look at $\delta\psi_1$: the constraint $\delta\psi_1 + \delta\psi_1^* = 0$ is satisfied by the dynamics $i\frac{d}{dt}\delta\psi_1 = b_1 u$ since b_1 and u are real. Thus we have only to consider $\Im(\delta\psi_1)$ that obeys to $\frac{d}{dt}\Im(\delta\psi_1) = -b_1 u$. The linearized dynamics are described by one scalar differential equation and $n-1$ complex differential equations:

$$\frac{d}{dt}\Im(\delta\psi_1) = -b_1 u, \qquad i\frac{d}{dt}\delta\psi_k = \omega_k\delta\psi_k + b_k u, \quad k = 2, \ldots, n. \tag{3.6}$$

We have the following theorem

Theorem 3.2.1. *The linear time-invariant system (3.6) is controllable if, and only if, for all $k \in \{1, \ldots, n\}$, $b_k \neq 0$ and for all $k_1, k_2 \in \{1, \ldots, n\}$ such that $k_1 \neq k_2$, $|\omega_{k_1}| \neq |\omega_{k_2}|$ ($\omega_1 = 0$).*

Proof. The proof given here is not the simplest one (on can for example use the Kalman criterion, see, e.g., [38]). Nevertheless, for any $T > 0$, it is constructive and provides open-loop controls $[0, T] \ni t \mapsto u(t) \in \mathbb{R}$, steering between the initial state

$$(\Im(\delta\psi_1^0), \delta\psi_2^0, \ldots, \delta\psi_n^0) = (\Im(\delta\psi_1), \delta\psi_2, \ldots, \delta\psi_n)_{t=0}$$

to the final state

$$(\Im(\delta\psi_1^T), \delta\psi_2^T, \ldots, \delta\psi_n^T) = (\Im(\delta\psi_1), \delta\psi_2, \ldots, \delta\psi_n)_{t=T}\,.$$

The construction of such feedforward controls is directly inspired from flatness based motion planing [60, 44].

If one of the b_k's is zero, then trivially the system is not controllable (the associated dynamics for $\delta\psi_k$ is independent from the control u). Now, let us assume that all b_k's are non-zero and set

$$x = -\frac{\Im(\delta\psi_1)}{b_1}, \qquad z_k = \frac{\delta\psi_k}{b_k} \quad k = 2, \ldots, n.$$

Then (3.6) reads

$$\frac{d}{dt}x = u, \qquad i\frac{d}{dt}z_k = \omega_k z_k + u \quad k = 2, \dots, n. \tag{3.7}$$

If for some $k_1, k_2 \in \{1, \dots, n\}$ with $k_1 < k_2$, $\omega_{k_1} = \pm\omega_{k_2}$, then

- for $k_1 = 0$, $\omega_{k_2} = 0$ and the variable $\xi = iz_{k_2} - x$ satisfies $\frac{d}{dt}\xi = 0$;
- for $k_1 > 0$ and $\omega_{k_1} = \omega_{k_2}$, the variable $\xi = z_{k_1} - z_{k_2}$ satisfies $i\frac{d}{dt}\xi = \omega_{k_1}\xi$;
- for $k_1 > 0$ and $\omega_{k_1} = -\omega_{k_2}$,the variable $\xi = z_{k_1} + z_{k_2}^*$ satisfies $i\frac{d}{dt}\xi = \omega_{k_1}\xi$.

Thus in any of these 3 cases, $|\xi|$ is independent of t and the system is not controllable. We have proved that $b_k \neq 0$ and $|\omega_{k_1}| \neq |\omega_{k_2}|$ are necessary controllability conditions.

Let us prove now that these conditions are sufficient. Denote by $(x^0, z_1^0, \dots, z_n^0)$ and $(x^T, z_1^T, \dots, z_n^T)$ the initial and final states. The general solution of the under-determined system (3.7) reads

$$\begin{aligned}
u &= \tfrac{d}{dt}\left(\tfrac{d^2}{dt^2} + \omega_2^2\right)\dots\left(\tfrac{d^2}{dt^2} + \omega_n^2\right) y \\
x &= \left(\tfrac{d^2}{dt^2} + \omega_2^2\right)\dots\left(\tfrac{d^2}{dt^2} + \omega_n^2\right) y \\
z_2 &= -\tfrac{d}{dt}\left(\tfrac{d^2}{dt^2} + \omega_3^2\right)\dots\left(\tfrac{d^2}{dt^2} + \omega_n^2\right)\left(\omega_2 y + i\tfrac{d}{dt}y\right) \\
&\vdots \\
z_k &= -\tfrac{d}{dt}\left(\tfrac{d^2}{dt^2} + \omega_2^2\right)\dots\left(\tfrac{d^2}{dt^2} + \omega_{k-1}^2\right)\left(\tfrac{d^2}{dt^2} + \omega_{k+1}^2\right)\dots\left(\tfrac{d^2}{dt^2} + \omega_n^2\right)\left(\omega_k y + i\tfrac{d}{dt}y\right) \\
&\vdots \\
z_n &= -\tfrac{d}{dt}\left(\tfrac{d^2}{dt^2} + \omega_2^2\right)\dots\left(\tfrac{d^2}{dt^2} + \omega_{n-1}^2\right)\left(\omega_n y + i\tfrac{d}{dt}y\right)
\end{aligned}$$

where $t \mapsto y(t) \in \mathbb{R}$ is an arbitrary KC^{2n-1} function[1]. Since $\omega_{k_1}^2 \neq \omega_{k_2}^2$ as soon as $k_1 \neq k_2$, the above linear relationship relying $(x, z_2, \dots, z_n)$ to $(y, \frac{dy}{dt}, \dots, \frac{d^{2n}y}{dt^{2n-2}})$ is invertible. Thus y and all its derivative up to order $2n-2$ are also linear combinations of $(x, z_2, \dots, z_n)$: y is called the flat or Brunovsky output of (3.7). The initial and final states set the values of y and its derivatives up to order $2n-2$ at $t = 0$ and $t = T$. For $t \in]0, T[$, the values of y and its derivatives are free, the only constraint being the fact that y is KC^{2n-1}. Trivailly, one can find many KC^{2n-1} functions $y(t)$ with prescribed values of derivatives up to order $2n-2$ at $t = 0$ and $t = T$. Take for example a polynomial of degree $4n-1$ for $y(t)$. Then a (polynomial) control steering from the initial to the final states is given by

$$u = \tfrac{d}{dt}\left(\tfrac{d^2}{dt^2} + \omega_2^2\right)\dots\left(\tfrac{d^2}{dt^2} + \omega_n^2\right) y.$$

□

[1] A KC^m function f is a function that is continuously differentiable up to order $m-1$, its derivatives of order $m-1$, $f^{(m-1]}$, is piecewise differentiable and $f^{(m)}$ is piecewise continuous.

Adaptations of the formula used in the proof to $b_1 = 0$ are possible as shown by the exercise below

Exercice 3.2.2. *Take (3.6) with $n = 3$, $b_1 = 0$, $b_2 = b_3 = 1$ and $0 < \omega_2 < \omega_3$.*

1. *Assume that $\delta\psi_2(t)$, $\delta\psi_3(t)$ and $u(t)$ are defined by*

$$\delta\psi_2 = -\left(\tfrac{d^2}{dt^2} + \omega_3^2\right)\left(\omega_2 y + i\tfrac{d}{dt}y\right)$$
$$\delta\psi_3 = -\left(\tfrac{d^2}{dt^2} + \omega_2^2\right)\left(\omega_3 y + i\tfrac{d}{dt}y\right)$$
$$u = \left(\tfrac{d^2}{dt^2} + \omega_2^2\right)\left(\tfrac{d^2}{dt^2} + \omega_3^2\right) y$$

   with y a KC^4 time function. Prove that they automatically satisfy

$$i\frac{d}{dt}\delta\psi_2 = \omega_2\delta\psi_2 + u, \quad i\frac{d}{dt}\delta\psi_3 = \omega_3\delta\psi_3 + u.$$

2. *Express y and $\frac{d^2y}{dt^2}$ as linear combinations of $\Re(\delta\psi_2)$ and $\Re(\delta\psi_3)$,*

3. *Express $\frac{dy}{dt}$ and $\frac{d^3y}{dt^3}$ as linear combinations of $\Im(\delta\psi_2)$ and $\Im(\delta\psi_3)$.*

4. *Take $T > 0$ and an initial state $(\delta\psi_2, \delta\psi_3)_{t=0} = \alpha \in \mathbb{C}^2$. Construct for any final state $(\delta\psi_2, \delta\psi_3)_{t=T} = \beta \in \mathbb{C}^2$ an open-loop steering control $[0, T] \ni t \mapsto u(t) \in \mathbb{R}$ (hint: use for example a polynomial of degree 7 for y).*

Exercice 3.2.3. *Take $T > 0$, $\alpha, \beta \in \mathbb{C}^n$ and consider the linearized system*

$$i\frac{d}{dt}\delta\psi_k = \omega_k\delta\psi_k + b_k u, \quad k = 1, \dots, n$$

1. *Prove that it is controllable if, and only if, $b_k \neq 0$ and $|\omega_{k_1}|^2 \neq |\omega_{k_2}|^2$ for $k_1 \neq k_2$.*

2. *Prove that the problem*

$$\min_{\substack{i\frac{d}{dt}\delta\psi_k = \omega_k\delta\psi_k + b_k u,\ k \in \{1,\dots,n\},\ t \in [0,T] \\ (\delta\psi_1, \dots, \delta\psi_n)_{t=0} = \alpha,\ (\delta\psi_1, \dots, \delta\psi_n)_{t=T} = \beta \\ u \in L^2([0,T], \mathbb{R})}} \int_0^T u^2(t)dt$$

   admits a unique solution $u(t) = u_{T,\alpha,\beta}(t)$ that admits the following resonant form:

$$u_{T,a,b}(t) = \sum_{k=2}^{n} C_k(T, \alpha, \beta)\cos(\omega_k t) + S_k(T, \alpha, \beta)\sin(\omega_k t)$$

   where C_k and S_k are smooth functions of their arguments.

3. *Show that C_k and S_k admit the following asymptotic expansion for T large:*

$$C_k + iS_k = \frac{\beta_k e^{i\omega_k T} - \alpha_k}{2ib_k T} + o(\tfrac{1}{T}).$$

   Deduce that $\lim_{T\mapsto+\infty} \int_0^T (u_{T,a,b}(t))^2 dt = 0$.

3.3 Resonant control, rotating wave approximation

3.3.1 Multi-frequency averaging

Let us consider the finite dimensional system (3.1) with only one control $m = 1$ and the skew-Hermitian matrices $A_k = -iH_k$, $k = 0, 1$. Assume that the single scalar control is of small amplitude and admits an almost periodic time-dependence

$$u(t) = \epsilon \left(\sum_{j=1}^{r} \mathbf{u}_j e^{i\omega_j t} + \mathbf{u}_j^* e^{-i\omega_j t} \right) \tag{3.8}$$

where $\epsilon > 0$ is a small parameter, $\epsilon \mathbf{u}_j$ is the constant complex amplitude associated to the pulsation $\omega_j \geq 0$ and r stands for the number of independent pulsations ($\omega_j \neq \omega_k$ for $j \neq k$). We are interested in approximations, for ϵ tending to 0^+, of trajectories $t \mapsto |\psi_\epsilon\rangle_t$ of (3.1). Such approximations should be explicit and valid on time intervals of length $O(\frac{1}{\epsilon})$ (first order approximation) or $O(\frac{1}{\epsilon^2})$ (second order approximation). The wave function $|\psi_\epsilon\rangle$ obeys the following linear time-varying differential equation

$$\frac{d}{dt} |\psi_\epsilon\rangle = \left(A_0 + \epsilon \left(\sum_{j=1}^{r} \mathbf{u}_j e^{i\omega_j t} + \mathbf{u}_j^* e^{-i\omega_j t} \right) A_1 \right) |\psi_\epsilon\rangle \,. \tag{3.9}$$

Consider the following change of variables

$$|\psi_\epsilon\rangle_t = e^{A_0 t} |\phi_\epsilon\rangle_t \tag{3.10}$$

where $|\psi_\epsilon\rangle$ is replaced by $|\phi_\epsilon\rangle$. Through this change of variables, we put the system in the so-called "interaction frame":

$$\frac{d}{dt} |\phi_\epsilon\rangle = \epsilon B(t) |\phi_\epsilon\rangle \tag{3.11}$$

where $B(t)$ is a skew-Hermitian operator whose time-dependence is almost periodic[2]:

$$B(t) = \sum_{j=1}^{r} \mathbf{u}_j e^{i\omega_j t} e^{-A_0 t} A_1 e^{A_0 t} + \mathbf{u}_j^* e^{-i\omega_j t} e^{-A_0 t} A_1 e^{A_0 t}.$$

More precisely each entry of B is a linear combination of oscillating terms of the form $e^{i\omega' t}$ with $\omega' \geq 0$. This results from the spectral decomposition of A_0 to compute $e^{A_0 t}$. Thus one can always decompose $B(t)$ into a constant skew-Hermitian operator $\bar{B}$ and the time derivative of a bounded and almost periodic skew-Hermitian operator $\widetilde{B}(t)$ whose entries are linear combinations of $e^{i\omega' t}$ with $\omega' > 0$:

$$B(t) = \bar{B} + \frac{d}{dt} \widetilde{B}(t). \tag{3.12}$$

[2] An almost periodic time function f is equal by definition to $F(\varpi_1 t, \ldots, \varpi_p t)$ where the function F is a 2π-periodic function of each of its p arguments and the ϖ_j's form a set of p different pulsations.

Notice that we can always set $\widetilde{B}(t) = \frac{d}{dt}\widetilde{C}(t)$ where $\widetilde{C}$ is also an almost periodic skew-Hermitian operator. Then (3.11) reads $\frac{d}{dt}|\phi_\epsilon\rangle = \left(\epsilon\bar{B} + \epsilon\frac{d}{dt}\widetilde{B}\right)|\phi_\epsilon\rangle$ and suggests the following almost periodic change of variables

$$|\chi_\epsilon\rangle = (\mathbf{1} - \epsilon\widetilde{B}(t))\,|\phi_\epsilon\rangle \tag{3.13}$$

well defined for ϵ small enough and then close to identity. In the $|\chi_\epsilon\rangle$ frame, the dynamics reads

$$\frac{d}{dt}|\chi_\epsilon\rangle = \epsilon\left(\bar{B} - \epsilon\widetilde{B}\bar{B} - \epsilon\widetilde{B}\frac{d}{dt}\widetilde{B}\right)\left(\mathbf{1} - \epsilon\widetilde{B}\right)^{-1}|\chi_\epsilon\rangle\,.$$

Since $\widetilde{B}(t)$ is almost periodic and $\left(\mathbf{1} - \epsilon\widetilde{B}\right)^{-1} = \mathbf{1} + \epsilon\widetilde{B} + O(\epsilon^2)$, the dynamics of $|\chi_\epsilon\rangle$ reads

$$\frac{d}{dt}|\chi_\epsilon\rangle = \left(\epsilon\bar{B} + \epsilon^2[\bar{B}, \widetilde{B}(t)] - \epsilon^2\widetilde{B}(t)\frac{d}{dt}\widetilde{B}(t) + \epsilon^3 E(\epsilon, t)\right)|\chi_\epsilon\rangle$$

where the operator $E(\epsilon, t)$ is still almost periodic versus t but now its entries are no more linear combinations of time exponentials. The operator $\widetilde{B}(t)\frac{d}{dt}\widetilde{B}(t)$ is an almost periodic operator whose entries are linear combinations of oscillating time exponentials. Thus we have

$$\widetilde{B}(t)\frac{d}{dt}\widetilde{B}(t) = \bar{D} + \frac{d}{dt}\widetilde{D}(t)$$

where $\widetilde{D}(t)$ is almost periodic. With these notations we have

$$\frac{d}{dt}|\chi_\epsilon\rangle = \left(\epsilon\bar{B} - \epsilon^2\bar{D} + \epsilon^2\frac{d}{dt}\left([\bar{B}, \widetilde{C}(t)] - \widetilde{D}(t)\right) + \epsilon^3 E(\epsilon, t)\right)|\chi_\epsilon\rangle \tag{3.14}$$

where the skew-Hermitian operators $\bar{B}$ and $\bar{D}$ are constants and the other ones $\widetilde{C}$, $\widetilde{D}$, and E are almost periodic.

The first order approximation of $|\phi_\epsilon\rangle$ is given by the solution $\left|\phi_\epsilon^{1^{\rm st}}\right\rangle$ of

$$\frac{d}{dt}\left|\phi_\epsilon^{1^{\rm st}}\right\rangle = \epsilon\bar{B}\left|\phi_\epsilon^{1^{\rm st}}\right\rangle \tag{3.15}$$

where $\bar{B}$ can be interpreted as the averaged value of $B(t)$:

$$\bar{B} = \lim_{T\mapsto\infty}\frac{1}{T}\int_0^T B(t)\,dt = \lim_{T\mapsto\infty}\frac{1}{T}\int_0^T\left(\sum_{j=1}^r \mathbf{u}_j e^{i\omega_j t}e^{-A_0 t}A_1 e^{A_0 t} + \mathbf{u}_j^* e^{-i\omega_j t}e^{-A_0 t}A_1 e^{A_0 t}\right)dt.$$

Approximating $B(t)$ by $\bar{B}$ in (3.11) is called the Rotating Wave Approximation (RWA). The second order approximation reads then

$$\frac{d}{dt}\left|\phi_\epsilon^{2^{\rm nd}}\right\rangle = (\epsilon\bar{B} - \epsilon^2\bar{D})\left|\phi_\epsilon^{2^{\rm nd}}\right\rangle. \tag{3.16}$$

In (3.15) and (3.16), the operators $\epsilon\bar{B}$ and $\epsilon\bar{B} - \epsilon^2\bar{D}$ are skew-Hermitian: these approximate dynamics remain of Schrödinger type and are thus characterized by the approximate Hamiltonians

$$\bar{H}^{1^{\text{st}}} = i\epsilon\bar{B} \quad \text{and} \quad \bar{H}^{2^{\text{nd}}} = i(\epsilon\bar{B} - \epsilon^2\bar{D}).$$

A precise justification of the rotating wave approximation is given by the following lemma.

Lemma 3.3.1 (First order approximation)**.** *Consider the solution of (3.11) with initial condition $|\phi_\epsilon\rangle_0 = |\phi_a\rangle$ and denote by $\left|\phi_\epsilon^{1^{st}}\right\rangle$ the solution of (3.15) with the same initial condition, $\left|\phi_\epsilon^{1^{st}}\right\rangle_0 = |\phi_a\rangle$. Then, there exist $M > 0$ and $\eta > 0$ such that for all $\epsilon \in]0, \eta[$ we have*

$$\max_{t\in\left[0,\frac{1}{\epsilon}\right]} \left\| |\phi_\epsilon\rangle_t - \left|\phi_\epsilon^{1^{st}}\right\rangle_t \right\| \leq M\epsilon$$

Proof. Denote by $|\chi_\epsilon\rangle$ the solution of (3.14) with $|\chi_\epsilon\rangle_0 = (\mathbf{1} - \epsilon\widetilde{B}(0))\,|\phi_a\rangle$. According to (3.13), there exist $M_1 > 0$ and $\eta_1 > 0$, such that for all $\epsilon \in]0, \eta_1]$ and $t > 0$ we have $\||\chi_\epsilon\rangle_t - |\phi_\epsilon\rangle_t\| \leq M_1\epsilon$. But (3.14) admits the following form $\frac{d}{dt}|\chi_\epsilon\rangle = \left(\epsilon\bar{B} + \epsilon^2 F(t)\right)|\chi_\epsilon\rangle$ where the operator $F(t)$ is uniformly bounded versus t. Thus, exist $M_2 > 0$ and $\eta_2 > 0$ such that the solution $\left|\varphi_\epsilon^{1^{\text{st}}}\right\rangle$ of (3.16) with initial condition $(\mathbf{1} - \epsilon\widetilde{B}(0))\,|\phi_a\rangle$ satisfies, for all $\epsilon \in]0, \eta_2]$,

$$\max_{t\in\left[0,\frac{1}{\epsilon}\right]} \left\| \left|\varphi_\epsilon^{1^{\text{st}}}\right\rangle_t - |\chi_\epsilon\rangle_t \right\| \leq M_2\epsilon.$$

The propagator of (3.15) is unitary and thus

$$\left\| \left|\varphi_\epsilon^{1^{\text{st}}}\right\rangle_t - \left|\phi_\epsilon^{1^{\text{st}}}\right\rangle_t \right\| = \left\| \left|\varphi_\epsilon^{1^{\text{st}}}\right\rangle_0 - \left|\phi_\epsilon^{1^{\text{st}}}\right\rangle_0 \right\| = \epsilon \left\| \widetilde{B}(0)\,|\phi_a\rangle \right\|.$$

We conclude with the triangular inequality

$$\left\| |\phi_\epsilon\rangle_t - \left|\phi_\epsilon^{1^{\text{st}}}\right\rangle_t \right\| \leq \||\phi_\epsilon\rangle_t - |\chi_\epsilon\rangle_t\| + \left\| |\chi_\epsilon\rangle_t - \left|\varphi_\epsilon^{1^{\text{st}}}\right\rangle_t \right\| + \left\| \left|\varphi_\epsilon^{1^{\text{st}}}\right\rangle_t - \left|\phi_\epsilon^{1^{\text{st}}}\right\rangle_t \right\|.$$

□

The following lemma underlies the second order approximation:

Lemma 3.3.2 (Second order approximation)**.** *Consider the solution of (3.11) with initial condition $|\phi_\epsilon\rangle_0 = |\phi_a\rangle$ and denote by $\left|\phi_\epsilon^{2^{nd}}\right\rangle$ the solution of (3.16) with the same initial condition, $\left|\phi_\epsilon^{2^{nd}}\right\rangle_0 = |\phi_a\rangle$. Then, there exist $M > 0$ and $\eta > 0$ such that for all $\epsilon \in]0, \eta[$ we have*

$$\max_{t\in\left[0,\frac{1}{\epsilon^2}\right]} \left\| |\phi_\epsilon\rangle - \left|\phi_\epsilon^{2^{nd}}\right\rangle \right\|^2 \leq M\epsilon$$

Proof. As for the proof of lemma 3.3.1, we introduce $|\chi_\epsilon\rangle$, $\left|\varphi_\epsilon^{2^{\rm nd}}\right\rangle$ solution of (3.16) starting from $\left|\varphi_\epsilon^{2^{\rm nd}}\right\rangle_0 = (\mathbf{1} - \epsilon\widetilde{B}(0))\,|\phi_a\rangle$. Using similar arguments, it is then enough to prove that exit $M_3, \eta_3 > 0$ such that, for all $\epsilon \in]0, \eta_3[$, $\max_{t\in\left[0,\frac{1}{\epsilon}\right]} \left\| \left|\varphi_\epsilon^{2^{\rm nd}}\right\rangle_t - |\chi_\epsilon\rangle_t \right\| \leq M_3\epsilon$. This estimate is a direct consequence of the almost periodic change of variables

$$|\xi_\epsilon\rangle = \left(\mathbf{1} - \epsilon^2\left([\bar{B}, \widetilde{C}(t)] - \widetilde{D}(t)\right)\right)|\chi_\epsilon\rangle$$

that transforms (3.14) into

$$\frac{d}{dt}|\xi_\epsilon\rangle = \left(\epsilon\bar{B} - \epsilon^2\bar{D} + \epsilon^3 F(\epsilon, t)\right)|\xi_\epsilon\rangle$$

where F is almost periodic. This cancels the oscillating operator $\epsilon^2\frac{d}{dt}\left([\bar{B}, \widetilde{C}(t)] - \widetilde{D}(t)\right)$ appearing in (3.14): the equation satisfied by $|\xi_\epsilon\rangle$ and the second order approximation (3.16) differ only by third order almost periodic operator $\epsilon^3 F(\epsilon, t)$. □

Exercice 3.3.3. *The goal is to prove that, even if the amplitudes $\mathbf{u}_j$ are slowly varying, i.e., $\mathbf{u}_j = \mathbf{u}_j(\epsilon t)$ where $\tau \mapsto \mathbf{u}_j(\tau)$ is continuously differentiable, the first and second order approximations remain valid. We have then two time-dependancies for*

$$B(t,\tau) = \sum_{j=1}^{r} \mathbf{u}_j(\tau)e^{i\omega_j t}e^{-A_0 t}A_1 e^{A_0 t} + \mathbf{u}_j^*(\tau)e^{-i\omega_j t}e^{-A_0 t}A_1 e^{A_0 t}$$

with $\tau = \epsilon t$. Then $\frac{d}{dt}B = \frac{\partial B}{\partial t} + \epsilon\frac{\partial B}{\partial \tau}$.

1. *Extend the decomposition (3.12) to*

$$B(t,\tau) = \bar{B}(\tau) + \frac{\partial \widetilde{B}}{\partial t}(t,\tau)$$

where $\widetilde{B}(t,\tau)$ is t-almost periodic with zero mean in t (τ is fixed here).

2. *Show that the approximation lemma 3.3.1 is still valid where (3.15) is replaced by*

$$\frac{d}{dt}\left|\phi_\epsilon^{1^{st}}\right\rangle = \epsilon\bar{B}(\epsilon t)\left|\phi_\epsilon^{1^{st}}\right\rangle$$

3. *Show that the approximation lemma 3.3.2 is still valid where (3.16) is replaced by*

$$\frac{d}{dt}\left|\phi_\epsilon^{2^{nd}}\right\rangle = (\epsilon\bar{B}(\epsilon t) - \epsilon^2\bar{D}(\epsilon t))\left|\phi_\epsilon^{2^{nd}}\right\rangle$$

and where $\widetilde{B}(t,\tau)\frac{\partial \widetilde{B}}{\partial t}(t,\tau) = \bar{D}(\tau) + \frac{\partial \widetilde{D}}{\partial t}(t,\tau)$ with $\widetilde{D}(t,\tau)$ almost periodic versus t and with zero t-mean.

4. *Extend the above approximation lemma when $\tau \mapsto \mathbf{u}_j(\tau)$ is piecewise continuous and, on each interval where it remains continuous, it is also continuously differentiable ($\tau \mapsto \mathbf{u}_j(\tau)$ is made by the concatenation of continuously differentiable functions).*

3.3.2 The approximation recipes

Such first order and second order approximations extend without any difficulties to the case of m scalar oscillating controls in (3.1). They can be summarized as follows (without introducing the small parameter ϵ and the skew-Hermitian operators A_k). Consider the controlled Hamiltonian associated to $|\psi\rangle$

$$H = H_0 + \sum_{k=1}^{m} u_k H_k \tag{3.17}$$

with m oscillating real controls

$$u_k(t) = \sum_{j=1}^{r} \mathbf{u}_{k,j} e^{\omega_j t} + \mathbf{u}_{k,j}^* e^{-\omega_j t}$$

where $\mathbf{u}_{k,j}$ is the slowly varying complex amplitude associated to control number k and pulsation ω_j. In the sequel, all the computations are done assuming $\mathbf{u}_{k,j}$ constant. Nevertheless, the obtained approximate Hamiltionians given in (3.19) are also valid for slowly time-varying amplitudes.[3]

The interaction Hamiltonian

$$H_{\text{int}}(t) = \sum_{k,j} \left(\mathbf{u}_{k,j} e^{\omega_j t} + \mathbf{u}_{k,j}^* e^{-\omega_j t}\right) e^{iH_0 t} H_k e^{-iH_0 t} \tag{3.18}$$

is associated to the interaction frame via the unitary transformation $|\phi\rangle = e^{iH_0 t}|\psi\rangle$. It admits the decomposition

$$H_{\text{int}}(t) = H_{\text{rwa}}^{1^{\text{st}}} + \frac{d}{dt} I_{\text{osc}}(t)$$

where $H_{\text{rwa}}^{1^{\text{st}}}$ is the averaged Hamiltonian corresponding to the non-oscillating part of H_{int} (secular part) and I_{osc} is the time integral of the oscillating part. I_{osc} is an almost periodic Hermitian operator whose entries are linear combinations of oscillating time-exponentials. The Rotating Wave Approximation consists in approximating the time-varying Hamiltonian $H_{\text{int}}(t)$ by $H_{\text{rwa}}^{1^{\text{st}}}$. This approximation is valid when the amplitudes $\mathbf{u}_{k,j}$ are small. It is of first order. The second order approximation is then obtained by adding to $H_{\text{rwa}}^{1^{\text{st}}}$ a second order correction made by the averaged part J_{rwa} of the almost periodic Hamiltonian

$$i\left(\frac{d}{dt} I_{\text{osc}}(t)\right) I_{\text{osc}}(t) = J_{\text{rwa}} + \frac{d}{dt} J_{\text{osc}}(t)$$

with J_{osc} almost periodic. Notice J_{rwa} is also Hermitian since $\frac{d}{dt} I_{\text{osc}}^2 = \frac{d}{dt} I_{\text{osc}} I_{\text{osc}} + I_{\text{osc}} \frac{d}{dt} I_{\text{osc}}$. We can summarize these approximations as the following recipes:

$$H_{\text{rwa}}^{1^{\text{st}}} = \overline{H_{\text{int}}}, \quad H_{\text{rwa}}^{2^{\text{nd}}} = H_{\text{rwa}}^{1^{\text{st}}} - i\overline{\left(H_{\text{int}} - \overline{H_{\text{int}}}\right)\left(\int_t (H_{\text{int}} - \overline{H_{\text{int}}})\right)} \tag{3.19}$$

where the over-line means taking the average.

[3] More precisely and according to exercise 3.3.3, we can assume that each $\mathbf{u}_{k,j}$ is of small magnitude, admits a finite number of discontinuities and, between two successive discontinuities, is a slowly time varying function that is continuously differentiable.

3.3.3 2-level systems and Rabi oscillations

Let us consider the controlled 2-level system described by (2.1):

$$i\frac{d}{dt}|\psi\rangle = \left(\tfrac{\omega_{eg}}{2}\sigma_z + \tfrac{u(t)}{2}\sigma_x\right)|\psi\rangle$$

and assume that $u(t) = \mathbf{u}e^{i\omega_r t} + \mathbf{u}^* e^{-i\omega_r t}$ with the complex amplitude $\mathbf{u}$ such that $|\mathbf{u}| \ll \omega_{eg}$ and with pulsation ω_r close to ω_{eg}, i.e., $|\omega_{eg} - \omega_r| \ll \omega_{eg}$. Denote by $\Delta_r = \omega_{eg} - \omega_r$ the detuning between the control and the system then we get the standard form (3.17) with $m = 2$, $H_0 = \frac{\omega_r}{2}\sigma_z$, $u_1 H_1 = \frac{\Delta_r}{2}\sigma_z$ and $u_2 H_2 = \frac{\mathbf{u}e^{i\omega_r t} + \mathbf{u}^* e^{-i\omega_r t}}{2}\sigma_x$ with $\|H_0\|$ much larger than $\|u_1 H_1 + u_2 H_2\|$. A direct computation yields to the following interaction Hamiltonian defined by (3.18):

$$H_{\text{int}} = \frac{\Delta_r}{2}\sigma_z + \frac{\mathbf{u}e^{i\omega_r t} + \mathbf{u}^* e^{-i\omega_r t}}{2} e^{\frac{i\omega_r t}{2}\sigma_z}\sigma_x e^{-\frac{i\omega_r t}{2}\sigma_z}.$$

With the identities $e^{i\theta\sigma_z} = \cos\theta + i\sin\theta\sigma_z$ and $\sigma_z\sigma_x = i\sigma_y$ we get the formula

$$e^{i\theta\sigma_z}\sigma_x e^{-i\theta\sigma_z}\sigma_x = e^{2i\theta}\sigma_+ + e^{-2i\theta}\sigma_-$$

where $\sigma_+ = |e\rangle\langle g| = \frac{\sigma_x + i\sigma_y}{2}$ and $\sigma_- = |g\rangle\langle e| = \frac{\sigma_x - i\sigma_y}{2}$. Thus we have

$$H_{\text{int}} = \tfrac{\Delta_r}{2}\sigma_z + \tfrac{\mathbf{u}e^{2i\omega_r t} + \mathbf{u}^*}{2}\sigma_+ + \tfrac{\mathbf{u}^* e^{-2i\omega_r t} + \mathbf{u}}{2}\sigma_-$$

The decomposition of $H_{\text{int}} = H_{\text{rwa}}^{1\text{st}} + \frac{d}{dt}I_{\text{osc}}$ reads:

$$H_{\text{int}} = \underbrace{\tfrac{\Delta_r}{2}\sigma_z + \tfrac{\mathbf{u}^*}{2}\sigma_+ + \tfrac{\mathbf{u}}{2}\sigma_-}_{H_{\text{rwa}}^{1\text{st}}} + \underbrace{\tfrac{\mathbf{u}e^{2i\omega_r t}}{2}\sigma_+ + \tfrac{\mathbf{u}^* e^{-2i\omega_r t}}{2}\sigma_-}_{\frac{d}{dt}I_{\text{osc}}}.$$

Thus the first order approximation of any solution $|\psi\rangle$ of

$$i\frac{d}{dt}|\psi\rangle = \left(\tfrac{\omega_r + \Delta_r}{2}\sigma_z + \tfrac{\mathbf{u}e^{i\omega_r t} + \mathbf{u}^* e^{-i\omega_r t}}{2}\sigma_x\right)|\psi\rangle$$

is given by $e^{-i\frac{\omega_r t}{2}\sigma_z}|\phi\rangle$ where $|\phi\rangle$ is solution of the linear time-invariant equation

$$i\frac{d}{dt}|\phi\rangle = \left(\tfrac{\Delta_r}{2}\sigma_z + \tfrac{\mathbf{u}^*}{2}\sigma_+ + \tfrac{\mathbf{u}}{2}\sigma_-\right)|\phi\rangle, \quad |\phi\rangle_0 = |\psi\rangle_0. \tag{3.20}$$

According to (3.19) the second order approximation requires the computation of the secular term in $I_{\text{osc}}\frac{d}{dt}I_{\text{osc}}$. Since $I_{\text{osc}} = \frac{\mathbf{u}e^{2i\omega_r t}}{4i\omega_r}\sigma_+ - \frac{\mathbf{u}^* e^{-2i\omega_r t}}{4i\omega_r}\sigma_-$, we have

$$I_{\text{osc}}\frac{d}{dt}I_{\text{osc}} = \tfrac{|\mathbf{u}|^2}{8i\omega_r}\sigma_z$$

where we have also applied $\sigma_+^2 = \sigma_-^2 = 0$ and $\sigma_z = \sigma_+\sigma_- - \sigma_-\sigma_+$. The second order approximation resulting from (3.19) reads:

$$i\frac{d}{dt}|\phi\rangle = \left(\left(\tfrac{\Delta_r}{2} + \tfrac{|\mathbf{u}|^2}{8\omega_r}\right)\sigma_z + \tfrac{\mathbf{u}^*}{2}\sigma_+ + \tfrac{\mathbf{u}}{2}\sigma_-\right)|\phi\rangle, \quad |\phi\rangle_0 = |\psi\rangle_0. \tag{3.21}$$

We observe that (3.20) and (3.21) differ only by a correction of $\frac{|\mathbf{u}|^2}{4\omega_r}$ added to the detuning Δ_r. This correction is called the Bloch-Siegert shift.

Set $\mathbf{u} = \Omega_r e^{i\theta}$ and $\Delta_r' = \Delta_r + \frac{\Omega_r^2}{4\omega_r}$ with $\Omega_r > 0$ and θ real and constant. Then

$$\left(\left(\tfrac{\Delta_r}{2} + \tfrac{|\mathbf{u}|^2}{8\omega_r}\right)\sigma_z + \tfrac{\mathbf{u}^*}{2}\sigma_+ + \tfrac{\mathbf{u}}{2}\sigma_-\right) = \frac{\Omega_r}{2}\left(\cos\theta\sigma_x + \sin\theta\sigma_y\right) + \frac{\Delta_r'}{2}\sigma_z. \tag{3.22}$$

Set

$$\Omega_r' = \sqrt{\left(\Delta_r + \tfrac{\Omega_r^2}{4\omega_r}\right)^2 + \Omega_r^2}, \quad \sigma_r = \frac{\Omega_r\left(\cos\theta\sigma_x + \sin\theta\sigma_y\right) + \Delta_r'\sigma_z}{\Omega_r'}.$$

Then $\sigma_r^2 = \mathbf{1}$ and thus the solution of (3.21),

$$|\phi\rangle_t = e^{-i\frac{\Omega_r' t}{2}\sigma_r}|\phi\rangle_0 = \cos\left(\tfrac{\Omega_r' t}{2}\right)|\phi\rangle_0 - i\sin\left(\tfrac{\Omega_r' t}{2}\right)\sigma_r|\phi\rangle_0,$$

oscillates between $|\phi\rangle_0$ and $-i\sigma_r|\phi\rangle_0$ with the Rabi pulsation $\frac{\Omega_r'}{2}$.

For $\Delta_r = 0$ and neglecting second order terms in Ω_r, we have $\Omega_r' \approx \Omega_r$, $\Delta_r' \approx 0$ and $\sigma_r \approx \cos\theta\sigma_x + \sin\theta\sigma_y$. When $|\phi\rangle_0 = |g\rangle$ we see that, up-to second order terms, $|\phi\rangle_t$ oscillates between $|g\rangle$ and $e^{-i(\theta+\frac{\pi}{2})}|e\rangle$. With $\theta = -\frac{\pi}{2}$, we have

$$|\chi\rangle_t = \cos\left(\tfrac{\Omega_r t}{2}\right)|g\rangle + \sin\left(\tfrac{\Omega_r t}{2}\right)|e\rangle,$$

and we see that, with a constant amplitude $\mathbf{u} = \Omega_r e^{i\eta}$ for $t \in [0, T]$, we have the following transition, depending on the pulse-length $T > 0$:

- if $\Omega_r T = \pi$ then $|\phi\rangle_T = |e\rangle$ and we have a transition between the ground state to the excited one, together with stimulated absorption of a photon of energy ω_{eg}. If we measure the energy in the final state we always find E_e. This is a π-pulse in reference to the Bloch sphere interpretation of (3.21) (see sub-section 2.1.2).
- if $\Omega_r T = \frac{\pi}{2}$ then $|\phi\rangle_T = (|g\rangle + |e\rangle)/\sqrt{2}$ and the final state is a coherent superposition of $|g\rangle$ and $|e\rangle$. A measure of the energy of the final state yields either E_g or E_e with a probability of $1/2$ for both E_g and E_e. This is a $\frac{\pi}{2}$-pulse.

Since $|\psi\rangle = e^{-\frac{i\omega_r t}{2}\sigma_z}|\phi\rangle$, we see that a π-pulse transfers $|\psi\rangle$ from $|g\rangle$ at $t = 0$ to $e^{i\alpha}|e\rangle$ at $t = T = \frac{\pi}{\Omega_r}$ where the phase $\alpha \approx \frac{\omega_r}{\Omega_r}\pi$ is very large since $\Omega_r \ll \omega_r$. Similarly, a $\frac{\pi}{2}$-pulse, transfers $|\psi\rangle$ from $|g\rangle$ at $t = 0$ to $\frac{e^{-i\alpha}|g\rangle + e^{i\alpha}|e\rangle}{\sqrt{2}}$ at $t = T = \frac{\pi}{2\Omega_r}$ with a very large relative half-phase $\alpha \approx \frac{\omega_r}{2\Omega_r}\pi$. Thus, this kind of pulses is well adapted when the initial state, $|\psi\rangle_0$, and final state, $|\psi\rangle_T$, are characterized by $|\langle\psi|g\rangle|^2$ and $|\langle\psi|e\rangle|^2$ where these large phases disappear. One speak then of populations since $|\langle\psi|g\rangle|^2$ (resp. $|\langle\psi|e\rangle|^2$) is the probability to find E_g (resp. E_e) when we measure the energy of the isolated system $H_0 = E_g|g\rangle\langle g| + E_e|e\rangle\langle e|$.

Exercice 3.3.4. *Take the first order approximation (3.20) with $\Delta_r = 0$ and $\mathbf{u} \in \mathbb{C}$ as control.*

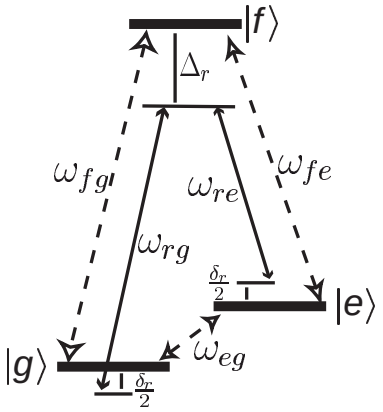

Figure 3.1: Raman transition for a Λ-level system ($\delta_r < 0$ and $\Delta_r > 0$ on the figure).

1. *Set* $\Theta_r = \frac{\Omega_r}{2}T$. *Show that the solution at* T *of the propagator* $U_t \in SU(2)$, $i\frac{d}{dt}U = \frac{\Omega_r(\cos\theta\sigma_x + \sin\theta\sigma_y)}{2}U$, $U_0 = \mathbf{1}$ *is given by*

$$U_T = \cos\Theta_r \mathbf{1} - i\sin\Theta_r\left(\cos\theta\sigma_x + \sin\theta\sigma_y\right),$$

2. *Take a wave function* $\left|\bar{\phi}\right\rangle$. *Show that exist* Ω_r *and* θ *such that* $U_T\left|g\right\rangle = e^{i\alpha}\left|\bar{\phi}\right\rangle$, *where* α *is some global phase.*

3. *Prove that for any given two wave functions* $\left|\phi_a\right\rangle$ *and* $\left|\phi_b\right\rangle$ *exists a piece-wise constant control* $[0, 2T] \ni t \mapsto \mathbf{u}(t) \in \mathbb{C}$ *such that the solution of* (3.20) *with* $\left|\phi\right\rangle_0 = \left|\phi_a\right\rangle$ *and* $\Delta_r = 0$ *satisfies* $\left|\phi\right\rangle_T = e^{i\beta}\left|\psi_b\right\rangle$ *for some global phase* β.

4. *Generalize the above question when* $\left|\phi\right\rangle$ *obeys the second order approximation* (3.21) *with* Δ_r *as additional control.*

3.3.4 Λ-systems and Raman transition

This transition strategy is used for Λ-systems (see section 2.5) where the additional sate $|f\rangle$ admits an energy E_f much greater than E_g and E_e. However, we will see that the averaged Hamiltonian is very similar to the one describing Rabi oscillations and the state $|f\rangle$ can be ignored. The transition from $|g\rangle$ to $|e\rangle$ is no more performed via a quasi-resonant control with a single frequency close to $\omega_{eg} = E_e - E_g$, but with a control based on two frequencies ω_{rg} and ω_{re}, in a neighborhood of $\omega_{fg} = E_f - E_g$ and $\omega_{fe} = E_f - E_e$, with $\omega_{rg} - \omega_{re}$ close to ω_{eg}. Such transitions result from a nonlinear phenomena and second order perturbations. The main practical advantage comes from the fact that ω_{re} and ω_{rg} are in many examples optical frequencies (around 10^{15} rad/s) whereas ω_{eg} is a radio frequency (around 10^{10} rad/s). The wave length of the laser generating u is around 1 μm and thus spacial resolution is much better with optical waves than with radio-frequency ones.

Take the 3-level system ($|g\rangle$, $|e\rangle$ and $|f\rangle$ of energy E_g, E_e and E_f) of figure 3.1. The atomic pulsations are denoted as follows:

$$\omega_{fg} = E_f - E_g,\ \omega_{fe} = E_f - E_e,\ \omega_{eg} = E_e - E_g.$$

We assume a Hamiltonian of the form

$$H = E_g |g\rangle\langle g| + E_e |e\rangle\langle e| + E_f |f\rangle\langle f| + \frac{u}{2}\Big(\mu_g(|g\rangle\langle f| + |f\rangle\langle g|) + \mu_e(|e\rangle\langle f| + |f\rangle\langle e|)\Big)$$

where μ_g and μ_e are coupling coefficients with the electromagnetic field described by $u(t)$, a quasi-resonant control defined by the constant complex amplitudes $\mathbf{u}_g$ and $\mathbf{u}_e$,

$$u = \mathbf{u}_g e^{i\omega_{rg}t} + \mathbf{u}_g^* e^{-i\omega_{rg}t} + \mathbf{u}_e e^{i\omega_{re}t} + \mathbf{u}_e^* e^{-i\omega_{re}t}$$

and where the pulsation ω_{rg} and ω_{re} are close to ω_{fg} and ω_{fe}. According to Figure 3.1 set

$$\omega_{fg} = \omega_{rg} + \Delta_r - \tfrac{\delta_r}{2}, \quad \omega_{fe} = \omega_{re} + \Delta_r + \tfrac{\delta_r}{2},$$

and assume that

$$\Big(\max(|\mu_g|,|\mu_e|)\max(|\mathbf{u}_g|,|\mathbf{u}_e|)\Big) \text{ and } |\delta_r| \ll \min\left(\omega_{rg}, \omega_{re}, \omega_{fg}, \omega_{fe}, |\Delta_r|, |\omega_{re} - \omega_{rg} + \Delta_r|, |\omega_{re} - \omega_{rg} - \Delta_r|\right).$$

In the interaction frame (passage from $|\psi\rangle$ where $i\frac{d}{dt}|\psi\rangle = H(t)|\psi\rangle$ to $|\phi\rangle$),

$$|\psi\rangle = \Big(e^{-i\left(E_g + \frac{\delta_r}{2}\right)t} |g\rangle\langle g| + e^{-i\left(E_e - \frac{\delta_r}{2}\right)t} |e\rangle\langle e| + e^{-iE_f t} |f\rangle\langle f|\Big) |\phi\rangle$$

the Hamiltonian becomes ($i\frac{d}{dt}|\phi\rangle = H_{\text{int}}(t)|\phi\rangle$):

$$\begin{aligned} H_{\text{int}} &= \tfrac{\delta_r}{2}(|e\rangle\langle e| - |g\rangle\langle g|) \\ &+ \mu_g \left(\mathbf{u}_g e^{i\omega_{rg}t} + \mathbf{u}_e e^{i\omega_{re}t} + \mathbf{u}_g^* e^{-i\omega_{rg}t} + \mathbf{u}_e^* e^{-i\omega_{re}t}\right)\left(e^{i(\omega_{rg}+\Delta_r)t}|g\rangle\langle f| + e^{-i(\omega_{rg}+\Delta_r)t}|f\rangle\langle g|\right) \\ &+ \mu_e \left(\mathbf{u}_g e^{i\omega_{rg}t} + \mathbf{u}_e e^{i\omega_{re}t} + \mathbf{u}_g^* e^{-i\omega_{rg}t} + \mathbf{u}_e^* e^{-i\omega_{re}t}\right)\left(e^{i(\omega_{re}+\Delta_r)t}|e\rangle\langle f| + e^{-i(\omega_{re}+\Delta_r)t}|f\rangle\langle e|\right). \end{aligned}$$

It is clear from (3.19), that $H^{1^{\text{st}}}_{\text{rwa}} = \frac{\delta_r}{2}(|e\rangle\langle e| - |g\rangle\langle g|)$ and thus second order terms should be considered and $H^{2^{\text{nd}}}_{\text{rwa}}$ has to be computed for a meaning full approximation. Simple but tedious computations show that $\int(H_{\text{int}} - H^{1^{\text{st}}}_{\text{rwa}})$ (the time primitive of zero mean) is given by

$$\begin{aligned} &\frac{\mu_g}{2}\left(\frac{\mathbf{u}_g e^{i(2\omega_{rg}+\Delta_r)t}}{i(2\omega_{rg}+\Delta_r)} + \frac{\mathbf{u}_e e^{i(\omega_{rg}+\omega_{re}+\Delta_r)t}}{i(\omega_{rg}+\omega_{re}+\Delta_r)} + \frac{\mathbf{u}_g^* e^{i\Delta_r t}}{i\Delta_r} + \frac{\mathbf{u}_e^* e^{i(\omega_{rg}-\omega_{re}+\Delta_r)t}}{i(\omega_{rg}-\omega_{re}+\Delta_r)}\right)|g\rangle\langle f| \\ &\quad + \frac{\mu_e}{2}\left(\frac{\mathbf{u}_g e^{i(\omega_{rg}+\omega_{re}+\Delta_r)t}}{i(\omega_{rg}+\omega_{re}+\Delta_r)} + \frac{\mathbf{u}_e e^{i(2\omega_{re}+\Delta_r)t}}{i(2\omega_{re}+\Delta_r)} + \frac{\mathbf{u}_g^* e^{i(\omega_{re}-\omega_{rg}+\Delta_r)t}}{i(\omega_{re}-\omega_{rg}+\Delta_r)} + \frac{\mathbf{u}_e^* e^{i\Delta_r t}}{i\Delta_r}\right)|e\rangle\langle f| \\ &\quad - \frac{\mu_g}{2}\left(\frac{\mathbf{u}_g^* e^{-i(2\omega_{rg}+\Delta_r)t}}{i(2\omega_{rg}+\Delta_r)} + \frac{\mathbf{u}_e^* e^{-i(\omega_{rg}+\omega_{re}+\Delta_r)t}}{i(\omega_{rg}+\omega_{re}+\Delta_r)} + \frac{\mathbf{u}_g e^{-i\Delta_r t}}{i\Delta_r} + \frac{\mathbf{u}_e e^{-i(\omega_{rg}-\omega_{re}+\Delta_r)t}}{i(\omega_{rg}-\omega_{re}+\Delta_r)}\right)|f\rangle\langle g| \\ &\quad - \frac{\mu_e}{2}\left(\frac{\mathbf{u}_g^* e^{-i(\omega_{rg}+\omega_{re}+\Delta_r)t}}{i(\omega_{rg}+\omega_{re}+\Delta_r)} + \frac{\mathbf{u}_e^* e^{-i(2\omega_{re}+\Delta_r)t}}{i(2\omega_{re}+\Delta_r)} + \frac{\mathbf{u}_g e^{-i(\omega_{re}-\omega_{rg}+\Delta_r)t}}{i(\omega_{re}-\omega_{rg}+\Delta_r)} + \frac{\mathbf{u}_e e^{-i\Delta_r t}}{i\Delta_r}\right)|f\rangle\langle e|. \end{aligned}$$

The non oscillating terms of $i\left(\int_t\left(H_{\rm int}-H_{\rm rwa}^{\rm 1st}\right)\right)\left(H_{\rm int}-H_{\rm rwa}^{\rm 1st}\right)$ are then given by simple but tedious computations:

$$\begin{aligned}H_{\rm rwa}^{\rm 2nd} &= \tfrac{\mu_g\mu_e}{4}\left(\tfrac{1}{\omega_{rg}+\omega_{re}+\Delta_r}+\tfrac{1}{\Delta_r}\right)\left(\mathbf{u}_g^*\mathbf{u}_e\left|g\right\rangle\left\langle e\right|+\mathbf{u}_g\mathbf{u}_e^*\left|e\right\rangle\left\langle g\right|\right)+\tfrac{\delta_r}{2}\left(\left|e\right\rangle\left\langle e\right|-\left|g\right\rangle\left\langle g\right|\right)\\ &+\tfrac{\mu_g^2}{4}\left(\tfrac{|\mathbf{u}_g|^2}{2\omega_{rg}+\Delta_r}+\tfrac{|\mathbf{u}_g|^2}{\Delta_r}+\tfrac{|\mathbf{u}_e|^2}{\omega_{rg}-\omega_{re}+\Delta_r}\right)\left|g\right\rangle\left\langle g\right|+\tfrac{\mu_e^2}{4}\left(\tfrac{|\mathbf{u}_e|^2}{2\omega_{re}+\Delta_r}+\tfrac{|\mathbf{u}_e|^2}{\Delta_r}+\tfrac{|\mathbf{u}_g|^2}{\omega_{re}-\omega_{rg}+\Delta_r}\right)\left|e\right\rangle\left\langle e\right|\\ &-\tfrac{1}{4}\left(\tfrac{\mu_g^2|\mathbf{u}_g|^2}{2\omega_{rg}+\Delta_r}+\tfrac{\mu_e^2|\mathbf{u}_e|^2}{2\omega_{re}+\Delta_r}+\tfrac{\mu_g^2|\mathbf{u}_g|^2+\mu_e^2|\mathbf{u}_e|^2}{\omega_{rg}+\omega_{re}+\Delta_r}+\tfrac{\mu_g^2|\mathbf{u}_g|^2+\mu_e^2|\mathbf{u}_e|^2}{\Delta_r}+\tfrac{\mu_g^2|\mathbf{u}_g|^2}{\omega_{re}-\omega_{rg}+\Delta_r}+\tfrac{\mu_e^2|\mathbf{u}_e|^2}{\omega_{rg}-\omega_{re}+\Delta_r}\right)\left|f\right\rangle\left\langle f\right|.\end{aligned} \tag{3.23}$$

This expression simplfies if we assume additionnally that

$$|\Delta_r|,\ |\omega_{re}-\omega_{rg}+\Delta_r|,\ |\omega_{re}-\omega_{rg}-\Delta_r| \ll \omega_{rg},\ \omega_{re},\ \omega_{fg},\ \omega_{fe}.$$

With these additional assumptions we have 3 time-scales:

1. The slow one associated to δ_r, $\mu_g|\mathbf{u}_g|$, $\mu_g|\mathbf{u}_e|$, $\mu_e|\mathbf{u}_g|$ and $\mu_e|\mathbf{u}_e|$
2. The intermediate one attached to Δ_r, $|\omega_{re}-\omega_{rg}+\Delta_r|$ and $|\omega_{re}-\omega_{rg}-\Delta_r|$
3. The fast one related to ω_{rg}, ω_{re}, ω_{fg} and ω_{fe}.

We have then the following approximation of the average Hamiltonian

$$\begin{aligned}H_{\rm rwa}^{\rm 2nd} &\approx \tfrac{\mu_g\mu_e\mathbf{u}_g^*\mathbf{u}_e}{4\Delta_r}\left|g\right\rangle\left\langle e\right|+\tfrac{\mu_g\mu_e\mathbf{u}_g\mathbf{u}_e^*}{4\Delta_r}\left|e\right\rangle\left\langle g\right|+\tfrac{\delta_r}{2}\left(\left|e\right\rangle\left\langle e\right|-\left|g\right\rangle\left\langle g\right|\right)\\ &+\tfrac{\mu_g^2}{4}\left(\tfrac{|\mathbf{u}_g|^2}{\Delta_r}+\tfrac{|\mathbf{u}_e|^2}{\omega_{rg}-\omega_{re}+\Delta_r}\right)\left|g\right\rangle\left\langle g\right|+\tfrac{\mu_e^2}{4}\left(\tfrac{|\mathbf{u}_e|^2}{\Delta_r}+\tfrac{|\mathbf{u}_g|^2}{\omega_{re}-\omega_{rg}+\Delta_r}\right)\left|e\right\rangle\left\langle e\right|\\ &-\tfrac{1}{4}\left(\tfrac{\mu_g^2|\mathbf{u}_g|^2+\mu_e^2|\mathbf{u}_e|^2}{\Delta_r}+\tfrac{\mu_g^2|\mathbf{u}_g|^2}{\omega_{re}-\omega_{rg}+\Delta_r}+\tfrac{\mu_e^2|\mathbf{u}_e|^2}{\omega_{rg}-\omega_{re}+\Delta_r}\right)\left|f\right\rangle\left\langle f\right|.\end{aligned}$$

If $\langle\phi|f\rangle_0 = 0$ then $\langle\phi|f\rangle_t = 0$ up to third order terms: the space $\mathrm{span}\{|g\rangle, |e\rangle\}$ and $\mathrm{span}\{|f\rangle\}$ are invariant space of $H_{\rm rwa}^{\rm 2nd}$. Thus, if the initial state belongs to $\mathrm{span}\{|g\rangle, |e\rangle\}$, we can forget the $|f\rangle\langle f|$ term in $H_{\rm rwa}^{\rm 2nd}$ (restriction of the dynamics to this invariant sub-space) and we get a 2-level Hamiltonian, called Raman Hamiltonian, that lives on $\mathrm{span}\{|g\rangle, |e\rangle\}$:

$$\begin{aligned}H_{\rm Raman} &= \tfrac{\mu_g\mu_e\mathbf{u}_g^*\mathbf{u}_e}{4\Delta_r}\left|g\right\rangle\left\langle e\right|+\tfrac{\mu_g\mu_e\mathbf{u}_g\mathbf{u}_e^*}{4\Delta_r}\left|e\right\rangle\left\langle g\right|+\tfrac{\delta_r}{2}\left(\left|e\right\rangle\left\langle e\right|-\left|g\right\rangle\left\langle g\right|\right)\\ &+\tfrac{\mu_g^2}{4}\left(\tfrac{|\mathbf{u}_g|^2}{\Delta_r}+\tfrac{|\mathbf{u}_e|^2}{\omega_{rg}-\omega_{re}+\Delta_r}\right)\left|g\right\rangle\left\langle g\right|+\tfrac{\mu_e^2}{4}\left(\tfrac{|\mathbf{u}_e|^2}{\Delta_r}+\tfrac{|\mathbf{u}_g|^2}{\omega_{re}-\omega_{rg}+\Delta_r}\right)\left|e\right\rangle\left\langle e\right|.\end{aligned} \tag{3.24}$$

that is similar (up to a global phase shift) to the average Hamiltonian underlying Rabi oscillations (3.22) with

$$\begin{aligned}\Delta_r' &= \delta_r+\tfrac{\mu_e^2}{4}\left(\tfrac{|\mathbf{u}_e|^2}{\Delta_r}+\tfrac{|\mathbf{u}_g|^2}{\omega_{re}-\omega_{rg}+\Delta_r}\right)-\tfrac{\mu_g^2}{4}\left(\tfrac{|\mathbf{u}_g|^2}{\Delta_r}+\tfrac{|\mathbf{u}_e|^2}{\omega_{rg}-\omega_{re}+\Delta_r}\right)\\ \Omega_r e^{i\theta} &= \tfrac{\mu_g\mu_e\mathbf{u}_g^*\mathbf{u}_e}{2\Delta_r}\end{aligned}$$

During such Raman pulses, the intermediate state $|f\rangle$ remains almost empty (i.e. $\langle\psi|f\rangle \approx 0$) and thus, as physicists say, the life time of $|f\rangle$ does not require to be long. This point should be studied in more details: in parallel to the three existing time-scales, we have to consider Γ, the inverse of the life time of $|f\rangle$; it seems, but we do not find any precise justification, that, if Γ and Δ_r are of same magnitude order, the approximations remain valid and there is no need to consider the instability of $|f\rangle$. This could also be true even if $|\Delta_r| \ll \Gamma \ll \omega_{fg}, \omega_{fe}$.

To tackle such questions, one has to consider non-conservative dynamics for $|\psi\rangle$ and to take into account decoherence effects due to the coupling of $|f\rangle$ with the environment, coupling leading to a finite life-time. The incorporation into the $|\psi\rangle$-dynamics of such irreversible effects, is analogue to the incorporation of friction and viscous effects in classical Hamiltonian dynamics. In the second part we presents such models to described open quantum systems (see also chapter 4 of [33] for a tutorial exposure and [17, 4] for more mathematical presentations).

3.3.5 Jaynes-Cummings model

Consider the Jaynes-Cummings Hamiltonian defined in (2.12) that governes the dynamics of $|\psi\rangle$,

$$i\frac{d}{dt}|\psi\rangle = \left(\tfrac{\omega_{eg}}{2}\sigma_z + \omega_c\left(a^\dagger a + \tfrac{1}{2}\right) + u(a+a^\dagger) + i\tfrac{\Omega}{2}\sigma_x(a^\dagger - a)\right)|\psi\rangle\,.$$

Assume that $u(t) = \mathbf{u}e^{i\omega_r t} + \mathbf{u}^* e^{-i\omega_r t}$ where the complex amplitude $\mathbf{u}$ is constant. Define the following detunings

$$\Delta_c = \omega_c - \omega_r, \quad \Delta_{eg} = \omega_{eg} - \omega_r$$

and assume that

$$|\Delta_c|,\ |\Delta_{eg}|,\ |\Omega|, |\mathbf{u}| \ll \omega_{eg},\ \omega_c,\ \omega_r.$$

Then $H_{JC} = H_0 + \epsilon H_1$ where ϵ is a small parameter and

$$\begin{aligned} H_0 &= \tfrac{\omega_r}{2}\sigma_z + \omega_r\left(a^\dagger a + \tfrac{1}{2}\right) \\ \epsilon H_1 &= \left(\tfrac{\Delta_{eg}}{2}\sigma_z + \Delta_c\left(a^\dagger a + \tfrac{1}{2}\right) + (\mathbf{u}e^{i\omega_r t} + \mathbf{u}^* e^{-i\omega_r t})(a+a^\dagger) + i\tfrac{\Omega}{2}\sigma_x(a^\dagger - a)\right). \end{aligned}$$

Even if we the system is infinite dimensional, we apply here heuristically the rotating wave approximation summarized in Subsection 3.3.2. First we have to compute the Hamiltonian in the interaction frame via the following change of variables $|\psi\rangle \mapsto |\phi\rangle$:

$$|\psi\rangle = e^{-i\omega_r t\left(a^\dagger a + \frac{1}{2}\right)} e^{\frac{-i\omega_r t}{2}\sigma_z}|\phi\rangle$$

We get the following interaction Hamiltonian ($\sigma_+ = |e\rangle\langle g|$, $\sigma_- = |g\rangle\langle e|$)

$$\begin{aligned} H_{\text{int}} = \tfrac{\Delta_{eg}}{2}\sigma_z + \Delta_c\left(a^\dagger a + \tfrac{1}{2}\right) + \left(\mathbf{u}e^{i\omega_r t} + \mathbf{u}^* e^{-i\omega_r t}\right)\left(e^{-i\omega_r t}a + e^{i\omega_r t}a^\dagger\right) \\ + i\tfrac{\Omega}{2}(e^{-i\omega_r t}\sigma_- + e^{i\omega_r t}\sigma_+)(e^{i\omega_r t}a^\dagger - e^{-i\omega_r t}a) \end{aligned}$$

where we have applied the following identities (see sections 2.1.1 and 2.2):

$$e^{\frac{i\theta}{2}\sigma_z}\ \sigma_x e^{-\frac{i\theta}{2}\sigma_z} = e^{-i\theta}\sigma_- + e^{i\theta t}\sigma_+, \quad e^{i\theta\left(a^\dagger a+\frac{1}{2}\right)}\ a\ e^{-i\theta\left(a^\dagger a+\frac{1}{2}\right)} = e^{-i\theta}a$$

The secular part of H_{int} is given by

$$H^{1^{\text{st}}}_{\text{rwa}} = \tfrac{\Delta_{eg}}{2}\sigma_z + \Delta_c\left(a^\dagger a + \tfrac{1}{2}\right) + \mathbf{u}a + \mathbf{u}^* a^\dagger + i\tfrac{\Omega}{2}(\sigma_- a^\dagger - \sigma_+ a) \tag{3.25}$$

and its oscillating part by

$$H_{\text{int}} - H^{1^{\text{st}}}_{\text{rwa}} = \mathbf{u}e^{2i\omega_r t}a^\dagger + \mathbf{u}^* e^{-2i\omega_r t}a + i\tfrac{\Omega}{2}(e^{2i\omega_r t}\sigma_+ a^\dagger - e^{-2i\omega_r t}\sigma_- a).$$

Then we have

$$\int_t (H_{\text{int}} - H^{1^{\text{st}}}_{\text{rwa}}) = \tfrac{1}{2i\omega_r}\left(\mathbf{u}e^{2i\omega_r t}a^\dagger - \mathbf{u}^* e^{-2i\omega_r t}a + i\tfrac{\Omega}{2}(e^{2i\omega_r t}\sigma_+ a^\dagger + e^{-2i\omega_r t}\sigma_- a)\right)$$

and, following (3.19), the second order approximation reads

$$\begin{aligned} H^{2^{\text{nd}}}_{\text{rwa}} = \tfrac{\Delta_{eg}+\frac{\Omega^2}{8\omega_r}}{2}\sigma_z + \Delta_c\left(a^\dagger a + \tfrac{1}{2}\right) + \mathbf{u}a + \mathbf{u}^* a^\dagger + i\tfrac{\Omega}{2}(\sigma_- a^\dagger - \sigma_+ a) \\ + i\tfrac{\Omega}{4\omega_r}(\mathbf{u}\sigma_- - \mathbf{u}^*\sigma_+) + \tfrac{\Omega^2}{8\omega_r}\sigma_z a^\dagger a - \tfrac{\Omega^2}{16\omega_r} - \tfrac{|\mathbf{u}|^2}{2\omega_r} \end{aligned} \tag{3.26}$$

(use $[a, a^\dagger] = 1$, $\sigma_+\sigma_- = |e\rangle\langle e|$ and $\sigma_-\sigma_+ = |g\rangle\langle g|$).

Consider now that the average Hamiltonian $H^{1^{\text{st}}}_{\text{rwa}}$ defined by (3.25) with $\mathbf{u} \in \mathbb{C}$ as control. It splits into $H_0 + v_1 H_1 + v_2 H_2$ where $\mathbf{u} = \frac{1}{2}(v_1 + iv_2)$ with $v_1, v_2 \in \mathbb{R}$ and

$$H_0 = \tfrac{\Delta_{eg}}{2}\sigma_z + \Delta_c(X^2+P^2) - \frac{\Omega}{2}(X\sigma_y + P\sigma_x), \qquad H_1 = \tfrac{a+a^\dagger}{2} = X, \quad H_2 = \tfrac{a-a^\dagger}{2i} = P. \tag{3.27}$$

With the commutation rules for the Pauli matrices $\sigma_{x,y,z}$ and the Heisenberg commutation relation $[X, P] = \frac{i}{2}$, the Lie algebra spanned by iH_0, iH_1 and iH_2 is of infinite dimension. Thus, it is natural to wish, according to a heuristic use of Theorem 3.1.9, that this system is controllable. To fix the problem, it is useful to translate it into the partial differential equations language where powerful tools exist for studying linear and nonlinear controllability (see, e.g., the recent book [25]). The controlled system $i\frac{d}{dt}|\phi\rangle = (H_0 + v_1 H_1 + v_2 H_2)|\phi\rangle$ reads as a system of two partial differential equations affine in the two scalar controls $u_1 = \sqrt{2}v_1$ and $u_2 = \sqrt{2}v_2$. The quantum state $|\phi\rangle$ is described by two elements of $L^2(\mathbb{R}, \mathbb{C})$, ϕ_g and ϕ_e, whose time evolution is given by

$$\begin{aligned} i\frac{\partial\phi_g}{\partial t} &= -\tfrac{\Delta_c}{2}\frac{\partial^2\phi_g}{\partial x^2} + \left(\frac{\Delta_c x^2 - \Delta_{eg}}{2}\right)\phi_g + \left(u_1 x + iu_2\frac{\partial}{\partial x}\right)\phi_g - i\tfrac{\Omega}{2\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)\phi_e \\ i\frac{\partial\phi_e}{\partial t} &= -\tfrac{\Delta_c}{2}\frac{\partial^2\phi_e}{\partial x^2} + \left(\frac{\Delta_c x^2 + \Delta_{eg}}{2}\right)\phi_e + \left(u_1 x + iu_2\frac{\partial}{\partial x}\right)\phi_e - i\tfrac{\Omega}{2\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)\phi_g \end{aligned} \tag{3.28}$$

since X stands for $\frac{x}{\sqrt{2}}$ and P for $-\frac{i}{\sqrt{2}}\frac{\partial}{\partial x}$. An open question is the controllability on the set of functions (ϕ_g, ϕ_e) defined up to a global phase and such that $\|\phi_g\|_{L^2} + \|\phi_e\|_{L^2} = 1$. In a first step, one can take $\Delta_c = 0$ (which is not a limitation in fact) and $\Delta_{eg} = 0$ (which is a strict sub-case).

Exercice 3.3.5. *Consider* $i\frac{d}{dt}|\psi\rangle = H_0 + v_1 H_1 + v_2 H_1$ *with* H_0, H_1 *and* H_2 *given by* (3.27) *with* $\Delta_{eg} = \Delta_c = 0$, $\Omega > 0$ *and* (v_1, v_2) *as control. The system is therefore given by*

$$i\frac{d}{dt}|\psi\rangle = \left(i\tfrac{\Omega}{2}(\sigma_- a^\dagger - \sigma_+ a) + \mathbf{u}a^\dagger + \mathbf{u}^* a\right)|\psi\rangle$$

with $u = \frac{v_1 + iv_2}{2}$.

1. *Set* $\mathbf{v} \in \mathbb{C}$ *solution of* $\frac{d}{dt}\mathbf{v} = -i\mathbf{u}$ *and consider the following change of frame* $|\phi\rangle = D_{-\mathbf{v}}|\psi\rangle$ *with the displacement operator* $D_{-\mathbf{v}} = e^{-\mathbf{v}a^\dagger + \mathbf{v}^* a}$. *Show that, up to a global phase change, we have*

$$i\frac{d}{dt}|\phi\rangle = \left(\tfrac{i\Omega}{2}\left(\sigma_- a^\dagger - \sigma_+ a\right) + \left(\tilde{\mathbf{u}}\sigma_+ + \tilde{\mathbf{u}}^*\sigma_-\right)\right)|\phi\rangle$$

   with $\tilde{\mathbf{u}} = i\frac{\Omega}{2}\mathbf{v}$.

2. *Take the orthonormal basis* $\{|g,n\rangle, |e,n\rangle\}$ *with* $n \in \mathbb{N}$ *being the photon number and where for instance* $|g,n\rangle$ *stands for the tensor product* $|g\rangle \otimes |n\rangle$. *Set* $|\phi\rangle = \sum_n \phi_{g,n}|g,n\rangle + \phi_{e,n}|e,n\rangle$ *with* $\phi_{g,n}, \phi_{e,n} \in \mathbb{C}$ *depending on* t *and* $\sum_n |\phi_{g,n}|^2 + |\phi_{e,n}|^2 = 1$. *Show that, for* $n \geq 0$

$$i\frac{d}{dt}\phi_{g,n+1} = i\frac{\Omega}{2}\sqrt{n+1}\phi_{e,n} + \tilde{\mathbf{u}}^*\phi_{e,n+1}, \quad i\frac{d}{dt}\phi_{e,n} = -i\tfrac{\Omega}{2}\sqrt{n+1}\phi_{g,n+1} + \tilde{\mathbf{u}}\phi_{g,n}$$

   and $i\frac{d}{dt}\phi_{g,0} = \tilde{\mathbf{u}}^*\phi_{e,0}$.

3. *Assume that* $|\phi\rangle_0 = |g,0\rangle$. *Construct an open-loop control* $[0,T] \ni t \mapsto \tilde{\mathbf{u}}(t)$ *such that* $|\phi\rangle_T = |g,1\rangle$ *(hint: take* $u = \bar{u}\delta(t)$ *and adjust the constants* $\bar{u}$ *and* $T > 0$, $\delta(t)$ *Dirac distribution at* 0*).*

4. *Generalize the above open-loop control when the goal state* $|\phi\rangle_T$ *is* $|g,n\rangle$ *with any arbitrary photon number* n.

3.3.6 A single trapped ion and the Law-Eberly method

It is a composite system with a quantum state similar to the previous subsection: $|\psi\rangle$ belongs to $\mathbb{C}^2 \otimes L^2(\mathbb{R},\mathbb{C})$ and the Hamiltonian reads

$$\omega\left(a^\dagger a + \tfrac{1}{2}\right) + \frac{\omega_{eg}}{2}\sigma_z + \left(\mathbf{u}e^{i(\omega_l t - \eta(a+a^\dagger))} + \mathbf{u}^* e^{-i(\omega_l t - \eta(a+a^\dagger))}\right)\sigma_x$$

where η is called the Lamb-Dicke parameter and is of small magnitude in general. Furthermore, the control is an electromagnetic wave of complex amplitude $\mathbf{u}$ and with a phase $\omega_l(t - x/c)$ depending on the spatial coordinate x. Such x-dependence ensures the impulsion conservation: when the ion absorbs a photon, its energy changes (increase of $\hbar\omega_l$) but also its impulsion captures the photon impulsion $\hbar k = \hbar\frac{\omega_l}{c}$. Such impulsion exchanges

excite the vibration mode inside the trap described here as a simple harmonic oscillator. The ion vibration are quantized, each quantum being called a phonon. The scales are as follows:

$$|\omega_l - \omega_{eg}| \ll \omega_{eg}, \quad \omega \ll \omega_{eg}, \quad |\mathbf{u}| \ll \omega_{eg}, \quad \left|\frac{d}{dt}\mathbf{u}\right| \ll \omega_{eg}|\mathbf{u}|.$$

In the "laser frame", $|\psi\rangle = e^{-\frac{i\omega_l t}{2}\sigma_z}|\phi\rangle$, the Hamiltonian becomes:

$$\omega\left(a^\dagger a + \tfrac{1}{2}\right) + \frac{\omega_{eg}-\omega_l}{2}\sigma_z + \left(\mathbf{u}e^{2i\omega_l t}e^{-i\eta(a+a^\dagger)} + \mathbf{u}^* e^{i\eta(a+a^\dagger)}\right)|e\rangle\langle g| + \left(\mathbf{u}e^{-i\eta(a+a^\dagger)} + \mathbf{u}^* e^{-2i\omega_l t}e^{i\eta(a+a^\dagger)}\right)|g\rangle\langle e|$$

Even if the system is infinite dimensional, we apply here heuristically the rotating wave approximation summarized in subsection 3.3.2. This just consists in neglecting highly oscillating terms due to $e^{\pm 2i\omega_l t}$. It yields the following average Hamiltonian (corresponding to the first order approximation $H^{1^{\text{st}}}_{\text{rwa}}$ defined in (3.19))

$$H^{1^{\text{st}}}_{\text{rwa}} = \omega\left(a^\dagger a + \tfrac{1}{2}\right) + \tfrac{\Delta}{2}\sigma_z + \mathbf{u}e^{-i\eta(a+a^\dagger)}|g\rangle\langle e| + \mathbf{u}^* e^{i\eta(a+a^\dagger)}|e\rangle\langle g| \tag{3.29}$$

with $\Delta = \omega_{eg} - \omega_l$ the laser/atom detuning. The Schrödinger equation $i\frac{d}{dt}|\psi\rangle = H^{1^{\text{st}}}_{\text{rwa}}|\psi\rangle$ is a partial differential system on the two components (ψ_g, ψ_e):

$$\begin{aligned} i\frac{\partial\psi_g}{\partial t} &= \tfrac{\omega}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_g - \tfrac{\Delta}{2}\psi_g + \mathbf{u}e^{-i\sqrt{2}\eta x}\psi_e \\ i\frac{\partial\psi_e}{\partial t} &= \mathbf{u}^* e^{i\sqrt{2}\eta x}\psi_g + \tfrac{\omega}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_e + \tfrac{\Delta}{2}\psi_e. \end{aligned} \tag{3.30}$$

Here $\mathbf{u} \in \mathbb{C}$ is the control input. In [30] this system is proved to be approximately controllable for (ψ_g, ψ_e) on the unit sphere of $(L^2)^2$. The proof proposed in [30] relies on the Law-Eberly proof of spectral controllability for a secular approximation when u is a superposition of three mono-chromatic plane waves: pulsation ω_{eg} (ion electronic transition) and amplitude $\mathbf{u}$; pulsation $\omega_{eg} - \omega$ (red shift by a vibration quantum) and amplitude $\mathbf{u}_r$; pulsation $\omega_{eg} + \omega$ (blue shift by a vibration quantum) and amplitude $\mathbf{u}_b$.

With this control, the Hamiltonian reads

$$\begin{aligned} H =& \omega\left(a^\dagger a + \tfrac{1}{2}\right) + \frac{\omega_{eg}}{2}\sigma_z + \left(\mathbf{u}e^{i(\omega_{eg}t - \eta(a+a^\dagger))} + \mathbf{u}^* e^{-i(\omega_{eg}t - \eta(a+a^\dagger))}\right)\sigma_x \\ &+ \left(\mathbf{u}_b e^{i((\omega_{eg}+\omega)t - \eta_b(a+a^\dagger))} + \mathbf{u}_b^* e^{-i((\omega_{eg}+\omega)t - \eta_b(a+a^\dagger))}\right)\sigma_x \\ &+ \left(\mathbf{u}_r e^{i((\omega_{eg}-\omega)t - \eta_r(a+a^\dagger))} + \mathbf{u}_r^* e^{-i((\omega_{eg}-\omega)t - \eta_r(a+a^\dagger))}\right)\sigma_x. \end{aligned}$$

We still have $\omega \ll \omega_{eg}$. The Lamb-Dicke parameters $|\eta|, |\eta_b|, |\eta_r| \ll 1$ are almost identical. The amplitudes vary very slowly:

$$\left|\frac{d}{dt}\mathbf{u}\right| \ll \omega|\mathbf{u}|, \quad \left|\frac{d}{dt}\mathbf{u}_r\right| \ll \omega|\mathbf{u}_r|, \quad \left|\frac{d}{dt}\mathbf{u}_b\right| \ll \omega|\mathbf{u}_b|.$$

In the interaction frame, $|\psi\rangle$ is replaced by $|\phi\rangle$ according to

$$|\psi\rangle = e^{-i\omega t\left(a^\dagger a+\frac{1}{2}\right)} e^{\frac{-i\omega_{eg}t}{2}\sigma_z} |\phi\rangle .$$

The Hamiltonian becomes

$$\begin{aligned}
&e^{i\omega t\left(a^\dagger a\right)} \left(\mathbf{u} e^{i\omega_{eg}t} e^{-i\eta(a+a^\dagger)} + \mathbf{u}^* e^{-i\omega_{eg}t} e^{i\eta(a+a^\dagger)}\right) \\
&\qquad\qquad e^{-i\omega t\left(a^\dagger a\right)} \left(e^{i\omega_{eg}t} |e\rangle\langle g| + e^{-i\omega_{eg}t} |g\rangle\langle e|\right) \\
&\quad + e^{i\omega t\left(a^\dagger a\right)} \left(\mathbf{u}_b e^{i(\omega_{eg}+\omega)t} e^{-i\eta_b(a+a^\dagger)} + \mathbf{u}_b^* e^{-i(\omega_{eg}+\omega)t} e^{i\eta_b(a+a^\dagger)}\right) \\
&\qquad\qquad e^{-i\omega t\left(a^\dagger a\right)} \left(e^{i\omega_{eg}t} |e\rangle\langle g| + e^{-i\omega_{eg}t} |g\rangle\langle e|\right) \\
&\quad + e^{i\omega t\left(a^\dagger a\right)} \left(\mathbf{u}_r e^{i(\omega_{eg}-\omega)t} e^{-i\eta_r(a+a^\dagger)} + \mathbf{u}_r^* e^{-i(\omega_{eg}-\omega)t} e^{i\eta_r(a+a^\dagger)}\right) \\
&\qquad\qquad e^{-i\omega t\left(a^\dagger a\right)} \left(e^{i\omega_{eg}t} |e\rangle\langle g| + e^{-i\omega_{eg}t} |g\rangle\langle e|\right)
\end{aligned}$$

With the approximation $e^{i\epsilon(a+a^\dagger)} \approx 1 + i\epsilon(a + a^\dagger)$ for $\epsilon = \pm\eta, \eta_b, \eta_r$, the Hamiltonian becomes (up to second order terms in ϵ),

$$\begin{aligned}
&\left(\mathbf{u} e^{i\omega_{eg}t}(1 - i\eta(e^{-i\omega t}a + e^{i\omega t}a^\dagger)) + \mathbf{u}^* e^{-i\omega_{eg}t}(1 + i\eta(e^{-i\omega t}a + e^{i\omega t}a^\dagger))\right) \\
&\qquad\qquad \left(e^{i\omega_{eg}t} |e\rangle\langle g| + e^{-i\omega_{eg}t} |g\rangle\langle e|\right) \\
&\quad + \left(\mathbf{u}_b e^{i(\omega_{eg}+\omega)t}(1 - i\eta_b(e^{-i\omega t}a + e^{i\omega t}a^\dagger)) + \mathbf{u}_b^* e^{-i(\omega_{eg}+\omega)t}(1 + i\eta_b(e^{-i\omega t}a + e^{i\omega t}a^\dagger))\right) \\
&\qquad\qquad \left(e^{i\omega_{eg}t} |e\rangle\langle g| + e^{-i\omega_{eg}t} |g\rangle\langle e|\right) \\
&\quad + \left(\mathbf{u}_r e^{i(\omega_{eg}-\omega)t}(1 - i\eta_r(e^{-i\omega t}a + e^{i\omega t}a^\dagger)) + \mathbf{u}_r^* e^{-i(\omega_{eg}-\omega)t}(1 + i\eta_r(e^{-i\omega t}a + e^{i\omega t}a^\dagger))\right) \\
&\qquad\qquad \left(e^{i\omega_{eg}t} |e\rangle\langle g| + e^{-i\omega_{eg}t} |g\rangle\langle e|\right)
\end{aligned}$$

The oscillating terms (with pulsations $2\omega_{eg}$, $2\omega_{eg} \pm \omega$, $2(\omega_{eg} \pm \omega)$ and $\pm\omega$) have a zero average. The mean Hamiltonian, illustrated on Figure 3.2, reads

$$\bar{H} = \mathbf{u} |g\rangle\langle e| + \mathbf{u}^* |e\rangle\langle g| + \overline{\mathbf{u}}_b a |g\rangle\langle e| + \overline{\mathbf{u}}_b^* a^\dagger |e\rangle\langle g| + \overline{\mathbf{u}}_r a^\dagger |g\rangle\langle e| + \overline{\mathbf{u}}_r^* a |e\rangle\langle g|$$

where we have set $\overline{\mathbf{u}}_b = -i\eta_b \mathbf{u}_b$ and $\overline{\mathbf{u}}_r = -i\eta_r \mathbf{u}_r$. The above Hamiltonian is "valid" as soon as $|\eta|, |\eta_b|, |\eta_r| \ll 1$ and

$$|\mathbf{u}|, |\mathbf{u}_b|, |\mathbf{u}_r| \ll \omega, \quad \left|\frac{d}{dt}\mathbf{u}\right| \ll \omega|\mathbf{u}|, \quad \left|\frac{d}{dt}\mathbf{u}_b\right| \ll \omega|\mathbf{u}_b|, \quad \left|\frac{d}{dt}\mathbf{u}_r\right| \ll \omega|\mathbf{u}_r|.$$

To interpret the structure of the different operators building this average Hamiltonian, physicists have a nice mnemonic trick based on energy conservation. Take for example $a |g\rangle\langle e|$ attached to the control $\overline{\mathbf{u}}_b$, i.e. to the blue shifted photon of pulsation $\omega_{eg} + \omega$. The operator $|g\rangle\langle e|$ corresponds to the quantum jump from $|e\rangle$ to $|g\rangle$ whereas the operator a is the destruction of one phonon. Thus $a |g\rangle\langle e|$ is the simultaneous jump from $|e\rangle$ to $|g\rangle$ (energy change of ω_{eg}) with destruction of one phonon (energy change of ω). The emitted

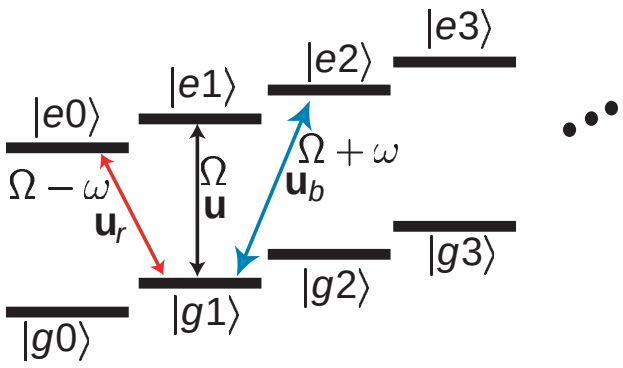

Figure 3.2: a trapped ion submitted to three mono-chromatic plane waves of pulsations ω_{eg}, $\omega_{eg}-\omega$ and $\omega_{eg}+\omega$.

photon has to take away the total energy lost by the system, i.e. $\omega_{eg}+\omega$. Its pulsation is then $\omega_{eg}+\omega$ and corresponds thus to $\overline{\mathbf{u}}_b$. We understand why $a^\dagger |g\rangle \langle e|$ is associated to $\mathbf{u}_r$: the system loses ω_{eg} during the jump from $|e\rangle$ to $|g\rangle$; at the same time, it wins ω, the phonon energy; the emitted photon takes away $\omega_{eg}-\omega$ and thus corresponds to $\overline{\mathbf{u}}_r$. This point is illustrated on figure 3.2 describing the different first order transitions between the different states of definite energy.

The dynamics $i\frac{d}{dt}|\phi\rangle = \bar{H}|\phi\rangle$ depends linearly on 6 scalar controls: it is a drift-less system of infinite dimension (non-holonomic system of infinite dimension). The two underlying partial differential equations are

$$
\begin{aligned}
i\frac{\partial \phi_g}{\partial t} &= \left(\mathbf{u} + \frac{\overline{\mathbf{u}}_b}{\sqrt{2}}\left(x+\frac{\partial}{\partial x}\right) + \frac{\overline{\mathbf{u}}_r}{\sqrt{2}}\left(x-\frac{\partial}{\partial x}\right)\right)\phi_e \\
i\frac{\partial \phi_e}{\partial t} &= \left(\mathbf{u}^* + \frac{\overline{\mathbf{u}}_b^*}{\sqrt{2}}\left(x-\frac{\partial}{\partial x}\right) + \frac{\overline{\mathbf{u}}_r^*}{\sqrt{2}}\left(x+\frac{\partial}{\partial x}\right)\right)\phi_g
\end{aligned}
$$

We write the above dynamics in the eigenbasis, $\{|g,n\rangle, |e,n\rangle\}_{n\in\mathbb{N}}$, of the operator $\omega\left(a^\dagger a + \frac{1}{2}\right) + \frac{\omega_{eg}}{2}\sigma_z$, which is the tensor product of the eigenbasis of the harmonic oscillator, $(|n\rangle)_{n\in\mathbb{N}}$, and that of the 2-level system, $(|g\rangle, |e\rangle)$. In this basis, the dynamics can be written as

$$
\begin{aligned}
i\frac{d}{dt}\phi_{g,n} &= \mathbf{u}\phi_{e,n} + \overline{\mathbf{u}}_r\sqrt{n}\phi_{e,n-1} + \overline{\mathbf{u}}_b\sqrt{n+1}\phi_{e,n+1} \\
i\frac{d}{dt}\phi_{e,n} &= \mathbf{u}^*\phi_{g,n} + \overline{\mathbf{u}}_r^*\sqrt{n+1}\phi_{g,n+1} + \overline{\mathbf{u}}_b^*\sqrt{n}\phi_{g,n-1}
\end{aligned}
$$

with $|\phi\rangle = \sum_{n=0}^{+\infty}\phi_{g,n}|g,n\rangle + \phi_{e,n}|e,n\rangle$ and $\sum_{n=0}^{+\infty}|\phi_{g,n}|^2 + |\phi_{e,n}|^2 = 1$.

Law and Eberly [43] have proved that it is always possible (and in any arbitrary time $T>0$) to steer $|\psi\rangle$ from any finite linear superposition of $\{|g,n\rangle, |e,n\rangle\}_{n\in\mathbb{N}}$ at $t=0$, to any other finite linear superposition at time $t=T$ (spectral controllability). They need only two controls $\mathbf{u}$ and $\overline{\mathbf{u}}_b$ (resp. $\mathbf{u}$ and $\overline{\mathbf{u}}_r$): $\overline{\mathbf{u}}_r$ (resp. $\overline{\mathbf{u}}_b$) remains zero and the supports of $\mathbf{u}$

and $\overline{\mathbf{u}}_b$ (resp. $\mathbf{u}$ and $\overline{\mathbf{u}}_r$) do not overlap. This spectral controllability implies approximate controllability.

Let us detail now the main idea behind the Law-Eberly method to prove spectral controllability. Take $n > 0$ and denote by $\mathcal{H}_n$ the truncation to n-phonon space:

$$\mathcal{H}_n = \text{span}\left\{|g,0\rangle, |e,0\rangle, \ldots, |g,n\rangle, |e,n\rangle\right\}$$

Take $n > 0$, an initial condition $|\phi\rangle_0 \in \mathcal{H}_n$ and $T > 0$. Then for $t \in [0, \frac{T}{2}]$ the control

$$\overline{\mathbf{u}}_r(t) = \overline{\mathbf{u}}_b(t) = 0, \quad \overline{\mathbf{u}}(t) = \tfrac{2i}{T} \arctan \left| \tfrac{\phi_{e,n}(0)}{\phi_{g,n}(0)} \right| e^{i \arg(\phi_{g,n}(0)\phi^*_{e,n}(0))}$$

ensures that $\phi_{e,n}(T/2) = 0$. For $t \in [\frac{T}{2}, T]$, the control

$$\overline{\mathbf{u}}_b(t) = \overline{\mathbf{u}}(t) = 0, \quad \overline{\mathbf{u}}_r(t) = \tfrac{2i}{T\sqrt{n}} \arctan \left| \frac{\phi_{g,n}(\frac{T}{2})}{\phi_{e,n-1}(\frac{T}{2})} \right| e^{i \arg\left(\phi_{g,n}(\frac{T}{2})\phi^*_{e,n-1}(\frac{T}{2})\right)}$$

ensures that $\phi_{e,n}(t) \equiv 0$ and that $\phi_{g,n}(T) = 0$. Thus with this two-pulse control the first one on $\overline{\mathbf{u}}$, the second one on $\overline{\mathbf{u}}_r$ we have $|\phi\rangle_T \in \mathcal{H}_{n-1}$.

After n iterations of this two-pulse process $|\phi\rangle_{nT}$ belongs to $\mathcal{H}_0$ and then for $t \in [nT, (n+\frac{1}{2})T]$ the control

$$\overline{\mathbf{u}}_r(t) = \overline{\mathbf{u}}_b(t) = 0, \quad \overline{\mathbf{u}}(t) = \tfrac{2i}{T} \arctan \left| \tfrac{\phi_{e,0}(nT)}{\phi_{g,0}(nT)} \right| e^{i \arg\left(\phi_{g,0}(nT)\phi^*_{e,0}(nT)\right)}$$

guaranties that $|\phi\rangle_{(n+\frac{1}{2})T} = e^{i\theta}|g,0\rangle$.

Up to a global phase, we can steer, in any arbitrary time and with a piecewise constant control, any element of $\mathcal{H}_n$ to $|g,0\rangle$. Since the system is driftless ($t \mapsto -t$ and $(\overline{\mathbf{u}}, \overline{\mathbf{u}}_b, \overline{\mathbf{u}}_r) \mapsto -(\overline{\mathbf{u}}, \overline{\mathbf{u}}_b, \overline{\mathbf{u}}_r)$ leave the system unchanged) we can easily reverse the time and thus can also steer $|g,0\rangle$ to any element of $\mathcal{H}_n$. To steer $|\phi\rangle$ form any initial state in $\mathcal{H}_n$ to any final state also in $\mathcal{H}_n$, it is enough to steer the initial state to $|g,0\rangle$ and then to steer $|g,0\rangle$ to the final state. To summarize: on can always steer, with piecewise constant controls and in an arbitrary short time, any finite linear superposition of $(|g,\nu\rangle, |e,\nu\rangle)_{\nu\geq 0}$ to any other one.

3.3.7 Two trapped ions

Let us consider two ions catched in the same trap and coupled to one of the two vibration modes, the center of mass mode of frequency ω (see [33, chapitre 8] for detailed explanations and modeling assumptions). Considerations similar to the ones developed in the previous subsection yield the following average Hamiltonian

$$\begin{aligned}(\mathbf{u}_1 + \mathbf{u}_{1b}a + \mathbf{u}_{1r}a^\dagger)(|g\rangle\langle e|)_1 + (\mathbf{u}_1^* + \mathbf{u}_{1b}^* a^\dagger + \mathbf{u}_{1r}^* a)(|e\rangle\langle g|)_1 \\ + (\mathbf{u}_2 + \mathbf{u}_{2b}a + \mathbf{u}_{2r}a^\dagger)(|g\rangle\langle e|)_2 + (\mathbf{u}_2^* + \mathbf{u}_{2b}^* a^\dagger + \mathbf{u}_{2r}^* a)(|e\rangle\langle g|)_2\end{aligned}$$

where the indices 1 and 2 are relative to ion number 1 and ion number 2, each of them having its own control, u_1 and u_2 that are superpositions of three mono-chromatic plane waves: pulsation ω_{eg} with amplitudes $\mathbf{u}_1$ and $\mathbf{u}_2$; pulsation $\omega_{eg}+\omega$ with amplitudes proportional to $\mathbf{u}_{1b}$ and $\mathbf{u}_{2b}$; pulsation $\omega_{eg}-\omega$ with amplitudes proportional to $\mathbf{u}_{1r}$ and $\mathbf{u}_{2r}$.

The quantum state $|\phi\rangle$ is described by 4 elements of $L^2(\mathbb{R},\mathbb{C})$, $(\psi_{gg},\psi_{ge},\psi_{eg},\psi_{ee})$. They satisfy the following partial differential equations:

$$
\begin{aligned}
i\frac{\partial}{\partial t}\phi_{gg} &= \left(\mathbf{u}_1+\frac{\mathbf{u}_{1r}}{\sqrt{2}}\left(x-\frac{\partial}{\partial x}\right)+\frac{\mathbf{u}_{1b}}{\sqrt{2}}\left(x+\frac{\partial}{\partial x}\right)\right)\phi_{eg}\\
&\quad+\left(\mathbf{u}_2+\frac{\mathbf{u}_{2r}}{\sqrt{2}}\left(x-\frac{\partial}{\partial x}\right)+\frac{\mathbf{u}_{2b}}{\sqrt{2}}\left(x+\frac{\partial}{\partial x}\right)\right)\phi_{ge}\\
i\frac{\partial}{\partial t}\phi_{eg} &= \left(\mathbf{u}_1^*+\frac{\mathbf{u}_{1r}^*}{\sqrt{2}}\left(x+\frac{\partial}{\partial x}\right)+\frac{\mathbf{u}_{1b}^*}{\sqrt{2}}\left(x-\frac{\partial}{\partial x}\right)\right)\phi_{gg}\\
&\quad+\left(\mathbf{u}_2+\frac{\mathbf{u}_{2r}}{\sqrt{2}}\left(x-\frac{\partial}{\partial x}\right)+\frac{\mathbf{u}_{2b}}{\sqrt{2}}\left(x+\frac{\partial}{\partial x}\right)\right)\phi_{ee}\\
i\frac{\partial}{\partial t}\phi_{ge} &= \left(\mathbf{u}_1+\frac{\mathbf{u}_{1r}}{\sqrt{2}}\left(x-\frac{\partial}{\partial x}\right)+\frac{\mathbf{u}_{1b}}{\sqrt{2}}\left(x+\frac{\partial}{\partial x}\right)\right)\phi_{ee}\\
&\quad+\left(\mathbf{u}_2^*+\frac{\mathbf{u}_{2r}^*}{\sqrt{2}}\left(x+\frac{\partial}{\partial x}\right)+\frac{\mathbf{u}_{2b}^*}{\sqrt{2}}\left(x-\frac{\partial}{\partial x}\right)\right)\phi_{gg}\\
i\frac{\partial}{\partial t}\phi_{ee} &= \left(\mathbf{u}_1^*+\frac{\mathbf{u}_{1r}^*}{\sqrt{2}}\left(x+\frac{\partial}{\partial x}\right)+\frac{\mathbf{u}_{1b}^*}{\sqrt{2}}\left(x-\frac{\partial}{\partial x}\right)\right)\phi_{ge}\\
&\quad+\left(\mathbf{u}_2^*+\frac{\mathbf{u}_{2r}^*}{\sqrt{2}}\left(x+\frac{\partial}{\partial x}\right)+\frac{\mathbf{u}_{2b}^*}{\sqrt{2}}\left(x-\frac{\partial}{\partial x}\right)\right)\phi_{eg}
\end{aligned}
$$

We conjecture that this system is controllable (at least spectrally controllable and thus approximately controllable).

We recall here a 4-pulse sequence (see also [33]) that steers in finite time $|\psi\rangle$ from $|gg,0\rangle$ at $t=0$, ions in ground states and 0 phonon, to the entangled state (Bell state) at $t=4T$,

$$\frac{|gg,0\rangle+|ee,0\rangle}{\sqrt{2}},$$

a coherent superposition of $|gg,0\rangle$ and $|ee,0\rangle$ (ions in excited states with 0 phonon). One proceeds in 4 successive pulses of duration $T>0$ and where only one of the 6 controls is different form zero:

1. $\pi/2$-pulse on $\mathbf{u}_{b1}$: only $\mathbf{u}_{b1}$ differs from 0 and is equal to $-i\frac{\pi}{T}$; the Hamiltonian (with this particular control) leaves invariant the sub-space spanned by $|gg,0\rangle$ and $|eg,1\rangle$; since the initial state is $|gg,0\rangle$, we have thus a simple $\pi/2$-pulse of Rabi type; it ends with $|\psi\rangle=\frac{|gg,0\rangle+|eg,1\rangle}{\sqrt{2}}$, exactly.

2. π-pulse on $\mathbf{u}_2$: we apply $\mathbf{u}_2=-i\frac{2\pi}{T}$ and start with $|\psi\rangle=\frac{|gg,0\rangle+|eg,1\rangle}{\sqrt{2}}$; we finish the pulse with $|\psi\rangle=\frac{|ge,0\rangle+|ee,1\rangle}{\sqrt{2}}$

3. π-pulse on $\mathbf{u}_{b2}$: set $\mathbf{u}_{b2} = -i\frac{2\pi}{T}$; $|\psi\rangle$ is steered from $\frac{|ge,0\rangle+|ee,1\rangle}{\sqrt{2}}$ to $\frac{|ge,0\rangle-|eg,0\rangle}{\sqrt{2}}$ since the state $|ge,0\rangle$ is not touched by the control $\mathbf{u}_{b2}$.

4. π-pulse on $\mathbf{u}_1$: we apply $\mathbf{u}_1 = -i\frac{2\pi}{T}$; $|\psi\rangle$ is steered from $\frac{|ge,0\rangle-|eg,0\rangle}{\sqrt{2}}$ to $\frac{|ee,0\rangle+|gg,0\rangle}{\sqrt{2}}$, the target Bell state.

These kinds of open-loop controls have been tested experimentally to generate entangled quantum states and also quantum gates. They are made through a succession of pulses where only a single control is non-zero.

3.4 Adiabatic control

3.4.1 Time-adiabatic approximation without gap conditions

We first recall the quantum version of adiabatic invariance. We restrict here the exposure to finite dimension and without the exponentially precise estimations. However we give here the simplest version of a time-adiabatic approximation result without any gap conditions. All the details can be found in the recent book of Teufel [66] with extension to infinite dimensional case.

Theorem 3.4.1. *Take $m+1$ Hermitian matrices $n \times n$: $H_0, \ldots, H_m$. For $u \in \mathbb{R}^m$ set $H(u) := H_0 + \sum_{k=1}^m u_k\, H_k$. Assume that u is a slowly varying time-function: $u = u(s)$ with $s = \epsilon t \in [0,1]$ and ϵ a small positive parameter. Consider a solution $\left[0, \frac{1}{\epsilon}\right] \ni t \mapsto |\psi\rangle_t^\epsilon$ of*

$$i\frac{d}{dt}\,|\psi\rangle_t^\epsilon = H(u(\epsilon t))\,|\psi\rangle_t^\epsilon\,.$$

Take $[0,s] \ni s \mapsto P(s)$ a family of orthogonal projectors such that for each $s \in [0,1]$, $H(u(s))P(s) = E(s)P(s)$ where $E(s)$ is an eigenvalue of $H(u(s))$. Assume that $[0,s] \ni s \mapsto H(u(s))$ is C^2, $[0,s] \ni s \mapsto P(s)$ is C^2 and that, for almost all $s \in [0,1]$, $P(s)$ is the orthogonal projector on the eigen-space associated to the eigen-value $E(s)$. Then

$$\lim_{\epsilon \mapsto 0^+} \left(\sup_{t \in [0, \frac{1}{\epsilon}]} \left| \|P(\epsilon t)\,|\psi\rangle_t^\epsilon\,\|^2 - \|P(0)\,|\psi\rangle_0^\epsilon\,\|^2 \right| \right) = 0.$$

This theorem is a finite dimensional version of Theorem 6.2, page 175 in [66] where, for simplicity sake, we have removed the so-called adiabatic Hamiltonian and adiabatic propagator that intertwines the spectral subspace of the slowly time-dependent Hamiltonian $H(u(\epsilon t))$.

This theorem implies that the solution of $i\frac{d}{dt}\,|\psi\rangle = H\left(u(\frac{t}{T})\right)|\psi\rangle$ follows the spectral decomposition of $H\left(u(\frac{t}{T})\right)$ as soon as T is large enough and when $H\left(u(\frac{t}{T})\right)$ does not admit multiple eigenvalues (non-degenerate spectrum): apply the above theorem with $P = P_k$ where P_k is the orthogonal projection on the k'th eigenstate of H to conclude that the population on state $|k\rangle$ is approximatively constant. If, for instance, $|\psi\rangle$ starts at $t = 0$

in the ground state and if $u(0) = u(1)$ then $|\psi\rangle$ returns at $t = T$, up to a global phase (related to the Berry phase [62]), to the same ground state.

Whenever, for some value of s, the spectrum of $H(u(s))$ becomes degenerate the above theorem says that the populations follow the smooth decomposition versus s of $H(u(s))$. For example, assume that the spectrum of H is not degenerate except at $\bar{s}$ where only two eigenvalues become identical: for all s we assume that the n eigenvalues of $H(u(s))$ are labeled according to their order

$$E_1(s) < E_2(s) < \ldots < E_{\bar{k}}(s) \leq E_{\bar{k}+1}(s) < E_{k+2}(s) < \ldots < E_n(s)$$

and $E_{\bar{k}}(s) = E_{\bar{k}+1}(s)$ only when $s = \bar{s}$ for some $\bar{k} \in \{1, \ldots, n\}$. Since $s \mapsto H(u(s))$ is smooth, exists always a spectral decomposition of $H(u(s))$ that is smooth versus s (this comes from the fact that the spectral decomposition of a Hermitian matrix depends smoothly on its entries). Thus we have only two cases:

1. the non-crossing case where $s \mapsto E_{\bar{k}}(s)$ and $s \mapsto E_{\bar{k}+1}(s)$ are smooth functions
2. the crossing case where

$$s \mapsto \begin{cases} E_{\bar{k}}(s), & \text{for } s \leq \bar{s}; \\ E_{\bar{k}+1}(s), & \text{for } s \geq \bar{s}. \end{cases} \quad \text{and} \quad s \mapsto \begin{cases} E_{\bar{k}+1}(s), & \text{for } s \leq \bar{s}; \\ E_{\bar{k}}(s), & \text{for } s \geq \bar{s}. \end{cases}$$

   are smooth functions.

In the non-crossing case the projectors that satisfy the theorem's assumption are the orthogonal projectors $P_k(s)$ on the k'th eigen-direction associated to $E_k(s)$. In the crossing case, the projectors on the eigenspaces associated to $E_{\bar{k}}$ and $E_{\bar{k}+1}$ have to be exchanged when s passes through $\bar{s}$ to guaranty at least the continuity of $P_{\bar{k}}(s)$ and $P_{\bar{k}+1}(s)$: for $s < \bar{s}$, $P_{\bar{k}}$ (resp. $P_{\bar{k}+1}$ is the projector of the eigenspace associated to $E_{\bar{k}}$ (resp. $E_{\bar{k}+1}$); for $s > \bar{s}$, $P_{\bar{k}}$ (resp. $P_{\bar{k}+1}$) is the projector of the eigenspace associated to $E_{\bar{k}+1}$ (resp. $E_{\bar{k}}$); for $s = \bar{s}$, $P_{\bar{k}}$ and $P_{\bar{k}+1}$ are extended by continuity and correspond to orthogonal projectors on two orthogonal eigen-directions that span the eigenspace of dimension two associated to $E_{\bar{k}}(\bar{s}) = E_{\bar{k}+1}(\bar{s})$.

3.4.2 Adiabatic motion on the Bloch sphere

Let us take a 2-level system. Since we do not care for global phase, we will use the Bloch vector of Subsection 2.1.2:

$$\frac{d}{dt}\vec{M} = (u\vec{i} + v\vec{j} + w\vec{k}) \times \vec{M}$$

where we assume that $\vec{B} = (u\vec{i} + v\vec{j} + w\vec{k})$, a vector in $\mathbb{R}^3$, is the control (in magnetic resonance, $\vec{B}$ is the magnetic field). We set $\omega \in \mathbb{R}$ and $\vec{B} = \omega\vec{b}$ where $\vec{b}$ is a unitary vector in $\mathbb{R}^3$. Thus we have

$$\frac{d}{dt}\vec{M} = \omega\vec{b} \times \vec{M}, \quad \text{with, as control input, } \omega \in \mathbb{R}, \vec{b} \in \mathbb{S}^2.$$

Assume now that $\vec{B}$ varies slowly: we take $T > 0$ large (i.e., $\omega T \gg 1$), and set $\omega(t) = \varpi\left(\frac{t}{T}\right)$, $\vec{b}(t) = \vec{\beta}\left(\frac{t}{T}\right)$ where ϖ and $\vec{\beta}$ depend regularly on $s = \frac{t}{T} \in [0,1]$. Assume that, at $t = 0$, $\vec{M}_0 = \vec{\beta}(0)$. If, for any $s \in [0,1]$, $\varpi(s) > 0$, then the trajectory of $\vec{M}$ with the above control $\vec{B}$ verifies: $\vec{M}(t) \approx \vec{\beta}\left(\frac{t}{T}\right)$, i.e. $\vec{M}$ follows adiabatically the direction of $\vec{B}$. If $\vec{b}(T) = \vec{b}(0)$, i.e., if the control $\vec{B}$ makes a loop between 0 and T ($\beta(0) = \beta(1)$) then $\vec{M}$ follows the same loop (in direction).

To justify this point, it suffices to consider $|\psi\rangle$ that obeys the Schrödinger equation $i\frac{d}{dt}|\psi\rangle = \left(\frac{u}{2}\sigma_x + \frac{v}{2}\sigma_y + \frac{w}{2}\sigma_z\right)|\psi\rangle$ and to apply the adiabatic theorem of the previous subsection. The absence of spectrum degeneracy results from the fact that ϖ never vanishes and remains always strictly positive. The initial condition $\vec{M}_0 = \vec{\beta}(0)$ corresponds to $|\psi\rangle_0$ in the ground state of $\frac{u(0)}{2}\sigma_x + \frac{v(0)}{2}\sigma_y + \frac{w(0)}{2}\sigma_z$. Thus $|\psi\rangle_t$ follows the ground state of $\frac{u(t)}{2}\sigma_x + \frac{v(t)}{2}\sigma_y + \frac{w(t)}{2}\sigma_z$, i.e., $\vec{M}(t)$ follows $\vec{\beta}\left(\frac{t}{T}\right)$.

The assumption concerning the non degeneracy of the spectrum is important. If it is not satisfied, $|\psi\rangle_t$ can jump smoothly from one branch to another branch when some eigenvalues cross. In order to understand this phenomenon (analogue to monodromy), assume that $\varpi(s)$ vanishes only once at $\bar{s} \in]0,1[$ with $\varpi(s) > 0$ (resp. < 0) for $s \in [0, \bar{s}[$ (resp. $s \in]\bar{s}, 1]$). Then, around $t = \bar{s}T$, $|\psi\rangle_t$ changes smoothly from the ground state to the excited state of $H(t)$, since their energies coincide for $t = \bar{s}T$. With such a choice for ϖ, $\vec{B}$ performs a loop if, additionally $\vec{b}(0) = -\vec{b}(1)$ and $\varpi(0) = -\varpi(1)$, whereas $|\psi\rangle_t$ does not. It starts from the ground state at $t = 0$ and ends on the excited state at $t = T$. In fact, $\vec{M}(t)$ follows adiabatically the direction of $\vec{B}(t)$ for $t \in [0, \bar{s}T]$ and then the direction of $-\vec{B}(t)$ for $t \in [\bar{s}T, T]$. Such quasi-static motion planing method is particularly robust and widely used in practice. We refer to [72, 2] for related control theoretic results. In the following subsections we detail some important examples.

3.4.3 Stimulated Raman Adiabatic Passage (STIRAP)

Consider the Λ-system of Figure 2.2. Its control Hamiltonian reads

$$H = \omega_g |g\rangle\langle g| + \omega_e |e\rangle\langle e| + \omega_f |f\rangle\langle f| + u\mu_{gf}(|g\rangle\langle f| + |f\rangle\langle g|) + u\mu_{ef}(|e\rangle\langle f| + |f\rangle\langle e|).$$

Assume $\omega_{gf} = \omega_f - \omega_g > \omega_{ef} = \omega_f - \omega_e > 0$ and an oscillatory and small control involving perfect resonances with transitions $g \leftrightarrow f$ and $e \leftrightarrow f$:

$$u = u_{gf}\cos(\omega_{gf}t) + u_{ef}\cos(\omega_{ef}t)$$

with slowly varying small real amplitudes u_{gf} and u_{ef}. Put the system in the interaction frame via the unitary transformation $e^{-it(\omega_g|g\rangle\langle g| + \omega_e|e\rangle\langle e| + \omega_f|f\rangle\langle f|)}$. Use the rotating wave approximation (order 1 in (3.19)) to get the average Hamiltonian

$$H_{\text{rwa}} = \frac{\Omega_{gf}}{2}(|g\rangle\langle f| + |f\rangle\langle g|) + \frac{\Omega_{ef}}{2}(|e\rangle\langle f| + |f\rangle\langle e|)$$

with slowly varying Rabi pulsations $\Omega_{gf} = \mu_{gf}u_{gf}$ and $\Omega_{ef} = \mu_{ef}u_{ef}$.

Let us now analyze the dependence of the spectral decomposition of $H_{\rm rwa}$ versus the two parameters Ω_{gf} and Ω_{ef}. When $\Omega_{gf}^2+\Omega_{ef}^2\neq 0$, spectrum of $H_{\rm rwa}$ admits three distinct eigne-values:

$$\Omega_- = -\frac{\sqrt{\Omega_{gf}^2+\Omega_{ef}^2}}{2}, \quad \Omega_0 = 0, \quad \Omega_+ = \frac{\sqrt{\Omega_{gf}^2+\Omega_{ef}^2}}{2}$$

associated to the following eigen-vectors :

$$\begin{aligned}
|-\rangle &= \frac{\Omega_{gf}}{\sqrt{2(\Omega_{gf}^2+\Omega_{ef}^2)}}\,|g\rangle + \frac{\Omega_{ef}}{\sqrt{2(\Omega_{gf}^2+\Omega_{ef}^2)}}\,|e\rangle - \frac{1}{\sqrt{2}}\,|f\rangle \\
|0\rangle &= \frac{-\Omega_{ef}}{\sqrt{\Omega_{gf}^2+\Omega_{ef}^2}}\,|g\rangle + \frac{\Omega_{gf}}{\sqrt{\Omega_{gf}^2+\Omega_{ef}^2}}\,|e\rangle \\
|+\rangle &= \frac{\Omega_{gf}}{\sqrt{2(\Omega_{gf}^2+\Omega_{ef}^2)}}\,|g\rangle + \frac{\Omega_{ef}}{\sqrt{2(\Omega_{gf}^2+\Omega_{ef}^2)}}\,|e\rangle + \frac{1}{\sqrt{2}}\,|f\rangle\,.
\end{aligned}$$

Assume now that the Rabi pulsation depends on $s\in[0,\frac{3\pi}{2}]$ according to the following formula

$$\Omega_{gf}(s) = \begin{cases} \bar\Omega_g\cos^2 s, & \text{for } s\in[\frac{\pi}{2},\frac{3\pi}{2}]; \\ 0, & \text{elsewhere.}\end{cases}, \qquad \Omega_{ef}(s) = \begin{cases} \bar\Omega_e\sin^2 s, & \text{for } s\in[0,\pi]; \\ 0, & \text{elsewhere.}\end{cases}$$

with $\bar\Omega_g>0$ and $\bar\Omega_e>0$ constant parameter. With such s dependence, we have three analytic branches of the spectral decomposition:

- for $s\in]0,\frac{\pi}{2}[$ we have

$$\begin{aligned}
\Omega_-(s) = -\bar\Omega_e\sin s \text{ with } |-\rangle_s &= \tfrac{|e\rangle-|f\rangle}{\sqrt{2}}. \\
\Omega_0 = 0 \text{ with } |0\rangle_s &= -\,|g\rangle \\
\Omega_+(s) = \bar\Omega_e\sin s \text{ with } |+\rangle_s &= \tfrac{|e\rangle+|f\rangle}{\sqrt{2}}..
\end{aligned}$$

- for $s\in]\frac{\pi}{2},\pi[$ we have

$$\begin{aligned}
\Omega_-(s) = -\sqrt{\bar\Omega_g^2\cos^4 s+\bar\Omega_e^2\sin^4 s} \text{ with } |-\rangle_s &= \tfrac{\bar\Omega_g\cos^2 s|g\rangle+\bar\Omega_e\sin^2 s|e\rangle}{\sqrt{2(\bar\Omega_g^2\cos^4 s+\bar\Omega_e^2\sin^4 s)}} - \tfrac{1}{\sqrt{2}}\,|f\rangle \\
\Omega_0 = 0 \text{ with } |0\rangle_s &= \tfrac{-\bar\Omega_e\sin^2 s|g\rangle+\bar\Omega_g\cos^2 s|e\rangle}{\sqrt{\bar\Omega_g^2\cos^4 s+\bar\Omega_e^2\sin^4 s}} \\
\Omega_+(s) = \sqrt{\bar\Omega_g^2\cos^4 s+\bar\Omega_e^2\sin^4 s} \text{ with } |+\rangle_s &= \tfrac{\bar\Omega_g\cos^2 s|g\rangle+\bar\Omega_e\sin^2 s|e\rangle}{\sqrt{2(\bar\Omega_g^2\cos^4 s+\bar\Omega_e^2\sin^4 s)}} + \tfrac{1}{\sqrt{2}}\,|f\rangle\,.
\end{aligned}$$

- for $s\in]\pi,\frac{3\pi}{2}[$ we have

$$\begin{aligned}
\Omega_-(s) = -\bar\Omega_g|\cos s| \text{ with } |-\rangle_s &= \tfrac{|g\rangle-|f\rangle}{\sqrt{2}}. \\
\Omega_0 = 0 \text{ with } |0\rangle_s &= |e\rangle \\
\Omega_+(s) = \bar\Omega_g|\cos s| \text{ with } |+\rangle_s &= \tfrac{|g\rangle+|f\rangle}{\sqrt{2}}..
\end{aligned}$$

Let us consider the eigenvalue Ω_0: it is associated to the projector $P_0(s)$ on $|0\rangle_s$ that depends smoothly on $s \in [0, \frac{3\pi}{2}]$ as shown by the concatenation of the above formula on the three intervals $]0, \frac{\pi}{2}[$, $]\frac{\pi}{2}, \pi[$ and $]\pi, \frac{3\pi}{2}[$. Thus assume that $|\psi\rangle_0 = |g\rangle$ then adiabatic Theorem 3.4.1 shows that, for $\epsilon > 0$ small enough, the solution of $i\frac{d}{dt}|\psi\rangle = H_{\rm rwa}|\psi\rangle$ with the time-varying control amplitudes

$$[0, \tfrac{3\pi}{2\epsilon}] \ni t \mapsto (u_{fg}, u_{ef}) = \left(\frac{\Omega_{gf}(\epsilon t)}{\mu_{gf}}, \frac{\Omega_{ef}(\epsilon t)}{\mu_{ef}}\right)$$

is given approximatively by

$$|\psi\rangle_t \approx e^{i\theta_t}|0\rangle_{\epsilon t} = e^{i\theta_t}\begin{cases} -|g\rangle, & \text{for } t \in [0, \frac{\pi}{2\epsilon}]; \\ \frac{-\bar\Omega_e \sin^2(\epsilon t)|g\rangle + \bar\Omega_g \cos^2(\epsilon t)|e\rangle}{\sqrt{\bar\Omega_g^2 \cos^4(\epsilon t) + \bar\Omega_e^2 \sin^4(\epsilon t)}}, & \text{for } t \in [\frac{\pi}{2\epsilon}, \frac{\pi}{\epsilon}]; \\ |e\rangle, & \text{for } t \in [\frac{\pi}{\epsilon}, \frac{3\pi}{2\epsilon}]; \end{cases}$$

where θ_t is a time-varying global phase. Thus at the final time $t = \frac{3\pi}{2\epsilon}$, $|\psi\rangle$ coincides, up to a global phase to $|e\rangle$. It is surprising that during this adiabatic passage from $|g\rangle$ to $|e\rangle$ the control u_{ef} driving the transition $e \leftrightarrow f$ is turned on first whereas the control u_{gf} driving transition $g \leftrightarrow f$ is turned on later. It is also very interesting that the precise knowledge of the coupling parameter μ_{gf} and μ_{ef} is not necessary. However the precise knowledge of the transition frequencies ω_{gf} and ω_{ef} is required. Such adiabatic control strategies are widely used (see, e.g., the recent review [40]).

Exercice 3.4.2. *Design an adiabatic passage $s \mapsto (\Omega_{gf}(s), \Omega_{ef}(s))$ from $|g\rangle$ to $\frac{-|g\rangle+|e\rangle}{\sqrt{2}}$, up to a global phase.*

3.4.4 Chirped pulse for a 2-level system

Let us start with $H = \frac{\omega_{eg}}{2}\sigma_z + \frac{u}{2}\sigma_x$ considered in Section 2.1 and the quasi-resonant control ($|\omega_r - \omega_{eg}| \ll \omega_{eg}$)

$$u(t) = v\left(e^{i(\omega_r t+\theta)} + e^{-i(\omega_r t+\theta)}\right)$$

where $v, \theta \in \mathbb{R}$, $|v|$ and $|\frac{d\theta}{dt}|$ are small and slowly varying

$$|v|, \left|\tfrac{d\theta}{dt}\right| \ll \omega_{eg}, \quad \left|\tfrac{dv}{dt}\right| \ll \omega_{eg}|v|, \quad \left|\tfrac{d^2\theta}{dt^2}\right| \ll \omega_{eg}\left|\tfrac{d\theta}{dt}\right|.$$

Following similar computations to those of Subsection 3.3.3, consider the following change of frame $|\psi\rangle = e^{-i\frac{\omega_r t+\theta}{2}\sigma_z}|\phi\rangle$. Then $i\frac{d}{dt}\psi = H|\psi\rangle$ becomes

$$i\frac{d}{dt}|\phi\rangle = \left(\tfrac{\omega_{eg}-\omega_r-\frac{d}{dt}\theta}{2}\sigma_z + \tfrac{ve^{2i(\omega_r t+\theta)}+v}{2}\sigma_+ + \tfrac{ve^{-2i(\omega_r t-\theta)}+v}{2}\sigma_-\right)|\phi\rangle.$$

With $\Delta_r = \omega_{eg} - \omega_r$ and $w = -\frac{d}{dt}\theta$ and using the first order approximation (see (3.19) with $H^{1^{\rm st}}_{\rm rwa}$) we get the following averaged control Hamiltonian

$$H_{\rm chirp} = \tfrac{\Delta_r+w}{2}\sigma_z + \tfrac{v}{2}\sigma_x$$

where (v, w) are two real control inputs. Take three constant parameters $a > |\Delta_r|$, $b > 0$, $0 < \epsilon \ll a, b$. Set

$$w = a\cos(\epsilon t), \quad v = b\sin^2(\epsilon t).$$

Set $s = \epsilon t$ varying in $[0, \pi]$. These explicit expressions are not essential. Only the shape of $s \mapsto w(s)$ and of $s \mapsto v(s)$ are important here: w decreases regularly from a to $-a$; v is a bump function that remains strictly positive for $s \in]0, \pi[$ and that vanishes with its derivatives at $s = 0$ and $s = \pi$.

The spectral decomposition of H_{chirp} for $s \in]0, \pi[$ is standard with two distinct and opposite eigenvalues.

$$\Omega_- = -\frac{\sqrt{(\Delta_r+w)^2+v^2}}{2} \text{ with } |-\rangle = \frac{\cos\alpha\,|g\rangle - (1-\sin\alpha)\,|e\rangle}{\sqrt{2(1-\sin\alpha)}}$$
$$\Omega_+ = \frac{\sqrt{(\Delta_r+w)^2+v^2}}{2} \text{ with } |+\rangle = \frac{(1-\sin\alpha)\,|g\rangle + \cos\alpha\,|e\rangle}{\sqrt{2(1-\sin\alpha)}}$$

where $\alpha \in]\frac{-\pi}{2}, \frac{\pi}{2}[$ is defined by $\tan\alpha = \frac{\Delta_r+w}{v}$. Since $\lim_{s\mapsto 0^+}\alpha = \frac{\pi}{2}$ and $\lim_{s\mapsto\pi^-}\alpha = -\frac{\pi}{2}$

$$\lim_{s\mapsto 0^+}|-\rangle_s = |g\rangle\,, \quad \lim_{s\mapsto 0^+}|+\rangle_s = |e\rangle\,, \quad \lim_{s\mapsto\pi^-}|-\rangle_s = -|e\rangle\,, \quad \lim_{s\mapsto\pi^-}|+\rangle_s = |g\rangle\,.$$

Consequently the adiabatic approximation of Theorem 3.4.1 implies that the solution $|\phi\rangle$ of

$$i\frac{d}{dt}|\phi\rangle = \left(\frac{\Delta_r+a\cos(\epsilon t)}{2}\sigma_z + \frac{b\sin^2(\epsilon t)}{2}\sigma_x\right)|\phi\rangle\,, \quad |\phi\rangle_{t=0} = |g\rangle$$

is given approximatively, for ϵ small and $t \in [0, \frac{\pi}{\epsilon}]$, by

$$|\phi\rangle_t = e^{i\vartheta_t}\,|-\rangle_{s=\epsilon t}$$

with ϑ_t a time-varying global phase. Thus for $t = \frac{\pi}{\epsilon}$, $|\psi\rangle$ coincides with $|e\rangle$ up to a global phase. Notice the remarkable robustness of such adiabatic control strategy. We do not need to know precisely neither the detuning Δ_r nor the chirp and control amplitudes a and b. This means in particular that such adiabatic chirp control from g to e is insensitive to all parameters appearing in a 2-level system.

Such adiabatic chirp passage can be extended to any ladder configuration that is slightly an-harmonic.

3.5 Optimal control

Take the n-level system $i\frac{d}{dt}|\psi\rangle = (H_0 + \sum_{k=1}^m u_k H_k)\,|\psi\rangle$, initial and final states $|\psi_a\rangle$ and $|\psi_b\rangle$ and a transition time $T > 0$ ($\langle\psi_a|\psi_a\rangle = \langle\psi_b|\psi_b\rangle = 1$). Assume that this system is controllable, i.e. according to Theorem 3.1.9, the Lie algebra generated by $\{iH_0, iH_1, \ldots, iH_k\}$ contains $su(n)$. We are looking for optimal controls $[0, T] \ni t \mapsto u(t)$

minimizing $\int_0^T (\sum_{k=1}^m u_k^2)$ and steering $|\psi\rangle$ from $|\psi_a\rangle$ at $t=0$ to $|\psi_b\rangle$ at $t=T$. Thus we are considering the following problem

$$\begin{array}{c} \min \\ u_k \in L^2([0,T],\mathbb{R}),\ k=1,\dots,m \\ i\frac{d}{dt}|\psi\rangle = (H_0 + \sum_{k=1}^m u_k H_k)|\psi\rangle\,,\ t\in(0,T) \\ |\psi\rangle_{t=0} = |\psi_a\rangle\,,\ |\langle\psi_b|\psi\rangle|^2_{t=T} = 1 \end{array} \quad \frac{1}{2}\int_0^T \left(\sum_{k=1}^m u_k^2(t)\right) dt \tag{3.31}$$

for given T, $|\psi_a\rangle$ and $|\psi_b\rangle$ ($\langle\psi_a|\psi_a\rangle = \langle\psi_b|\psi_b\rangle = 1$). Notice that $|\langle\psi_b|\psi\rangle|^2 = 1$ means that $|\psi\rangle_T = e^{i\theta}|\psi_b\rangle$ where $\theta\in\mathbb{R}$ is an arbitrary global phase.

Since the initial and final constraints are difficult to satisfy simultaneously from a numerical point of view, we will consider also the second problem where the final constraint is relaxed

$$\begin{array}{c} \min \\ u_k \in L^2([0,T],\mathbb{R}),\ k=1,\dots,m \\ i\frac{d}{dt}|\psi\rangle = (H_0 + \sum_{k=1}^m u_k H_k)|\psi\rangle\,,\ t\in(0,T) \\ |\psi\rangle_{t=0} = |\psi_a\rangle \end{array} \quad \frac{1}{2}\int_0^T \left(\sum_{k=1}^m u_k^2(t)\right) dt + \frac{\alpha}{2}(1-|\langle\psi_b|\psi\rangle|^2_T) \tag{3.32}$$

with the positive penalization coefficient $\alpha>0$. Notice that for α large this problem tends to the original one (3.31).

3.5.1 First order stationary condition

The first order conditions recalled in Appendix G yield to the following set of necessary conditions. Notice that the adjoint state can be seen as a Ket, denoted by $|p\rangle\in\mathbb{C}^n$ (of constant length but different of one in general) since it satisfies the same Schrödinger equations as $|\psi\rangle$.

For problem (3.31), the first order stationary conditions read:

$$\left\{\begin{array}{c} i\frac{d}{dt}|\psi\rangle = (H_0 + \sum_{k=1}^m u_k H_k)|\psi\rangle\,,\ t\in(0,T) \\ i\frac{d}{dt}|p\rangle = (H_0 + \sum_{k=1}^m u_k H_k)|p\rangle\,,\ t\in(0,T) \\ u_k = -\Im\Big(\langle p|H_k|\psi\rangle\Big), k=1,\dots,m,\quad t\in(0,T) \\ |\psi\rangle_{t=0} = |\psi_a\rangle\,,\ |\langle\psi_b|\psi\rangle|^2_{t=T} = 1 \end{array}\right. \tag{3.33}$$

For the relaxed problem (3.32), the first order stationary conditions read:

$$\left\{\begin{array}{c} i\frac{d}{dt}|\psi\rangle = (H_0 + \sum_{k=1}^m u_k H_k)|\psi\rangle\,,\ t\in(0,T) \\ i\frac{d}{dt}|p\rangle = (H_0 + \sum_{k=1}^m u_k H_k)|p\rangle\,,\ t\in(0,T) \\ u_k = -\Im\Big(\langle p|H_k|\psi\rangle\Big), k=1,\dots,m,\quad t\in(0,T) \\ |\psi\rangle_{t=0} = |\psi_a\rangle\,,\ |p\rangle_{t=T} = -\alpha\langle\psi_b|\psi\rangle_{t=T}\ |\psi_b\rangle\,. \end{array}\right. \tag{3.34}$$

These optimality conditions differ only by the boundary conditions at $t = 0$ and $t = T$: the common part

$$\begin{array}{c} i\frac{d}{dt}\left|\psi\right\rangle = \left(H_0 + \sum_{k=1}^m u_k H_k\right)\left|\psi\right\rangle,\ t \in (0,T) \\ i\frac{d}{dt}\left|p\right\rangle = \left(H_0 + \sum_{k=1}^m u_k H_k\right)\left|p\right\rangle,\ t \in (0,T) \\ u_k = -\Im\left(\langle p|H_k|\psi\rangle\right), k = 1,\dots,m,\quad t \in (0,T) \end{array}$$

is a Hamiltonian system with $\left|\psi\right\rangle$ and $\left|p\right\rangle$ being the conjugate variables. The underlying Hamiltonian function is given by : $\overline{\mathbb{H}}(\left|\psi\right\rangle, \left|p\right\rangle) = \min_{u\in\mathbb{R}^m} \mathbb{H}(\left|\psi\right\rangle, \left|p\right\rangle, u)$ where

$$\mathbb{H}(\left|\psi\right\rangle, \left|p\right\rangle, u) = \frac{1}{2}\left(\sum_{k=1}^m u_k^2\right) + \Im\left(\left\langle p\right| H_0 + \sum_{k=1}^m u_k H_k \left|\psi\right\rangle\right). \tag{3.35}$$

Thus for any solutions $(\left|\psi\right\rangle, \left|p\right\rangle, u)$ of (3.33) or (3.34), $\mathbb{H}(\left|\psi\right\rangle, \left|p\right\rangle, u)$ is independent of t. Notice that

$$\overline{\mathbb{H}}(\left|\psi\right\rangle, \left|p\right\rangle) = \Im\left(\left\langle p\right| H_0 \left|\psi\right\rangle\right) - \frac{1}{2}\left(\sum_{k=1}^m \Im\Big(\langle p|H_k|\psi\rangle\Big)^2\right).$$

3.5.2 Monotone numerical scheme

For the relaxed problem (3.32) a general monotone iteration scheme exists. Defining the cost function

$$J(u) = \frac{1}{2}\int_0^T \left(\sum_{k=1}^m u_k^2(t)\right)\, dt + \frac{\alpha}{2}(1 - |\langle\psi_b|\psi_u\rangle|_T^2)$$

where $\left|\psi_u\right\rangle$ denotes the solution of $i\frac{d}{dt}\left|\psi\right\rangle = \left(H_0 + \sum_{k=1}^m u_k H_k\right)\left|\psi\right\rangle$ starting from $\left|\psi_a\right\rangle$, and starting from an initial guess $u^0 \in L^2([0,T],\mathbb{R}^m)$, this scheme generates a sequence of controls $u^\nu \in L^2([0,T],\mathbb{R}^m)$, $\nu = 1,2,\dots$, such that the cost $J(u^\nu)$ is decreasing, $J(u^{\nu+1}) \leq J(u^\nu)$

This scheme does not guaranty in general the convergence to an optimal solution. But applied on several example with a correct tuning of the penalization coefficient α, it produces interesting controls with $\left|\psi\right\rangle_T$ close to $\left|\psi_b\right\rangle$. Such monotonic schemes have been proposed for quantum systems in [65] for the first time (see also [73] for a slightly different version). We follow here the presentation of [9] which also provides an extension to infinite dimensional case. See also [19] for much earlier results on optimal control in infinite dimensional cases.

Take $u, v \in L^2([0,T],\mathbb{R}^m)$, denote by $P = \left|\psi_b\right\rangle\left\langle\psi_b\right|$ the orthogonal projector on $\left|\psi_b\right\rangle$, then

$$J(u) - J(v) = -\frac{\alpha\Big(\langle\psi_u - \psi_v|P|\psi_u - \psi_v\rangle_T + \langle\psi_u - \psi_v|P|\psi_v\rangle_T + \langle\psi_v|P|\psi_u - \psi_v\rangle_T\Big)}{2} + \int_0^T \frac{\sum_{k=1}^m (u_k - v_k)(u_k + v_k)}{2}.$$

Denote by $|p_v\rangle$ the adjoint associated to v, i.e. the solution of the backward systems

$$i\frac{d}{dt}|p_v\rangle = \left(H_0 + \sum_{k=1}^{m} v_k H_k\right)|p_v\rangle\,,\ |p_v\rangle_T = -\alpha P\,|\psi_v\rangle_T\,.$$

We have

$$i\frac{d}{dt}(|\psi_u\rangle - |\psi_v\rangle) = \left(H_0 + \sum_{k=1}^{m} v_k H_k\right)(|\psi_u\rangle - |\psi_v\rangle) + \left(\sum_{k=1}^{m}(u_k - v_k)H_k\right)|\psi_u\rangle\,.$$

We consider the Hermitian product of this relation with the adjoint state $|p_v\rangle$:

$$\left\langle p_v \left| \tfrac{d(\psi_u-\psi_v)}{dt} \right\rangle = \left\langle p_v \left| \tfrac{H_0+\sum_{k=1}^m v_k H_k}{i} \right| \psi_u - \psi_v \right\rangle + \left\langle p_v \left| \tfrac{\sum_{k=1}^m (u_k-v_k)H_k}{i} \right| \psi_u \right\rangle.$$

An integration by parts yields

$$\begin{aligned}\int_0^T \left\langle p_v \left| \tfrac{d(\psi_u-\psi_v)}{dt} \right\rangle &= \langle p_v|\psi_u-\psi_v\rangle_T - \langle p_v|\psi_u-\psi_v\rangle_0 - \int_0^T \left\langle \tfrac{dp_v}{dt} \right| \psi_u-\psi_v\rangle \\ &= -\alpha\langle\psi_v|P|\psi_u-\psi_v\rangle_T + \int_0^T \left\langle p_v \left| \tfrac{H_0+\sum_{k=1}^m v_k H_k}{i} \right| \psi_u-\psi_v \right\rangle\end{aligned}$$

since $|\psi_v\rangle_0 = |\psi_u\rangle_0$, $|p_v\rangle_T = -\alpha P\,|\psi_v\rangle_T$ and $\frac{d}{dt}\langle p_v| = -\langle p_v|\left(\frac{H_0+\sum_{k=1}^m v_k H_k}{i}\right)$. We get:

$$-\alpha\langle\psi_v|P|\psi_u-\psi_v\rangle_T = \int_0^T \left\langle p_v \left| \tfrac{\sum_{k=1}^m (u_k-v_k)H_k}{i} \right| \psi_u \right\rangle.$$

Thus $\alpha\Re\left(\langle\psi_v|P|\psi_u-\psi_v\rangle_T\right) = -\int_0^T \Im\left(\langle p_v\,|\sum_{k=1}^m (u_k-v_k)H_k|\,\psi_u\rangle\right)$. Finally we have

$$\begin{aligned}J(u) - J(v) = &-\tfrac{\alpha}{2}\left(\langle\psi_u-\psi_v|P|\psi_u-\psi_v\rangle\right)_T \\ &+ \tfrac{1}{2}\sum_{k=1}^{m}\left(\int_0^T (u_k-v_k)\left(u_k+v_k+2\Im\left(\langle p_v\,|H_k|\,\psi_u\rangle\right)\right)\,dt\right).\end{aligned}$$

If each u_k satisfies $u_k = -\Im\left(\langle p_v\,|H_k|\,\psi_u\rangle\right)$ for all $t \in [0,T)$ we have

$$J(u) - J(v) = -\tfrac{\alpha}{2}\left(\langle\psi_u-\psi_v|P|\psi_u-\psi_v\rangle\right)_T - \tfrac{1}{2}\sum_{k=1}^{m}\left(\int_0^T (u_k-v_k)^2\right)$$

and thus $J(u) \le J(v)$.

These computations suggest the following iteration scheme. Assume that, at step ν, we have computed the control u^ν, the associated quantum state $|\psi^\nu\rangle = |\psi_{u^\nu}\rangle$ and its adjoint $|p^\nu\rangle = |p_{u^\nu}\rangle$. We get their new time values $u^{\nu+1}$, $|\psi^{\nu+1}\rangle$ and $|p^{\nu+1}\rangle$ in two steps:

1. Imposing $u_k^{\nu+1} = -\Im\left(\langle p^\nu | H_k | \psi^{\nu+1}\rangle\right)$ is just a feedback; one get $u^{\nu+1}$ just by a forward integration of the nonlinear Schrödinger equation,

$$i\frac{d}{dt}|\psi\rangle = \left(H_0 - \sum_{k=1}^{m} \Im\left(\langle p_{u^\nu} | H_k | \psi\rangle\right) H_k\right)|\psi\rangle\,, \quad |\psi\rangle_0 = |\psi_a\rangle\,,$$

that provides $[0,T] \ni t \mapsto |\psi^{\nu+1}\rangle$ and the m new controls $u_k^{\nu+1}$.

2. Backward integration from $t = T$ to $t = 0$ of

$$i\frac{d}{dt}|p\rangle = \left(H_0 + \sum_{k=1}^{m} u_k^{\nu+1}(t) H_k\right)|p\rangle\,, \quad |p\rangle_T = -\alpha\langle\psi_b|\psi^{\nu+1}\rangle_T\,|\psi_b\rangle$$

yields to the new adjoint trajectory $[0,T] \ni t \mapsto |p^{\nu+1}\rangle$.

3.5.3 Optimality and resonance

This section is strongly inspired from [16] where the results presented in this subsection have been published. We will consider here the special class of driftless systems with n energy levels $|1\rangle$, $\ldots|n\rangle$ forming an orthonormal frame of $\mathbb{C}^n$. Denote by I a subset of $\{1,\ldots,n\}^2$ such that if $(k,l) \in I$, then $(l,k) \in I$ (symmetric) and $l \neq k$ (no diagonal). The controlled Schrödinger equation admits the following form:

$$i\frac{d}{dt}|\psi\rangle = \left(\sum_{(k,l)\in I} \mu_{kl}\mathbf{u}_{kl}\,|k\rangle\langle l|\right)|\psi\rangle$$

where the complex $\mathbf{u}_{kl} = \mathbf{u}_{lk}^*$ are the controls and $\mu_{kl} = \mu_{lk}$ are real strictly positive parameters. As shown in Section 3.3 (see, e.g., the trapped ions models), these driftless structures are obtained after the application of a rotating wave approximation and assuming perfect resonance (negligible detuning) and slowly varying complex amplitudes $\mathbf{u}_{kl}$ ($(k,l) \in I$). We consider here population transfer: the initial and final states are only characterized by the positive real numbers $|\langle k|\psi\rangle|^2$. This problem reads

$$\begin{array}{c} \min \\ \mathbf{u}_{k,l} \in L^2([0,T],\mathbb{C}),\ (k,l) \in I \\ i\frac{d}{dt}|\psi\rangle = \left(\sum_{(k,l)\in I} \mu_{kl}\mathbf{u}_{kl}\,|k\rangle\langle l|\right)|\psi\rangle\,,\ t \in (0,T) \\ |\langle k|\psi\rangle|^2_{t=0} = a_k^2,\ |\langle k|\psi\rangle|^2_{t=T} = b_k^2,\ k = 1,\ldots,n \end{array} \quad \frac{1}{2}\int_0^T \left(\sum_{(k,l)\in I} |\mathbf{u}_{kl}|^2(t)\right) dt \qquad (3.36)$$

for given T, $a_k \geq 0$ and $b_k \geq 0$ ($\sum_{k=1}^n a_k^2 = \sum_{k=1}^n b_k^2 = 1$).

We will prove here below that this problem admits the same minimal cost as the following reduced problem

$$\begin{array}{l}\min \\ v_{k,l} \in L^2([0,T],\mathbb{R}),\ v_{kl} = -v_{l,k},\ (k,l) \in I \\ \frac{d}{dt}\left|\phi\right\rangle = \left(\sum_{(k,l)\in I} \mu_{kl} v_{kl} \left|k\right\rangle \left\langle l\right|\right) \left|\phi\right\rangle,\ t \in (0,T) \\ \left\langle k|\phi\right\rangle|_{t=0} = a_k,\ \left\langle k|\phi\right\rangle_{t=T} = b_k,\ k = 1,\dots,n \end{array} \quad \frac{1}{2}\int_0^T \left(\sum_{(k,l)\in I} |v_{kl}|^2(t)\right) dt \tag{3.37}$$

where the components of $|\psi\rangle = |\phi\rangle$ remain real, the $\mathbf{u}_{kl}$'s are purely imaginary, $\mathbf{u}_{kl} = iv_{kl}$ ($v_{kl} \in \mathbb{R}$ with $v_{kl} = -v_{lk}$), the initial and final values of $|\phi\rangle$ are given. Thus an optimal solution $\mathbf{u}_{kl}$ of (3.36) is provided by an optimal solution v_{kl} of (3.37) via $\mathbf{u}_{kl} = iv_{kl}$, i.e. all the $\mathbf{u}_{kl}$'s have the same common phase (modulo π). When we go back to resulting physical control,

$$u_{kl}(t) = \mathbf{u}_{kl}(t)e^{i(\omega_k-\omega_l)t} + \mathbf{u}_{kl}^*(t)e^{-i(\omega_k-\omega_l)t} = -2v_{kl}(t)\sin\left((\omega_k-\omega_l)t\right),$$

it is in resonance with the frequency transition between $|k\rangle$ and $|l\rangle$. It contains only amplitude modulations (up to a π phase-shift since v_{kl} can pass through zero). For such driftless quantum systems, optimal population transfer is achieved by resonant controls.

Denote by θ, the n-uple of angles $(\theta_1, \theta_2, \dots, \theta_n)$ where each angle θ_k is defined modulo 2π. Associated to any $\theta = (\theta_1, \theta_2, \dots, \theta_n)$ consider the following transformations on $|\psi\rangle$ and $\mathbf{u}_{kl}$ defined by

$$\begin{aligned} |\psi\rangle \mapsto \left|\psi^\theta\right\rangle &= \left(\sum_{k=1}^n e^{i\theta_k} \left|k\right\rangle \left\langle k\right|\right) |\psi\rangle \\ \mathbf{u}_{kl} \mapsto \mathbf{u}_{kl}^\theta &= e^{i(\theta_k-\theta_l)}\mathbf{u}_{kl}. \end{aligned}$$

Simple computations show that these transformations leave the cost and the constraints of (3.36) unchanged (invariance versus phase choices for each $|k\rangle$).

Take $\mathbf{u}_{kl} \in L^2([0,T],\mathbb{C})$ and $[0,T] \ni t \mapsto |\psi\rangle \in \mathbb{C}^n$ satisfying the constraints of (3.36). Choose $\theta = (\theta_1, \dots, \theta_n)$ such that $\left\langle k|\psi^\theta\right\rangle_{t=0} = a_k$ for $k = 1, \dots, n$. Then $\left|\psi^\theta\right\rangle$ and $\mathbf{u}_{kl}^\theta$ satisfy

$$\begin{array}{c} i\frac{d}{dt}\left|\psi^\theta\right\rangle = \left(\sum_{(k,l)\in I} \mu_{kl}\mathbf{u}_{kl}^\theta \left|k\right\rangle \left\langle l\right|\right) \left|\psi^\theta\right\rangle,\ t \in (0,T) \\ \left\langle k|\psi^\theta\right\rangle_{t=0} = a_k,\ \left|\left\langle k|\psi^\theta\right\rangle\right|^2_{t=T} = b_k^2,\ k = 1,\dots,n. \end{array}$$

Since $|\mathbf{u}_{kl}| = |\mathbf{u}_{kl}^\theta|$ the cost $\frac{1}{2}\int_0^T \left(\sum_{(k,l)\in I} |\mathbf{u}_{kl}|^2(t)\right) dt$ is unchanged. In the sequel, we remove the superscript θ and assume that $|\psi\rangle$ and $\mathbf{u}_{kl}$ satisfy

$$i\frac{d}{dt}\left|\psi\right\rangle = \left(\sum_{(k,l)\in I} \mu_{kl}\mathbf{u}_{kl} \left|k\right\rangle \left\langle l\right|\right) \left|\psi\right\rangle,\ \left\langle k|\psi\right\rangle_{t=0} = a_k,\ \left|\left\langle k|\psi\right\rangle\right|^2_{t=T} = b_k^2.$$

Set $\psi_k = \langle k|\psi\rangle$. The evolution of population k, $|\psi_k|^2$, is given by

$$\frac{d}{dt}(|\psi_k|^2) = \sum_{l \mid (k,l)\in I} \mu_{kl} \frac{\mathbf{u}_{kl}\mathbf{z}_{kl}^* - \mathbf{u}_{kl}^*\mathbf{z}_{kl}}{i}$$

where we have set $\mathbf{z}_{kl} = \psi_k\psi_l^* \in \mathbb{C}$. The evolution of the direction of ψ_k in the complex plane is governed by

$$\psi_k^*\frac{d}{dt}\psi_k - \psi_k\frac{d}{dt}\psi_k^* = \sum_{l \mid (k,l)\in I} \mu_{kl} \frac{\mathbf{u}_{kl}\mathbf{z}_{kl}^* + \mathbf{u}_{kl}^*\mathbf{z}_{kl}}{i}.$$

When $z_{kl} \neq 0$, we can decompose $\mathbf{u}_{kl}$ as a real superposition of $\frac{\mathbf{z}_{kl}}{|\mathbf{z}_{kl}|}$ and $i\frac{\mathbf{z}_{kl}}{|\mathbf{z}_{kl}|}$:

$$\mathbf{u}_{kl} = \left(\tfrac{\mathbf{u}_{kl}\mathbf{z}_{kl}^* + \mathbf{u}_{kl}^*\mathbf{z}_{kl}}{2|\mathbf{z}_{kl}|}\right)\frac{\mathbf{z}_{kl}}{|\mathbf{z}_{kl}|} + \left(\tfrac{\mathbf{u}_{kl}\mathbf{z}_{kl}^* - \mathbf{u}_{kl}^*\mathbf{z}_{kl}}{2i|\mathbf{z}_{kl}|}\right)\frac{i\mathbf{z}_{kl}}{|\mathbf{z}_{kl}|}.$$

For $(k,l) \in I$ consider the real value function v_{kl} defined by

$$v_{kl}(t) = \begin{cases} 0, & \text{if } \mathbf{z}_{kl}(t) = 0; \\ \frac{\mathbf{u}_{kl}(t)\mathbf{z}_{kl}^*(t) - \mathbf{u}_{kl}^*(t)\mathbf{z}_{kl}(t)}{2i|\mathbf{z}_{kl}(t)|}, & \text{if } \mathbf{z}_{kl}(t) \neq 0;. \end{cases}$$

We have $v_{kl} = -v_{lk}$ since $\mathbf{u}_{kl}^* = \mathbf{u}_{lk}$ and $z_{kl}^* = z_{lk}$. When $\mathbf{z}_{kl} \neq 0$, v_{kl} is the component of $\mathbf{u}_{kl}$ along $i\frac{\mathbf{z}_{kl}}{|\mathbf{z}_{kl}|}$, consequently $|v_{kl}| \leq |\mathbf{u}_{kl}|$ (the later is true even if $\mathbf{z}_{kl} = 0$). Thus each v_{kl} belongs to $L^2([0,T],\mathbb{R})$. Consider now the solution $|\phi\rangle$ of the initial value problem

$$\frac{d}{dt}\phi_k = \sum_{l \mid (k,l)\in I} \mu_{kl} v_{kl}\phi_l, \quad \phi_k(0) = a_k, \quad k = 1,\dots,n.$$

Since the $|\psi_k|$'s satisfy the same differential system with the same initial condition, we have $\phi_k = |\psi_k|$. Then $|\phi\rangle = \sum_{k=1}^n \phi_k |k\rangle$ satisfies the final condition $\langle k|\phi\rangle = b_k$ of (3.37).

To summarize: starting from complex controls $\mathbf{u}_{kl} \in L^2([0,T],\mathbb{C})$ satisfying the constraints of problem (3.36), we have constructed real controls $v_{kl} \in L^2([0,T],\mathbb{C})$ satisfying the constraints of the reduced problem (3.37); the cost associated to $\mathbf{u}_{kl}$ is larger than the cost associated to v_{kl} since $|v_{kl}| \leq |u_{kl}|$. Moreover for all k, $|\langle k|\psi\rangle| = \langle k|\phi\rangle$ and thus the components of $|\phi\rangle$ remain always positive.

The first order stationary conditions of the full problem (3.36) are:

$$\begin{gathered} i\tfrac{d}{dt}|\psi\rangle = \left(\textstyle\sum_{(k,l)\in I} \mu_{kl}\mathbf{u}_{kl}|k\rangle\langle l|\right)|\psi\rangle\,, \ t \in (0,T) \\ i\tfrac{d}{dt}|p\rangle = \left(\textstyle\sum_{(k,l)\in I} \mu_{kl}\mathbf{u}_{kl}|k\rangle\langle l|\right)|p\rangle\,, \ t \in (0,T) \\ \mathbf{u}_{kl} = i\mu_{kl}\left(\langle p|l\rangle\langle k|\psi\rangle - \langle\psi|l\rangle\langle k|p\rangle\right), \ t \in (0,T), \ (k,l) \in I \\ |\langle k|\psi\rangle|_{t=0}^2 = a_k^2, \ |\langle k|\psi\rangle|_{t=T}^2 = b_k^2, \ k = 1,\dots,n \end{gathered} \tag{3.38}$$

The first order stationary conditions of the reduced problem (3.37) are:

$$\begin{array}{c} \frac{d}{dt}\left|\phi\right\rangle = \left(\sum_{(k,l)\in I}\mu_{kl}v_{kl}\left|k\right\rangle\left\langle l\right|\right)\left|\phi\right\rangle\,,\ t\in(0,T) \\ \frac{d}{dt}\left|p\right\rangle = \left(\sum_{(k,l)\in I}\mu_{kl}v_{kl}\left|k\right\rangle\left\langle l\right|\right)\left|p\right\rangle\,,\ t\in(0,T) \\ v_{kl} = \mu_{kl}\left(\langle p|l\rangle\langle k|\phi\rangle - \langle\phi|l\rangle\langle k|p\rangle\right),\ t\in(0,T),\ (k,l)\in I \\ \langle k|\phi\rangle_{t=0} = a_k,\ \langle k|\phi\rangle_{t=T} = b_k,\ k=1,\ldots,n \end{array} \tag{3.39}$$

where $\left|\phi\right\rangle$, $\left|p\right\rangle$ belong to $\mathbb{R}^n$ and the v_{kl}'s to $\mathbb{R}$. For $n=2$ and $n=3$, problem (3.37) has been completely solved in [15]. For $n=4$, no general solution has been proposed, up to now. The following exercise considers two simple cases for $n=2,3$.

Exercice 3.5.1. *Consider the reduced problem (3.37) and its stationary conditions (3.39)*

1. *Show that the Hamiltonian $\mathbb{H}$ and the minimized Hamiltonian $\overline{\mathbb{H}}$ read:*

$$\mathbb{H}(\left|\phi\right\rangle,\left|p\right\rangle,v) = \sum_{(k,l)\in I}\tfrac{1}{2}v_{kl}^2 + \mu_{kl}v_{kl}\langle k|p\rangle\langle l|\phi\rangle$$

$$\overline{\mathbb{H}}(\left|\phi\right\rangle,\left|p\right\rangle) = -\sum_{(k,l)\in I}\tfrac{\mu_{kl}^2}{2}\Big(\langle p|l\rangle\langle k|\phi\rangle - \langle\phi|l\rangle\langle k|p\rangle\Big)^2$$

   (hint: remember that $v_{kl}=-v_{lk}$).

2. *For $n=2$, $\mu_{12}=\mu>0$, $(a_1,a_2)=(\cos\alpha,\sin\alpha)$ with $\alpha\in[0,\frac{\pi}{2}]$ and $(b_1,b_2)=(\cos\beta,\sin\beta)$ with $\beta\in[0,\frac{\pi}{2}]$, solve (3.39).*

3. *For $n=3$, $\mu_{12}=\mu_{23}=\mu_{31}=\mu>0$, solve (3.39).*

3.5.4 Cheap control and quantum control landscapes

For the quantum system $i\frac{d}{dt}\left|\psi\right\rangle = (H_0+\sum_{k=1}^m u_kH_k)\left|\psi\right\rangle$ with initial condition $\left|\psi_a\right\rangle\in\mathbb{R}^n$, we are here interested in finding an open-loop control $[0,T]\ni\mapsto u(t)\in\mathbb{R}^m$ ($T>0$ is given) that maximizes the expectation value $\langle\psi|O|\psi\rangle_T$ of an observable O at the final time T. This question can be seen as a cheap control problem since it is associated to the following problem:

$$\begin{array}{c} \max \qquad \langle\psi|O|\psi\rangle_T \\ u_k\in L^2([0,T],\mathbb{R}),\ k=1,\ldots,m \\ i\frac{d}{dt}\left|\psi\right\rangle = (H_0+\sum_{k=1}^m u_kH_k)\left|\psi\right\rangle\,,\ t\in(0,T) \\ \left|\psi\right\rangle_{t=0} = \left|\psi_a\right\rangle \end{array} \tag{3.40}$$

where O is a Hermitian operator.

It is shown in [35] how to find experimentally u that maximizes $\langle\psi|O|\psi\rangle_T$ without knowing H_0 nor the H_k's . The procedure is simple when we have at our disposal an experimental setup that provides $\langle\psi|O|\psi\rangle_T$ once you have prescribed the open-loop control $[0,T]\mapsto u(t)$. It consists in parameterizing the function $u(t)$ through the parameters $s\in$

$\mathbb{R}^{n_s}$, $n_s > 0$, and then to maximize an appropriate $J(s)$, via an external loop manipulating s between each experiment and based on past experimental values of $\langle\psi|O|\psi\rangle_T$. The cost function $s \mapsto J(s)$ is given by $J(s) = \langle\psi^s|O|\psi^s\rangle_T$ where $|\psi^s\rangle_T$ is the solution at time T associated to the control u defined by s.

It is noticed in [35] (see also [58] for more recent results) that the mapping $s \mapsto J(s)$ does not admit, in general, local maxima that are not global ones: the landscape $(s, J(s))$ is monotone. Optimization algorithms cannot get trapped and should produce control parameters $\bar{s}$ for which J reaches it maximum $\bar{J} = J(\bar{s})$. The absence of local and non-global maxima is related to controllability and can be explained as follows.

Assume $i\frac{d}{dt}|\psi\rangle = (H_0 + \sum_{k=1}^m u_k H_k)|\psi\rangle$ is operator controllable (see Definition 3.1.6 and Theorem 3.1.9). Denote by $|\psi^u\rangle$ the solution of

$$i\frac{d}{dt}|\psi\rangle = (H_0 + \sum_{k=1}^m u_k H_k)|\psi\rangle, \quad |\psi\rangle_0 = |\psi_a\rangle$$

and by $\delta\psi$ the state of the first variation around $|\psi^u\rangle$ due to a variation δu of u:

$$i\frac{d}{dt}\delta\psi = (H_0 + \sum_{k=1}^m u_k H_k)\delta\psi + \sum_{k=1}^m \delta u_k H_k |\psi_u\rangle, \quad \delta\psi_0 = 0.$$

Controllability ensures that, for almost all controls $[0, T] \ni t \mapsto u(t)$, this first order system is controllable: this means that the linear mapping $\delta u \mapsto \delta\psi_T$ is onto (see, e.g., [24]) and thus its image contains the orthogonal space of $|\psi\rangle_T$. Assume that such u is parameterized by s. Associated to a variation δs of s is attached a variation δu of u and we can consider the following linear mapping

$$\mathbb{R}^{n_s} \ni \delta s \mapsto \delta\psi_T.$$

When $n_s > 2n-1$ and for a generic parameterization, this mapping still remains onto and thus its image contains $|\psi\rangle^\perp$.

Since the critical points of the mapping $|\psi\rangle \mapsto \langle\psi|O|\psi\rangle$ from the unit sphere of $\mathbb{C}^n$ to $\mathbb{R}$ are either global minima (smallest eigenvalue of O), global maxima (largest eigenvalue of O) or saddle points (intermediate eigenvalues of O), we conclude that for a generic parameterization s with $n_s \geq 2n-1$, the critical points of $s \mapsto \langle\psi^{u^s}|O|\psi^{u^s}\rangle_t$ are either global minima, global maxima or saddle points. The above arguments are heuristic. They can be made rigorous and are valid for any controllable system $\frac{d}{dt}x = f(x, u)$ where the maxima of $J(x)$ are all global ones (see, e.g., [59] for a more rigorous exposure in the context of open quantum systems). Note that, in the above analysis we have assumed no constraint on the control law and therefore the L^2-norm of the control can explode when we try to solve the optimization problem. As soon as we add a penalization on the control's norm, we lose the above property and local minima might appear.

3.6 Lyapunov Control

Take $i\frac{d}{dt}|\psi\rangle = (H_0 + \sum_{k=1}^m u_k H_k)|\psi\rangle$ with m scalar control(s) and $|\psi\rangle$ on the unit sphere of $\mathbb{C}^n$. Since $|\psi\rangle$ and $e^{i\theta(t)}|\psi\rangle$ describe the same physical state for any global phase $t \mapsto$

$\theta(t) \in \mathbb{R}$, the two states $|\psi_1\rangle$ and $|\psi_2\rangle$ are identified when there exists $\theta \in \mathbb{R}$ such that $|\psi_1\rangle = \exp(i\theta)\,|\psi_2\rangle$. To take into account such non trivial geometry, we add an additional control ω corresponding to $\frac{d}{dt}\theta$. Thus we consider the following control system

$$i\frac{d}{dt}|\psi\rangle = \left(H_0 + \omega\mathbf{1} + \sum_{k=1}^{m} u_k H_k\right)|\psi\rangle \tag{3.41}$$

where $\omega \in \mathbb{R}$ is a new control playing the role of a gauge degree of freedom. We can choose it arbitrarily without changing the physical quantities attached to $|\psi\rangle$. With such additional fictitious control ω, we will assume in the sequel that the state space is the unit sphere of $\mathbb{C}^n$ and the dynamics given by (3.41) admit $m+1$ independent real controls $(u_1, \ldots, u_m)$ and ω.

Assume that we have at our disposal a static feedback law, i.e., $m+1$ real functions $(f_k(|\psi\rangle))_{0\leq k\leq m}$, such that the closed-loop system

$$i\frac{d}{dt}|\psi\rangle = \left(H_0 + f_0(|\psi\rangle)\mathbf{1} + \sum_{k=1}^{m} f_k(|\psi\rangle)H_k\right)|\psi\rangle$$

converges asymptotically towards an eigen-state $\left|\bar{\psi}\right\rangle$ of H_0. These feedback laws can not be used in real-time on the physical system: we need to measure $|\psi\rangle$ at each time and any measurement process has a back-action for the quantum system; this back-action is not taken into account with such Schrödinger type models (see Chapter 4). Nevertheless, we can exploit such convergence to get open-loop controls $[0,T] \ni t \mapsto u(t)$ steering the system from $|\psi_a\rangle$ at $t=0$ to an arbitrary small neighborhood of $\left|\bar{\psi}\right\rangle$ at $t=T$: if $|\psi_a\rangle$ belongs to the attraction region of $\left|\bar{\psi}\right\rangle$, then the numerical integration over $[0,T]$ of the above non-linear Schrödinger equation, provides $t \mapsto u_k(t) = f_k(|\psi\rangle_t)$ such that, for T large enough, $|\psi\rangle_T$ is close to $\left|\bar{\psi}\right\rangle$. It is then sufficient to store the control trajectories $[0,T] \in t \mapsto u_k(t)$ as approximately steering open-loop controls from $|\psi_a\rangle$ to $\left|\bar{\psi}\right\rangle$.

The goal of this section is to propose a systematic method to construct such feedback laws (i.e., the functions f_0 and f_k) based on control Lyapunov techniques that goes back to the seminal paper [36] and that has been studied for quantum systems in [56, 55, 10, 63].

3.6.1 Tracking and state preparation

Consider a reference trajectory $t \mapsto (|\psi^r\rangle(t), \omega^r(t), u_1^r(t), \ldots, u_m^r(t))$, i.e., a smooth solution of (3.41) $i\frac{d}{dt}|\psi^r\rangle = (H_0 + \sum_k u_k^r H_k + \omega^r)\,|\psi^r\rangle$. Take the following real function $\mathcal{L}(|\psi_r\rangle, |\psi\rangle)$ ($\Re$ stands for real part):

$$\mathcal{L}(|\psi_r\rangle, |\psi\rangle) = 1 - \Re(\langle\psi_r|\psi\rangle) \tag{3.42}$$

$\mathcal{L}$ is positive for all $|\psi_r\rangle, |\psi\rangle \in \mathbb{C}^n$ of length 1. It vanishes only when $|\psi_r\rangle = |\psi\rangle$. Thus $\mathcal{L}$ measures a distance between $|\psi\rangle$ and the reference $|\psi_r\rangle$. Simple computations show that $\mathcal{L}$ is a control Lyapunov function when $|\psi\rangle$ satisfies (3.41) ($\Im$ stands for imaginary part)

$$\frac{d}{dt}\mathcal{L} = -\sum_{k=1}^{m}(u_k - u_k^r)\Im(\langle\psi_r|H_k|\psi\rangle) - (\omega - \omega^r)\Im(\langle\psi_r|\psi\rangle).$$

With the following static time-varying feedback

$$u_k = u_k^r + a_k \Im(\langle \psi_r | H_k | \psi \rangle) \text{ for } k = 1, \dots, m, \quad \omega = \omega^r + b \Im(\langle \psi_r | \psi \rangle) \tag{3.43}$$

($a_k > 0$ and $b > 0$ real parameters), we ensure that $\frac{d}{dt}\mathcal{L} \leq 0$.

Let us detail the important case where the reference trajectory corresponds to an equilibrium: $u_k^r = 0$, $\omega^r = -\bar{\omega}$ and $|\psi^r\rangle = |\bar{\psi}\rangle$ where $|\bar{\psi}\rangle$ is an eigenstate of H_0 associated to the eigenvalue $\bar{\omega}$: $H_0 |\bar{\psi}\rangle = \bar{\omega} |\bar{\psi}\rangle$. Then (3.43) becomes a static-state feedback:

$$u_k = a_k \Im(\langle \bar{\psi} | H_k | \psi \rangle) \text{ for } k = 1, \dots, m, \quad \omega = -\bar{\omega} + b \Im(\langle \bar{\psi} | \psi \rangle). \tag{3.44}$$

The following result is proved in [55] :

Theorem 3.6.1. *Consider the control Schrödinger equation (3.41) with $m = 1$ and $|\bar{\psi}\rangle$ an eigenstate of H_0 of energy $\bar{\omega}$. Take the static state feedback (3.44) with parameters a_1 and b strictly positive. If the linearized system around the steady-state $|\psi\rangle = |\bar{\psi}\rangle$ and stationary controls $\omega = -\bar{\omega}$ and $u_k = 0$ is controllable, then for all initial conditions $|\psi\rangle_0$ on the unit sphere of $\mathbb{C}^n$ except $-|\bar{\psi}\rangle$, the solution of the closed-loop system (3.41) with (3.44) converges asymptotically towards $|\bar{\psi}\rangle$. Moreover the closed-loop system admits only two equilibria: $|\bar{\psi}\rangle$ which is exponentially stable and $-|\bar{\psi}\rangle$ which is exponentially unstable.*

It is reasonable to guess that the same result is true for m arbitrary. For the tracking feedback, it seems also reasonable (following the original idea of [36]) that local controllability around the reference trajectory implies the convergence of (3.41) with (3.43) towards the reference trajectory (see, e.g., [63] for related results).

When the linearization around the steady-state $|\bar{\psi}\rangle$ is not controllable, application of Lasalle's invariance principle (see Appendix E and also [55]) indicates a strong lack of convergence. In this case it is still possible to overcome this convergence deficiency with implicit Lyapunov function as shown in [10] and to recover asymptotic convergence.

One can also think of other possibilities to overcome this lack of controllability for the linearized system. For instance, one can think of applying the tracking controller (3.43) around a time-periodic reference trajectory passing through the goal steady-state $|\bar{\psi}\rangle$. Roughly speaking, when the linearization around this periodic reference trajectory is controllable, we should recover asymptotic convergence of the tracking error towards zero. Then the open-loop steering control is obtained by stopping the closed-loop simulation at some discrete times of the form $NT_r + \bar{t}$ where T_r is the period of $|\psi_r\rangle$, $\bar{t} \in [0, T_r[$ is such that $|\psi_r\rangle_{\bar{t}} = |\bar{\psi}\rangle$ and N is an integer large enough to have $|\psi\rangle_{NT+\bar{t}}$ close enough to $|\psi_r\rangle_{NT+\bar{t}} = |\bar{\psi}\rangle$.

When $H_0 = 0$, it is very simple to generate periodic trajectories passing through any goal state $|\bar{\psi}\rangle$: take $\omega_r = 0$ and $t \mapsto u_k^r(t)$ odd and T_r-periodic. Then any solution of $i\frac{d}{dt}|\psi\rangle = (\sum_{k=1}^m u_k(t) H_k)|\psi\rangle$ is T_r periodic (use the time reversal symmetry implied by odd reference controls $u_k^r(t)$). Take for $|\psi_r\rangle$ the solution starting at $|\bar{\psi}\rangle$. Moreover if the Lie algebra spanned by $\{iH_1, \dots, iH_m\}$ contain $su(n)$, then there exists a lot of odd and T_r-periodic functions $u_k^r(t)$ such that the linearization around $|\psi_r\rangle$ is controllable

(see [24, 25]). Therefore the tracking controller should ensure convergence towards the reference trajectory.

Notice finally that the tracking feedback formulae (3.43) extend directly to infinite dimensional systems for which convergence based on Lasalle's invariance principle is much harder to obtain. We refer to [49, 12] for such stabilization results.

3.6.2 Tracking and quantum gate design

Let us consider the propagator $U_t \in U(n)$ associated to (3.41):

$$i\frac{d}{dt}U = \left(H_0 + \omega \mathbf{1} + \sum_{k=1}^{m} u_k H_k\right) U. \tag{3.45}$$

Notice the invariance of (3.45) versus right translation in $U(n)$ (the unitary group over $\mathbb{C}^n$): if U_t is solution of (3.45), then for any $V \in U(n)$, $U_t V$ is also a solution of (3.45) with the same controls ω and u.

Consider a reference trajectory $t \mapsto (U_t^r, \omega^r(t), u_1^r(t), \ldots, u_m^r(t))$ satisfying (3.45). The operator counterpart of the control Lyapunov function for $|\psi\rangle$ becomes now

$$\mathcal{L}(U, U^r) = n - \Re\left(\mathrm{Tr}\left(U^\dagger U^r\right)\right). \tag{3.46}$$

It is invariant versus right translation ($\mathcal{L}(U, U^r) = \mathcal{L}(UV, U^r V)$ for any $V \in U(n)$) and we have

$$\frac{d}{dt}\mathcal{L} = \sum_{k=1}^{m}(u_k - u_k^r)\Im\left(\mathrm{Tr}\left(U^\dagger H_k U^r\right)\right) + (\omega - \omega^r)\Im\left(\mathrm{Tr}\left(U^\dagger U^r\right)\right).$$

We deduce the following tracking feedback:

$$u_k = u_k^r - a_k\Im\left(\mathrm{Tr}\left(U^\dagger H_k U^r\right)\right) \text{ for } k = 1, \ldots, m, \quad \omega = \omega^r - b\Im\left(\mathrm{Tr}\left(U^\dagger U^r\right)\right) \tag{3.47}$$

($a_k > 0$ and $b > 0$ parameters).

Consider now the driftless case where $H_0 = 0$ and assume that we are looking for an open-loop control $[0, T] \ni t \mapsto (\omega, u_1, \ldots, u_m)$ steering U from $\mathbf{1}$ to $\bar{U}$, where $\bar{U}$ is some target unitary transformation of $U(n)$ describing a quantum gate. As proposed in [63], reference trajectories associated to reference controls ω^r and u_k^r that are simultaneously periodic and odd functions of times yield automatically to time-periodic reference operator trajectories U^r just by integration of $i\frac{d}{dt}U^r = (\omega^r \mathbf{1} + \sum_{k=1}^m u_k^r(t) H_k)\, U^r$. Take then the solution starting from $\bar{U}$ and compute the steering control by numerical integration of the closed-loop system (3.45) with the tracking feedback (3.47). As soon as the Lie algebra generated by the iH_k's contains $su(n)$, a generic choice for ω^r and u_k^r yields to a reference trajectory around which the linearized system is controllable and then, as shown in [63], the tracking controller is asymptotically stable if the initial tracking error $\mathcal{L}_{t=0}$ is strictly less than 2.

The above formulation does not directly extend to infinite dimensional case: the formula (3.46) defining the control Lyapunov function is indefinite for $n = \infty$. Nevertheless we

can consider a finite dimensional truncation by considering the following control Lyapunov function

$$\mathcal{L}(U^r, U) = n_P - \Re\left(\mathrm{Tr}\left(PU^\dagger U^r P\right)\right)$$

where P is an orthogonal projector on a finite dimensional subspace of dimension n_P. Since

$$\frac{d}{dt}\mathcal{L} = \sum_{k=1}^{m}(u_k - u_k^r)\Im\left(\mathrm{Tr}\left(PU^\dagger H_k U^r\right)\right) + (\omega - \omega^r)\Im\left(\mathrm{Tr}\left(PU^\dagger U^r P\right)\right),$$

the tracking feedback reads:

$$u_k = u_k^r - a_k\Im\left(\mathrm{Tr}\left(PU^\dagger H_k U^r P\right)\right) \text{ for } k = 1, \ldots, m, \quad \omega = \omega^r - b\Im\left(\mathrm{Tr}\left(PU^\dagger U^r P\right)\right).$$

It will be interesting to test such Lyapunov strategy for the generation of a C-not gate on the two trapped ions system described in Subsection 3.3.7 and to compare it with the sequence of pulses described in [33][Chapter 8, Subsection 8.4.3] corresponding to the Cirac-Zoller gate. To apply this tracking feedback, it suffices to choose

- P as the orthogonal projector on the "computational space" of dimension $n_P = 4$ and spanned by $|gg, 0\rangle$, $|ge, 0\rangle$, $|eg, 0\rangle$ and $|ee, 0\rangle$ (0-phonon subspace),

$$P = |gg, 0\rangle\langle gg, 0| + |ge, 0\rangle\langle ge, 0| + |eg, 0\rangle\langle eg, 0| + |ee, 0\rangle\langle ee, 0|;$$

- $\bar{U}$ such that its restriction to the computational space, $P\bar{U}P$, coincides with the C-not gate; take for example

$$\bar{U} = |gg, 0\rangle\langle gg, 0| + |ge, 0\rangle\langle ge, 0| + |eg, 0\rangle\langle ee, 0| + |ee, 0\rangle\langle eg, 0| + \mathbf{1} - P;$$

- U^r as a periodic trajectory passing through $\bar{U}$ and associated to odd and periodic references for the complex amplitudes $\mathbf{u}_1$, $\mathbf{u}_{1b}$, $\mathbf{u}_{1r}$, $\mathbf{u}_2$, $\mathbf{u}_{2b}$ and $\mathbf{u}_{2r}$.

When $\mathcal{L}$ vanishes $PU^\dagger U^r P = P$ and, whenever U^r passes through $\bar{U}$, the restriction of U to the computational space coincides with the C-not gate $PUP = P\bar{U}P$.

Chapter 4

Models of Open Systems

4.1 Quantum measurement

Whenever talking about the quantum state of a system, we refer to an observer's knowledge about a system. More precisely, it is the knowledge of the observer about the outcome of the future measurements on the system.

Such information theoretical definition of the state of a physical system may appear unfamiliar and uncomfortable as for instance, the observers with different knowledge may assign different states, simultaneously, to a single system. The most natural way to talk about the consistency of these assigned states is to define a *common state of maximal knowledge* as a common pure state. Indeed, considering the collection $\{\rho_j\}$ of different density matrices assigned by different observers to a same physical system, we call the *common state of maximal knowledge* a pure state defined by a wave function $|\psi\rangle$ such that there exists an $\epsilon > 0$ for which, $\rho_j - \epsilon |\psi\rangle \langle\psi|$ is a positive operator, i.e. ρ_j is the mixture of $|\psi\rangle$ with some other states. From a system theoretical point of view, we can think of this common state of maximal knowledge as the actual state of the system and the density matrix ρ_j is the filtering state encoding the information gained by an observer j.

Another consequence of such definition of the quantum state is that any measurement of the system, which leads to obtaining information on the system, necessarily changes the state of the system. This is known as the projection postulate. Through this section, we provide a brief overview of important measurement paradigms for quantum systems and the two next sections are devoted to some concrete examples. This chapter is strongly inspired from [33], [71] and [64].

4.1.1 Projective measurement

The projective measurement is the traditional description of measurement in quantum mechanics. Indeed, assume the measurement of a physical quantity O to which we can assign a Hermitian operator (observable) $\mathcal{O}$ defined on $\mathcal{H}$ the Hilbert space of the system.

We start by diagonalizing the operator as

$$\mathcal{O} = \sum_{\nu} \lambda_\nu P_\nu,$$

where λ_ν's are the eigenvalues of $\mathcal{O}$, which are all real and different, and P_ν is the projection operator over the associated eigenspace. Note that, in general, the spectrum of the operator $\mathcal{O}$ can be degenerate and therefore the projection operator P_ν is not necessarily a rank-1 operator.

When we measure $\mathcal{O}$, the result will be necessarily one of the eigenvalues λ_ν. Moreover, an outcome λ_ν of the measurement implies an instantaneous projection of the state of our knowledge through the associated projection operator. We also talk of the conditional state of the system as it is conditioned on the measurement outcome. Indeed, assuming that our state of knowledge at time t is given by the density matrix[1] ρ , measurement of the physical observable $\mathcal{O}$ at time t can be formulated as below:

1. The probability of obtaining the value λ_ν is given by $p_\nu = \mathrm{Tr}\,(\rho P_\nu)$; note that $\sum_\nu p_\nu = 1$ as $\sum_\nu P_\nu = \mathbf{1}_{\mathcal{H}}$ ($\mathbf{1}_{\mathcal{H}}$ represents the identity operator of $\mathcal{H}$).

2. After the measurement, the conditional (a posteriori) state of the system given the outcome λ_ν is
$$\rho_+ = \frac{P_\nu\ \rho\ P_\nu}{p_\nu}.$$

Here, ρ_+ denotes the state of the system just after the measurement. Furthermore, we have assumed that the evolution, from other causes, of the system during the measurement process is not significant and can be neglected.

A particular feature of the projective measurement is that, if the same measurement is immediately repeated, then the same result is guaranteed. Indeed, the probability of obtaining the same result λ_ν for the second measurement of the observable $\mathcal{O}$ is given by

$$\mathrm{Tr}\,(P_\nu \rho_+) = \mathrm{Tr}\,(P_\nu\ \rho\ P_\nu)/p_\nu = 1,$$

where we have applied the fact that $P_\nu P_\nu = P_\nu$.

For pure states (encoding the common state of maximal knowledge), $\rho = |\psi\rangle\langle\psi|$, the projective measurement can be more simply expressed as

$$p_\nu = \langle\psi|\, P_\nu\, |\psi\rangle\,,$$
$$\psi_+ = \frac{P_\nu \psi}{\sqrt{p_\nu}}.$$

Finally, the particular case of a projective measurement where the eigenvalues $\{\lambda_\nu\}$ are non-degenerate, and therefore the eigenprojections P_ν are rank-1 operators, is called a *von Neumann measurement*.

[1] ρ is an operator on $\mathcal{H}$, Hermitian, positive and of trace 1. Thus $\mathrm{Tr}\left(\rho^2\right) \leq 1$ with equality only when ρ is an orthogonal projector on some pure quantum state $|\psi\rangle$, i.e., $\rho = |\psi\rangle\langle\psi|$.

4.1.2 Positive Operator Valued Measure (POVM)

The projective measurements are, generally, inadequate for describing real measurements, as the experimenter never directly measures the system of interest. In fact, the system of interest (for instance an atom or a quantized electromagnetic field) interacts with its environment (electromagnetic field or a probe atom), and the experimenter observes the effect of the system on the environment (the radiated field or the probe atom).

In order to formulate such measurement paradigm, we need to consider the quantum state in a larger Hilbert space consisting of the system and the measurement apparatus (also called the meter). Indeed, we consider a total initial state (before the measurement process) for the system together with the meter, which is given by a separable wavefunction

$$|\Psi\rangle = |\psi_S\rangle \otimes |\theta_M\rangle$$

living on the total Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_M$. The measurement process consists in a unitary evolution of the whole state (leading to a non-separable–entangled– state) followed by a projective von Neumann measurement of the measurement apparatus. Let us denote by $U_{S,M}$ the unitary evolution entangling the state of the system to that of the meter, and by $\mathcal{O}_M = \mathbf{1}_S \otimes \left(\sum_\nu \lambda_\nu P_\nu\right)$ the measured observable for the meter. Here, the projection operator P_ν is a rank-1 projection in $\mathcal{H}_M$ over the eigenstate $|\lambda_\nu\rangle \in \mathcal{H}_M$: $P_\nu = |\lambda_\nu\rangle\langle\lambda_\nu|$. The measurement procedure can be formulated as below (see section A.3 for an introduction to tensor product)

1. The probability of obtaining the value λ_ν is given by $p_\nu = \langle\psi_S|\mathcal{M}_\nu^\dagger\mathcal{M}_\nu|\psi_S\rangle$ where $\mathcal{M}_\nu$ is an operator defined on $\mathcal{H}_S$, the Hilbert space of the system, by

$$\big(\mathcal{M}_\nu|\psi_S\rangle\big) \otimes |\lambda_\nu\rangle = \big(\mathbf{1}_S \otimes P_\nu\big)U_{S,M}\big(|\psi_S\rangle \otimes |\theta_M\rangle\big).$$

   Thus we have

$$U_{S,M}\big(|\psi_S\rangle \otimes |\theta_M\rangle\big) = \sum_\nu \big(\mathcal{M}_\nu|\psi_S\rangle\big) \otimes |\lambda_\nu\rangle.$$

   Note that $\sum_\nu p_\nu = 1$ as

$$\sum_\nu \langle\psi_S|\mathcal{M}_\nu^\dagger\mathcal{M}_\nu|\psi_S\rangle = \big(|\psi_S\rangle \otimes |\theta_M\rangle\big)^\dagger U_{S,M}^\dagger \left(\sum_\nu \mathbf{1}_H \otimes P_\nu\right) U_{S,M}\big(|\psi_S\rangle \otimes |\theta_M\rangle\big) = 1, \tag{4.1}$$

   where we have used $\sum_\nu |\lambda_\nu\rangle\langle\lambda_\nu| = \mathbf{1}_M$ and $U_{S,M}^\dagger U_{S,M} = \mathbf{1}_{SM}$.

2. After the measurement, the conditional (a posteriori) state of the system given the outcome λ_ν is

$$|\psi_S\rangle_+ = \frac{\mathcal{M}_\nu|\psi_S\rangle}{\sqrt{p_\nu}}.$$

The operators $\mathcal{M}_\nu$ are called the *measurement operators* (see appendix B).

This can also be extended to the case of a mixed state where the probability of obtaining the value λ_ν is simply given by $p_\nu = \mathrm{Tr}\left(\mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger\right)$ and the conditional state given the outcome λ_ν is

$$\rho_+ = \mathbb{M}_\nu(\rho) := \frac{\mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger}{\mathrm{Tr}\left(\mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger\right)}, \tag{4.2}$$

with $\mathbb{M}_\nu$ a nonlinear super-operator (it sends an operator to an operator) on $\mathcal{H}_S$. Indeed, through the computations of (4.1), $\sum_\nu \mathcal{M}_\nu^\dagger \mathcal{M}_\nu = \mathbf{1}_S$ and this, together with the positiveness of the operators $\mathcal{M}_\nu^\dagger \mathcal{M}_\nu$, are the only conditions for the set $\{\mathcal{M}_\nu\}$ to define a *Positive Operator Valued Measure* (POVM).

Also, one can define the *Generalized POVM* as the case where the initial state of the meter is not a pure state or that the projective measurement of the meter is not a von Neumann measurement (see [71, chapter 1] for a tutorial exposure of quantum measurement).

4.1.3 Quantum Non-Demolition (QND) measurement

Before anything, we need that the measurement of the meter observable $\mathcal{O}_M$ after the interaction between the system and the meter encodes some information on the system S itself. This imposes some constraints on unitary transformation $U_{S,M}$ considered in the previous subsection:

$$U_{S,M} \left|\Psi\right\rangle = U_{S,M}\big(\left|\psi_S\right\rangle \otimes \left|\theta_M\right\rangle\big).$$

Assume that such unitary transformation $U_{S,M}$ results from a Hamiltonian $H = H_S + H_M + H_{SM}$ where H_S and H_M describe, respectively, the evolutions of the system and the meter and H_{SM} denotes the system-meter interaction Hamiltonian. Then $U_{S,M}$ is the propagator generated by H during the interaction interval of length τ between S and M (for time-invariant H, we have $U_{S,M} = e^{-i\tau H}$). It is clear that a necessary condition for the influence of S on $\mathcal{O}_M$ just after the interaction is that $[H, \mathcal{O}_M] \neq 0$. Otherwise $\mathcal{O}_M U_{S,M} = U_{S,M} \mathcal{O}_M$. Using the spectral decomposition $\mathcal{O}_M = \sum_\nu \lambda_\nu \mathbf{1}_S \otimes \left|\lambda_\nu\right\rangle\!\left\langle\lambda_\nu\right|$ (see previous subsection), we have for any ν,

$$\mathcal{O}_M U_{S,M}\big(\left|\psi_S\right\rangle \otimes \left|\lambda_\nu\right\rangle\big) = U_{S,M}\mathcal{O}_M\big(\left|\psi_S\right\rangle \otimes \left|\lambda_\nu\right\rangle\big) = \lambda_\nu U_{S,M}\big(\left|\psi_S\right\rangle \otimes \left|\theta_M\right\rangle\big).$$

Thus, necessarily $U_{S,M}\big(\left|\psi_S\right\rangle \otimes \left|\lambda_\nu\right\rangle\big) = \big(U_\nu \left|\psi_S\right\rangle\big) \otimes \left|\lambda_\nu\right\rangle$ where U_ν is a unitary transformation on $\mathcal{H}_S$ only. With $\left|\theta_M\right\rangle = \sum_\nu \theta_\nu \left|\lambda_\nu\right\rangle$, we get, for any $\left|\psi_S\right\rangle$,

$$U_{S,M}\big(\left|\psi_S\right\rangle \otimes \left|\theta_M\right\rangle\big) = \sum_\nu \theta_\nu \big(U_\nu \left|\psi_S\right\rangle\big) \otimes \left|\lambda_\nu\right\rangle$$

Then measurement operators $\mathcal{M}_\nu$ are equal to $\theta_\nu U_\nu$. The probability to get measurement outcome ν, $\left\langle\psi_S\right|\mathcal{M}_\nu^\dagger \mathcal{M}_\nu\left|\psi_S\right\rangle = |\theta_\nu|^2$, is completely independent of systems state $\left|\psi_S\right\rangle$. This means that the measurement statistics for the meter observable $\mathcal{O}_M$ does not encode any

information on the system S and therefore $[H, \mathcal{O}_M]$ must not vanish. When $H_M = 0$, this necessary condition reads $[H_{SM}, \mathcal{O}_M] \neq 0$.

Let us consider the measurement of a physical observable $\mathcal{O}_S$ defined for the system S, through its coupling with a meter M with a von Neumann measurements of an observable $\mathcal{O}_M$ on the meter. The essential condition for a measurement process of $\mathcal{O}_S$ to be quantum non-demolition (abbreviated as QND) is that the measurement should not affect the eigenstates of $\mathcal{O}_S$ when $\mathcal{O}_S$ admits a non degenerate spectrum (other-wise we have to consider the eigenspace instead of the eigenstate). A sufficient but not necessary condition for this is

$$[H, \mathcal{O}_S] = 0$$

Under this condition $\mathcal{O}_S$ and $U_{S,M}$ commute. For eigenstate $|\mu\rangle$ of $\mathcal{O}_S$ associated to eigenvalue μ, we have

$$\mathcal{O}_S U_{S,M}\big(\,|\mu\rangle \otimes |\theta_M\rangle\,\big) = U_{S,M}\mathcal{O}_S\big(\,|\mu\rangle \otimes |\theta_M\rangle\,\big) = \mu U_{S,M}\big(\,|\mu\rangle \otimes |\theta_M\rangle\,\big).$$

Exercice 4.1.1. *Prove that the above formula implies* $U_{S,M}\big(\,|\mu\rangle \otimes |\theta_M\rangle\,\big) = |\mu\rangle \otimes \big(U_\mu\,|\theta_M\rangle\,\big)$ *where* U_μ *is a unitary operator on* $\mathcal{H}_M$ *only:* $U_{S,M}$ *does not entangle eigenstates of* $\mathcal{O}_S$ *with the meter.*

With the measurement operators $\mathcal{M}_\nu$, we also have

$$U_{S,M}\big(\,|\mu\rangle \otimes |\theta_M\rangle\,\big) = \sum_\nu \mathcal{M}_\nu\,|\mu\rangle \otimes |\lambda_\nu\rangle\,.$$

Thus necessarily, using exercise 4.1.1 each $\mathcal{M}_\nu\,|\mu\rangle$ is colinear to $|\mu\rangle$. Whatever the measurement outcome ν is, the conditional state provided by (4.2) remains unchanged: $\rho_+ = \mathbb{M}_\nu(\rho)$ when $\rho = |\mu\rangle\langle\mu|$. When the spectrum of $\mathcal{O}_S$ is degenerate and P_μ is the projector on the eigenspace associated to the eigenvalue μ of $\mathcal{O}_S$, this invariance reads: for all ν, $\mathcal{M}_\nu P_\mu = P_\mu \mathcal{M}_\nu$. Any eigenspace of $\mathcal{O}_S$ is invariant with respect to all the $\mathcal{M}_\nu$'s.

4.1.4 Stochastic process attached to a POVM

To any POVM defined by a set of measurement operators $(\mathcal{M}_\nu)$ on $\mathcal{H}_S$, is attached a stochastic process. This process admits the set $\{\rho\}$ of density operators on $\mathcal{H}_S$ as state space. It is defined by the transition rules:

$$\rho_+ = \frac{\mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger}{\mathrm{Tr}\left(\mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger\right)} \text{ with probability } p_\nu = \mathrm{Tr}\left(\mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger\right). \tag{4.3}$$

For any observable A on $\mathcal{H}_S$, its conditional expectation value after the transition knowing the state ρ just before the transition is given by

$$\mathbb{E}\left(\mathrm{Tr}\left(A\rho_+\right)/\rho\right) = \mathrm{Tr}\left(A\mathbb{K}\rho\right) \tag{4.4}$$

where the linear map $\mathbb{K}\rho = \sum_\nu \mathcal{M}_\nu \rho \mathcal{M}_\nu^\dagger$ is a Kraus map (see appendix B).

Assume that this POVM provides a QND measurement of an observable $\mathcal{O}_S$ on $\mathcal{H}_S$. Then the orthogonal projector $P_{\mathcal{O}_S}$ on any eigenspace of $\mathcal{O}_S$, yields to a martingale $\mathrm{Tr}\,(\rho P_{\mathcal{O}_S})$:

$$\mathbb{E}\left(\mathrm{Tr}\,(P_{\mathcal{O}_S}\rho_+)/\rho\right) = \mathrm{Tr}\,(P_{\mathcal{O}_S}\rho)$$

since $P_{\mathcal{O}_S}$ is a stationary point of the dual Kraus map $\mathbb{K}^*$: $\mathbb{K}^* P_{\mathcal{O}_S} = \sum_\nu \mathcal{M}_\nu^\dagger P_{\mathcal{O}_S} \mathcal{M}_\nu = P_{\mathcal{O}_S}$. Moreover, if $P_{\mathcal{O}_S}$ is of rank one, then it corresponds to a stationary state $\bar{\rho} = P_{\mathcal{O}_S}$ of the Markov process (4.3): for all ν, $\mathcal{M}_\nu \bar{\rho} \mathcal{M}_\nu^\dagger = \mathrm{Tr}\left(\mathcal{M}_\nu \bar{\rho} \mathcal{M}_\nu^\dagger\right)\bar{\rho}$.

Exercice 4.1.2. *Prove that for a QND measurement of a system observable $\mathcal{O}_S$, the random process $Tr(\rho\mathcal{O}_S)$ is also a martingale.*

4.2 A discrete-time system: the photon-box

This section is devoted to the case study of a photon box consisting of a cavity quantum electrodynamic setup developed within Laboratoire Kastler-Brossel (LKB) at École Normale Supérieure.

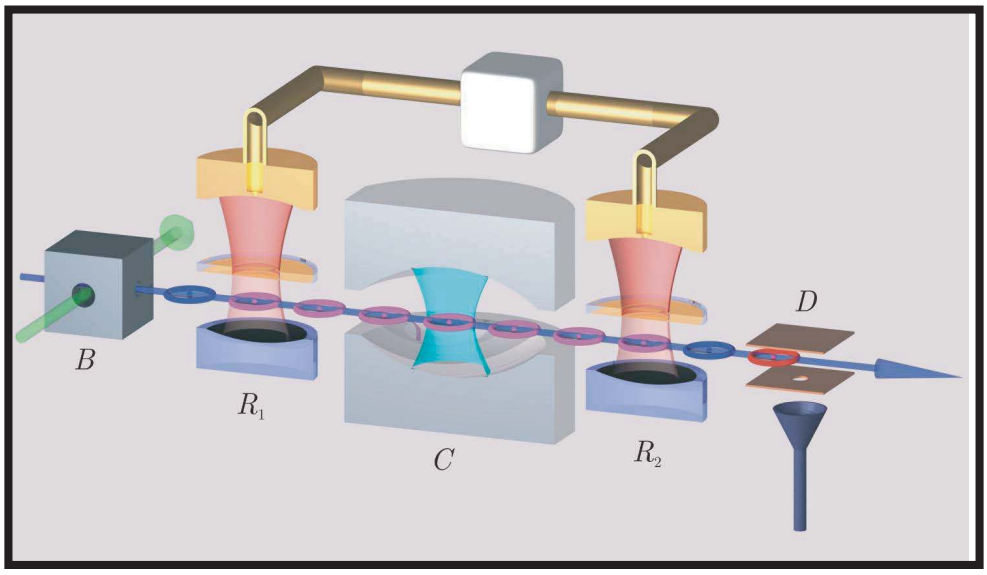

Figure 4.1: the ENS photon box; atoms get out box B one by one, undergo then a first Rabi pulse in Ramsey zone R_1, become entangled with electromagnetic field trapped in C, undergo a second Rabi pulse in Ramsey zone R_2 and finally are measured in the detector D.

4.2.1 The Markov chain model

Here S corresponds to a quantized trapped mode inside the cavity. It is described by a wave function $|\psi\rangle$ in the Hilbert space $\mathcal{H}_S$ (see section 2.2)

$$\mathcal{H}_S = \left\{ \sum_{n=0}^{\infty} \psi_n |n\rangle \ \mid (\psi_n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

where $|n\rangle$ represents the Fock state associated to exactly n photons inside the cavity and $l^2(\mathbb{C})$ is the space of square summable sequences in $\mathbb{C}$ ($\sum_{n=0}^{\infty} |\psi_n|^2 = 1$). The meter M is associated to atoms : $\mathcal{H}_M = \mathbb{C}^2$, each atom admits two-level and is described by a wave function $c_g |g\rangle + c_e |e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$.

Let us follow an atom leaving B where it is prepared in state $|g\rangle$. It is symbolized by a small horizontal and blue torus in figure 4.2. When atom comes out B, the state of the composite system atom/field is separable and is denoted by $|\Psi\rangle_B \in \mathcal{H}_M \otimes \mathcal{H}_S$

$$|\Psi\rangle_B = |g\rangle \otimes |\psi\rangle . \tag{4.5}$$

When atom comes out the first Ramsey zone R_1 (red torus between R_1 and C), the state remains separable but has changed to

$$|\Psi\rangle_{R_1} = (U_{R_1} \otimes \mathbf{1}) |\Psi\rangle_B = (U_{R_1} |g\rangle) \otimes |\psi\rangle \tag{4.6}$$

where the unitary transformation performed in R_1 only affects the atom:

$$U_{R_1} = e^{-i\frac{\theta_1}{2}(x_1\sigma_x + y_1\sigma_y + z_1\sigma_z)} = \cos(\tfrac{\theta_1}{2}) - i\sin(\tfrac{\theta_1}{2})(x_1\sigma_x + y_1\sigma_y + z_1\sigma_z) \tag{4.7}$$

corresponds, in the Bloch sphere representation, to a rotation of angle θ_1 around the oriented axis defined by the unit-length vector $x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k}$ ($x_1^2 + y_1^2 + z_1^2 = 1$), see section 2.1.2.

When atom comes out cavity C, the state does not remain separable: atom and field becomes entangled and the state is described by

$$|\Psi\rangle_C = U_C |\Psi\rangle_{R_1} \tag{4.8}$$

where the unitary transformation U_C on $\mathcal{H}_M \otimes \mathcal{H}_S$ is associated to a Jaynes-Cumming Hamiltonian for describing the atom/field inter-action:

$$H_C = \tfrac{\Delta}{2}\sigma_z + i\tfrac{\Omega}{2}(\sigma_- a^\dagger - \sigma_+ a) \tag{4.9}$$

is the Jaynes-Cumming Hamiltonian after the RWA approximation ($\Delta = \omega_{eg} - \omega_c$ de-tuning between atom and cavity field, Ω the vacuum Rabi pulsation, see section 3.3.5 and (3.25) with $\mathbf{u} = 0$, $\omega_r = \omega_c$ and $\Delta_c = 0$ and $\Delta_{eg} = \Delta$). The precise form of U_C is given in next sub-section for resonant and dispersive cases.

When atom comes out second Ramsey zone R_2, the state becomes

$$|\Psi\rangle_{R_2} = (U_{R_2} \otimes \mathbf{1}) |\Psi\rangle_C$$

where U_{R_2} is similar to U_{R_1} but with different parameters θ_2, x_2, y_2, z_2,

$$U_{R_2} = e^{-i\frac{\theta_2}{2}(x_2\sigma_x + y_2\sigma_y + z_2\sigma_z)} = \cos(\tfrac{\theta_2}{2}) - i\sin(\tfrac{\theta_2}{2})(x_2\sigma_x + y_2\sigma_y + z_2\sigma_z). \tag{4.10}$$

This means that, just before the measurement in D, the state is given by

$$|\Psi\rangle_{R_2} = U|g\rangle \otimes |\psi\rangle = |g\rangle \otimes \mathcal{M}_g|\psi\rangle + |e\rangle \otimes \mathcal{M}_e|\psi\rangle \tag{4.11}$$

where $U = U_{R_2}U_C U_{R_1}$ is the total unitary transformation defining the linear measurement operators $\mathcal{M}_g$ and $\mathcal{M}_e$ on $\mathcal{H}_S$.

Denote by $s \in \{g, e\}$ the measurement outcome in detector D: with probability $p_s = \langle\psi|\mathcal{M}_s^\dagger\mathcal{M}_s|\psi\rangle$ we get s. Just after the measurement outcome s, the state becomes separable. It has partially collapsed to

$$|\Psi\rangle_D = \tfrac{1}{\sqrt{p_s}}\ |s\rangle \otimes (\mathcal{M}_s|\psi\rangle) = \frac{|s\rangle \otimes (\mathcal{M}_s|\psi\rangle)}{\sqrt{\langle\psi|\mathcal{M}_s^\dagger\mathcal{M}_s|\psi\rangle}}.$$

We have in front of a Markov process: after the complete passage of an atom, the cavity state initially equal to $|\psi\rangle$ undergoes an irreversible and stochastic jump to $|\psi\rangle_+$ driven by $\mathcal{M}_g$ and $\mathcal{M}_e$ defined by unitary operator $U = U_{R_2}U_C U_{R_1}$ and (4.11):

$$|\psi\rangle_+ = \begin{cases} \frac{\mathcal{M}_g|\psi\rangle}{\sqrt{\langle\psi|\mathcal{M}_g^\dagger\mathcal{M}_g|\psi\rangle}}, & \text{with probability } p_g = \langle\psi|\mathcal{M}_g^\dagger\mathcal{M}_g|\psi\rangle; \\ \frac{\mathcal{M}_e|\psi\rangle}{\sqrt{\langle\psi|\mathcal{M}_e^\dagger\mathcal{M}_e|\psi\rangle}}, & \text{with probability } p_e = \langle\psi|\mathcal{M}_e^\dagger\mathcal{M}_e|\psi\rangle. \end{cases} \tag{4.12}$$

For the density matrix formulation we have thus

$$\rho_+ = \begin{cases} \mathbb{M}_g(\rho) = \frac{\mathcal{M}_g\rho\mathcal{M}_g}{\mathrm{Tr}\left(\mathcal{M}_g\rho\mathcal{M}_g^\dagger\right)}, & \text{with probability } p_g = \mathrm{Tr}\left(\mathcal{M}_g\rho\mathcal{M}_g^\dagger\right); \\ \mathbb{M}_e(\rho) = \frac{\mathcal{M}_e\rho\mathcal{M}_e}{\mathrm{Tr}\left(\mathcal{M}_e\rho\mathcal{M}_e^\dagger\right)}, & \text{with probability } p_e = \mathrm{Tr}\left(\mathcal{M}_e\rho\mathcal{M}_e^\dagger\right). \end{cases} \tag{4.13}$$

Exercice 4.2.1. *Consider that $\mathcal{M}_g$ and $\mathcal{M}_e$ defined by (4.11). Show that, for any density matrix ρ the operator (defining a Kraus map, see appendix B)*

$$\mathcal{M}_g\rho\mathcal{M}_g^\dagger + \mathcal{M}_e\rho\mathcal{M}_e^\dagger$$

does not depend on $(\theta_2, x_2, y_2, z_2)$, the parameters of the second Ramsey pulse U_{R_2}.

4.2.2 The Jaynes-Cumming propagator

In the resonant case, $\Delta = 0$. The atom/cavity propagator U_C based on Jaynes-Cumming Hamiltonian (4.9) admits the following form (see [33] for the detailed derivations including Gaussian radial dependence of the quantized mode and atom velocity):

$$U_C = |g\rangle\langle g| \cos\left(\tfrac{\Theta}{2}\sqrt{N}\right) + |e\rangle\langle e| \cos\left(\tfrac{\Theta}{2}\sqrt{N+1}\right) + |g\rangle\langle e| \left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{N}\right)}{\sqrt{N}}\right) a^\dagger - |e\rangle\langle g| a \left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{N}\right)}{\sqrt{N}}\right) \quad (4.14)$$

where $N = a^\dagger a$ is the photon number operator, the adjustable parameter Θ being the Rabi angle with zero photon.

In the dispersive case, $|\Delta| \gg |\Omega|$, U_C based on Jaynes-Cumming Hamiltonian (4.9) admits the following form (see [33] for the detailed derivations based on adiabatic invariance):

$$U_C = |g\rangle\langle g| e^{-i\phi(N)} + |e\rangle\langle e| e^{i\phi(N+1)} \quad (4.15)$$

where the dephasing $\phi(N)$ depends on the photon number and can be approximated by a linear real function: $\phi(N) = \vartheta_0 + \vartheta N$, the phases ϑ_0 and ϑ being adjustable parameters.

The exercise below can be seen as a simplified derivation of the above formulae for U_C.

Exercice 4.2.2. *Let us assume that the Jaynes-Cumming propagator U_C admits the following form*

$$U_C = e^{-i\tau\left(\frac{\Delta\left(|e\rangle\langle e| - |g\rangle\langle g|\right)}{2} + i\frac{\Omega\left(|g\rangle\langle e| a^\dagger - |e\rangle\langle g| a\right)}{2}\right)}$$

where τ is an interaction time.

1. *Show by recurrence on integer k that*

$$\Big(\Delta\big(|e\rangle\langle e| - |g\rangle\langle g|\big) + i\Omega\big(|g\rangle\langle e| a^\dagger - |e\rangle\langle g| a\big)\Big)^{2k} = |e\rangle\langle e| \left(\Delta^2 + (N+1)\Omega^2\right)^k + |g\rangle\langle g| \left(\Delta^2 + N\Omega^2\right)^k$$

and that

$$\Big(\Delta\big(|e\rangle\langle e| - |g\rangle\langle g|\big) + i\Omega\big(|g\rangle\langle e| a^\dagger - |e\rangle\langle g| a\big)\Big)^{2k+1} = |e\rangle\langle e| \Delta \left(\Delta^2 + (N+1)\Omega^2\right)^k - |g\rangle\langle g| \Delta \left(\Delta^2 + N\Omega^2\right)^k + i\Omega\Big(|g\rangle\langle e| \left(\Delta^2 + N\Omega^2\right)^k a^\dagger - |e\rangle\langle g| a \left(\Delta^2 + N\Omega^2\right)^k\Big).$$

2. *Deduce that*

$$U_C = |g\rangle\langle g| \left(\cos\left(\frac{\tau\sqrt{\Delta^2+N\Omega^2}}{2}\right) + i\frac{\Delta \sin\left(\frac{\tau\sqrt{\Delta^2+N\Omega^2}}{2}\right)}{\sqrt{\Delta^2+N\Omega^2}} \right) + |e\rangle\langle e| \left(\cos\left(\frac{\tau\sqrt{\Delta^2+(N+1)\Omega^2}}{2}\right) - i\frac{\Delta \sin\left(\frac{\tau\sqrt{\Delta^2+(N+1)\Omega^2}}{2}\right)}{\sqrt{\Delta^2+(N+1)\Omega^2}} \right) + |g\rangle\langle e| \left(\frac{\Omega \sin\left(\frac{\tau\sqrt{\Delta^2+N\Omega^2}}{2}\right)}{\sqrt{\Delta^2+N\Omega^2}} \right) a^\dagger - |e\rangle\langle g| a \left(\frac{\Omega \sin\left(\frac{\tau\sqrt{\Delta^2+N\Omega^2}}{2}\right)}{\sqrt{\Delta^2+N\Omega^2}} \right) \quad (4.16)$$

where $N = a^\dagger a$ the photon-number operator (a is the photon annihilator operator).

3. *In the resonant case, $\Delta = 0$, express the vacuum Rabi angle Θ appearing in (4.14) with respect to Ω and τ.*

4. *In the dispersive case, $|\Delta| \gg |\Omega|$, and when the interaction time τ is large, $\Delta\tau \sim \left(\frac{\Delta}{\Omega}\right)^2$, show that, up to first order terms in Ω/Δ, we get*

$$e^{-i\tau\left(\frac{\Delta\left(|e\rangle\langle e| - |g\rangle\langle g|\right)}{2} + i\frac{\Omega\left(|g\rangle\langle e| a^\dagger - |e\rangle\langle g| a\right)}{2}\right)} = |g\rangle\langle g| e^{i\left(\frac{\Delta\tau}{2} + \frac{\Omega^2\tau}{4\Delta}N\right)} + |e\rangle\langle e| e^{-i\left(\frac{\Delta\tau}{2} + \frac{\Omega^2\tau}{4\Delta}(N+1)\right)}.$$

Express the phases ϑ_0 and ϑ appearing in (4.15) with respect to τ, Δ and Ω.

4.2.3 The resonant case

Let us detail the operators $\mathcal{M}_g$ and $\mathcal{M}_e$ defined in (4.11) when U_C is given by (4.14), $U_{R_1} = e^{-i\frac{\theta_1}{2}\sigma_y}$ and $U_{R_2} = \mathbf{1}$. Since $U_{R_1} = \cos\left(\frac{\theta_1}{2}\right) + \sin\left(\frac{\theta_1}{2}\right)\left(|g\rangle\langle e| - |e\rangle\langle g|\right)$, $|\Psi\rangle_{R_1}$ given by (4.6) reads:

$$|\Psi\rangle_{R_1} = \left(\cos\left(\frac{\theta_1}{2}\right)|g\rangle - \sin\left(\frac{\theta_1}{2}\right)|e\rangle\right) \otimes |\psi\rangle .$$

Then $|\Psi\rangle_C$ given by (4.8) becomes

$$\begin{aligned}|\Psi\rangle_C = \cos\left(\tfrac{\theta_1}{2}\right)&\left(|g\rangle\otimes\cos\left(\tfrac{\Theta}{2}\sqrt{N}\right)|\psi\rangle - |e\rangle\otimes a\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{N}\right)}{\sqrt{N}}\right)|\psi\rangle\right)\\ &-\sin\left(\tfrac{\theta_1}{2}\right)\left(|e\rangle\otimes\cos\left(\tfrac{\Theta}{2}\sqrt{N+1}\right)|\psi\rangle + |g\rangle\otimes\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{N}\right)}{\sqrt{N}}\right)a^\dagger|\psi\rangle\right)\\ &= |g\rangle\otimes\left(\cos\left(\tfrac{\theta_1}{2}\right)\cos\left(\tfrac{\Theta}{2}\sqrt{N}\right) - \sin\left(\tfrac{\theta_1}{2}\right)\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{N}\right)}{\sqrt{N}}\right)a^\dagger\right)|\psi\rangle\\ &\quad - |e\rangle\otimes\left(\sin\left(\tfrac{\theta_1}{2}\right)\cos\left(\tfrac{\Theta}{2}\sqrt{N+1}\right) + \cos\left(\tfrac{\theta_1}{2}\right)a\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{N}\right)}{\sqrt{N}}\right)\right)|\psi\rangle\,.\end{aligned}$$

Since $U_{R_2}=\mathbf{1}$, $|\Psi\rangle_C = |\Psi\rangle_{R_2}$. The measurement operators are thus given by

$$\begin{aligned}\mathcal{M}_g &= \cos\left(\tfrac{\theta_1}{2}\right)\cos\left(\tfrac{\Theta}{2}\sqrt{N}\right) - \sin\left(\tfrac{\theta_1}{2}\right)\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{N}\right)}{\sqrt{N}}\right)a^\dagger\\ \mathcal{M}_e &= -\sin\left(\tfrac{\theta_1}{2}\right)\cos\left(\tfrac{\Theta}{2}\sqrt{N+1}\right) - \cos\left(\tfrac{\theta_1}{2}\right)a\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{N}\right)}{\sqrt{N}}\right)\end{aligned} \tag{4.17}$$

Exercice 4.2.3. *Verify that the operators (measurement operators) given by (4.17) satisfy* $\mathcal{M}_g^\dagger\mathcal{M}_g + \mathcal{M}_e^\dagger\mathcal{M}_e = \mathbf{1}$ *(hint: use,* $N = a^\dagger a$*,* $a\ f(N) = f(N+1)\ a$ *and* $a^\dagger f(N) = f(N-1)\ a^\dagger$*).*

4.2.4 The dispersive case

Let us detailed the measurement operators $\mathcal{M}_g$ and $\mathcal{M}_e$ defined in (4.11) when U_C is given by (4.15), $U_{R_1} = e^{-i\frac{\pi}{4}\sigma_y}$ and $U_{R_2} = e^{-i\frac{\pi}{4}(-\sin\eta\sigma_x+\cos\eta\sigma_y)}$ (with angle η chosen below). Since $U_{R_1} = \frac{|g\rangle\langle e|-|e\rangle\langle g|}{\sqrt{2}}$, $|\Psi\rangle_{R_1}$ given by (4.6) reads:

$$|\Psi\rangle_{R_1} = \frac{|g\rangle - |e\rangle}{\sqrt{2}}\otimes|\psi\rangle\,.$$

Then $|\Psi\rangle_C$ given by (4.8) becomes

$$|\Psi\rangle_C = \tfrac{1}{\sqrt{2}}|g\rangle\otimes e^{-i\phi(N)}|\psi\rangle - \tfrac{1}{\sqrt{2}}|e\rangle\otimes e^{i\phi(N+1)}|\psi\rangle\,.$$

Since $U_{R_2} = \frac{1}{\sqrt{2}}\left(\mathbf{1} + e^{i\eta}|g\rangle\langle e| - e^{-i\eta}|e\rangle\langle g|\right)$, we have

$$\begin{aligned}2|\Psi\rangle_{R_2} &= \left(|g\rangle - e^{-i\eta}|e\rangle\right)\otimes e^{-i\phi(N)}|\psi\rangle - \left(e^{i\eta}|g\rangle + |e\rangle\right)\otimes e^{i\phi(N+1)}|\psi\rangle\\ &= |g\rangle\otimes\left(e^{-i\phi(N)} - e^{i(\eta+\phi(N+1))}\right)|\psi\rangle - |e\rangle\otimes\left(e^{-i(\eta+\phi(N))} + e^{i\phi(N+1)}\right)|\psi\rangle\end{aligned}$$

where $\phi(N) = \vartheta_0 + N\vartheta$. Take φ_0 an arbitrary phase and set $\eta = 2(\varphi_0 - \vartheta_0) - \vartheta - \pi$. Then the measurement operators are given by the simple formulae

$$\mathcal{M}_g = \cos(\varphi_0 + N\vartheta), \quad \mathcal{M}_e = \sin(\varphi_0 + N\vartheta) \tag{4.18}$$

where we have removed the irrelevant global phase factors $e^{i(\varphi_0-\vartheta_0)}$ for $\mathcal{M}_g$ and $e^{i(\vartheta_0-\varphi_0+\pi/2)}$ for $\mathcal{M}_e$. In the Fock basis $\{|n\rangle\}_0^\infty$), the operator $\mathcal{M}_g$ (resp. $\mathcal{M}_e$) is diagonal with diagonal elements $\cos(n\vartheta+\varphi_0)$ (resp. $\sin(n\vartheta+\varphi_0)$. We note in particular that $\mathcal{M}_g^\dagger\mathcal{M}_g+\mathcal{M}_e^\dagger\mathcal{M}_e = \mathbf{1}$.

Exercice 4.2.4. *Take $\mathcal{M}_g$ and $\mathcal{M}_e$ defined by (4.11) with U_C given by (4.15) with ϕ an arbitrary real value function.*

1. *Show that any Fock state $|n\rangle$ is an eigenvector of $\mathcal{M}_g$ and $\mathcal{M}_e$, whatever U_{R_1} and U_{R_2} are.*

2. *Deduce from preceding question that, for any density operator ρ, any integer n and any Ramsey pulses U_{R_1} and U_{R_2}, we have*

$$\langle n|\mathcal{M}_g\rho\mathcal{M}_g^\dagger|n\rangle + \langle n|\mathcal{M}_e\rho\mathcal{M}_e^\dagger|n\rangle = \langle n|\rho|n\rangle.$$

3. *What does-it mean for the Markov chain associated to such $\mathcal{M}_g$ and $\mathcal{M}_e$ and defined by (4.13).*

4.2.5 Measurement uncertainties and a Bayesian filter

This sub-section is directly inspired from [28]. Let us consider now the situation where the atom passes through the cavity but we do not detect it after the second Ramsey zone. To describe the cavity state we have to use mixed states and thus density matrix ρ and the operator $\mathbb{M}_g$ and $\mathbb{M}_e$ defined in (4.2). Having no knowledge on whether the atom ends up in the state $|g\rangle$ or $|e\rangle$, the best we can say about the cavity state (our knowledge of the system) after the passage of the atom is its expectation value:

$$\rho_+ = p_g\mathbb{M}_g\rho + p_e\mathbb{M}_e\rho = \mathcal{M}_g\rho\mathcal{M}_g^\dagger + \mathcal{M}_e\rho\mathcal{M}_e^\dagger. \tag{4.19}$$

The above map, sending ρ to ρ_+, defines the Kraus representation for a linear quantum operation (see the Appendix B for a definition and properties of linear quantum operations).

Now consider the case where we realize the atom detection but we are uncertain about its result. Indeed, in practice, the detection process is not perfect and we need to take into account at least three kinds of uncertainties:

- the atom preparation process is itself a random process following a Poisson law; indeed the pulses carrying the atoms that pass through the setup might be empty of atoms; we note the occupancy rate of the pulses by $\eta_a \in]0, 1]$ (η_a is about 0.4 for the LKB experimental setup);

- the atom detector is imperfect and can miss a certain percentage of the atoms; we denote the detector's efficiency by $\eta_d \in]0,1]$ (η_d is about 0.8 for the LKB experimental setup);
- the atom detector is not fault-free and the result of the measurement (atom in the state $|g\rangle$ or $|e\rangle$) can be interchanged; we denote the fault rate by $\eta_f \in [0, 1/2]$ (η_f is about 0.1 for the LKB experimental setup).

Whenever realizing the atom detection, we can achieve three results: 1- the atom is in $|g\rangle$, 2- the atom in $|e\rangle$, 3-the detector does not detect any atom. For each situation we may have various possibilities:

Atom in $|g\rangle$: Either the atom is actually in the state $|e\rangle$ and the detector has made a mistake by detecting it in $|g\rangle$ (this happens with a probability p_g^f to be determined) or the atom is really in the state $|g\rangle$ (this happens with probability $1 - p_g^f$). Indeed, the conditional probability of having the atom in $|e\rangle$ while the detection result has been $|g\rangle$ may be computed through the Bayesian formula and is given by:

$$p_g^f = \frac{\eta_f p_e}{\eta_f p_e + (1-\eta_f) p_g},$$

where $p_g = \mathrm{Tr}\left(\mathcal{M}_g \rho \mathcal{M}_g^\dagger\right)$ and $p_e = \mathrm{Tr}\left(\mathcal{M}_e \rho \mathcal{M}_e^\dagger\right)$.

Also, the conditional evolution of the density matrix (as our knowledge on the cavity state conditioned on the measurement result) is given as follows:

$$\begin{aligned}\rho_+ &= p_g^f \mathbb{M}_e \rho + (1-p_g^f)\mathbb{M}_g \rho \\ &= \frac{\eta_f}{\eta_f p_e + (1-\eta_f) p_g} \mathcal{M}_e \rho \mathcal{M}_e^\dagger + \frac{1-\eta_f}{\eta_f p_e + (1-\eta_f) p_g} \mathcal{M}_g \rho \mathcal{M}_g^\dagger.\end{aligned}$$

Atom in $|e\rangle$: Exactly in the same way, the conditional evolution of the density matrix is given as follows:

$$\rho_+ = \frac{\eta_f}{\eta_f p_g + (1-\eta_f) p_e} \mathcal{M}_g \rho \mathcal{M}_g^\dagger + \frac{1-\eta_f}{\eta_f p_g + (1-\eta_f) p_e} \mathcal{M}_e \rho \mathcal{M}_e^\dagger.$$

No atom detected: Either the pulse has been empty (this happens with a probability p_{na} to be determined) or there has been an atom which has not been detected by the detector (this happens with the probability $1 - p_{\mathrm{na}}$). Indeed, the conditional probability of having an empty pulse while no atom has been detected by the detector can be computed through the Bayes rule and is given by:

$$p_{\mathrm{na}} = \frac{1-\eta_a}{\eta_a(1-\eta_d) + (1-\eta_a)} = \frac{1-\eta_a}{1-\eta_a\eta_d}.$$

In such case the density matrix remains untouched. The complementary situation corresponding to an undetected atom leads to an evolution of the density matrix

through the Kraus representation (4.19). Finally, the conditional evolution of the density matrix (conditioned on the result of the measurement indicating no detected atoms) is given as follows:

$$\begin{aligned}\rho_+ &= p_{\text{na}}\ \rho + (1-p_{\text{na}})(\mathcal{M}_g\rho\mathcal{M}_g^\dagger + \mathcal{M}_e\rho\mathcal{M}_e^\dagger)\\ &= \frac{1-\eta_a}{1-\eta_a\eta_d}\rho + \frac{\eta_a(1-\eta_d)}{1-\eta_a\eta_d}(\mathcal{M}_g\rho\mathcal{M}_g^\dagger + \mathcal{M}_e\rho\mathcal{M}_e^\dagger).\end{aligned}$$

Here, still, we have a Kraus representation for a linear quantum operation.

4.2.6 Relaxation as an unread measurement

Additionally to the above uncertainties in the measurement process, one needs to consider the relaxation of the system due to its coupling to the environment to obtain a complete model for the open system. Two main sources of relaxation can be considered here. A first source concerns the photon loss phenomenon caused by their absorption by the environment (the mirrors in particular). The second source concerns the photon gain phenomenon due to the coupling of the field with a reservoir of non-zero temperature ($T \approx 0.8K$). Denoting by κ_- and by κ_+, respectively the photon loss and the photon gain rate, and assuming that the environment is in thermal equilibrium at temperature T, we have (k_b denoting the Boltzmann constant and ω_c the cavity's resonance frequency),

$$\kappa_+ = \kappa_- e^{-\frac{\hbar\omega_c}{k_bT}}.$$

We refer to [33, Chapter 4, Page 187] for more details. By defining n_{th} as the average number of thermal photons per mode at frequency ω_c, given by Planck's law:

$$n_{\text{th}} = \frac{1}{e^{\frac{\hbar\omega_c}{k_bT}} - 1},$$

we can express both κ_- and κ_+ in term of unique cavity rate κ:

$$\kappa_- = \kappa(1+n_{\text{th}}), \qquad \kappa_+ = \kappa n_{\text{th}}.$$

Note that, here the dominant phenomenon is the photon loss as we work in low temperature regime and therefore $n_{\text{th}} \ll 1$ ($n_{\text{th}} \approx 0.05$ for the LKB experiment). We start therefore by investigating the relaxation caused by the photon loss, which can be modeled through a measurement operator $\mathcal{M}_{\text{loss}}$, proportional to the photon annihilation operator a. Indeed, considering τ_a the duration of a pulse (time interval between the passage of the two atoms), this measurement operator $\mathcal{M}_{\text{loss}}$ can be written as $\sqrt{\kappa_-\tau_a}a$ so that the probability of losing a photon during the current pulse is given by (we neglect the possibility of losing many photons at a same pulse as it admits a very small probability)

$$P_{\text{loss}} = \text{Tr}\left(\mathcal{M}_{\text{loss}}^\dagger\mathcal{M}_{\text{loss}}\rho\right) = \kappa_-\tau_a\text{Tr}\left(a^\dagger a\rho\right) = \kappa_-\tau_a\text{Tr}\left(\mathbf{N}\rho\right).$$

This natural expression indicates that the probability of the photon loss is proportional to the duration of the pulse and to the mean number of photons in the cavity. Here, we assume moreover that the pulse duration is much smaller than the cavity decay time $T_{\text{cav}} = 1/\kappa$ ($\tau_a \ll T_{\text{cav}}$). For the LKB experimental setup, the pulse duration τ_a is about $85e-06$ seconds and T_{cav} is about $13e-02$ seconds and therefore this assumption is clearly satisfied.

Let us assume now that we dispose of an instrument allowing us to the detect this photon loss when it happens. As soon as we detect a photon loss, the cavity density matrix ρ evolves drastically as follows:

$$\rho_+ = \frac{\mathcal{M}_{\text{loss}}\rho\mathcal{M}_{\text{loss}}^\dagger}{\text{Tr}\left(\mathcal{M}_{\text{loss}}\rho\mathcal{M}_{\text{loss}}^\dagger\right)} = \frac{a\rho a^\dagger}{\text{Tr}\left(\mathbf{N}\rho\right)},$$

recalling that this loss happens with a small probability of $\text{Tr}\left(\mathbf{N}\rho\right)(1+n_{\text{th}})\tau_a/\tau_{\text{cav}}$. Now, let us consider the situation where we do not detect any photon loss. A first impression would be that the density matrix should not change. This is not correct and the fact that we do not detect any photon, actually, updates our information on the system as it privileges the probability of having a fewer number of photons in the cavity. In order to have a more clear idea of the situation, let us assume that, similarly to the photon loss case, we associate a measurement operator $\mathcal{M}_{\text{no-loss}}$ to the phenomenon of not detecting a photon loss. Let us now find this jump operator.

In order to have a well-defined POVM measurement, we need to have

$$\mathcal{M}_{\text{loss}}^\dagger\mathcal{M}_{\text{loss}} + \mathcal{M}_{\text{no-loss}}^\dagger\mathcal{M}_{\text{no-loss}} = \mathbf{1}. \tag{4.20}$$

This, in particular, forbids the possibility of having $\mathcal{M}_{\text{no-loss}} = \mathbf{1}$. Indeed, a possible solution, up to the first order in τ_a/T_{cav}, is given by:

$$\mathcal{M}_{\text{no-loss}} = \mathbf{1} - (1+n_{\text{th}})\tau_a\frac{\tau_a}{2T_{\text{cav}}}a^\dagger a.$$

Noting now that, we actually de not dispose of a measurement instrument indicating the loss of the photons, the evolution of the density matrix is given by the following Kraus representation:

$$\rho_+ = \mathcal{M}_{\text{loss}}\rho\mathcal{M}_{\text{loss}}^\dagger + \mathcal{M}_{\text{no-loss}}\rho\mathcal{M}_{\text{no-loss}}^\dagger = \rho + (1+n_{\text{th}})\frac{\tau_a}{T_{\text{cav}}}\left(a\rho a^\dagger - \frac{1}{2}a^\dagger a\rho - \frac{1}{2}\rho a^\dagger a\right),$$

where we have still neglected the second order terms in τ_a/T_{cav}.

The photon gain phenomenon can be treated exactly in the same way and through the measurement operator $\mathcal{M}_{\text{gain}} = \sqrt{\kappa_+\tau_a}a^\dagger$ proportional to the photon creation operator. The total evolution can be therefore written as follows:

$$\frac{\rho_+ - \rho}{\tau_a} = -\frac{\kappa(1+n_{\text{th}})}{2}\left(a^\dagger a\rho + \rho a^\dagger a - 2a\rho a^\dagger\right) - \frac{\kappa n_{\text{th}}}{2}\left(aa^\dagger\rho + \rho aa^\dagger - 2a^\dagger\rho a\right).$$

4.3 Continuous-time models

Through this section, we consider a quantum system for which the quantum jumps happen continuously in time. As a simple prototype, we will start by considering a two-level atom interacting with a quasi-resonant external optical field. In practice, the atom can also interact with the vacuum modes of the free radiation field. This leads to spontaneous jumps of the atom from its excited state $|e\rangle$ to the ground state $|g\rangle$ together with the emission of a photon in a random direction (see Figure 4.2). Such spontaneous emission might happen at any random time and therefore we deal with a time-continuous random process. Through the next Subsection, we assume that we dispose of photo-detectors allowing us to detect all the spontaneously emitted photons. We propose then a time-continuous stochastic model for the evolution of the systems density matrix living in the 2-dimensional Hilbert space spanned by $|g\rangle$ and $|e\rangle$.

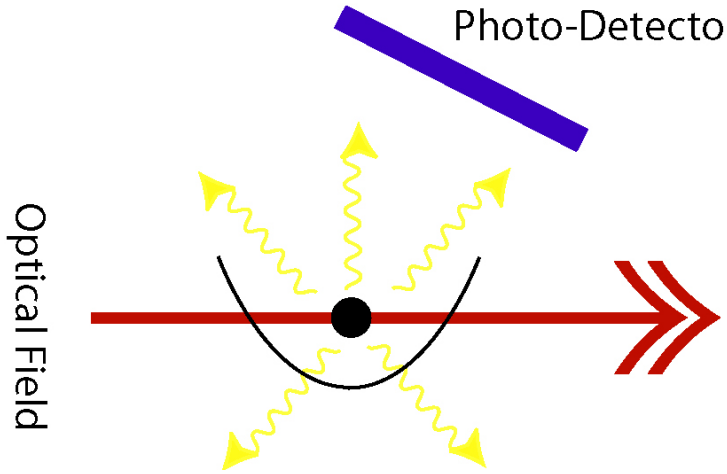

Figure 4.2: A single atom within a Paul trap is addressed by an external optical field and the spontaneously emitted photons are detected by surrounding photodetectors.

4.3.1 Spontaneous emission, quantum Monte Carlo trajectories and Lindblad equation

In order to model the evolution of the 2-level atom (state described by 2×2 density matrix ρ) in presence of the spontaneous emission, we start by dividing the time into small slices of duration dt as in a numerical scheme. During the time interval $[t, t + dt[$, either we detect an emitted photon or none. Indeed, the probability of detecting a photon must be proportional to the population of the excited state $\mathrm{Tr}\left(\rho\,|e\rangle\,\langle e|\right) = \langle e|\rho|e\rangle$ as well as the length of the time interval:

$$P_{\text{jump}} = \Gamma\langle e|\rho|e\rangle dt.$$

Here Γ is the decay rate of the system which is equivalent to the inverse of the atomic lifetime of the excited state $|e\rangle$. The measurement operator corresponding to the jump phenomenon is, actually, given by

$$\mathcal{M}_{\text{jump}} = \sqrt{\Gamma dt}\ \sigma_-,$$

where $\sigma_- = |g\rangle\langle e|$. This indicates the natural fact that, as soon as we detect a photon, the density matrix collapses into the ground state $|g\rangle\langle g|$. Now, let us consider the situation where the photodetectors do not detect any photon. As in Section 4.2.6, a first impression would be that the only evolution of the density matrix should be its unitary evolution due to the Hamiltonian terms. However, this impression is not correct as the fact of not detecting any photon, actually, updates our information on the system as it implies a smaller probability for the atom to be in the excited state. Indeed, if we denote by $\mathcal{M}_{\text{no-jump}}$ the measurement operator corresponding to the phenomenon of not detecting a photon, then in order to have a well-posed POVM, we need to have

$$\mathcal{M}_{\text{jump}}^\dagger \mathcal{M}_{\text{jump}} + \mathcal{M}_{\text{no-jump}}^\dagger \mathcal{M}_{\text{no-jump}} = \mathbf{1}. \tag{4.21}$$

As $\mathcal{M}_{\text{jump}}^\dagger \mathcal{M}_{\text{jump}}$ is of order Γdt, the measurement operator $\mathcal{M}_{\text{no-jump}}$ is necessarily of order $\mathbf{1}$ with respect to Γdt. Indeed, we can write the following generic expression for $\mathcal{M}_{\text{no-jump}}$:

$$\mathcal{M}_{\text{no-jump}} = \mathbf{1} - \Gamma dt\mathcal{A} - i\Gamma dt\mathcal{B}$$

where $\mathcal{A}$ and $\mathcal{B}$ are Hermitian matrices in $\mathbb{C}^2$. Replacing this inside (4.21) and neglecting the second order terms in Γdt, we have

$$\mathcal{A} = \frac{1}{2}\sigma_+\sigma_-,$$

where $\sigma_+ = \sigma_-^\dagger = |e\rangle\langle g|$. Therefore, whenever no jump is detected the density matrix follows the following dynamics (we neglect the second order terms in dt):

$$\begin{aligned}\rho(t+dt) &= \frac{\mathcal{M}_{\text{no-jump}}\rho\mathcal{M}_{\text{no-jump}}^\dagger}{\text{Tr}\left(\mathcal{M}_{\text{no-jump}}\rho(t)\mathcal{M}_{\text{no-jump}}^\dagger\right)} \\ &= \rho(t) - dt\frac{\Gamma}{2}(\sigma_+\sigma_-\rho(t) + \rho(t)\sigma_+\sigma_-) + dt\Gamma\text{Tr}\left(\sigma_-\rho(t)\sigma_+\right)\rho(t) - i\ dt\ \Gamma[\mathcal{B},\rho(t)],\end{aligned}$$

where $[\mathcal{B},\rho(t)]$ denotes the commutator between $\mathcal{B}$ and $\rho(t)$. This can be written as

$$\frac{\rho(t+dt)-\rho(t)}{dt} = -\frac{\Gamma}{2}(\sigma_+\sigma_-\rho(t) + \rho(t)\sigma_+\sigma_-) + \Gamma\text{Tr}\left(\sigma_-\rho(t)\sigma_+\right)\rho(t) - i\ \Gamma[\mathcal{B},\rho(t)].$$

We note here that the above generic computation does not allow us to give an explicit formulation of $\mathcal{B}$. However, as it is a Hermitian matrix, the second part of the dynamics corresponds to a unitary evolution. Indeed, the operator $\Gamma\mathcal{B}$ can be added to the

Hamiltonian of the system (that we have not considered through the above dynamics) as a perturbative Hamiltonian due to the coupling to the vacuum modes of the free radiation field. This perturbation implies a relaxation-induced shift in the energy levels of the atom which is known as the Lamb shifts of the atomic levels. In order to summarize, the stochastic evolution of the atom's density matrix is modeled through the following trajectories, known as *quantum Monte Carlo* trajectories:

$$\rho(t+dt) = \begin{cases} \dfrac{\sigma_-\rho(t)\sigma_+}{\mathrm{Tr}\,(\sigma_-\rho(t)\sigma_+)} = |g\rangle\langle g| & \text{with probability } dt\Gamma\mathrm{Tr}\,(\sigma_-\rho(t)\sigma_+), \\ \rho(t) - i\ dt\ \Gamma[H(t),\rho(t)] - dt\dfrac{\Gamma}{2}(\sigma_+\sigma_-\rho(t) + \rho(t)\sigma_+\sigma_-) + dt\Gamma\mathrm{Tr}\,(\sigma_-\rho(t)\sigma_+)\rho(t) & \\ \qquad\qquad \text{with probability } 1 - dt\Gamma\mathrm{Tr}\,(\sigma_-\rho(t)\sigma_+), \end{cases}$$

where $H(t)$ includes all the Hamiltonian terms. We can combine these two possibilities by applying a Poisson process: in any given time interval $[t, t+dt[$, we define dN_t such that it is unity with probability $\Gamma\mathrm{Tr}\,(\sigma_-\rho(t)\sigma_+)dt$ and zero otherwise. In particular, we have for the conditional average over all possible photo-detection histories

$$\mathbb{E}\,(dN_t) = \Gamma\mathrm{Tr}\,(\sigma_-\rho(t)\sigma_+)dt. \tag{4.22}$$

Very formally, the above dynamics can be represented through the following Ito *stochastic master equation*:

$$d\rho = -i[H(t),\rho]dt - \frac{\Gamma}{2}(\sigma_+\sigma_-\rho + \rho\sigma_+\sigma_-)dt + \Gamma\mathrm{Tr}\,(\sigma_-\rho\sigma_+)\rho dt + \left(\frac{\sigma_-\rho\sigma_+}{\mathrm{Tr}\,(\sigma_-\rho\sigma_+)} - \rho\right)dN_t. \tag{4.23}$$

Now, we can think of the situation where we dispose of a statistical ensemble of identical two-level atoms with no mutual interactions. Assuming that anyone of these two-level atoms obeys the above jump dynamics, we can consider the dynamics of the ensemble average. Noting the statistical independence of dN_t and $\rho(t)$, together with the expression (4.22) for $\mathbb{E}\,(dN_t)$, we get the following average dynamics:

$$\frac{d\rho}{dt} = -i[H(t),\rho] + \frac{\Gamma}{2}\left(\sigma_-\rho\sigma_+ - \frac{1}{2}\sigma_+\sigma_-\rho - \frac{1}{2}\rho\sigma_+\sigma_-\right), \tag{4.24}$$

where (by an abuse of notations) ρ actually stands for the expectation value of ρ in the above jump dynamics. The equation (4.24) is called the *Lindblad master equation* modeling the average dynamics.

4.3.2 Λ-system

We consider here another quantum system consisting of an atom with three energy levels: an excited state $|e\rangle$ and two ground states $|g_1\rangle$ and $|g_2\rangle$. The excited state $|e\rangle$ is assumed to be highly unstable (very short atomic lifetime) and jumps towards one of the two ground states (assumed to be metastable) by emitting spontaneously a photon (of relevant energy)

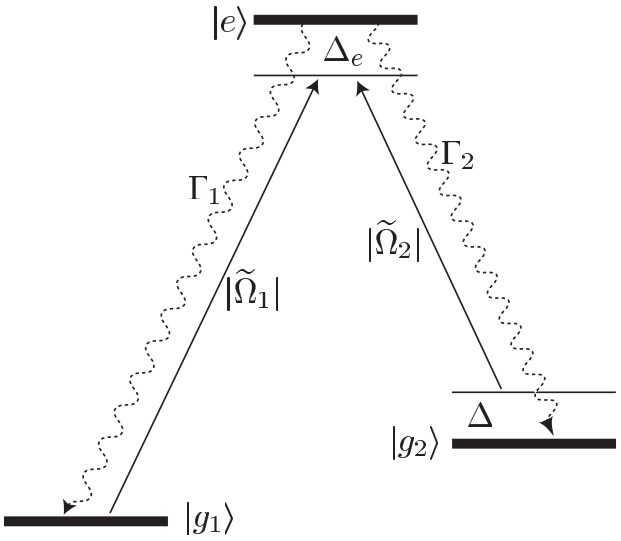

Figure 4.3: Relevant energy levels, transitions and decoherence rates for the Λ-system.

in a random direction. Considering the energy structure of Figure 4.3, the stochastic master equation modeling the quantum Monte-Carlo trajectories of the Λ-system is given as follows:

$$\begin{aligned} d\rho = &- i[H_0 + u(t)H_1, \rho]dt \\ &- \frac{1}{2}(Q_1^\dagger Q_1 \rho + \rho Q_1^\dagger Q_1)dt + \mathrm{Tr}\left(Q_1 \rho Q_1^\dagger\right)\rho dt + \left(\frac{Q_1 \rho Q_1^\dagger}{\mathrm{Tr}\left(Q_1 \rho Q_1^\dagger\right)} - \rho\right) dN_t^1 \\ &- \frac{1}{2}(Q_2^\dagger Q_2 \rho + \rho Q_2^\dagger Q_2)dt + \mathrm{Tr}\left(Q_2 \rho Q_2^\dagger\right)\rho dt + \left(\frac{Q_2 \rho Q_2^\dagger}{\mathrm{Tr}\left(Q_2 \rho Q_2^\dagger\right)} - \rho\right) dN_t^2, \end{aligned} \tag{4.25}$$

where

$$\begin{aligned} H_0 &= E_e \left|e\right\rangle \left\langle e\right| + E_{g1} \left|g_1\right\rangle \left\langle g_1\right| + E_{g2} \left|g_2\right\rangle \left\langle g_2\right|, \\ H_1 &= \mu_1(\left|g_1\right\rangle \left\langle e\right| + \left|e\right\rangle \left\langle g_1\right|) + \mu_2(\left|g_2\right\rangle \left\langle e\right| + \left|e\right\rangle \left\langle g_2\right|), \\ Q_1 &= \sqrt{\Gamma_1} \left|g_1\right\rangle \left\langle e\right|, \qquad Q_2 = \sqrt{\Gamma_2} \left|g_2\right\rangle \left\langle e\right|, \end{aligned}$$

and where dN_t^1 and dN_t^2 are independent Poisson increments with averages

$$\mathbb{E}\left(dN_t^1\right) = \mathrm{Tr}\left(Q_1 \rho Q_1^\dagger\right)dt, \qquad \mathbb{E}\left(dN_t^2\right) = \mathrm{Tr}\left(Q_2 \rho Q_2^\dagger\right)dt.$$

The control field $u(t)$ is quasi-resonant with the transition frequencies and is given by

$$u(t) = u_1 e^{i(\omega_1 - \Delta_e)t} + u_1^* e^{-i(\omega_1 - \Delta_e)t} + u_2 e^{i(\omega_2 - \Delta_e - \Delta)t} + u_2^* e^{-i(\omega_2 - \Delta_e - \Delta)t}, \tag{4.26}$$

where $\omega_1 = E_e - E_{g1}$ and $\omega_2 = E_e - E_{g2}$, u_1 and u_2 are slowly varying complex amplitudes and Δ_e and Δ are small detuning terms. We have three time scales here:

- the very fast time-scale associated to the optical frequencies ω_1 and ω_2;
- the fast time-scale associated to the lifetimes of the excited state's transitions, Γ_1 and Γ_2;
- the slow time-scale associated to the laser amplitudes $|\mu_1 u_1|$ and $|\mu_2 u_2|$.

We are interested here by the slow time-scale of system (4.25) where the control $u(t)$ is given by (4.26) with the following time-scales separation:

$$|\mu_k u_k| \ll \Gamma_{k'} \ll \omega_{k''} \qquad \text{and} \qquad \left|\frac{d}{dt}u_k\right| \ll \Gamma_{k'}|u_k|, \qquad k, k', k'' \in \{1, 2\}.$$

Through this section, we are going to perform the elimination of the fast time-scales for the averaged Lindblad master equation and next, we derive the associated quantum Monte-Carlo trajectories from the reduced Lindblad master equation. The Lindblad equation modeling the average dynamics of the stochastic master equation (4.25) is given by

$$\frac{d\rho}{dt} = -i[H_0 + u(t)H_1, \rho] + \frac{1}{2}\sum_{k=1}^{2}\left(2Q_k\rho Q_k^\dagger - Q_k^\dagger Q_k\rho - \rho Q_k^\dagger Q_k\right). \tag{4.27}$$

Elimination of the fastest time-scales is standard. It corresponds to the rotating wave approximation and can be justified by averaging techniques. Indeed, by passing to the rotating frame by considering the change of variables

$$\rho(t) \to e^{iH_0 t}\rho(t)e^{-iH_0 t}$$

and removing the highly oscillating terms of frequencies $2\omega_1$ and $2\omega_2$, we obtain the Lindblad equation:

$$\frac{d}{dt}\rho = -i[\tilde{H}, \rho] + \frac{1}{2}\sum_{k=1}^{2}(2Q_k\rho Q_k^\dagger - Q_k^\dagger Q_k\rho - \rho Q_k^\dagger Q_k). \tag{4.28}$$

Elimination of the highly oscillating part in $ue^{iH_0 t}H_1 e^{-iH_0 t}$, yields to the averaged Hamiltonian $\tilde{H}$

$$\tilde{H} = \frac{\Delta}{2}(|g_2\rangle\langle g_2| - |g_1\rangle\langle g_1|) + \left(\Delta_e + \frac{\Delta}{2}\right)(|g_1\rangle\langle g_1| + |g_2\rangle\langle g_2|) + \Omega_1|g_1\rangle\langle e| + \Omega_1^*|e\rangle\langle g_1| + \Omega_2|g_2\rangle\langle e| + \Omega_2^*|e\rangle\langle g_2|. \tag{4.29}$$

where $\Omega_k = \mu_k u_k$ are the slowly varying complex Rabi amplitudes. Now, we have the time-scales separation:

$$|\Delta_e|, |\Delta|, |\Omega_k| \ll \Gamma_{k'} \quad \text{and} \quad \left|\frac{d}{dt}\Omega_k\right| \ll \Gamma_{k'}|\Omega_k|, \qquad k, k' \in \{1, 2\}.$$

The slow/fast reduction of the next subsection, based on singular perturbation techniques, allows us to remove the other fast dynamics.

4.3.3 Slow/fast dynamics and model reduction

This sub-section is directly inspired from [54]. We take $\Gamma_k = \overline{\Gamma}_k/\epsilon$ where ϵ is a small positive parameter and $\overline{\Gamma}_k$'s are of the same order as $\tilde{H}$. Thus we have a master equation with the following structure:

$$\frac{d}{dt}\rho = -i[\tilde{H},\rho] + \sum_{k=1}^{2} \frac{\overline{\Gamma}_k}{2\epsilon}(2\sigma_k\rho\sigma_k^\dagger - \sigma_k^\dagger\sigma_k\rho - \rho\sigma_k^\dagger\sigma_k), \tag{4.30}$$

where $\sigma_k = |g_k\rangle\langle e|$.

Define, with $P = |e\rangle\langle e|$,

$$\rho_f = P\rho + \rho P - P\rho P \quad , \quad \rho_s = (1-P)\rho(1-P) + \frac{1}{\overline{\Gamma}_1 + \overline{\Gamma}_2} \sum_{k=1}^{2} \overline{\Gamma}_k \ \sigma_k\rho\sigma_k^\dagger. \tag{4.31}$$

We have

$$\rho = \rho_s + \rho_f - \frac{1}{\overline{\Gamma}_1 + \overline{\Gamma}_2} \sum_{k=1}^{2} \overline{\Gamma}_k \ \sigma_k\rho_f\sigma_k^\dagger \tag{4.32}$$

and therefore $\rho \mapsto (\rho_f, \rho_s)$ is a bijective map. This map is a sort of "change of variables" decoupling the slow part from the fast part of the dynamics. Note that, in the slow part, ρ_s, we have somehow removed the fast dynamics associated with the optical state $|e\rangle$. Moreover, note that, contrarily to ρ_f, the slow part ρ_s is still a well-defined density matrix. This change of variable leads to a standard form:

$$\frac{d}{dt}\rho_f = -\frac{(\overline{\Gamma}_1 + \overline{\Gamma}_2)}{2\epsilon}(\rho_f + P\rho_f P) - i(P[\tilde{H},\rho] + [\tilde{H},\rho]P - P[\tilde{H},\rho]P), \tag{4.33}$$

$$i\frac{d}{dt}\rho_s = (1-P)[\tilde{H},\rho](1-P) + \frac{1}{\overline{\Gamma}_1 + \overline{\Gamma}_2} \sum_{k=1}^{2} \overline{\Gamma}_k\sigma_k[\tilde{H},\rho]\sigma_k^\dagger. \tag{4.34}$$

where $\frac{1}{\epsilon}$ only appears in first equation defining $\frac{d}{dt}\rho_f$. Therefore ρ_f is associated with the fast part of the dynamics and ρ_s represents the slow part. The fast part is asymptotically stable because $-\frac{(\overline{\Gamma}_1+\overline{\Gamma}_2)}{2\epsilon}(\rho_f + P\rho_f P)$ defines a negative definite super-operator on the space of Hermitian operators: $\mathrm{Tr}\left(-(\rho_f + P\rho_f P)\rho_f\right) = -(\|\rho_f\|^2 + \|P\rho_f P\|^2)$, and therefore $\mathrm{Tr}\left(\rho_f^2\right)$ defines a strict quadratic Lyapunov function. Moreover the inverse of this super-operator $X \mapsto X + PXP$ is

$$X \mapsto X - \frac{1}{2}PXP. \tag{4.35}$$

Here we can apply the slow manifold approximation (D.1) described in the Appendix D. Computing the first order terms, we find the following approximation for ρ_f with respect to ρ_s:

$$\rho_f = \frac{-2i\epsilon}{\overline{\Gamma}_1 + \overline{\Gamma}_2} \ (P\tilde{H}\rho_s - \rho_s\tilde{H}P) + O(\epsilon^2). \tag{4.36}$$

where we have also applied $P\rho_s = \rho_s P = 0$. Inserting now the equations (4.32) into the equation (4.34), we get:

$$\frac{d}{dt}\rho_s = -i(1-P)[\tilde{H},\rho_s](1-P) - i(1-P)[\tilde{H},\rho_f](1-P) + \frac{i}{\overline{\Gamma}_1+\overline{\Gamma}_2}(1-P)\sum_{k=1}^{2}\overline{\Gamma}_k[\tilde{H},\sigma_k\rho_f\sigma_k^\dagger](1-P) - \frac{i}{\overline{\Gamma}_1+\overline{\Gamma}_2}\sum_{k=1}^{2}\overline{\Gamma}_k\ \sigma_k[\tilde{H},\rho_f]\sigma_k^\dagger,$$

where we have used the relations

$$\sigma_k\sigma_l = 0, \quad \sigma_k^\dagger\sigma_k = P = |e\rangle\langle e|, \quad P\sigma_k = 0, \quad \sigma_k P = \sigma_k, \quad \forall k,l \in \{1,2\}.$$

and $\sigma_k\rho_s = \rho_s\sigma_k^\dagger = 0$.

Applying now the first order approximation (4.36), and after some simple but tedious computations, we have

$$\frac{d}{dt}\rho_s = -i(1-P)[\tilde{H},\rho_s](1-P) - \frac{2\epsilon}{\overline{\Gamma}_1+\overline{\Gamma}_2}\Big((1-P)\tilde{H}P\tilde{H}(1-P)\rho_s + \rho_s(1-P)\tilde{H}P\tilde{H}(1-P)\Big) + \frac{4\epsilon}{\left(\overline{\Gamma}_1+\overline{\Gamma}_2\right)^2}\sum_{k=1}^{2}\overline{\Gamma}_k\sigma_k\tilde{H}\rho_s\tilde{H}\sigma_k^\dagger + O(\epsilon^2). \tag{4.37}$$

We use here the identities $\sigma_k\rho_s = \rho_s\sigma_k^\dagger = 0$ and $\sigma_k P = \sigma_k$. Continuing the computations, we get

$$\frac{d}{dt}\rho_s = -i[\overline{H},\rho_s] + \frac{\epsilon}{2}\sum_{k=1}^{2}\left(2\overline{Q}_k\rho_s\overline{Q}_k^\dagger - \overline{Q}_k^\dagger\overline{Q}_k\rho_s - \rho_s\overline{Q}_k^\dagger\overline{Q}_k\right) \tag{4.38}$$

where we have defined

$$\overline{H} = (1-P)\tilde{H}(1-P) \qquad \text{and} \qquad \overline{Q}_k = \frac{2\sqrt{\overline{\Gamma}_k}}{\overline{\Gamma}_1+\overline{\Gamma}_2}(1-P)\sigma_k\tilde{H}(1-P). \tag{4.39}$$

We have the following theorem:

Theorem 4.3.1. *Consider ρ the solution of the Lindblad master equation (4.30) with $0 < \epsilon \ll 1$ and ρ_s the solution of the slow master equation (4.38) with (4.39). Assume for the initial states $\|\rho(0) - \rho_s(0)\| = \sqrt{Tr((\rho(0)-\rho_s(0))(\rho(0)-\rho_s(0)))} = O(\epsilon)$. Then*

$$\|\rho(t) - \rho_s(t)\| = \sqrt{Tr((\rho(t)-\rho_s(t))(\rho(t)-\rho_s(t)))} = O(\epsilon)$$

on a time scale $t \sim 1/\epsilon$.

The above approximation is stronger than the one encountered usually in singular-perturbation techniques: approximation errors of order $O(\epsilon)$ are valid not only for the usual bounded time scale $t \sim 1$ (see Appendix D) but also for $t \sim 1/\epsilon$ that is unbounded as ϵ tends to 0^+.

Proof. Applying (4.32) and the singular perturbation theory of the appendix D, we have $\rho(t) = \widetilde{\rho}_s(t) + O(\epsilon)$ where

$$\frac{d}{dt}\widetilde{\rho}_s = -i[\overline{H}, \widetilde{\rho}_s] + \frac{\epsilon}{2}\sum_{k=1}^{2}\left(2\overline{Q}_k\widetilde{\rho}_s\overline{Q}_k^\dagger - \overline{Q}_k^\dagger\overline{Q}_k\widetilde{\rho}_s - \widetilde{\rho}_s\overline{Q}_k^\dagger\overline{Q}_k\right) + O(\epsilon^2), \quad \widetilde{\rho}_s(0) = \rho(0). \tag{4.40}$$

Denoting by $\widehat{\delta\rho_s} = \widetilde{\rho}_s - \rho_s$, we have

$$\frac{d}{dt}\mathrm{Tr}\left(\widehat{\delta\rho_s}^{\,2}\right) \leq 2\epsilon\sum_{k=1}^{2}\left(\mathrm{Tr}\left(\overline{Q}_k\widehat{\delta\rho_s}\overline{Q}_k^\dagger\widehat{\delta\rho_s}\right) - \mathrm{Tr}\left(\overline{Q}_k^\dagger\overline{Q}_k\widehat{\delta\rho_s}^{\,2}\right)\right) + \mathrm{Tr}\left(O(\epsilon^2)\widehat{\delta\rho_s}\right).$$

We apply now the following Cauchy-Schwarz inequalities:

$$\begin{aligned}
\left|\mathrm{Tr}\left(\overline{Q}_k\widehat{\delta\rho_s}\overline{Q}_k^\dagger\widehat{\delta\rho_s}\right)\right| &\leq \left|\mathrm{Tr}\left(\overline{Q}_k^\dagger\overline{Q}_k\widehat{\delta\rho_s}^{\,2}\right)\right|^{1/2}\left|\mathrm{Tr}\left(\overline{Q}_k\overline{Q}_k^\dagger\widehat{\delta\rho_s}^{\,2}\right)\right|^{1/2} \\
&\leq \left|\mathrm{Tr}\left(Q_k^\dagger Q_k Q_k^\dagger Q_k\right)\right|^{1/4}\left|\mathrm{Tr}\left(Q_k Q_k^\dagger Q_k Q_k^\dagger\right)\right|^{1/4}\left|\mathrm{Tr}\left(\widehat{\delta\rho_s}^{\,4}\right)\right|^{1/2}, \\
\left|\mathrm{Tr}\left(\overline{Q}_k\overline{Q}_k^\dagger\widehat{\delta\rho_s}^{\,2}\right)\right| &\leq \left|\mathrm{Tr}\left(Q_k^\dagger Q_k Q_k^\dagger Q_k\right)\right|^{1/2}\left|\mathrm{Tr}\left(\widehat{\delta\rho_s}^{\,4}\right)\right|^{1/2}, \\
\left|\mathrm{Tr}\left(O(\epsilon^2)\widehat{\delta\rho_s}\right)\right| &\leq \left|\mathrm{Tr}\left(O(\epsilon^4)\right)\right|^{1/2}\left|\mathrm{Tr}\left(\widehat{\delta\rho_s}^{\,2}\right)\right|^{1/2} \leq O(\epsilon^2)\left|\mathrm{Tr}\left(\widehat{\delta\rho_s}^{\,2}\right)\right|^{1/2}.
\end{aligned}$$

Taking

$$C_1 = 4\max_{k=1,2}\left|\mathrm{Tr}\left(Q_k^\dagger Q_k Q_k^\dagger Q_k\right)\right|^{1/2},$$

and as $\mathrm{Tr}\left(Q_k^\dagger Q_k Q_k^\dagger Q_k\right) = \mathrm{Tr}\left(Q_k Q_k^\dagger Q_k Q_k^\dagger\right)$, we have a constant $C_2 > 0$ such that,

$$\mathrm{Tr}\left(\widehat{\delta\rho_s}^{\,2}(t)\right) \leq \mathrm{Tr}\left(\widehat{\delta\rho_s}^{\,2}(0)\right) + \epsilon C_1\int_0^t \mathrm{Tr}^{\frac{1}{2}}\left[\widehat{\delta\rho_s}^{\,4}(\tau)\right]d\tau + \epsilon^2 C_2\int_0^t \mathrm{Tr}^{\frac{1}{2}}\left[\widehat{\delta\rho_s}^{\,2}(\tau)\right]d\tau.$$

Note that, $\widehat{\delta\rho_s}^{\,2}$ being non-negative, we have

$$\mathrm{Tr}^{\frac{1}{2}}\left[\widehat{\delta\rho_s}^{\,4}(\tau)\right] \leq \mathrm{Tr}\left[\widehat{\delta\rho_s}^{\,2}(\tau)\right].$$

Therefore, noting $\xi = \sqrt{\mathrm{Tr}\left[\widehat{\delta\rho_s}^{\,2}(\tau)\right]}$, we have

$$\xi^2(t) \leq \xi^2(0) + \epsilon C_1\int_0^t \xi^2(\tau)d\tau + \epsilon^2 C_2\int_0^t \xi(\tau)d\tau.$$

Denoting $\zeta = \xi(t) + \frac{C_2}{2C_1}\epsilon$, some simple computations lead to

$$\zeta^2(t) \leq 2\xi^2(0) + 2\epsilon C_1 \int_0^t \zeta^2(\tau)d\tau - \frac{C_2^2}{2C_1}\epsilon^3\ t + \frac{C_2^2}{2C_1^2}\epsilon^2 \leq \xi^2(0) + \frac{C_2^2}{2C_1^2}\epsilon^2 + 2\epsilon C_1 \int_0^t \zeta^2(\tau)d\tau.$$

Applying the Gronwall lemma, we have $\zeta^2(t) \leq \left[\xi^2(0) + \frac{C_2^2}{2C_1^2}\epsilon^2\right] e^{2\epsilon C_1 t}$.

Noting that, by the Theorem's assumption, $\xi(0) = O(\epsilon)$, we have $\zeta(t) = O(\epsilon)$ on a time scale of $t \sim 1/\epsilon$. As $\xi(t) = \zeta(t) + O(\epsilon)$, this trivially finishes the proof. □

From a practical point of view, the main result of this section is as follows. The correct slow approximation (also called by physicists adiabatic approximation) of the system described by

$$\frac{d}{dt}\rho = -i[\tilde{H}, \rho] + \frac{1}{2}\sum_{k=1}^{2}\left(2Q_k\rho Q_k^\dagger - Q_k^\dagger Q_k\rho - \rho Q_k^\dagger Q_k\right)$$

with $Q_k = \sqrt{\Gamma_k}\,|g_k\rangle\langle e|$ and where the Γ_k's are much larger than $\tilde{H}$, is given by

$$\frac{d}{dt}\rho_s = -i[H_s, \rho_s] + \frac{1}{2}\sum_{k=1}^{2}\left(2Q_{s,k}\rho_s Q_{s,k}^\dagger - Q_{s,k}^\dagger Q_{s,k}\rho_s - \rho_s Q_{s,k}^\dagger Q_{s,k}\right) \tag{4.41}$$

where ρ_s is the density operator associated with the space spanned by the $|g_1\rangle$ and $|g_2\rangle$, and where the slow Hamiltonian and the slow jump operators are ($P = |e\rangle\langle e|$)

$$H_s = (1-P)\tilde{H}(1-P) \quad \text{and} \quad Q_{s,k} = \frac{2}{\Gamma_1+\Gamma_2}Q_k\tilde{H}(1-P), \qquad k \in \{1,2\}.$$

4.3.4 Physical interpretation and reduced Monte-Carlo trajectories

In this section, we provide a physical interpretation of the last section's result for the particular Hamiltonian of the Λ-system (4.29). We get

$$H_s = \frac{\Delta}{2}(|g_2\rangle\langle g_2| - |g_1\rangle\langle g_1|) + (\Delta_e + \frac{\Delta}{2})(|g_1\rangle\langle g_1| + |g_2\rangle\langle g_2|).$$

and

$$Q_{s,k} = 2\sqrt{\Gamma_k}\frac{\sqrt{|\Omega_1|^2+|\Omega_2|^2}}{\Gamma_1+\Gamma_2}\,|g_k\rangle\langle b_\Omega| \quad \text{with} \quad |b_\Omega\rangle = \frac{\Omega_1|g_1\rangle + \Omega_2|g_2\rangle}{\sqrt{|\Omega_1|^2+|\Omega_2|^2}}.$$

However, as we can restrict ourselves to the 2-dimensional Hilbert space spanned by $|g_1\rangle$ and $|g_2\rangle$, the second part of the Hamiltonian H_s can be removed as it only implies a global phase change.

From the Lindblad master equation (4.41) and the above expressions, we can guess the associated reduced stochastic master equation modeling the evolution of a single Λ-system:

$$\begin{aligned} d\rho_s &= -i\frac{\Delta}{2}\left[|g_2\rangle\langle g_2| - |g_1\rangle\langle g_1|, \rho_s\right]dt \\ &- \frac{1}{2}\left(Q_{s,1}^\dagger Q_{s,1}\rho_s + \rho_s Q_{s,1}^\dagger Q_{s,1}\right)dt + \mathrm{Tr}\left(Q_{s,1}\rho_s Q_{s,1}^\dagger\right)\rho_s dt + \left(\frac{Q_{s,1}\rho_s Q_{s,1}^\dagger}{\mathrm{Tr}\left(Q_{s,1}\rho_s Q_{s,1}^\dagger\right)} - \rho_s\right)dN_t^{s,1} \\ &- \frac{1}{2}\left(Q_{s,2}^\dagger Q_{s,2}\rho_s + \rho_s Q_{s,2}^\dagger Q_{s,2}\right)dt + \mathrm{Tr}\left(Q_{s,2}\rho_s Q_{s,2}^\dagger\right)\rho_s dt + \left(\frac{Q_{s,2}\rho_s Q_{s,2}^\dagger}{\mathrm{Tr}\left(Q_{s,2}\rho_s Q_{s,2}^\dagger\right)} - \rho_s\right)dN_t^{s,2}. \end{aligned} \tag{4.42}$$

Here $dN_t^{s,1}$ and $dN_t^{s,2}$ are independent Poisson increments with averages

$$\begin{aligned} \mathbb{E}\left(dN_t^{s,1}\right) &= \mathrm{Tr}\left(Q_{s,1}\rho_s Q_{s,1}^\dagger\right)dt = 4\Gamma_1\frac{|\Omega_1|^2+|\Omega_2|^2}{(\Gamma_1+\Gamma_2)^2}\mathrm{Tr}\left(|b_\Omega\rangle\langle b_\Omega|\,\rho_s\right)dt \\ \mathbb{E}\left(dN_t^{s,2}\right) &= \mathrm{Tr}\left(Q_{s,2}\rho_s Q_{s,2}^\dagger\right)dt = 4\Gamma_2\frac{|\Omega_1|^2+|\Omega_2|^2}{(\Gamma_1+\Gamma_2)^2}\mathrm{Tr}\left(|b_\Omega\rangle\langle b_\Omega|\,\rho_s\right)dt. \end{aligned}$$

We can interpret this stochastic master equation through the associated Monte-Carlo trajectories. Defining

$$\gamma_k = 4\Gamma_k\frac{|\Omega_1|^2+|\Omega_2|^2}{(\Gamma_1+\Gamma_2)^2}, \qquad k\in\{1,2\},$$

the evolution through the time interval $(t, t+dt)$ can be interpreted as below:

- ρ_s jumps into the ground state $|g_1\rangle\langle g_1|$ with probability $dt\gamma_1\mathrm{Tr}\left(|b_\Omega\rangle\langle b_\Omega|\,\rho_s(t)\right)$;
- or it jumps into the ground state $|g_2\rangle\langle g_2|$ with probability $dt\gamma_2\mathrm{Tr}\left(|b_\Omega\rangle\langle b_\Omega|\,\rho_s(t)\right)$;
- or finally, it evolves through the dynamics

$$\begin{aligned} \frac{d}{dt}\rho_s = -i\frac{\Delta}{2}&\left[|g_2\rangle\langle g_2| - |g_1\rangle\langle g_1|, \rho_s\right] \\ &- \frac{(\gamma_1+\gamma_2)}{2}\Big(|b_\Omega\rangle\langle b_\Omega|\,\rho_s + \rho_s\,|b_\Omega\rangle\langle b_\Omega| - 2\mathrm{Tr}\left(|b_\Omega\rangle\langle b_\Omega|\,\rho_s\right)\rho_s\Big), \end{aligned}$$

  with probability $1 - dt(\gamma_1+\gamma_2)\mathrm{Tr}\left(|b_\Omega\rangle\langle b_\Omega|\,\rho_s(t)\right)$.

This interpretation shows, in particular, why in the physics literature the state $|b_\Omega\rangle$ is often called the bright state and the orthogonal state

$$|d_\Omega\rangle = \frac{\Omega_2^*}{\sqrt{|\Omega_1|^2+|\Omega_2|^2}}|g_1\rangle - \frac{\Omega_1^*}{\sqrt{|\Omega_1|^2+|\Omega_2|^2}}|g_2\rangle$$

is called the dark state. Indeed, the probability of jumping towards one of the ground states by emitting a photon is proportional to the population of the bright state $|b_\Omega\rangle$. Therefore, whenever the system is in the state $|d_\Omega\rangle$, no photon will be emitted: hence the name of the dark state.

As it can be seen easily, whenever no detuning is admitted ($\Delta = 0$), the dark state $|d_\Omega\rangle\langle d_\Omega|$ is the only equilibrium state of the slow dynamics. In fact, we can even prove the following result:

Theorem 4.3.2. *Whenever $\Delta = 0$, the density matrix ρ_s, solution of the stochastic master equation (4.42), converges almost surely towards the dark state $|d_\Omega\rangle\langle d_\Omega|$.*

Remark 4.3.3. *The phenomenon of converging towards the dark state is often referred as the coherent population trapping in the physics literature. The target state can be controlled via the ratio Ω_1/Ω_2. The case $\Omega_2 = 0$ ($|d_\Omega\rangle = |g_2\rangle$) corresponds to the optical pumping phenomena.*

Proof. We consider the Markov process:

$$f_t = \mathrm{Tr}\left(|d_\Omega\rangle\langle d_\Omega|\,\rho(t)\right).$$

We can easily compute the evolution of the expectation value of f_t:

$$\frac{d}{dt}\mathbb{E}\left(f_t\right) = \frac{\gamma_1|\Omega_2|^2 + \gamma_2|\Omega_1|^2}{|\Omega_1|^2 + |\Omega_2|^2}\left(1 - \mathbb{E}\left(f_t\right)\right). \tag{4.43}$$

This, together with the fact that $f_t \in [0,1]$, implies that

$$\lim_{t\to\infty}\mathbb{E}\left(f_t\right) = 1.$$

This, together with the Markov inequality (see the Appendix H), proves the convergence in probability of f_t towards 1. However, in order to prove the almost sure convergence, we need to apply a stronger result. The relation (4.43) implies that the process f_t is a submartingale. The process f_t being a bounded submartingale, together with a time-continuous version of the Theorem H.0.10, implies that the random process f_t almost surely converges towards a random variable $\bar{f} \in [0,1]$. We can apply, now, the dominated convergence theorem:

$$\mathbb{E}\left(\bar{f}\right) = \mathbb{E}\left(\lim_{t\to\infty} f_t\right) = \lim_{t\to\infty}\mathbb{E}\left(f_t\right) = 1.$$

This implies that $\bar{f} \equiv 1$ almost surely and ends the proof of the Theorem. □

Before ending this section, we note that the passage from the Lindblad master equation to the associated Monte-Carlo trajectories is not a rigorous passage. One may find many other stochastic master equations leading to this same Lindblad equation in average. However, exploring the physical experiments and in particular the coherent population trapping phenomena, we can see that the considered stochastic master equation must be the one that incorporates the experimental evidences.

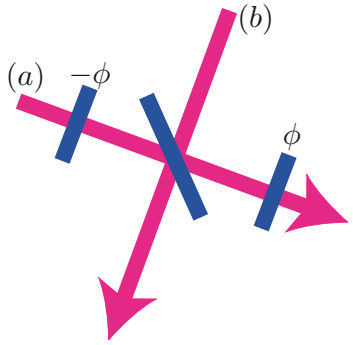

Figure 4.4: A beam-splitter with adjustable phase ϕ. A semi-reflecting plate with an angle of $\pi/4$ with respect to the beams (a) and (b), is sandwiched between two retarding plates inducing opposite phase shifts on beam (a).

4.4 Homodyne detection

In this section, we study another measurement setup allowing the measurement of other physical observables (different from photon number) for a scattered filed. The main idea for such experimental setup is to use a beam-splitter to induce a coupling between the field to be measured and a local oscillator (a coherent field). We start this section by analyzing the action of the beam-splitter. Next, we will show how the created coupling can be used to perform the measurement of various field quadratures. Finally, we will finish by exploring the associated stochastic master equation for a particular example consisting of the dispersive measurement of an atom through its entanglement to an off-resonant coherent field.

4.4.1 Quantum beam-splitter and quadrature measurement

Here, we consider two fields coupled through a beam-splitter as shown in figure 4.4.1. Let us note the annihilation operators of the two considered fields by a and b. The beam-splitter's action on the fields can be formulated through an interaction Hamiltonian H_{ab} acting on the Hilbert space consisting of the tensor product of the two Fock spaces corresponding to the two fields. Describing a linear interaction, the Hamiltonian H_{ab} is made of two terms associated to single-photon exchange between the two modes:

$$H_{ab}(t) = -\frac{g(t)}{2}\left(e^{-i\phi}a\otimes b^\dagger + e^{i\phi}a^\dagger\otimes b\right),$$

where $g(t)$ is a slowly varying real function of time with a support of length τ (duration of the coupling between the two modes). Furthermore the phase ϕ is adjusted through the retarding plates (between which the beam-splitter is sandwiched) inducing opposite phase shifts on the first beam.

Let us adopt the Heisenberg point of view and compute the evolution of the operators a and b under the action of this Hamiltonian. Denoting by (the integral being carried over the whole support of duration τ)

$$U = e^{-i\int H_{ab}(t)dt},$$

the unitary evolution operator, we have

$$\begin{aligned} U^\dagger(a\otimes \mathbf{1})U &= \cos(\theta/2)a\otimes\mathbf{1} + ie^{i\phi}\sin(\theta/2)\mathbf{1}\otimes b,\\ U^\dagger(\mathbf{1}\otimes b)U &= ie^{-i\phi}\sin(\theta/2)a\otimes\mathbf{1} + \cos(\theta/2)\mathbf{1}\otimes b, \end{aligned}$$

where $\theta = \int g(t)dt$ and where we have applied the Cambell-Baker-Hausdroff formula.

Furthermore, noting that $U^\dagger(\theta,\phi) = U(-\theta,\phi)$, we have

$$\begin{aligned} U(a\otimes \mathbf{1})U^\dagger &= \cos(\theta/2)a\otimes\mathbf{1} - ie^{i\phi}\sin(\theta/2)\mathbf{1}\otimes b,\\ U(\mathbf{1}\otimes b)U^\dagger &= -ie^{-i\phi}\sin(\theta/2)a + cos(\theta/2)b. \end{aligned} \tag{4.44}$$

Noting that, the action of the beam-splitter on the dark (vacuum state $|0\rangle\otimes|0\rangle$) is the identity, we now have all the elements to pass to the Schrödinger picture and compute for instance the output fields when the inputs are coherent fields. Indeed, we can write

$$U|\alpha\rangle\otimes|\beta\rangle = UD_\alpha\otimes D_\beta|0\rangle\otimes|0\rangle = UD_\alpha\otimes D_\beta U^\dagger|0\rangle\otimes|0\rangle,$$

where D_α and D_β are, respectively, the displacement operators for the fields a and b ($D_\alpha = \exp(\alpha a^\dagger - \alpha^* a)$). Applying the operator identity $Uf(a)U^\dagger = f(UaU^\dagger)$ for analytic functions f, we have

$$UD_\alpha U^\dagger = e^{\alpha Ua^\dagger U^\dagger - \alpha^* UaU^\dagger}.$$

Therefore, we can write

$$\begin{aligned} UD_\alpha\otimes D_\beta U^\dagger &= UD_\alpha\otimes\mathbf{1}U^\dagger U\mathbf{1}\otimes D_\beta U^\dagger\\ &= \exp(\alpha U(a^\dagger\otimes\mathbf{1})U^\dagger - \alpha^* U(a\otimes\mathbf{1})U^\dagger)\exp(\beta U(\mathbf{1}\otimes b^\dagger)U^\dagger - \beta^* U(\mathbf{1}\otimes b)U^\dagger)\\ &= \exp(\left(\alpha\cos(\theta/2) + i\beta e^{i\phi}\sin(\theta/2)\right)a^\dagger\otimes\mathbf{1} - \left(\alpha^*\cos(\theta/2) - i\beta^* e^{-i\phi}\sin(\theta/2)\right)a\otimes\mathbf{1})\\ &\quad\times\exp(\left(\beta\cos(\theta/2) + i\alpha e^{-i\phi}\sin(\theta/2)\right)\mathbf{1}\otimes b^\dagger - \left(\beta^*\cos(\theta/2) - i\alpha^* e^{i\phi}\sin(\theta/2)\right)\mathbf{1}\otimes b). \end{aligned}$$

Thus

$$U|\alpha\rangle\otimes|\beta\rangle = \left|\alpha\cos(\theta/2) + i\beta e^{i\phi}\sin(\theta/2)\right\rangle\otimes\left|\beta\cos(\theta/2) + i\alpha e^{-i\phi}\sin(\theta/2)\right\rangle. \tag{4.45}$$

Now, let us assume that we dispose of two photodetectors aligned with the two beams (a) and (b) allowing to measure the photon flux after the beam-splitter (see Figure 4.4.1). Furthermore, we assume that the field to be measured, (a), admits the density operator ρ_a while the field (b) is a coherent field $|\beta\rangle$, of real amplitude $\beta\in\mathbb{R}$, playing the role of a local oscillator.

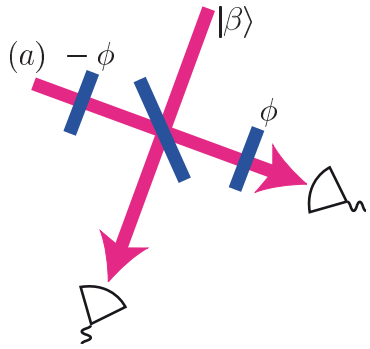

Figure 4.5: Homodyne measurement of various field quadratures

The photodetectors allow us to measure the photon number operators $a^\dagger a \otimes \mathbf{1}$ and $\mathbf{1} \otimes b^\dagger b$ for each incident field. Returning to the Heisenberg picture, we can easily compute the evolution of these operators through the action of the beam-splitter:

$$U^\dagger(a^\dagger a \otimes \mathbf{1})U = \cos^2(\theta/2)a^\dagger a \otimes \mathbf{1} + \sin^2(\theta/2)\mathbf{1} \otimes b^\dagger b + i\sin(\theta/2)\cos(\theta/2)\left(e^{i\phi}a^\dagger \otimes b - e^{-i\phi}a \otimes b^\dagger\right),$$
$$U^\dagger(\mathbf{1} \otimes b^\dagger b)U = \sin^2(\theta/2)a^\dagger a \otimes \mathbf{1} + \cos^2(\theta/2)\mathbf{1} \otimes b^\dagger b - i\sin(\theta/2)\cos(\theta/2)\left(e^{i\phi}a^\dagger \otimes b - e^{-i\phi}a \otimes b^\dagger\right).$$

Taking $\theta = \pi/2$ (balanced homodyne detection) and computing the difference between the photon flux signals, we have

$$\mathrm{Tr}\left((\rho_a \otimes |\beta\rangle\langle\beta|)U^\dagger(a^\dagger a \otimes \mathbf{1})U\right) - \mathrm{Tr}\left((\rho_a \otimes |\beta\rangle\langle\beta|)U^\dagger(\mathbf{1} \otimes b^\dagger b)U\right) =$$
$$i\mathrm{Tr}\left((\rho_a \otimes |\beta\rangle\langle\beta|)\left(e^{i\phi}a^\dagger \otimes b - e^{-i\phi}a \otimes b^\dagger\right)\right) = i\beta\mathrm{Tr}\left(\rho_a\left(e^{i\phi}a^\dagger - e^{-i\phi}a\right)\right).$$

Therefore, the difference between the photon flux signals provide a measurement of the field quadrature

$$X_{\phi+\pi/2} = e^{i(\phi+\pi/2)}a^\dagger + e^{-i(\phi+\pi/2)}a.$$

Through the following subsection we will consider an example showing how this homodyne procedure can be used to perform a dispersive measurement of an atom inside an off-resonant cavity.

4.4.2 Homodyne detection and quantum trajectories

In this subsection we consider the experimental scheme of Figure 4.4.2. The system consists of an atom put inside a low-Q cavity composed of two mirrors. The cavity is driven by a resonant coherent field $|\beta\rangle$ and the cavity field is damped through the output mirror (admitting an important transmission rate). Let us assume that, after a rotating wave approximation that, the Jaynes-Cummings type Hamiltonian is given by

$$H(t) \otimes \mathbf{1} + i\frac{\chi}{2}(L^\dagger \otimes a - L \otimes a^\dagger)$$

where $H(t)$ is a time-dependent Hamiltonian acting on the atom Hilbert space and $i\chi(L^\dagger \otimes a - L \otimes a^\dagger)$ is the atom-cavity interaction Hamiltonian. Here L is any operator (not necessarily hermitian) acting only on the atom. We do not necessarily assume that the atom is a two-level system: it could admit any arbitrary number of levels. Furthermore, taking for the homodyne measurement scheme $\theta = \pi/2$ and $\phi = -\pi/2$ and applying the relations (4.44), we find that the jump operators associated to the two photo-detectors are respectively given by $\sqrt{\gamma/2}(a - \beta/\sqrt{\gamma})$ and $\sqrt{\gamma/2}(a + \beta/\sqrt{\gamma})$ (here β stands for $\beta\mathbf{1}$), where γ is the damping rate caused by the transmission through the output mirror. The quantum Monte-Carlo trajectories are given by the following stochastic master equation:

$$d\xi = -i[H(t),\xi]dt + \frac{\chi}{2}[L^\dagger a - La^\dagger),\xi]dt - \frac{\gamma}{2}\left(a^\dagger a\xi + \xi a^\dagger a - 2\mathrm{Tr}\left(a\xi a^\dagger\right)\xi\right)dt$$
$$+\left(\frac{(a-\beta/\sqrt{\gamma})\xi(a^\dagger-\beta^*/\sqrt{\gamma})}{\mathrm{Tr}\left((a-\beta/\sqrt{\gamma})\xi(a^\dagger-\beta^*/\sqrt{\gamma})\right)} - \xi\right)dN_1 + \left(\frac{(a+\beta/\sqrt{\gamma})\xi(a^\dagger+\beta^*/\sqrt{\gamma})}{\mathrm{Tr}\left((a+\beta/\sqrt{\gamma})\xi(a^\dagger+\beta^*/\sqrt{\gamma})\right)} - \xi\right)dN_2,$$

where $\xi \in \mathcal{H}_S \otimes \mathcal{H}_c$ is the total state of the system and the cavity and where we have removed the tensor products for simplicity of notations. Furthermore, dN_1 and dN_2 are Poisson processes with mean values

$$\mathbb{E}\left(dN_1\right) = \frac{\gamma}{2}\mathrm{Tr}\left((a-\beta/\sqrt{\gamma})\xi(a^\dagger-\beta^*/\sqrt{\gamma})\right)dt \quad \text{and} \quad \mathbb{E}\left(dN_2\right) = \frac{\gamma}{2}\mathrm{Tr}\left((a+\beta/\sqrt{\gamma})\xi(a^\dagger+\beta^*/\sqrt{\gamma})\right)dt.$$

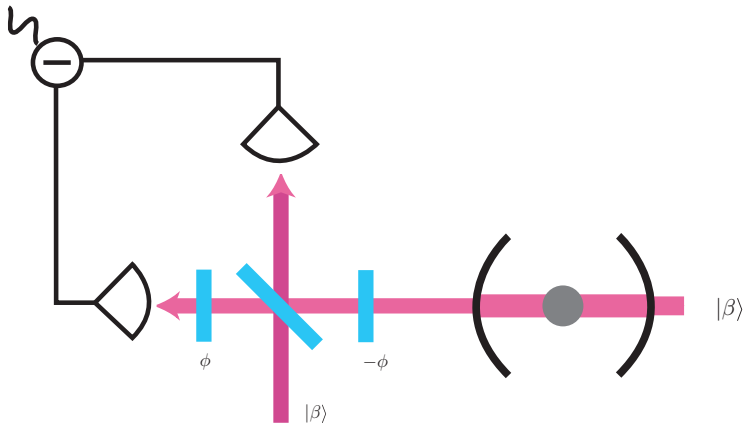

Figure 4.6: Homodyne measurement of an atom: the atom is put inside a cavity composed of two mirrors, one of which has an important transmission rate and serves as the output mirror; this cavity is driven by a coherent field $|\beta\rangle$ in resonance with the cavity.

Consider now the limit where $|H(t)|, \chi \ll \gamma$, so that the damping rate through the

output mirror becomes large:

$$\left|\frac{\|H(t)\|_{L_t^\infty}}{\gamma}\right| \sim \left|\frac{\chi}{\gamma}\right| \sim \epsilon \ll 1. \tag{4.46}$$

These assumptions on the time-scales allow us to apply an adiabatic elimination method based on singular perturbation theory (see the Appendix D). Indeed, as the damping is assumed to be strong, the probability for the cavity to contain more than 1 photon is small. Following the same analysis as in [70], we can write for the state of the system ξ: $\xi = \rho_0 \otimes |0\rangle\, 0 + (\rho_1 \otimes |1\rangle\langle 0| + \rho_1^\dagger \otimes |0\rangle\langle 1|) + O(\epsilon^2)$.

Let us start by analyzing the no-jump dynamics ($dN_1 = dN_2 = 0$). The dynamics for ρ_0 and ρ_1 become

$$\begin{aligned}\frac{d}{dt}\rho_0 &= -i[H(t),\rho_0] + \frac{\chi}{2}(L^\dagger\rho_1 + \rho_1^\dagger L) + \gamma O(\epsilon^2),\\ \frac{d}{dt}\rho_1 &= -i[H(t),\rho_1] - \frac{\chi}{2}L\rho_0 - \frac{\gamma}{2}\rho_1 + O(\epsilon^2).\end{aligned}$$

The time-scale separation (4.46) implies that this system is written in a standard Tikhonov form (see the Appendix D), the state ρ_1 is the exponentially stable fast part and can be slaved to the slow part ρ_0:

$$\rho_1 = -\frac{\chi}{\gamma}L\rho_0 + O(\epsilon^2). \tag{4.47}$$

By inserting (4.47) into the dynamics of ρ_0, we get the following dynamics for the slow part:

$$\frac{d}{dt}\rho_0 = -i[H(t),\rho_0] - \frac{\chi^2}{2\gamma}(L^\dagger L\rho_0 + \rho_0 L^\dagger L) + \gamma O(\epsilon^2).$$

Noting that (4.48) does not conserve the trace of ρ_0, by re-normalizing it we find for the no-jump slow dynamics

$$\frac{d}{dt}\rho_0 = -i[H(t),\rho_0] - \frac{\chi^2}{2\gamma}(L^\dagger L\rho_0 + \rho_0 L^\dagger L - 2\mathrm{Tr}\left(L\rho_0 L^\dagger\right)\rho_0) + \gamma O(\epsilon^2). \tag{4.48}$$

Now, we also note that the jump operators $\sqrt{\gamma/2}(a - \beta/\sqrt{\gamma})$ and $\sqrt{\gamma/2}(a + \beta/\sqrt{\gamma})$, associated to the two photo-detectors of Figure 4.4.2, yield to the jump operators

$$\frac{\chi}{\sqrt{2\gamma}}(L+\alpha) \quad \text{and} \quad \frac{\chi}{\sqrt{2\gamma}}(L-\alpha),$$

where $\alpha = \sqrt{\gamma}/\chi\beta$. Therefore the reduced stochastic master equation is given as follows (we replace ρ_0 by ρ for simplicity sakes)

$$\begin{aligned}d\rho = &-i[H(t),\rho]dt - \frac{\chi^2}{2\gamma}(L^\dagger L\rho + \rho L^\dagger L - 2\mathrm{Tr}\left(L\rho L^\dagger\right)\rho)\\ &+\left(\frac{(L+\alpha)\rho(L^\dagger+\alpha^*)}{\mathrm{Tr}\left((L+\alpha)\rho(L^\dagger+\alpha^*)\right)} - \rho\right)dN_1 + \left(\frac{(L-\alpha)\rho(L^\dagger-\alpha^*)}{\mathrm{Tr}\left((L-\alpha)\rho(L^\dagger-\alpha^*)\right)} - \rho\right)dN_2,\end{aligned}$$

where

$$\mathbb{E}\,(dN_1) = \frac{\chi^2}{2\gamma}\mathrm{Tr}\left((L+\alpha)\xi(L^\dagger+\alpha^*)\right)dt \quad \text{and} \quad \mathbb{E}\,(dN_2) = \frac{\chi^2}{2\gamma}\mathrm{Tr}\left((L-\alpha)\xi(L^\dagger-\alpha^*)\right)dt.$$

Now taking the limit of large amplitudes $|\alpha| \to \infty$, we can replace dN_1 and dN_2 by

$$dN_1 \to \mathbb{E}\,(dN_1)\,dt + \sqrt{\mathbb{E}\,(dN_1)}dW_1,$$
$$dN_2 \to \mathbb{E}\,(dN_2)\,dt + \sqrt{\mathbb{E}\,(dN_2)}dW_2,$$

where dW_1 and dW_2 are independent Wiener processes and where the limit is in the sense of the probability distribution laws. By a Wong-Zakai type theorem, the solution of the stochastic master equation can be well approximated by

$$d\rho = -i[H(t),\rho]dt - \frac{\chi^2}{2\gamma}(L^\dagger L\rho + \rho L^\dagger L - 2L\rho L^\dagger) + \frac{\chi}{\sqrt{\gamma}}\left(e^{i\phi_\alpha}L\rho + e^{-i\phi_\alpha}\rho L^\dagger - 2\mathrm{Tr}\left((e^{i\phi_\alpha}L + e^{-i\phi_\alpha}L^\dagger)\rho\rho\right)\right)dW,$$

where ϕ_α is the argument of α as a complex number and $dW = (dW_1 + dW_2)/\sqrt{2}$ is a Wiener process. In order to obtain the above master equation, we have only considered the lowest order terms in $|\alpha|^{-1}$.

Chapter 5

Control of Open Systems

In the aim of achieving a robust processing of quantum information, one of the main tasks is to prepare and to protect various quantum states. Through the last 15 years, the application of quantum feedback paradigms has been investigated by many physicists [69, 67, 27, 32, 50] as a possible solution for this robust preparation. However, most of these efforts have remained at a theoretical level and have not been able to be give rise to successful experiments. This is essentially due to the necessity of simulating, in parallel to the system, a quantum filter [13] providing an estimate of the state of the system based on the historic of quantum jumps induced by the measurement process. Indeed, it is, in general, difficult to perform such simulations in real time. For an up-to date introduction to quantum measurement and feedback for continuous time quantum system see [71].

Through this chapter, we consider the two prototypes of quantum systems whose models were studied within the last chapter. We start by the time-discrete photon-box model and we present a feedback algorithm stabilizing an arbitrary photon number state (Fock state). Here, the time-discreteness of the model allows to perform the computations of the quantum filter and of the feedback law in real time. This time-discrete feedback has been proposed in [28] and a first mathematical convergence analysis is given in [6]. The exposure below relies on these two references. Next, we consider the time-continuous Λ-system considered in Subsection 4.3.2. We present then a feedback algorithm introduced in [53] allowing to synchronize the laser fields with the atomic transitions. Here, by avoiding to simulate any quantum filter, we consider a simple feedback which only uses the time of the last quantum jump and that is compatible to a real-time implementation. Let us point out that, up to now, none of these two feedback schemes have been tested experimentally.

5.1 Discrete-time system

Through this section we consider the photon-box experiment of Figure 4.2 with the dispersive measurement protocol of the Subsection 4.2.4 defining the operators $\mathcal{M}_g$ and $\mathcal{M}_e$ appearing in the Markov process (4.13). Furthermore, we consider the manipulation of the cavity field by injecting into the mode a coherent field pulse generated by a resonant

microwave source (see Figure 5.1). This manipulation can be modeled through the unitary displacement operator $D_\alpha = \exp(\alpha a^\dagger - \alpha^* a)$ (see subsection 2.2):

$$\rho_+ = D_\alpha \rho D_\alpha^\dagger.$$

We describe the state of the quantum filter, just after the detection of the atom number

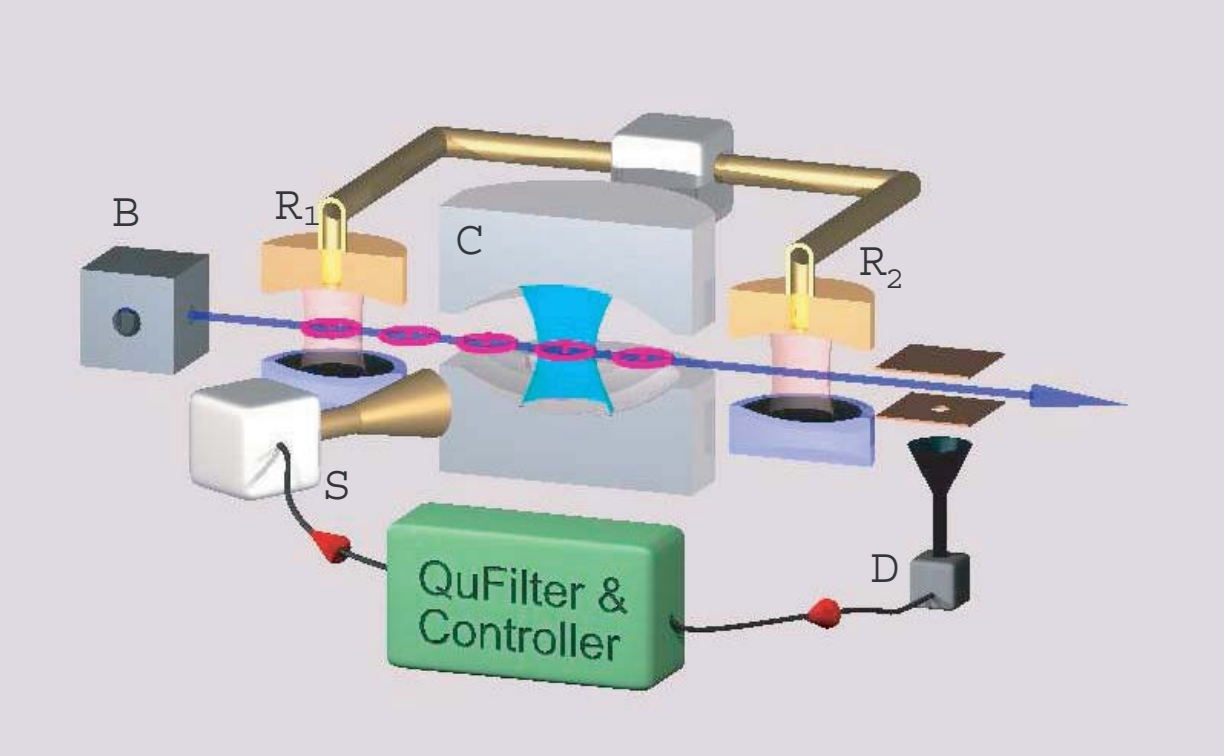

Figure 5.1: A schematic of the closed-loop system borrowed from [28]: an appropriate coherent field pulse whose amplitude and phase are computed as a function of the quantum filter state is injected between two atom passages.

$k-1$, by ρ_k^{est}. This state indicates our knowledge of the system as the observer who has access to the measurement outputs with all its uncertainties and imperfections. Also, we describe a common state of maximal knowledge for the cavity mode by ρ_k. This state denotes the knowledge of a perfect observer having access to all possible quantum jumps due to the measurement or the relaxation. This state of maximal knowledge is necessarily a projector and can also be described by a wavefunction. However, for simplicity sakes, we forget about the wavefunction description and we only consider the density matrix language.

Everywhere through this section, except for the Subsection 5.1.5, we are going to neglect the uncertainties of Subsection 4.2.5 and the relaxations of Subsection 4.2.6. Therefore, the evolution of the quantum filter state ρ_k^{est} must coincide with that of the common state

of maximal knowledge ρ_k (modeled by the associated quantum Monte-Carlo trajectories):

$$\begin{aligned}\rho_{k+1} =& \mathbb{M}_{s_k}(\rho_{k+\frac{1}{2}}), \qquad \rho_{k+\frac{1}{2}} = \mathbb{D}_{\alpha_k}\rho_k,\\ \rho^{\text{est}}_{k+1} =& \mathbb{M}_{s_k}(\rho^{\text{est}}_{k+\frac{1}{2}}), \qquad \rho^{\text{est}}_{k+\frac{1}{2}} = \mathbb{D}_{\alpha_k}\rho^{\text{est}}_k,\end{aligned}$$

where,

- $s_k \in \{g, e\}$, $\mathbb{M}_g(\rho) = \frac{\mathcal{M}_g\rho\mathcal{M}_g^\dagger}{\text{Tr}\left(\mathcal{M}_g\rho\mathcal{M}_g^\dagger\right)}$, $\mathbb{M}_e(\rho) = \frac{\mathcal{M}_e\rho\mathcal{M}_e^\dagger}{\text{Tr}\left(\mathcal{M}_e\rho\mathcal{M}_e^\dagger\right)}$ with operators $\mathcal{M}_g = \cos\left(\varphi_0 + \vartheta\mathbf{N}\right)$ and $\mathcal{M}_e = \sin\left(\varphi_0 + \vartheta\mathbf{N}\right)$ (φ_0, ϑ constant parameters).
- $\mathbb{D}_\alpha(\rho) = D_\alpha\rho D_\alpha^\dagger$. In open-loop, $\alpha = 0$, $D_0 = \mathbf{1}$ (identity operator) and $\mathbb{D}_0(\rho) = \rho$. Notice that $D_\alpha^\dagger = D_{-\alpha}$.
- s_k is a random variable taking the value g when the atom k is detected in g (resp. e when the atom k is detected in e) with probability

$$p_{g,k} = \text{Tr}\left(\mathcal{M}_g\rho_{k+\frac{1}{2}}\mathcal{M}_g^\dagger\right) \quad \left(\text{resp. } p_{e,k} = \text{Tr}\left(\mathcal{M}_e\rho_{k+\frac{1}{2}}\mathcal{M}_e^\dagger\right)\right). \tag{5.1}$$

Through the Subsection 5.1.1, we will study a sort of generic quantum separation principle stating that,

- if for the case where ρ^{est} coincides with ρ (this happens if ρ^{est} is initialized at the initial common state of maximal knowledge, $\rho^{\text{est}}_0 = \rho_0$) the feedback strategy ensures the almost sure convergence towards a particular pure state, then the same happens for the case of a generic ρ^{est}_0 different from ρ_0.

This statement will allow us through the following subsections 5.1.2, 5.1.3 and 5.1.4 to restrict ourselves to the case where $\rho^{\text{est}} \equiv \rho$.

Finally, the Subsection 5.1.5 is devoted to the numerical study of the feedback control in presence of the measurement limitations and the relaxation parameters. We note that in such situation, the dynamics of the filter equation ρ^{est} differ from those of ρ.

Everywhere through this Section, we are going to restrict ourselves to a modal approximation of the Hilbert space. Indeed, we note that the dispersive measurement of the Subsection 4.2.4 avoids any energy exchange between the field and the atoms. Therefore, by assuming that we apply controlled displacements of small amplitudes, we may hope that whenever we initialize the system with a small energy, the system will not reach states with very large number of photons. This means that we can restrict ourselves to the finite dimensional subspace spanned by the $n^{\max}+1$ first modes $\{|0\rangle, |1\rangle, \ldots, |n^{\max}\rangle\}$. Thus,

$$\mathbf{N} = \text{diag}(0, 1, \ldots, n^{\max}), \qquad a|0\rangle = 0, \quad a|n\rangle = \sqrt{n}\,|n-1\rangle \quad \text{for } n = 1, \ldots, n^{\max}.$$

The truncated creation operator $a^\dagger$ is the Hermitian conjugate of a. Notice that we still have $\mathbf{N} = a^\dagger a$, but truncation does not preserve the usual commutation $[a, a^\dagger] = 1$ (this is only valid when $n^{\max} = \infty$).

We will assume everywhere that the parameters φ_0, ϑ are chosen in order to have $\mathcal{M}_g$, $\mathcal{M}_e$ invertible and such that the spectrum of $\mathcal{M}_g^\dagger\mathcal{M}_g = \mathcal{M}_g^2$ and $\mathcal{M}_e^\dagger\mathcal{M}_e = \mathcal{M}_e^2$ are not degenerate.

5.1.1 A quantum separation principle

We consider here a generic discrete-time closed-loop quantum system together with its filter equation defined on the Hilbert space $\mathcal{H}$ of dimension $n^{\max}+1$:

$$\begin{aligned} \rho_{k+1} &= \mathbb{M}_{s_k}(\rho_{k+\frac{1}{2}}), \qquad & \rho_{k+\frac{1}{2}} &= \mathbb{U}_{\alpha_k}(\rho_k), \\ \rho^{\text{est}}_{k+1} &= \mathbb{M}_{s_k}(\rho^{\text{est}}_{k+\frac{1}{2}}), \qquad & \rho^{\text{est}}_{k+\frac{1}{2}} &= \mathbb{U}_{\alpha_k}(\rho^{\text{est}}_k), \end{aligned} \tag{5.2}$$

where $s_k \in \{1,\dots,m\}$ is a random variable taking the value $s \in \{1,\dots,m\}$ with probability $\text{Tr}\left(\mathcal{M}_s^\dagger \mathcal{M}_s \rho_{k+\frac{1}{2}}\right)$. Furthermore, $\mathbb{M}_s(\rho) = \mathcal{M}_s \rho \mathcal{M}_s^\dagger / \text{Tr}\left(\mathcal{M}_s \rho \mathcal{M}_s^\dagger\right)$, where the operators $\mathcal{M}_s$ are the Kraus operators for a POVM measurement satisfying $\sum_s \mathcal{M}_s^\dagger \mathcal{M}_s = \mathbf{1}$.

Finally $\mathbb{U}_\alpha(\rho) = \mathcal{U}_\alpha \rho \mathcal{U}_\alpha^\dagger$, where $\mathcal{U}_\alpha$ is a unitary operator depending on the feedback control $\alpha \in \mathbb{C}$.

We have the following theorem,

Theorem 5.1.1. *Consider any closed-loop system of the form (5.2), where the feedback law α_k is a function of the quantum filter: $\alpha_k = g(\rho_k^{est})$. Assume moreover that, whenever $\rho_0^{est} = \rho_0$ (so that the quantum filter coincides with the closed-loop dynamics, $\rho^{est} \equiv \rho$), the closed-loop system converges almost surely towards a fixed pure state $\bar{\rho}$. Then, for any choice of the initial state ρ_0^{est}, such that $ker\rho_0^{est} \subset ker\rho_0$, the trajectories of the system converge almost surely towards the same pure state: $\rho_k \to \bar{\rho}$.*

Remark 5.1.2. *One only needs to choose $\rho_0^{est} = \frac{1}{n^{\max}+1}\mathbf{1}_{(n^{\max}+1)\times(n^{\max}+1)}$, so that the assumption $ker\rho_0^{est} \subset ker\rho_0$ is satisfied for any ρ_0.*

Proof. The basic idea is based on the fact that $\mathbb{E}\left(\text{Tr}\left(\rho_k \bar{\rho}\right) \mid \rho_0, \rho_0^{\text{est}}\right)$ (where we take the expectation over all jump realizations) depends linearly on ρ_0 even though we are applying a feedback control. Indeed, the feedback law α_k depends only on the historic of the quantum jumps as well as the initialization of the quantum filter ρ_0^{est}. Therefore, we can write

$$\alpha_k = \alpha(\rho_0^{\text{est}}, s_0, \dots, s_{k-1}),$$

where $\{s_j\}_{j=0}^{k-1}$ denotes the sequence of k first jumps. Finally, through simple computations, we have

$$\mathbb{E}\left(\text{Tr}\left(\rho_k \bar{\rho}\right) \mid \rho_0, \rho_0^{\text{est}}\right) = \sum_{s_0,\dots,s_{k-1}} \widetilde{\mathbb{M}}_{s_{k-1}} \circ \mathbb{U}_{\alpha_{k-1}} \circ \dots \circ \widetilde{\mathbb{M}}_{s_0} \circ \mathbb{U}_{\alpha_0}(\rho_0,$$

where

$$\widetilde{\mathbb{M}}_s \rho = \mathcal{M}_s \rho \mathcal{M}_s^\dagger.$$

So, we easily have the linearity of $\mathbb{E}\left(\text{Tr}\left(\rho_k \bar{\rho}\right) \mid \rho_0, \rho_0^{\text{est}}\right)$ with respect to ρ_0.

At this point, we apply the assumption $\ker\rho_0^{\text{est}} \subset \ker\rho_0$ and therefore, one can find a constant $\gamma > 0$ and a well-defined density matrix ρ_0^c, such that

$$\rho_0^{\text{est}} = \gamma\rho_0 + (1-\gamma)\rho_0^c.$$

Now, considering the system (5.2) initialized at the state $(\rho_0^{\text{est}}, \rho_0^{\text{est}})$, we have by the assumptions of the Theorem and applying the dominated convergence theorem:

$$\lim_{k\to\infty} \mathbb{E}\left(\mathrm{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0^{\text{est}}, \rho_0^{\text{est}}\right) = 1.$$

By the linearity of $\mathbb{E}\left(\mathrm{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0, \rho_0^{\text{est}}\right)$ with respect to ρ_0, we have

$$\mathbb{E}\left(\mathrm{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0^{\text{est}}, \rho_0^{\text{est}}\right) = \gamma\mathbb{E}\left(\mathrm{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0, \rho_0^{\text{est}}\right) + (1-\gamma)\mathbb{E}\left(\mathrm{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0^c, \rho_0^{\text{est}}\right),$$

and as both $\mathbb{E}\left(\mathrm{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0, \rho_0^{\text{est}}\right)$ and $\mathbb{E}\left(\mathrm{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0^c, \rho_0^{\text{est}}\right)$ are less than or equal to one, we necessarily have that both of them converge to 1:

$$\lim_{k\to\infty} \mathbb{E}\left(\mathrm{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0, \rho_0^{\text{est}}\right) = 1.$$

This implies the almost sure convergence of the physical system towards the pure state $\bar{\rho}$. □

As stated previously, this theorem allows us to forget about the quantum filter state ρ^{est} through the following subsections 5.1.2, 5.1.3 and 5.1.4 and to restrict ourselves to the case where α_k can be directly chosen as a function of ρ_k.

Exercice 5.1.3. *Prove the following extension of theorem 5.1.1. Consider any closed-loop system of the form (5.2), where the feedback law α_k is a function of the quantum filter: $\alpha_k = g(\rho_k^{est})$. Assume moreover that, whenever $\rho_0^{est} = \rho_0$ (so that the quantum filter coincides with the closed-loop dynamics, $\rho^{est} \equiv \rho$), the closed-loop system converges almost surely towards a sub-space $P(\mathcal{H})$ defined by its orthogonal projector P ($Tr(\rho_k P)$ converges almost surely towards 1). Then, for any choice of the initial state ρ_0^{est}, such that $ker\rho_0^{est} \subset ker\rho_0$, the trajectories of the system converge almost surely towards the same sub-space, i.e., $Tr(\rho_k P)$ and $Tr(\rho_k^{est} P)$ converge almost surely to 1.*

5.1.2 Measurement: a non-deterministic preparation tool

Here, we consider the situation where no control is injected between two pulses and therefore the cavity state ρ_k follows the Markov chain dynamics

$$\rho_{k+1} = \mathbb{M}_{s_k}(\rho_k),$$

where s_k takes the value g (resp. e) with probability $P_{g,k} = \mathrm{Tr}\left(\mathcal{M}_g\rho_k\mathcal{M}_g^\dagger\right)$ (resp. with probability $P_{e,k} = \mathrm{Tr}\left(\mathcal{M}_e\rho_k\mathcal{M}_e^\dagger\right)$). We have the following theorem:

Theorem 5.1.4. *Consider the Markov process defined above with an initial density matrix ρ_0. Then*

- *for any $n \in \{0, \ldots, n^{\max}\}$, $Tr(\rho_k |n\rangle \langle n|) = \langle n| \rho_k |n\rangle$ is a martingale*

- *ρ_k converges with probability 1 to one of the $n^{\max}+1$ Fock state $|n\rangle\langle n|$ with $n \in \{0,\dots,n^{\max}\}$.*
- *the probability to converge towards the Fock state $|n\rangle\langle n|$ is given by $Tr(\rho_0|n\rangle\langle n|) = \langle n|\rho_0|n\rangle$.*

Proof. Let us prove that $\mathrm{Tr}(\rho_k|n\rangle\langle n|)$ is a martingale. Set $\xi = |n\rangle\langle n|$. We have

$$\begin{aligned}\mathbb{E}\left(\mathrm{Tr}\left(\xi\rho_{k+1}\right) \mid \rho_k\right) &= P_{g,k}\mathrm{Tr}\left(\xi\frac{\mathcal{M}_g\rho_k\mathcal{M}_g^\dagger}{P_{g,k}}\right) + P_{e,k}\mathrm{Tr}\left(\xi\frac{\mathcal{M}_e\rho_k\mathcal{M}_e^\dagger}{P_{e,k}}\right)\\ &= \mathrm{Tr}\left(\xi\mathcal{M}_g\rho_k\mathcal{M}_g^\dagger\right) + \mathrm{Tr}\left(\xi\mathcal{M}_e\rho_k\mathcal{M}_e^\dagger\right) = \mathrm{Tr}\left(\rho_k(\mathcal{M}_g^\dagger\xi\mathcal{M}_g + \mathcal{M}_e^\dagger\xi\mathcal{M}_e)\right).\end{aligned}$$

Since ξ commutes with $\mathcal{M}_g$ and $\mathcal{M}_e$ and $\mathcal{M}_g^\dagger\mathcal{M}_g+\mathcal{M}_e^\dagger\mathcal{M}_e = \mathbf{1}$, we have $\mathbb{E}\left(\mathrm{Tr}\left(\xi\rho_{k+1}\right) \mid \rho_k\right) = \mathrm{Tr}\left(\xi\rho_k\right)$. This implies that $\mathrm{Tr}\left(\rho_k|n\rangle\langle n|\right)$ is a martingale.

Now, we consider the following function

$$V_n(\rho) = f(\langle n|\rho|n\rangle),$$

where $f(x) = \frac{x+x^2}{2}$. Notice that f is 1-convexe, $f' \geq \frac{1}{2}$ on $[0,1]$ and satisfies

$$\forall(x,y,\theta)\in[0,1],\quad \theta f(x) + (1-\theta)f(y) = \tfrac{\theta(1-\theta)}{2}(x-y)^2 + f(\theta x + (1-\theta)y). \tag{5.3}$$

The function f is increasing and convex and $\langle n|\rho_k|n\rangle$ is a martingale. Thus $V_n(\rho_k)$ is a sub-martingale.

We apply the fact that

$$\langle n|\mathbb{M}_g(\rho)|n\rangle = \tfrac{\cos^2\varphi_n}{\mathrm{Tr}\left(\mathcal{M}_g\rho\mathcal{M}_g^\dagger\right)}\langle n|\rho|n\rangle,\quad \langle n|\mathbb{M}_e(\rho)|n\rangle = \tfrac{\sin^2\varphi_n}{\mathrm{Tr}\left(\mathcal{M}_e\rho\mathcal{M}_e^\dagger\right)}\langle n|\rho|n\rangle,$$

where $\varphi_n = \varphi_0 + n\vartheta$. Therefore, we have

$$\begin{aligned}\mathbb{E}\left(V_n(\rho_{k+1}) \mid \rho_k\right) = \mathrm{Tr}\left(\mathcal{M}_g\rho_k\mathcal{M}_g^\dagger\right) f\left(\tfrac{\cos^2\varphi_n}{\mathrm{Tr}\left(\mathcal{M}_g\rho_k\mathcal{M}_g^\dagger\right)}\langle n|\rho_k|n\rangle\right)\\ + \mathrm{Tr}\left(\mathcal{M}_e\rho_k\mathcal{M}_e^\dagger\right) f\left(\tfrac{\sin^2\varphi_n}{\mathrm{Tr}\left(\mathcal{M}_e\rho_k\mathcal{M}_e^\dagger\right)}\langle n|\rho_k|n\rangle\right)\end{aligned}$$

Then (5.3), together with

$$\theta = \mathrm{Tr}\left(\mathcal{M}_g\rho_k\mathcal{M}_g^\dagger\right),\ x = \tfrac{\cos^2\varphi_n}{\mathrm{Tr}\left(\mathcal{M}_g\rho_k\mathcal{M}_g^\dagger\right)}\langle n|\rho_k|n\rangle,\ y = \tfrac{\sin^2\varphi_n}{\mathrm{Tr}\left(\mathcal{M}_e\rho_k\mathcal{M}_e^\dagger\right)}\langle n|\rho_k|n\rangle$$

yields to

$$\mathbb{E}\left(V_n(\rho_{k+1}) \mid \rho_k\right) - V_n(\rho_k) = \tfrac{\mathrm{Tr}\left(\mathcal{M}_g\rho_k\mathcal{M}_g^\dagger\right)\mathrm{Tr}\left(\mathcal{M}_e\rho_k\mathcal{M}_e^\dagger\right)(\langle n|\rho_k|n\rangle)^2}{2}\left(\tfrac{\cos^2\varphi_n}{\mathrm{Tr}\left(\mathcal{M}_g\rho_k\mathcal{M}_g^\dagger\right)} - \tfrac{\sin^2\varphi_n}{\mathrm{Tr}\left(\mathcal{M}_e\rho_k\mathcal{M}_e^\dagger\right)}\right)^2.$$

Thus we re-discover that $V_n(\rho_k)$ is a sub-martingale, $\mathbb{E}\left(V_n(\rho_{k+1}) \mid \rho_k\right) \geq V_n(\rho_k)$. Moreover, we have also shown that $\mathbb{E}\left(V_n(\rho_{k+1}) \mid \rho_k\right) = V_n(\rho_k)$ implies that either $\langle n|\rho_k|n\rangle = 0$ or $\mathrm{Tr}\left(\mathcal{M}_g \rho_k \mathcal{M}_g^\dagger\right) = \cos^2\varphi_n$ (assumption $\mathcal{M}_g$ and $\mathcal{M}_e$ invertible is used here).

We apply now the Kushner's invariance theorem (Theorem H.0.13 recalled in the Appendix H) for the Markov process ρ_k and the non-negative super-martingale $1 - V_n(\rho_k)$. This theorem implies that the Markov process ρ_k converges in probability to the largest invariant subset of

$$\left\{\rho \mid \mathrm{Tr}\left(\mathcal{M}_g \rho \mathcal{M}_g^\dagger\right) = \cos^2\varphi_n \text{ or } \langle n|\rho|n\rangle = 0\right\}.$$

But the set $\{\rho \mid \langle n|\rho|n\rangle = 0\}$ is invariant. It remains thus to characterize the largest invariant subset included in $\left\{\rho \mid \mathrm{Tr}\left(\mathcal{M}_g \rho \mathcal{M}_g^\dagger\right) = \cos^2\varphi_n\right\}$. This invariant subset will be denoted by $\mathcal{X}_n$.

Take $\rho \in \mathcal{X}_n$. Invariance means that $\mathbb{M}_g(\rho)$ and $\mathbb{M}_e(\rho)$ belong to $\mathcal{X}_n$ (the fact that $\mathcal{M}_g$ and $\mathcal{M}_e$ are invertible ensures that probabilities to jump with $s = g$ or $s = e$ are strictly positive for any density matrix ρ). Consequently $\mathrm{Tr}\left(\mathcal{M}_g \mathbb{M}_g(\rho) \mathcal{M}_g^\dagger\right) = \mathrm{Tr}\left(\mathcal{M}_g \rho \mathcal{M}_g^\dagger\right) = \cos^2\varphi_n$. This means that $\mathrm{Tr}\left(\mathcal{M}_g^4 \rho\right) = \mathrm{Tr}^2\left(\mathcal{M}_g^2 \rho\right)$. By Cauchy-Schwartz inequality,

$$\mathrm{Tr}\left(\mathcal{M}_g^4 \rho\right) = \mathrm{Tr}\left(\mathcal{M}_g^4 \rho\right)\mathrm{Tr}\left(\rho\right) \geq \mathrm{Tr}^2\left(\mathcal{M}_g^2 \rho\right)$$

with equality if, and only if, $\mathcal{M}_g^4 \rho$ and ρ are co-linear. $\mathcal{M}_g^4$ being non-degenerate, ρ is necessarily a projector over an eigenstate of $\mathcal{M}_g^4$, i.e., $\rho = |m\rangle\langle m|$ for some $m \in \{0, \ldots, n^{\max}\}$. Since $\mathrm{Tr}\left(\mathcal{M}_g \rho \mathcal{M}_g^\dagger\right) = \cos^2\varphi_n > 0$, $m = n$ and thus $\mathcal{X}_n$ is reduced to $\{|n\rangle\langle n|\}$. Therefore the only possibilities for the ω-limit set are $\mathrm{Tr}\left(\rho\,|n\rangle\langle n|\right) = 0$ or 1 and

$$W_n(\rho_k) = \mathrm{Tr}\left(\rho_k\,|n\rangle\langle n|\right)(1 - \mathrm{Tr}\left(\rho_k\,|n\rangle\langle n|\right) \overset{k\to\infty}{\longrightarrow} 0 \qquad \text{in probability.}$$

The convergence in probability together with the fact that $W_n(\rho_k)$ is a positive bounded ($W_n \in [0,1]$) random process implies the convergence in expectation. Indeed

$$\begin{aligned}\limsup_{k\to\infty} \mathbb{E}\left(W_n(\rho_k)\right) &\leq \epsilon \limsup_{k\to\infty} \mathbb{P}(W_n(\rho_k) \leq \epsilon) + \limsup_{k\to\infty} \mathbb{P}(W_n(\rho_k) > \epsilon) \\ &\leq \epsilon + \limsup_{k\to\infty} \mathbb{P}(W_n(\rho_k) > \epsilon) \leq \epsilon,\end{aligned}$$

where for the last inequality, we have applied the convergence in probability of $W_n(\rho_k)$ towards 0. As the above inequality is valid for any $\epsilon > 0$, we have

$$\lim_{k\to\infty} \mathbb{E}\left(W_n(\rho_k)\right) = 0.$$

Furthermore, by the first part of the Theorem, we know that $\mathrm{Tr}\left(\rho_k\,|n\rangle\langle n|\right)$ is a bounded martingale and therefore by the Doob's first martingale convergence theorem (see the Theorem H.0.10 of the Appendix H), $\mathrm{Tr}\left(\rho_k\,|n\rangle\langle n|\right)$ converges almost surely towards a random variable $l_n^\infty \in [0,1]$. This implies that $W_n(\rho_k)$ converges almost surely towards the random variable $l_n^\infty(1 - l_n^\infty) \in [0,1]$. We apply now the dominated convergence theorem

$$\mathbb{E}\left(l_n^\infty(1 - l_n^\infty)\right) = \mathbb{E}\left(\lim_{k\to\infty} W_n(\rho_k)\right) = \lim_{k\to\infty} \mathbb{E}\left(W_n(\rho_k)\right) = 0.$$

This implies that $l_n^\infty(1-l_n^\infty)$ vanishes almost surely and therefore

$$W_n(\rho_k) = \mathrm{Tr}\left(\rho_k \left|n\right\rangle \left\langle n\right|\right)(1-\mathrm{Tr}\left(\rho_k \left|n\right\rangle \left\langle n\right|\right)) \overset{k\to\infty}{\longrightarrow} 0 \qquad \text{almost surely.}$$

As we can repeat this same analysis for any choice of $n \in \{0,1,\ldots,n^{\max}\}$, ρ_k converges almost surely to the set of of Fock states

$$\{\left|n\right\rangle \left\langle n\right| \ \mid n = 0,1,\ldots,n^{\max}\},$$

which ends the proof of the second part.

We have shown that the probability measure associated to the random variable ρ_k converges to the probability measure

$$\sum_{n=0}^{n^{\max}} p_n \delta(\left|n\right\rangle \left\langle n\right|),$$

where $\delta(\left|n\right\rangle \left\langle n\right|)$ denotes the Dirac distribution at $\left|n\right\rangle \left\langle n\right|$ and p_n is the probability of convergence towards $\left|n\right\rangle \left\langle n\right|$. In particular, we have

$$\mathbb{E}\left(\mathrm{Tr}\left(\left|n\right\rangle \left\langle n\right| \rho_k\right)\right) \overset{k\to\infty}{\longrightarrow} p_n.$$

But $\mathrm{Tr}\left(\left|n\right\rangle \left\langle n\right| \rho_k\right)$ is a martingale and $\mathbb{E}\left(\mathrm{Tr}\left(\left|n\right\rangle \left\langle n\right| \rho_k\right)\right) = \mathbb{E}\left(\mathrm{Tr}\left(\left|n\right\rangle \left\langle n\right| \rho_0\right)\right)$. Thus

$$p_n = \left\langle n\right| \rho_0 \left|n\right\rangle,$$

which ends the proof of the third and last part. □

The Theorem 5.1.4 implies that the QND measurement of the Subsection 4.2.4 can be seen as a Fock state preparation tool. However, this state preparation is non-deterministic as we can not be sure to converge towards a desired Fock state $\left|\bar{n}\right\rangle \left\langle \bar{n}\right|$. One way of removing this indeterminism is to repeat the QND measurement process by re-preparing the same initial state and re-launching the same measurement process up to reaching $\left|\bar{n}\right\rangle \left\langle \bar{n}\right|$. However this can take a lot of time and, whenever we remember that in reality we need to deal with the measurement uncertainties and the relaxations, a rapid deterministic convergence can be an important issue to keep an acceptable preparation fidelity (see the subsection 5.1.5). Through the next subsection we propose a feedback protocol to ensure a rapid and deterministic Fock state preparation.

Exercice 5.1.5 (Open-loop convergence in the resonance case)**.** *Consider the Markov chain $\rho_{k+1} = \mathbb{M}_{s_k}(\rho_k)$ where $s_k = g$ (resp. $s_k = e$) with probability $p_{g,k} = Tr\left(\mathcal{M}_g \rho_k \mathcal{M}_g^\dagger\right)$ (resp. $p_{e,k} = Tr\left(\mathcal{M}_e \rho_k \mathcal{M}_e^\dagger\right)$). The Kraus operator are given by (4.17) with $\theta_1 = 0$. The cavity state at step k is described by the density operator ρ_k.*

1. *Show that*
$$\mathbb{E}\left(Tr(N\rho_{k+1})/\rho_k\right) = Tr(N\rho_k) - Tr\left(\sin^2\left(\tfrac{\Theta}{2}\sqrt{N}\right)\rho_k\right).$$

2. *Assume that for any integer n, $\Theta\sqrt{n}/\pi$ is irrational. Then prove, using invariance principle (see appendix H), that almost surely ρ_k tends to the vacuum state $\left|0\right\rangle \left\langle 0\right|$ whatever its initial condition is.*

3. *When $\Theta\sqrt{n}/\pi$ is rational for some integer n, describes the possible ω-limits for ρ_k.*

5.1.3 Quantum feedback: ideal case

We reconsider the Markov chain

$$\rho_{k+1} = \mathbb{M}_{s_k}(\rho_{k+\frac{1}{2}}), \qquad \rho_{k+\frac{1}{2}} = \mathbb{D}_{\alpha_k}\rho_k, \tag{5.4}$$

where $s_k \in \{g, e\}$ with probabilities $p_{g,k} = \mathrm{Tr}\left(\mathcal{M}_g \rho_{k+\frac{1}{2}} \mathcal{M}_g^\dagger\right)$ and $p_{e,k} = \mathrm{Tr}\left(\mathcal{M}_e \rho_{k+\frac{1}{2}} \mathcal{M}_e^\dagger\right)$.

We aim to stabilize the Fock state with $\bar{n}$ photons characterized by the density operator $\bar{\rho} = |\bar{n}\rangle\langle\bar{n}|$. To this end we consider the above Markov chain and we choose the coherent feedback α_k such that the value of the fidelity $\mathrm{Tr}\,(\rho\bar{\rho})$ decreases when passing from $\rho_{k+\frac{1}{2}}$ to ρ_{k+1}. Note that, for $\alpha \in \mathbb{C}$ (encoding both the amplitude and the phase of the coherent pulse) of small enough amplitude, the Baker-Campbell-Hausdorff formula yields the following approximation

$$D_\alpha \rho D_\alpha^\dagger = e^{\alpha a^\dagger - \alpha^* a} \rho e^{-(\alpha a^\dagger - \alpha^* a)} = \rho + [\alpha a^\dagger - \alpha^* a, \rho] + O(|\alpha|^2). \tag{5.5}$$

Therefore, for α_k of small enough amplitude, we have

$$\mathrm{Tr}\,(\bar{\rho}\mathbb{D}_{\alpha_k}(\rho_k)) = \mathrm{Tr}\,(\bar{\rho}\rho_k) + \mathrm{Tr}\left(\bar{\rho}[\alpha_k a^\dagger - \alpha_k^* a, \rho_k]\right) + O(|\alpha_k|^2). \tag{5.6}$$

Thus the complex feedback

$$\alpha_k = \epsilon \mathrm{Tr}\,(\bar{\rho}[\rho_k, a]) \tag{5.7}$$

with a gain $\epsilon > 0$ small enough ensures that

$$\mathrm{Tr}\left(\bar{\rho}\rho_{k+\frac{1}{2}}\right) - \mathrm{Tr}\,(\bar{\rho}\rho_k) \geq \epsilon \left|\mathrm{Tr}\,(\bar{\rho}[\rho_k, a])\right|^2. \tag{5.8}$$

Furthermore, the conditional expectation of $\mathrm{Tr}\,(\bar{\rho}\rho_{k+1})$ knowing $\rho_{k+\frac{1}{2}}$ is given by

$$\mathbb{E}\left(\mathrm{Tr}\,(\bar{\rho}\rho_{k+1}) \mid \rho_{k+\frac{1}{2}}\right) = p_{g,k}\mathrm{Tr}\left(\frac{\bar{\rho}\mathcal{M}_g\rho_{k+\frac{1}{2}}\mathcal{M}_g^\dagger}{p_{g,k}}\right) + p_{e,k}\mathrm{Tr}\left(\frac{\bar{\rho}\mathcal{M}_e\rho_{k+\frac{1}{2}}\mathcal{M}_e^\dagger}{p_{e,k}}\right) = \mathrm{Tr}\left(\bar{\rho}\rho_{k+\frac{1}{2}}\right)$$

since $[\bar{\rho}, \mathcal{M}_g] = [\bar{\rho}, \mathcal{M}_e] = 0$ and $\mathcal{M}_g^\dagger\mathcal{M}_g + \mathcal{M}_e^\dagger\mathcal{M}_e = \mathbf{1}$. Thus

$$\mathbb{E}\left(\mathrm{Tr}\,(\bar{\rho}\rho_{k+1}) \mid \rho_k\right) = \mathbb{E}\left(\mathrm{Tr}\left(\bar{\rho}\rho_{k+\frac{1}{2}}\right) \mid \rho_k\right) \geq \mathrm{Tr}\,(\bar{\rho}\rho_k).$$

and consequently, the expectation value of $\mathrm{Tr}\,(\bar{\rho}\rho_k)$ increases at each sampling time:

$$\mathbb{E}\left(\mathrm{Tr}\,(\bar{\rho}\rho_{k+1})\right) \geq \mathbb{E}\left(\mathrm{Tr}\,(\bar{\rho}\rho_k)\right). \tag{5.9}$$

Considering the Markov process ρ_k, we have therefore shown that $\mathrm{Tr}\,(\bar{\rho}\rho_k)$ is a submartingale bounded from above by 1. This, together with Theorem H.0.10, implies that $\mathcal{V}(\rho_k)$ converges almost surely towards a random variable $\mathrm{fid}_\infty \in [0, 1]$. Furthermore by the dominated convergence theorem

$$\mathbb{E}\left(\mathrm{Tr}\,(\bar{\rho}\rho_k)\right) \nearrow \mathbb{E}\left(\mathrm{fid}_\infty\right) \qquad \text{as } k \to \infty.$$

A deeper analysis, based on Kushner's invariance principle (see the proof of the Theorem 5.1.6), implies that the only possible limits for $\mathrm{Tr}\,(\bar{\rho}\rho_k)$ are 0 or 1 (i.e. fid_∞ takes the values 0 or 1). Note that, if we were dealing with a deterministic system, this analysis would have been enough to ensure the almost global stabilization of ρ_k around $\bar{\rho}$: as the function $\mathrm{Tr}\,(\bar{\rho}\rho_k)$ is increasing, whenever $\mathrm{Tr}\,(\bar{\rho}\rho_0) > 0$ the only possible limit would have been $\mathrm{Tr}\,(\bar{\rho}\rho_k) \to 1$ which implies $\rho_k \to \bar{\rho}$. However, in the stochastic case we only have that the expectation $\mathbb{E}\,(\mathrm{Tr}\,(\bar{\rho}\rho_k))$ is increasing and therefore, even for an initial state ρ_0 such that $\mathrm{Tr}\,(\bar{\rho}\rho_0) < 1$, we can have a part of the trajectories that converge towards bad attractors on the level set $\mathrm{Tr}\,(\bar{\rho}\rho) = 0$. However, by rendering $\mathrm{Tr}\,(\bar{\rho}\rho_k)$ a submartingale we have increased the probability of converging towards $\bar{\rho}$.

In order to solve the problem of the bad attractors and ensuring the global stabilization, we suggest the following modification of the feedback law:

$$\alpha_k = \begin{cases} \epsilon \mathrm{Tr}\,(\bar{\rho}[\rho_k, a]) & \text{if } \mathrm{Tr}\,(\bar{\rho}\rho_k) \geq \eta \\ \underset{|\alpha| \leq \bar{\alpha}}{\mathrm{argmax}}\ \mathrm{Tr}\,(\bar{\rho}\mathbb{D}_\alpha(\rho_k)) & \text{if } \mathrm{Tr}\,(\bar{\rho}\rho_k) < \eta \end{cases} \tag{5.10}$$

where $\bar{\alpha} > 0$ is an arbitrary real constant and $\epsilon > 0$ and $\eta > 0$ are small enough.

In this subsection, we prove that the above feedback scheme (5.10) ensures the global asymptotic stabilization of the closed-loop system (5.4) around the target Fock state $\bar{\rho}$:

Theorem 5.1.6. *Consider the quantum system (5.4) with the switching feedback scheme (5.10). For small enough parameters $\epsilon, \eta > 0$ in the feedback scheme, the trajectories of (5.4) converge almost surely toward the target Fock state $\bar{\rho}$.*

Proof. The proof of Theorem 5.1.6 is based on the application of the Lyapunov-type function

$$V(\rho) = f(\mathrm{Tr}\,(\bar{\rho}\rho)),$$

where $f(x) = \frac{x+x^2}{2}$ has already been used during the proof of Theorem 5.1.4. The proof relies in 4 lemmas:

- in Lemma 5.1.7, we prove an inequality showing that, for small enough ϵ, $V(\rho_k)$ is a sub-martingale within $\mathcal{S}_{\geq\eta} = \{\rho \mid \mathrm{Tr}\,(\bar{\rho}\rho) \geq \eta\}$;
- in Lemma 5.1.8, we show that for small enough η, the trajectories starting within the set $\mathcal{S}_{<\eta} = \{\rho \mid \mathrm{Tr}\,(\bar{\rho}\rho) < \eta\}$ always reach in one step the set $\mathcal{S}_{\geq 2\eta} = \{\rho \mid \mathrm{Tr}\,(\bar{\rho}\rho) \geq 2\eta\}$;
- in Lemma 5.1.9, we show that the trajectories starting within the set $\mathcal{S}_{\geq 2\eta}$, will never hit the set $\mathcal{S}_{<\eta}$ with a uniformly non-zero probability $p_\eta > 0$;
- in Lemma 5.1.10, we combine the first step and the Kushner's invariance principle (see the Theorem H.0.13 of the Appendix H), to prove that almost all trajectories remaining inside $\mathcal{S}_{\geq\eta}$ converge towards $\bar{\rho}$.

The combination of the Lemmas 5.1.8, 5.1.9 and 5.1.10 shows then directly that ρ_k converges almost surely towards $\bar{\rho}$. We detail now these lemmas and their proofs. □

Lemma 5.1.7. *For $\epsilon > 0$ small enough and for ρ_k satisfying $Tr(\bar{\rho}\rho_k) \geq \eta$,*

$$\mathbb{E}\left(V(\rho_{k+1}) \mid \rho_k\right) \geq V(\rho_k) + \tfrac{\epsilon}{2}\left|Tr(\bar{\rho}\ [\rho_k, a])\right|^2 + \tfrac{p_{g,k}p_{e,k}}{2}\left(Tr(\bar{\rho}\mathbb{M}_g \circ \mathbb{D}_{\alpha_k}(\rho_k)) - Tr(\bar{\rho}\mathbb{M}_e \circ \mathbb{D}_{\alpha_k}(\rho_k))\right)^2. \quad (5.11)$$

Proof. We have

$$\mathbb{E}\left(V(\rho_{k+1}) \mid \rho_k\right) = p_{g,k} f\left(\mathrm{Tr}\left(\bar{\rho}\mathbb{M}_g \circ \mathbb{D}_{\alpha_k}(\rho_k)\right)\right) + p_{e,k} f\left(\mathrm{Tr}\left(\bar{\rho}\mathbb{M}_e \circ \mathbb{D}_{\alpha_k}(\rho_k)\right)\right).$$

By (5.3) we find

$$\mathbb{E}\left(V(\rho_{k+1}) \mid \rho_k\right) = f\left(\mathrm{Tr}\left(\bar{\rho}\mathbb{D}_{\alpha_k}(\rho_k)\right)\right) + \tfrac{p_{g,k}p_{e,k}}{2}\left(\mathrm{Tr}\left(\bar{\rho}\mathbb{M}_g \circ \mathbb{D}_{\alpha_k}(\rho_k)\right) - \mathrm{Tr}\left(\bar{\rho}\mathbb{M}_e \circ \mathbb{D}_{\alpha_k}(\rho_k)\right)\right)^2,$$

where we have applied the fact that $\mathcal{M}_g$ and $\mathcal{M}_e$ commute with $\bar{\rho}$ and that $\mathcal{M}_g^\dagger \mathcal{M}_g + \mathcal{M}_e^\dagger \mathcal{M}_e = \mathbf{1}$.

Moreover, since $\alpha_k = \epsilon \mathrm{Tr}\left(\bar{\rho}[\rho_k, a]\right)$, applying (5.6), we get

$$\mathrm{Tr}\left(\bar{\rho}\mathbb{D}_{\alpha_k}(\rho_k)\right) = \mathrm{Tr}\left(\bar{\rho}\rho_k\right) + 2\epsilon\left|\mathrm{Tr}\left(\bar{\rho}[\rho_k, a]\right)\right|^2 + O(\epsilon^2).$$

Thus for $\epsilon > 0$ small enough and uniformly in ρ_k

$$\mathrm{Tr}\left(\bar{\rho}\mathbb{D}_{\alpha_k}(\rho_k)\right) \geq \mathrm{Tr}\left(\bar{\rho}\ \rho_k\right) + \epsilon\left|\mathrm{Tr}\left(\bar{\rho}[\rho_k, a]\right)\right|^2.$$

Using the fact that f is increasing and $f(x+y) \geq f(x) + y/2$ for any $x, y > 0$, we get

$$f\left(\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_{\alpha_k}(\rho_k)\right)\right) \geq f((\mathrm{Tr}\left(\bar{\rho}\ \rho_k\right)) + \tfrac{\epsilon}{2}\left|\mathrm{Tr}\left(\bar{\rho}[\rho_k, a]\right)\right|^2.$$

This finishes the proof of Lemma 5.1.7 □

Lemma 5.1.8. *When $\eta > 0$ is small enough, any state ρ_k satisfying the inequality $Tr(\bar{\rho}\rho_k) < \eta$ yields a new state ρ_{k+1} such that $Tr(\bar{\rho}\rho_{k+1}) \geq 2\eta$.*

Proof. Let us first prove that for any density matrix ρ

$$\max_{|\alpha| \leq \bar{\alpha}} \mathrm{Tr}\left(\bar{\rho}\mathbb{D}_\alpha(\rho)\right) > 0. \quad (5.12)$$

If for some ρ, the above maximum is zero, then for all $\alpha \in \mathbb{C}$ (analyticity of $\mathbb{D}_\alpha$ with respect to $\Re(\alpha)$ and $\Im(\alpha)$):

$$\mathrm{Tr}\left(\bar{\rho}\mathbb{D}_\alpha(\rho)\right) = 0.$$

We can decompose ρ as a sum of projectors,

$$\rho = \sum_{\nu=1}^{m} \lambda_\nu \left|\psi_\nu\right\rangle \left\langle\psi_\nu\right|,$$

where λ_ν are strictly positive eigenvalues, $\sum_\nu \lambda_\nu = 1$, and ψ_ν are the associated normalized eigenstates of ρ, $1 \leq m \leq n^{\max}$. Since $\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_\alpha(\rho)\right) \equiv 0$ for all $\alpha \in \mathbb{C}$, we have for all

ν, $\langle\psi_\nu|\, D_\alpha|\bar{n}\rangle = 0$. Fixing one $\nu \in \{1,\dots,m\}$ and taking $\psi = \psi_\nu$ noting that $D_\alpha = \exp\left(\Re(\alpha)(a^\dagger - a) + i\Im(\alpha)(a^\dagger + a)\right)$ and deriving j times versus $\Re(\alpha)$ and $\Im(\alpha)$ around $\alpha = 0$ we get,

$$\left\langle \psi \mid (a^\dagger - a)^j |\bar{n}\right\rangle = \left\langle \psi \mid (a^\dagger + a)^j |\bar{n}\right\rangle = 0 \qquad \forall j \geq 0.$$

With $j = 0$, we get, $\langle\psi\,|\bar{n}\rangle = 0$. With $j = 1$ we get $\langle\psi\,|\bar{n}-1\rangle = \langle\psi\,|\bar{n}+1\rangle = 0$ since $a^\dagger\,|\bar{n}\rangle = \sqrt{\bar{n}+1}\,|\bar{n}+1\rangle$ and $a\,|\bar{n}\rangle = \sqrt{\bar{n}}\,|\bar{n}-1\rangle$. With $j = 2$ and using the null Hermitian products obtained for $j = 0$ and 1, we deduce that $\langle\psi\,|\bar{n}-2\rangle = \langle\psi\,|\bar{n}+2\rangle = 0$, since $aa^\dagger\,|\bar{n}\rangle$ and $a^\dagger a\,|\bar{n}\rangle$ are colinear to $|\bar{n}\rangle$. Similarly for any j and using the null Hermitian products obtained for $j' < j$, we deduce that $\langle\psi\mid \max(0,\bar{n}-j)\rangle = \langle\psi\mid \min(n^{\max},\bar{n}+j)\rangle = 0$. Thus, for any n, $\langle\psi|n\rangle = 0$, $|\psi\rangle = 0$ and we get a contradiction. We have therefore proved the relation (5.12).

Note, furthermore, that as the operators $\mathcal{M}_g$ and $\mathcal{M}_e$ are invertible, (5.12) also implies

$$F(\rho) = \min_{s\in\{g,e\}} \max_{|\alpha|\leq\bar{\alpha}} \mathrm{Tr}\left(\bar{\rho}\mathbb{M}_s \circ \mathbb{D}_\alpha(\rho)\right) > 0.$$

The map F is continuous with respect to ρ and as the space of the density matrices forms a compact set, there exists $\delta > 0$ such that $F(\rho) \geq \delta$ for any density matrix ρ. Taking now $\eta \leq \delta/2$ and ρ_k such that $\mathrm{Tr}\left(\bar{\rho}\rho_k\right) < \eta$, we necessarily have

$$\mathrm{Tr}\left(\bar{\rho}\rho_{k+1}\right) \geq \min_{s\in\{g,e\}} \max_{|\alpha|\leq\bar{\alpha}} \mathrm{Tr}\left(\bar{\rho}\mathbb{M}_s \circ \mathbb{D}_\alpha(\rho)\right) \geq \delta \geq 2\eta,$$

as the applied control field is given by $\alpha_k = \underset{|\alpha|\leq\bar{\alpha}}{\mathrm{argmax}}\ \mathrm{Tr}\left(\bar{\rho}\mathbb{D}_\alpha(\rho_k)\right)$. □

Lemma 5.1.9. *Whenever ρ_k satisfies $Tr(\bar{\rho}\rho_k) \geq 2\eta$, we have*

$$\mathbb{P}\left(\inf_{k'\geq k} Tr(\bar{\rho}\rho_{k'}) > \eta \mid\ Tr(\bar{\rho}\rho_k) \geq 2\eta\right) \geq p_\eta = \frac{\eta}{1-\eta} > 0.$$

Proof. We know from (5.9) that the process $1 - \mathrm{Tr}\left(\bar{\rho}\rho_k\right)$ is a supermartingale in the set $\mathcal{S}_{\geq\eta} = \{\rho \mid \mathrm{Tr}\left(\bar{\rho}\rho\right) \geq \eta\}$. Therefore, one only needs to use the Doob's inequality (see the Theorem H.0.11 of the Appendix H):

$$\mathbb{P}(\sup_{k'\geq k}(1 - \mathrm{Tr}\left(\bar{\rho}\rho_{k'}\right))) \geq 1-\eta \mid \rho_k \in \mathcal{S}_{\geq 2\eta}) \leq \frac{1 - \mathrm{Tr}\left(\bar{\rho}\rho_k\right)}{1-\eta} \leq \frac{1-2\eta}{1-\eta},$$

and thus

$$\begin{aligned}\mathbb{P}\left(\inf_{k'\geq k} \mathrm{Tr}\left(\bar{\rho}\rho_{k'}\right) > \eta \mid \mathrm{Tr}\left(\bar{\rho}\rho_k\right) \geq 2\eta\right) &= 1 - \mathbb{P}(\sup_{k'\geq k}(1 - \mathrm{Tr}\left(\bar{\rho}\rho_{k'}\right))) \geq 1-\eta \mid \mathrm{Tr}\left(\bar{\rho}\rho_k\right) \geq 2\eta) \\ &\geq 1 - \frac{1-2\eta}{1-\eta} = \frac{\eta}{1-\eta} = p_\eta.\end{aligned}$$

□

Lemma 5.1.10. *Almost all trajectories of (5.4) that never leave the set* $\mathcal{S}_{\geq\eta} = \{\rho \mid Tr\,(\bar{\rho}\rho) \geq \eta\}$ *converge towards the target state* $\bar{\rho}$.

Proof. We apply first the Kushner's invariance Theorem to the Markov process ρ_k with the sub-martingale function $V(\rho_k)$. It ensures convergence in probability towards $\mathcal{I}$ the largest invariant set attached to this sub-martingale (see Theorem H.0.13). Let us prove that $\mathcal{I}$ is reduced to $\{\bar{\rho}\}$.

By inequality (5.11), if ρ belongs to $\mathcal{I}$ then $\mathrm{Tr}\,(\bar{\rho}\,[\rho, a]) = 0$, i.e., $\alpha \equiv 0$ and also

$$\mathrm{Tr}\,(\bar{\rho}\,\mathbb{M}_g \circ \mathbb{D}_\alpha(\rho)) = \mathrm{Tr}\,(\bar{\rho}\,\mathbb{M}_e \circ \mathbb{D}_\alpha(\rho)).$$

Invariance associated $\alpha \equiv 0$ implies therefore

$$\mathrm{Tr}\,(\bar{\rho}\,\mathbb{M}_g(\rho)) = \mathrm{Tr}\,(\bar{\rho}\,\mathbb{M}_e(\rho)).$$

Then ρ satisfies

$$\mathrm{Tr}\left(\bar{\rho}M_g\rho M_g^\dagger\right)\mathrm{Tr}\left(M_e\rho M_e^\dagger\right) = \mathrm{Tr}\left(\bar{\rho}M_e\rho M_e^\dagger\right)\mathrm{Tr}\left(M_g\rho M_g^\dagger\right)$$

that reads, since $M_g^\dagger\bar{\rho}M_g = \cos^2\varphi_{\bar{n}}\ \bar{\rho}$, $M_e^\dagger\bar{\rho}M_e = \sin^2\varphi_{\bar{n}}\ \bar{\rho}$ (recalling that $\varphi_{\bar{n}} = \bar{n}\vartheta + \varphi_0$), and $\mathrm{Tr}\,(\bar{\rho}\rho) > 0$,

$$\cos^2\varphi_{\bar{n}}\mathrm{Tr}\left(M_e\rho M_e^\dagger\right) = \sin^2\varphi_{\bar{n}}\mathrm{Tr}\left(M_g\rho M_g^\dagger\right).$$

Since $\mathrm{Tr}\left(M_e\rho M_e^\dagger\right)+\mathrm{Tr}\left(M_g\rho M_g^\dagger\right) = 1$, we recover $\mathrm{Tr}\left(M_g\rho M_g^\dagger\right) = \cos^2\varphi_{\bar{n}}$ the same condition as the one appearing at the end of the proof of Theorem 5.1.4. Similar invariance arguments combined with $\mathrm{Tr}\,(\bar{\rho}\rho) > 0$ imply then $\rho = \bar{\rho}$. Thus $\mathcal{I}$ is reduced to $\{\bar{\rho}\}$.

Consider now the event $\mathcal{P}_{\geq\eta} = \{\forall k \geq 0,\ \mathrm{Tr}\,(\bar{\rho}\rho_k) \geq \eta\}\}$. Convergence of ρ_k in probability towards $\bar{\rho}$ means that

$$\forall\delta > 0, \quad \lim_{k\to\infty} \mathbb{P}\left(\|\rho_k - \bar{\rho}\| > \delta \mid \mathcal{P}_{\geq\eta}\right) = 0,$$

where $\|\cdot\|$ is any norm on the space of density matrices. The continuity of $\rho \mapsto \mathrm{Tr}\,(\bar{\rho}\rho)$ implies that, $\forall\delta > 0$,

$$\lim_{k\to\infty} \mathbb{P}\left(\mathrm{Tr}\,(\bar{\rho}\rho_k) < 1 - \delta \mid \mathcal{P}_{\geq\eta}\right) = 0.$$

As $0 \leq \mathrm{Tr}\,(\bar{\rho}\rho) \leq 1$, we have

$$1 \geq \mathbb{E}\left(\mathrm{Tr}\,(\bar{\rho}\rho_k) \mid \mathcal{P}_{\geq\eta}\right) \ \geq\ (1-\delta)\mathbb{P}\left(1 - \delta \leq \mathrm{Tr}\,(\bar{\rho}\rho_k) \mid \mathcal{P}_{\geq\eta}\right).$$

Thus

$$1 \geq \mathbb{E}\left(\mathrm{Tr}\,(\bar{\rho}\rho_k) \mid \mathcal{P}_{\geq\eta}\right) \ \geq\ (1-\delta)\Big(1 - \mathbb{P}\left(\mathrm{Tr}\,(\bar{\rho}\rho_k) < 1 - \delta \mid \mathcal{P}_{\geq\eta}\right)\Big).$$

and consequently, $\forall\delta > 0$, $\liminf_{k\to\infty} \mathbb{E}\left(\mathrm{Tr}\,(\bar{\rho}\rho_k) \mid \mathcal{P}_{\geq\eta}\right) \geq 1 - \delta$, i.e.,

$$\lim_{k\to\infty} \mathbb{E}\left(\mathrm{Tr}\,(\bar{\rho}\rho_k) \mid \mathcal{P}_{\geq\eta}\right) = 1.$$

The process $\mathrm{Tr}\,(\bar{\rho}\rho_k)$ being a submartingale, we know by Theorem H.0.10 that it converges, for almost all trajectories remaining in the set $\mathcal{S}_{\geq\eta}$, towards a random process fid_∞. By the dominated convergence Theorem, we obtain

$$\mathbb{E}\left(\mathrm{fid}_\infty\right) = \mathbb{E}\left(\lim_{k\to\infty} \mathrm{Tr}\left(\bar{\rho}\rho_k\right) \mid \mathcal{P}_{\geq\eta}\right) = \lim_{k\to\infty} \mathbb{E}\left(\mathrm{Tr}\left(\bar{\rho}\rho_k\right) \mid \mathcal{P}_{\geq\eta}\right) = 1.$$

This trivially proves that $\mathrm{fid}_\infty \equiv 1$ almost surely and finishes the proof of Lemma 5.1.10. □

5.1.4 Quantum feedback and feedback delay

We have proved that in the ideal case of perfect measurement and no environmentally induced relaxation, the feedback strategy of (5.10) ensures the deterministic preparation of an arbitrary Fock state. Through this subsection, we study a first source of imperfection: the delay. Indeed, in such an experimental setup, we have to take into account a delay of d steps between the measurement process and the feedback injection. Indeed, there are, constantly, d atoms flying between the photon box (the cavity) to be controlled and the atom-detector (typically $d = 5$). Therefore, in our feedback design, we do not have access to the measurement results for the d last atoms. For simplicity sakes, we will only consider the case of $d = 1$ in this section, noting that the general case can be treated exactly in the same manner (see the Remark 5.1.16 and the reference [6]).

In the case we have a delay of 1 atom between the measurement and the injection, the dynamics are given by

$$\rho_{k+1} = \mathbb{M}_{s_k}(\rho_{k+\frac{1}{2}}), \qquad \rho_{k+\frac{1}{2}} = \mathbb{D}_{\alpha_{k-1}}(\rho_k), \tag{5.13}$$

where $s_k \in \{g, e\}$ with probabilities $\mathrm{Tr}\left(\mathcal{M}_{s_k}^\dagger \mathcal{M}_{s_k} \rho_{k+\frac{1}{2}}\right)$.

A first source of problem is due to the fact that, the feedback α_{k-1} is chosen as a function of ρ_{k-1} and therefore, ρ_k does not define anymore a Markov chain. However, this can be fixed quite easily by extending the state space to $\chi_k = (\rho_k, \alpha_{k-1})$. As, for predicting χ_{k+1} we only need to know χ_k, it defines a Markov process.

Here, similarly to the previous section, we would like to choose the feedback law in order to ensure an increasing expectation for the fidelity $\mathrm{Tr}\,(\bar{\rho}\rho_k)$. However, we must note that the feedback α_k will act on the state ρ_{k+1} to reach ρ_{k+2} and we have not access to ρ_{k+1} when computing α_k. Therefore, we need to look for a feedback law α_k such that

$$\mathbb{E}\left(\mathrm{Tr}\left(\bar{\rho}\rho_{k+2}\right) \mid \chi_k\right) \geq \mathbb{E}\left(\mathrm{Tr}\left(\bar{\rho}\rho_{k+1}\right) \mid \chi_k\right). \tag{5.14}$$

Before anything, we note that

$$\mathbb{E}\left(\mathrm{Tr}\left(\bar{\rho}\rho_{k+2}\right) \mid \chi_k\right) = \mathbb{E}\left(\mathrm{Tr}\left(\bar{\rho}(\mathcal{M}_g^\dagger \rho_{k+3/2} \mathcal{M}_g + \mathcal{M}_e^\dagger \rho_{k+3/2} \mathcal{M}_e)\right) \mid \chi_k\right) = \mathbb{E}\left(\mathrm{Tr}\left(\bar{\rho}\rho_{k+3/2}\right) \mid \chi_k\right),$$

where we have applied the facts that $\mathcal{M}_{g,e}$ commute with $\bar{\rho}$ and that $\mathcal{M}_g^\dagger \mathcal{M}_g + \mathcal{M}_e^\dagger \mathcal{M}_e = \mathbf{1}$.

We go further in the computations and we have:

$$\begin{aligned}\mathbb{E}\left(\mathrm{Tr}\left(\bar{\rho}\rho_{k+3/2}\right) \mid \chi_k\right) &= p_{g,k}\mathbb{E}\left(\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_{\alpha_k}\circ\mathbb{M}_g\circ\mathbb{D}_{\alpha_{k-1}}(\rho_k)\right) \mid \chi_k\right)\\ &\quad + p_{e,k}\mathbb{E}\left(\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_{\alpha_k}\circ\mathbb{M}_e\circ\mathbb{D}_{\alpha_{k-1}}(\rho_k)\right) \mid \chi_k\right)\\ &= \mathbb{E}\left(\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_{\alpha_k}\circ\mathbb{K}_{\alpha_{k-1}}(\rho_k)\right) \mid \chi_k\right),\end{aligned}$$

where

$$\mathbb{K}_\alpha(\rho) = \mathcal{M}_g D_\alpha \rho D_\alpha^\dagger \mathcal{M}_g^\dagger + \mathcal{M}_e D_\alpha \rho D_\alpha^\dagger \mathcal{M}_e^\dagger.$$

Moreover

$$\mathbb{E}\left(\mathrm{Tr}\left(\bar{\rho}\rho_{k+1}\right) \mid \chi_k\right) = \mathbb{E}\left(\mathrm{Tr}\left(\bar{\rho}\ \mathbb{K}_{\alpha_{k-1}}(\rho_k)\right) \mid \chi_k\right).$$

Thus, in order to ensure the inequality (5.14), similarly to the previous subsection and applying the Baker-Campbell-Hausdorff formula, we only need to choose

$$\alpha_k = \epsilon \mathrm{Tr}\left(\bar{\rho}\ [\rho_k^{\mathrm{pred}}, a]\right),$$

where

$$\rho_k^{\mathrm{pred}} = \rho^{\mathrm{pred}}(\chi_k) := \mathbb{K}_{\alpha_{k-1}}(\rho_k),$$

is called the predictor's state as it can be seen as a prediction of the state ρ_{k+1}.

Similarly to the previous subsection, by applying stochastic Lyapunov techniques, we can prove that the above feedback strategy ensures the almost sure convergence towards the set of Fock states. While it increases the probability of converging towards the desired Fock state $\bar{\rho}$, it does not avoid the other Fock states. In the aim of avoiding these bad attractors, we propose the following modification

$$\alpha_k = \begin{cases} \epsilon \mathrm{Tr}\left(\bar{\rho}\ [\rho_k^{\mathrm{pred}}, a]\right) & \text{if } \mathrm{Tr}\left(\bar{\rho}\rho_k^{\mathrm{pred}}\right) \geq \eta \\ \underset{|\alpha|\leq\bar{\alpha}}{\mathrm{argmax}} \left(\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_\alpha(\rho_{g,k}^{\mathrm{pred}})\right)\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_\alpha(\rho_{e,k}^{\mathrm{pred}})\right)\right) & \text{if } \mathrm{Tr}\left(\bar{\rho}\rho_k^{\mathrm{pred}}\right) < \eta \end{cases} \tag{5.15}$$

with

$$\begin{cases} \rho_{g,k}^{\mathrm{pred}} = \mathcal{M}_g D_{\alpha_{k-1}} \rho_k D_{\alpha_{k-1}}^\dagger \mathcal{M}_g^\dagger \\ \rho_{e,k}^{\mathrm{pred}} = \mathcal{M}_e D_{\alpha_{k-1}} \rho_k D_{\alpha_{k-1}}^\dagger \mathcal{M}_e^\dagger \end{cases} \tag{5.16}$$

We note, in particular, that the form of the second feedback term in (5.15) for the case of $\mathrm{Tr}\left(\bar{\rho}\rho_k^{\mathrm{pred}}\right) < \eta$ is different from that of (5.10). In fact, in such a situation, we need to choose the feedback α_k such that whatever the (unread yet) result of the measurement s_k of the atom number k is, $\mathrm{Tr}\left(\bar{\rho}\rho_{k+1}^{\mathrm{pred}}\right)$ becomes greater (or equal) than η. This will be more clear in the proof of the Lemma 5.1.13.

Theorem 5.1.11. *Consider the Markov chain $\chi_k = (\rho_k, \alpha_{k-1})$ defined by the dynamics (5.13)-(5.15). It converges almost surely towards $(\bar{\rho}, 0)$.*

The proof of this theorem is very similar to that of Theorem 5.1.6 and is based on the Lyapunov function

$$V(\chi) = f\left(\mathrm{Tr}\left(\bar{\rho}\mathbb{K}_\alpha(\rho)\right)\right)$$

defined on the state $\chi = (\rho, \alpha)$ and where the function $f(x) = \frac{x+x^2}{2}$ is the same as in previous subsections. Four similar lemmas as the ones in the previous subsection ends the proof of Theorem 5.1.11.

Lemma 5.1.12. *For $\epsilon > 0$ small enough and for χ_k satisfying $Tr\left(\bar{\rho}\rho^{pred}(\chi_k)\right) \geq \eta$,*

$$\mathbb{E}\left(Tr\left(\bar{\rho}\rho^{pred}(\chi_{k+1})\right) \mid \chi_k\right) \geq Tr\left(\bar{\rho}\rho^{pred}(\chi_k)\right) + \epsilon \left| Tr\left(\bar{\rho}\ [\rho_k^{pred}, a]\right)\right|^2$$

and also

$$\mathbb{E}\left(V(\chi_{k+1}) \mid \chi_k\right) \geq V(\chi_k) + \tfrac{\epsilon}{2}\left| Tr\left(\bar{\rho}\ [\rho_k^{pred}, a]\right)\right|^2 + \tfrac{p_{g,k}p_{e,k}}{2}\left(Tr\left(\bar{\rho}\ \mathbb{D}_{\alpha_k}\circ\mathbb{M}_g\circ\mathbb{D}_{\alpha_{k-1}}(\rho_k)\right) - Tr\left(\bar{\rho}\ \mathbb{D}_{\alpha_k}\circ\mathbb{M}_e\circ\mathbb{D}_{\alpha_{k-1}}(\rho_k)\right)\right)^2. \quad (5.17)$$

Proof. Since $M_g^\dagger M_g + M_e^\dagger M_e = \mathbf{1}$ and $[\bar{\rho}, M_g] = [\bar{\rho}, M_e] = 0$, we have

$$\mathrm{Tr}\left(\bar{\rho}\ \mathbb{K}_{\alpha_k}(\rho_{k+1})\right) = \mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_{\alpha_k}(\rho_{k+1})\right).$$

Also, we have:

$$\mathbb{E}\left(f\left(\mathrm{Tr}\left(\bar{\rho}\ \mathbb{K}_{\alpha_k}(\rho_{k+1})\right)\right) \mid \chi_k\right) = p_{g,k} f\left(\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_{\alpha_k}\circ\mathbb{M}_g\circ\mathbb{D}_{\alpha_{k-1}}(\rho_k)\right)\right) + p_{e,k} f\left(\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_{\alpha_k}\circ\mathbb{M}_e\circ\mathbb{D}_{\alpha_{k-1}}(\rho_k)\right)\right).$$

The rest of the proof is exactly as in Lemma 5.1.7 and is based on the 1-convexity of the function f. □

Lemma 5.1.13. *When $\eta > 0$ is small enough, any state χ_k satisfying the inequality $Tr\left(\bar{\rho}\rho^{pred}(\chi_k)\right) < \eta$ yields a new state χ_{k+1} such that $Tr\left(\bar{\rho}\rho^{pred}(\chi_{k+1})\right) \geq 2\eta$.*

Proof. Since $\mathcal{M}_g$ and $\mathcal{M}_e$ are invertible, there exists $\zeta \in]0,1[$ such that, for any χ, $\mathrm{Tr}\left(\rho_g^{\mathrm{pred}}(\chi)\right) \geq \zeta$ and $\mathrm{Tr}\left(\rho_e^{\mathrm{pred}}(\chi)\right) \geq \zeta$ (ρ_g^{pred} and ρ_e^{pred} are defined in (5.16)). Denote by $\mathcal{X}_\zeta$ the compact set of Hermitian semi-definite positive matrices with trace in $[\zeta, 1]$: for any χ, $\rho_g^{\mathrm{pred}}(\chi)$ and $\rho_e^{\mathrm{pred}}(\chi)$ are in $\mathcal{X}_\zeta$. Let us prove first that, for any $\rho_g, \rho_e \in \mathcal{X}_\zeta$

$$\max_{|\alpha|\leq\bar{\alpha}} \left(\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_\alpha(\rho_g)\right)\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_\alpha(\rho_e)\right)\right) > 0. \quad (5.18)$$

If for some $\rho_g, \rho_e \in \mathcal{X}_\zeta$, the above maximum is zero, then for all $\alpha \in \mathbb{C}$ (analyticity of $\mathbb{D}_\alpha$ versus $\Re(\alpha)$ and $\Im(\alpha)$):

$$\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_\alpha(\rho_g)\right)\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_\alpha(\rho_e)\right) \equiv 0.$$

This implies that either $\mathrm{Tr}\,(\bar\rho\ \mathbb{D}_\alpha(\rho_g)) \equiv 0$ or $\mathrm{Tr}\,(\bar\rho\ \mathbb{D}_\alpha(\rho_e)) \equiv 0$ (if the product of two analytic functions is zero, one of them is zero). Let us assume for instance that, $\mathrm{Tr}\,(\bar\rho\ \mathbb{D}_\alpha(\rho_g)) \equiv 0$. Exactly as in the proof of Lemma 5.1.8, this implies a contradiction and therefore (5.18) holds true for any $\rho_g, \rho_e \in \mathcal{X}_\zeta$.

The rest of the proof is very similar to that of Lemma 5.1.8 and we leave it to the reader. □

Lemma 5.1.14. *Initializing the Markov process χ_k within the set $\{\chi \mid Tr\left(\bar\rho\rho^{pred}(\chi)\right) \geq 2\eta\}$, χ_k will never hit the set $\{\chi \mid Tr\left(\bar\rho\rho^{pred}(\chi)\right) < \eta\}$ with a probability larger than*

$$p_\eta = \frac{\eta}{1-\eta} > 0.$$

The proof of this Lemma is similar to that of Lemma 5.1.9 and is based on the application of Doob's inequality for the supermartingale $1 - \mathrm{Tr}\left(\bar\rho\rho^{\mathrm{pred}}(\chi_k)\right)$.

Lemma 5.1.15. *Sample paths χ_k remaining in the set $\{Tr\left(\bar\rho\rho^{pred}(\chi)\right) \geq \eta\}$ converge almost surely towards $\bar\chi$ as $k \to \infty$.*

The proof of this Lemma is similar to that of Lemma 5.1.10 and is based on the application of Kushner's invariance principle for the Markov process χ_k with the submartingale function $V(\chi_k)$. We leave the adaptation to the reader.

Remark 5.1.16. *The above analysis can be very easily extended to the case where we have d atom of delay between the measurement and the injection. The associated dynamics*

$$\rho_{k+1} = \mathbb{M}_{s_k}(\rho_{k+\frac{1}{2}}), \qquad \rho_{k+\frac{1}{2}} = \mathbb{D}_{\alpha_{k-d}}(\rho_k)$$

defines a Markov process for the extended state $\chi_k = (\rho_k, \alpha_{k-1}, \ldots, \alpha_{k-d})$. Similarly to (5.15), the feedback law

$$\alpha_k = \begin{cases} \epsilon\, Tr\left(\bar\rho\ [\rho_k^{pred}, a]\right) & \text{if } Tr\left(\bar\rho\rho_k^{pred}\right) \geq \eta \\ \underset{|\alpha|\leq\bar\alpha}{argmax}\left(Tr\left(\bar\rho\ \mathbb{D}_\alpha(\rho_{g,k}^{pred})\right) Tr\left(\bar\rho\ \mathbb{D}_\alpha(\rho_{e,k}^{pred})\right)\right) & \text{if } Tr\left(\bar\rho\rho_k^{pred}\right) < \eta \end{cases} \tag{5.19}$$

with

$$\begin{cases} \rho_k^{pred} = \mathbb{K}_{\alpha_{k-1}} \circ \ldots \circ \mathbb{K}_{\alpha_{k-d}}(\rho_k) \\ \rho_{g,k}^{pred} = \mathbb{K}_{\alpha_{k-1}} \circ \ldots \circ \mathbb{K}_{\alpha_{k-d+1}}(M_g D_{\alpha_{k-d}} \rho_k D_{\alpha_{k-d}}^\dagger M_g^\dagger) \\ \rho_{g,k}^{pred} = \mathbb{K}_{\alpha_{k-1}} \circ \ldots \circ \mathbb{K}_{\alpha_{k-d+1}}(M_e D_{\alpha_{k-d}} \rho_k D_{\alpha_{k-d}}^\dagger M_e^\dagger) \end{cases}$$

ensures the almost sure stabilization of χ_k around $(\bar\rho, 0, \ldots, 0)$. We refer to [6] for more details.

Remark 5.1.17. *The quantum separation principle of the Theorem 5.1.1 can be extended very easily to the above delayed case (we have implicitly assumed this above). We leave its proof to the readers as an exercise.*

5.1.5 Quantum feedback: realistic simulations

Through this subsection, we study all the other uncertainties and imperfections that lead to a difference between the dynamics of the system and the filter. For simplicity sakes we forget about the delay here and we leave the adaptation to the delayed case as an exercise to the reader.

Following the analysis of the Subsections 4.2.5 and 4.2.6, the quantum filter dynamics are given by

$$\rho_{k+1}^{\text{est}} = \mathbb{T} \circ \mathbb{B}_{s_k} \circ \mathbb{D}_{\alpha_k}(\rho_k^{\text{est}}),$$

where the $s_k \in \{g, e, u\}$ is a random variable denoting the result of measurement: atom in $|g\rangle$, atom in $|e\rangle$, or no atom detected. Furthermore $\mathbb{B}_s$ is the Bayesian filter given in Subsection 4.2.5:

$$\begin{aligned}
\mathbb{B}_g(\rho) &= \frac{1-\eta_f}{(1-\eta_f)p_g + \eta_f p_e}\mathcal{M}_g\rho\mathcal{M}_g^\dagger + \frac{\eta_f}{(1-\eta_f)p_g + \eta_f p_e}\mathcal{M}_e\rho\mathcal{M}_e^\dagger,\\
\mathbb{B}_e(\rho) &= \frac{1-\eta_f}{(1-\eta_f)p_e + \eta_f p_g}\mathcal{M}_e\rho\mathcal{M}_e^\dagger + \frac{\eta_f}{(1-\eta_f)p_e + \eta_f p_g}\mathcal{M}_g\rho\mathcal{M}_g^\dagger,\\
\mathbb{B}_u(\rho) &= \frac{1-\eta_a}{1-\eta_a\eta_d}\rho + \frac{\eta_a(1-\eta_d)}{1-\eta_a\eta_d}\left(\mathcal{M}_g\rho\mathcal{M}_g^\dagger + \mathcal{M}_e\rho\mathcal{M}_e^\dagger\right),
\end{aligned}$$

where $p_g = \text{Tr}\left(\mathcal{M}_g^\dagger\mathcal{M}_g\rho\right)$, $p_e = \text{Tr}\left(\mathcal{M}_e^\dagger\mathcal{M}_e\rho\right)$, η_f is the detection fault rate, η_a is the pulse occupation rate and η_d is the detection's efficiency rate.

Following the analysis of the Subsection 4.2.6, the super $\mathbb{T}$, modeling the decoherence, is given by:

$$\mathbb{T}(\rho) = \rho - \frac{\kappa\tau_a(1+n_{\text{th}})}{2}\left(a^\dagger a\rho + \rho a^\dagger a - 2a\rho a^\dagger\right) - \frac{\kappa\tau_a n_{\text{th}}}{2}\left(aa^\dagger\rho + \rho aa^\dagger - 2a^\dagger\rho a\right),$$

where κ is the cavity rate and n_{th} is the average number of thermal photons per mode.

Finally, the feedback law α_k is given by (5.19) where ρ_k must be replaced by ρ_k^{est}. Figures 5.2 and 5.3 correspond to the closed-loop simulation of a single trajectory for such feedback-scheme and with realistic imperfections (see [28] for more details):

$$\begin{aligned}
&\eta_f = \tfrac{1}{10},\ \eta_a = \tfrac{3}{10},\ \eta_d = \tfrac{8}{10},\ \kappa\tau_a = 6.4\ 10^{-4},\ n_{\text{th}} = \tfrac{5}{100},\\
&\qquad d = 5,\ \eta = \tfrac{1}{10},\ \bar{\alpha} = 1,\ n^{\max} = 9,\ \bar{n} = 3,\ \epsilon = \tfrac{1}{2\bar{n}+1},\ \vartheta = \tfrac{4}{10},\ \varphi_0 = \tfrac{\pi}{4} - \bar{n}\vartheta.
\end{aligned}$$

Note that, through these simulations the probability distribution for the random variables $s_k \in \{g, e, u\}$ are given by the simulation of the real system in parallel to the filter. In order to simulate the real system, we simulate its common state of maximal knowledge, assuming there is no uncertainty in the measurement and that the relaxation is also measured:

$$\rho_{k+1} = \mathbb{M}_{t_k} \circ \mathbb{M}_{r_k} \circ \mathbb{M}_{s_k} \circ \mathbb{D}_{\alpha_k}(\rho_k),$$

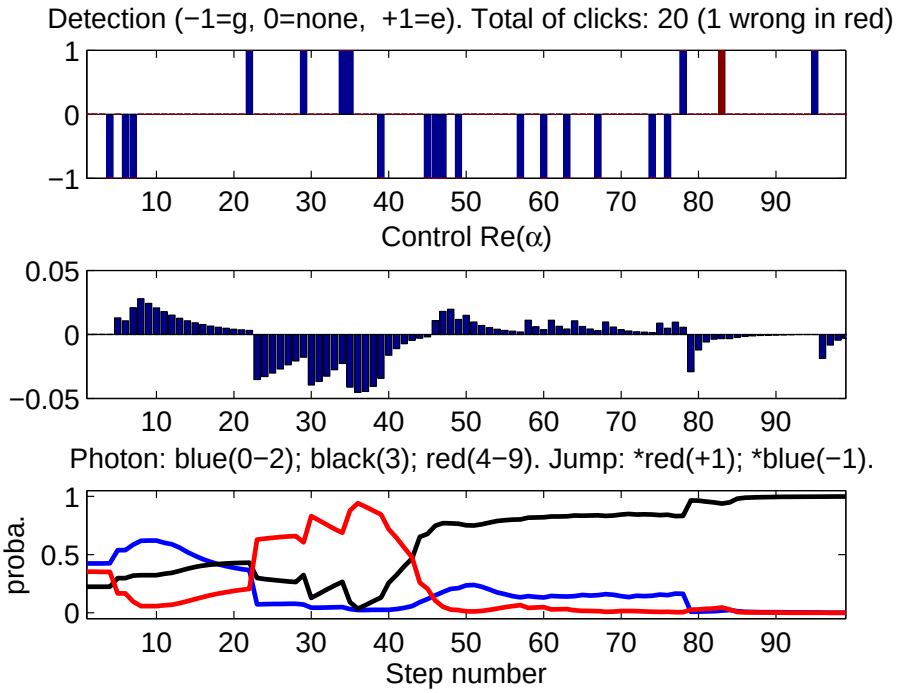

Figure 5.2: Closed-loop simulation of the photon-box system: a single realization with realistic parameters and uncertainties.

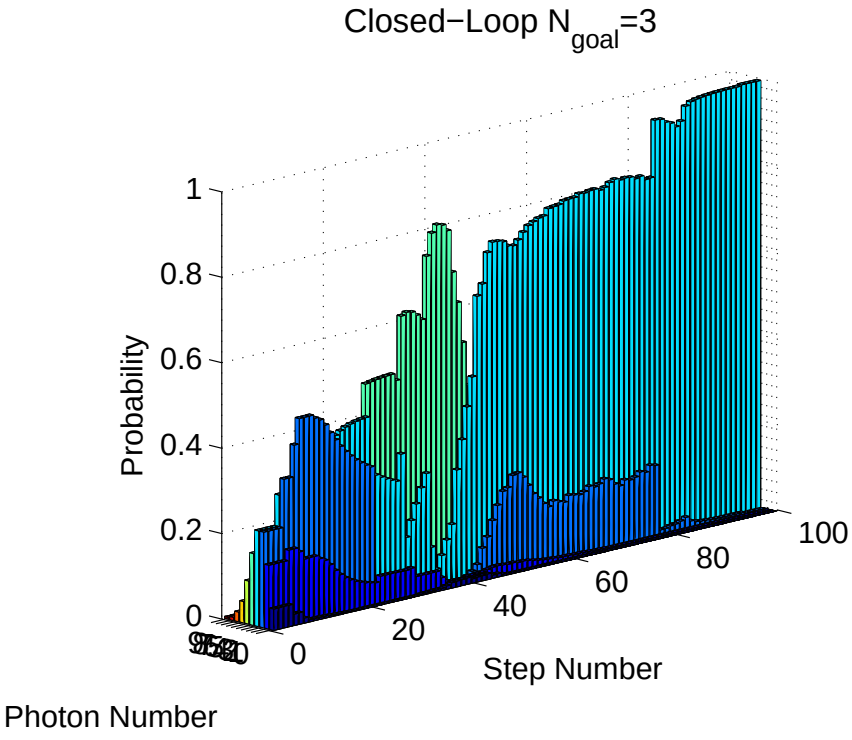

Figure 5.3: Convergence towards the 3-photon Fock state of the diagonal elements of the density matrix ρ (photon populations) .

where $s_k \in \{g, e, u\}$, $r_k \in \{\text{loss}, \text{non-loss}\}$ and $t_k \in \{\text{gain}, \text{no-gain}\}$ are random variables admitting probability distributions:

$$\begin{aligned}
\mathbb{P}(s_k = g) &= \eta_a \text{Tr}\left(\mathcal{M}_g^\dagger \mathcal{M}_g \mathbb{D}_{\alpha_k}(\rho_k)\right), \\
\mathbb{P}(s_k = e) &= \eta_a \text{Tr}\left(\mathcal{M}_e^\dagger \mathcal{M}_e \mathbb{D}_{\alpha_k}(\rho_k)\right), \\
\mathbb{P}(s_k = u) &= 1 - \eta_a, \\
\mathbb{P}(r_k = \text{loss}) &= \kappa(1 + n_{\text{th}})\tau_a \text{Tr}\left(a^\dagger a \rho\right), \\
\mathbb{P}(r_k = \text{no-loss}) &= 1 - \kappa(1 + n_{\text{th}})\tau_a \text{Tr}\left(a^\dagger a \rho\right), \\
\mathbb{P}(t_k = \text{gain}) &= \kappa n_{\text{th}} \tau_a \text{Tr}\left(a a^\dagger \rho\right), \\
\mathbb{P}(t_k = \text{no-gain}) &= 1 - \kappa n_{\text{th}} \tau_a \text{Tr}\left(a a^\dagger \rho\right).
\end{aligned}$$

These probability distributions are respectively associated to the Kraus operators $\sqrt{\eta_a}\mathcal{M}_g$, $\sqrt{\eta_a}\mathcal{M}_e$, $\sqrt{1-\eta_a}\ \mathbf{1}$, $\mathcal{M}_{\text{loss}}$, $\mathcal{M}_{\text{no-loss}}$, $\mathcal{M}_{\text{gain}}$ and $\mathcal{M}_{\text{no-gain}}$.

5.2 Continuous-time system

Through this section we consider the continuous-time system of the Section 4.3. The Λ-configuration of the subsection 4.3.2 and the physical phenomena of coherent population trapping, presented in Subsection 4.3.4, are the basis of atomic micro-clocks largely used within the GPS satellites.

Indeed, the SI second is defined to be "the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom" [1]. A primary frequency standard is a device that realize this definition. Indeed, the basic idea to achieve accurate and stable clocks is to synchronize a laser field with particular atomic transition frequencies.

In particular, the CPT-based atomic micro-clocks are based on the Λ-structure of the Subsection 4.3.2, where the two ground states $|g_1\rangle$ and $|g_2\rangle$ have their energy separation in the radio-frequency or microwave region, and the excited state $|e\rangle$ is coupled to the lower ones by optical transitions. In such clocks, we consider a population of identical quantum systems with few mutual interactions (a vapor cell) having reached its asymptotic statistical regime. As it can be seen through the Figure 5.4, the principle behind the mechanism of such clocks is that, whenever the detuning Δ (defined in Subsection 4.3.2) vanishes, the atoms converge rapidly to the dark state (coherent population trapping phenomena) and therefore the ensemble becomes transparent to the laser field (no photon absorption). However, as soon as the detuning Δ becomes non-zero, the system will keep absorbing the photons and irradiating them in a random direction through spontaneous emission. This yields to a drop of the transmission rate that can be measured via a photodetector aligned with the laser field. Therefore, we need to look for an algorithm allowing to tune the laser's frequency in order to reach the maximum transmission rate (or equivalently the minimum absorption rate). This can be done through an Extremum-Seeking feedback loop. A basic

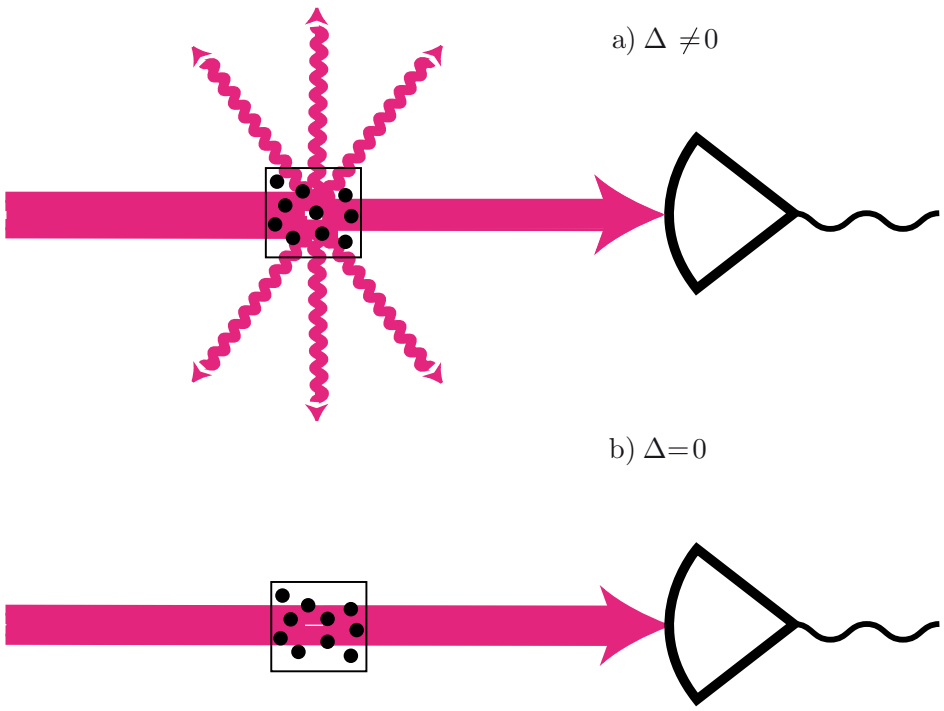

Figure 5.4: Basic principle behind the mechanism of the CPT-based atomic micro-clocks, a) whenever the detuning $\Delta \neq 0$ the vapor cell keeps absorbing the photons and irradiating them in a random direction through the spontaneous emission; b) no detuning leads to the transparency of the vapor cell with respect to the laser field; therefore we reach the maximum transmission rate.)

schematic of such a loop can be seen in Figure 5.5. Indeed for any Δ the absorption rate converges very rapidly towards a static value $f(\Delta)$ and the diagram of Figure 5.5 illustrates the algorithm to reach this minimum (at least approximately). Note that, in practice, we do not have access to the detuning Δ but to the laser frequencies. Having locked one of the laser frequencies, we will therefore tune the frequency of the other one to reach the minimum absorption rate.

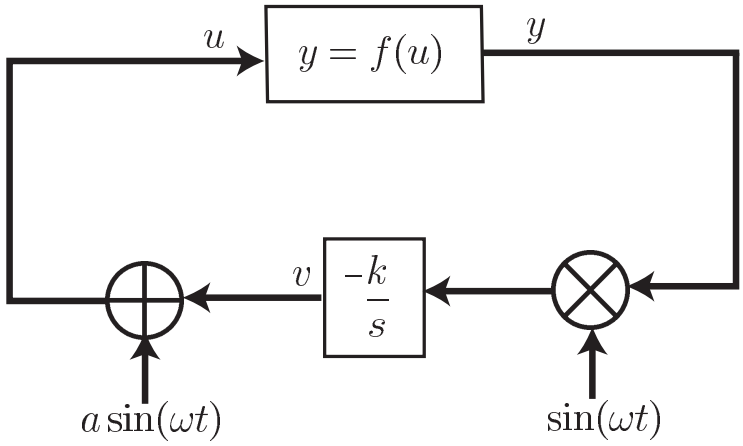

Figure 5.5: The basic extremum seeking feedback loop for a non-linear static system $y = f(u)$ ($s = \frac{d}{dt}$ is the Laplace variable and (k, a, ω) are constant design parameters).

The two main assumptions for any function $f(u)$ allowing us to apply such a feedback loop is that there exists a u^* such that $f'(u^*) = 0$ and $f''(u^*) > 0$. The function $f(u)$ admits, therefore, a local minima at u^*. We assume that, we start within a small neighborhood of this point and we are interested in improving the precision by getting as near as possible to u^*. Here, the input signal is perturbed with a sinusoid of frequency ω and small amplitude $a > 0$. Indeed, the dynamics associated to the diagram 5.5 can be written as:

$$\frac{d}{dt}v = -k\sin(\omega t)y, \qquad y = f(v + a\sin(\omega t)),$$

where $k > 0$ is a positive gain. Assuming that f is analytic and considering its Taylor expansion, we have

$$\frac{d}{dt}v = -k\sin(\omega t)\left(f(v) + a\sin(\omega t)f'(v) + a^2\ g(a, v, t)\right), \tag{5.20}$$

where $g(a, v, t)$ is a bounded $\frac{2\pi}{\omega}$-periodic function in t. By changing the time-scale to $\tau = \omega t$, we have

$$\frac{d}{d\tau}v = -\frac{k}{\omega}\sin(\tau)\left(f(v) + a\sin(\tau)f'(v) + a^2\ g(a, v, \tau)\right).$$

Assuming ω large enough, averaging over the period 2π of the system and removing the oscillating terms, we find

$$\frac{d}{d\tau}v_{\text{av}} = -\frac{ka}{2\omega}f'(v_{\text{av}}) - \frac{k\ a^2}{\omega}\ g_{\text{av}}(a, v_{\text{av}}).$$

However the system $\dot{w} = -\frac{ka}{2\omega}f'(w)$ is hyperbolically asymptotically stable around u^* (the assumptions $f'(u^*) = 0$ and $f''(u^*) > 0$). As $-\frac{k\ a^2}{\omega}\ g_{\text{av}}(a, w)$, is a small smooth perturbation of this vector field (a is assumed to be small and the derivative of g_{av} with respect to w at u^* is uniformly bounded for a in a small neighborhood of zero), the above system is also hyperbolically asymptotically stable around an equilibrium point in an $\mathcal{O}(a)$-neighborhood of u^*. Then by the averaging theorem, the system (5.20) admits a hyperbolically stable periodic orbit in an $\mathcal{O}(a + \frac{1}{\omega})$-neighborhood of u^*.

In contrast with the above scenario here we are interested in considering a single Λ-system (for instance a Λ-structured ion trapped within a Coulombian potential) instead of a vapor cell. In such a situation, we will not realize any change in the transmission rate even if the detuning $\Delta \neq 0$ (as the proportion of absorbed photons within the time unit can be neglected). However, by surrounding the dispositive by photodetectors and collecting the spontaneously emitted photons (or at least a part of them), we will get some information on the state of the system (see the Subsection 4.3.1 and the Figure 4.2). The collected information is no more in the form of a transmission or absorption rate but rather of the form of discrete click times of the photodetector indicating the arrival of an emitted photon. The question is whether how to integrate such information in a synchronization loop as the one presented in Figure 5.5. The next two subsections are based on [53]: they present and analyze the convergence of such a synchronization feedback loop.

5.2.1 Synchronizing feedback loop

We recall from the Subsection 4.3.4 that, whenever we apply near resonant laser fields and whenever we have the time-scale assumptions of the Subsection 4.3.2, the Lindblad master equation (and also the associated stochastic master equation) can be reduced (removing the stable fast dynamics associated to the excited state $|e\rangle$) to a master equation leaving on the 2-dimension Hilbert space spanned by $|g_1\rangle$ and $|g_2\rangle$. Recall that the dynamics of the slow part are given by the following stochastic master equation

$$\begin{aligned} d\rho = &-i\frac{\Delta}{2}\left[|g_2\rangle\langle g_2| - |g_1\rangle\langle g_1|, \rho\right]dt \\ &- \frac{1}{2}\left(L_1^\dagger L_1\rho + \rho L_1^\dagger L_1\right)dt + \text{Tr}\left(L_1\rho L_1^\dagger\right)\rho dt + \left(\frac{L_1\rho L_1^\dagger}{\text{Tr}\left(L_1\rho L_1^\dagger\right)} - \rho\right)dN_t^1 \\ &- \frac{1}{2}\left(L_2^\dagger L_2\rho + \rho L_2^\dagger L_2\right)dt + \text{Tr}\left(L_2\rho L_2^\dagger\right)\rho dt + \left(\frac{L_2\rho L_2^\dagger}{\text{Tr}\left(L_2\rho L_2^\dagger\right)} - \rho\right)dN_t^2 \end{aligned} \quad (5.21)$$

Here the Lindblad operators L_1 and L_2 are given by

$$L_k = 2\sqrt{\Gamma_k}\frac{\sqrt{|\Omega_1|^2+|\Omega|^2}}{\Gamma_1+\gamma_2}\left|g_k\right\rangle\left\langle b_\Omega\right|, \qquad b_\Omega = \frac{\Omega_1\left|g_1\right\rangle+\Omega_2\left|g_2\right\rangle}{\sqrt{|\Omega_1|^2+|\Omega_2|^2}}$$

and dN_t^1 and dN_t^2 are independent Poisson increments with averages

$$\mathbb{E}\left(dN_t^k\right) = \mathrm{Tr}\left(L_k\rho L_k^\dagger\right)dt = 4\Gamma_k\frac{|\Omega_1|^2+|\Omega_2|^2}{(\Gamma_1+\Gamma_2)^2}\mathrm{Tr}\left(\left|b_\Omega\right\rangle\left\langle b_\Omega\right|\rho\right)dt.$$

Here in the aim of the synchronization, we consider a modulation of the Rabi frequencies by replacing constant Ω_1 and Ω_2 with

$$\widetilde{\Omega}_1 = \Omega_1 + i\epsilon\Omega_2\cos(\omega t), \qquad \widetilde{\Omega}_2 = \Omega_2 - i\epsilon\Omega_1\cos(\omega t),$$

with $\epsilon \ll 1$ and $\omega \ll \Gamma_1, \Gamma_2$ the design parameters (furthermore as in Subsection 4.3.2 $\Omega_1, \Omega_2 \ll \Gamma_1, \Gamma_2$). Setting

$$\gamma_j = 4\frac{\Omega_1^2+\Omega_2^2}{(\Gamma_1+\Gamma_2)^2}\Gamma_j, \quad \text{for } j=1,2 \quad \text{and } \gamma = \gamma_1+\gamma_2,$$

If we replace Δ/γ by Δ, ω/γ by ω and γt by t in the stochastic master equation dynamics (5.21), we get the quantum Monte-Carlo trajectories in the $1/\gamma$ scale, the optical-pumping scale, that reads:

- In the absence of quantum jumps, the systems density matrix ρ evolves through the dynamics

$$\frac{d}{dt}\rho = -i\left[\frac{\Delta}{2}\sigma_z,\rho\right] - \frac{1}{2}\left\{\left|b+\iota\epsilon\cos(\omega t)d\right\rangle\left\langle b+\iota\epsilon\cos(\omega t)d\right|,\rho\right\} + \mathrm{Tr}\left(\left|b+\iota\epsilon\cos(\omega t)d\right\rangle\left\langle b+\iota\epsilon\cos(\omega t)d\right|\rho\right)\rho. \quad (5.22)$$

  with $|b\rangle = \cos\alpha\,|g_1\rangle + \sin\alpha\,|g_2\rangle$, $|d\rangle = -\sin\alpha\,|g_1\rangle + \cos\alpha\,|g_2\rangle$ ($\alpha \in \left[0,\frac{\pi}{2}\right]$ is the argument of $\Omega_1 + i\Omega_2$).

- At each time step dt the system may jump on the ground state $|g_j\rangle$ ($j=1,2$) with a probability given by

$$p_{\text{jump}}(\rho \to \left|g_j\right\rangle\left\langle g_j\right|) = \frac{\gamma_j}{\gamma_1+\gamma_2}\mathrm{Tr}\left(\left|b+i\epsilon\cos(\omega t)d\right\rangle\left\langle b+i\epsilon\cos(\omega t)d\right|\rho\right)dt \quad (5.23)$$

  This quantum jump leads to the emission of a photon that will be detected with certain efficiencies: $\eta_j \in (0,1]$ for the jumps to the state $|g_j\rangle$.

We assume a broad band detection process and thus the only information available with such measure is just the jump time. The type of jump (either to $|g_1\rangle$ or $|g_2\rangle$) is not available here. Thus the total jump probability reads

$$p_{\text{jump}} = \text{Tr}\left(|b + i\epsilon\cos(\omega t)d\rangle\langle b + i\epsilon\cos(\omega t)d|\,\rho\right)dt \tag{5.24}$$

After each jump, ρ coincides with $|g_1\rangle\langle g_1|$ or $|g_2\rangle\langle g_2|$.

We are interested in synchronizing the lasers with the system's frequencies and therefore make Δ converge to zero. Note that, in practice we have a certain knowledge of the transition frequency and therefore we can always tune our lasers so that the detuning $|\Delta|$ does not exceed a fixed threshold C.

The main strategy for the correction of the detuning is to wait for the matured quantum jumps (clicks of the photo-detector). This means that we choose a certain time constant $T \gg 1$ and if the distance between two jumps is more than T, we will correct the detuning according to the time when the second jump happens. Note that, one can easily show that these matured quantum jumps, almost surely, happen within a finite horizon.

Assume that $\epsilon \ll 1 \ll \omega$ and consider the following synchronization algorithm:

1. Start with a certain detuning Δ_0 with $|\Delta_0| \leq C$ and set the switching parameter $S = 0$ and the counter $N = 0$.

2. Wait for a first click and meanwhile evolve the switching parameter through $\frac{d}{dt}S = 1$.

3. If the click happens while $S \leq T$ then switch the parameter S to zero and go back to the step 2.

4. If the click happens while $S > T$ then switch the parameter S to zero, change the counter value to $N + 1$, correct the detuning Δ_N as follows:

$$\begin{cases} \Delta_{N+1} = \Delta_N - \delta\sin(2\alpha)\cos(\omega t) & \text{if } |\Delta_N - \delta\sin(2\alpha)\cos(\omega t)| \leq C, \\ \Delta_{N+1} = C, & \text{otherwise} \end{cases}$$

   and go back to the step 2.

Here, we have chosen the correction gain $\delta \ll 1$. We claim that, given any small ϵ, we can adjust the parameters ω large and δ small enough such that with the above algorithm, the detuning Δ_N converges in average to an $O(\epsilon^2)$-neighborhood of 0 with a deviation of order $O(\epsilon)$. Indeed, we have the following result

Theorem 5.2.1. *Consider the Monte-Carlo trajectories described by (5.22)-(5.23) where*

$$|b\rangle = \cos\alpha\,|g_1\rangle + \sin\alpha\,|g_2\rangle \qquad \textit{with} \quad 0 < \alpha < \frac{\pi}{2}. \tag{5.25}$$

Moreover, we assume perfect detection efficiency $\eta_1 = \eta_2 = 1$ and

$$\epsilon \ll 1, \qquad \frac{1}{\omega} \sim \epsilon^2. \tag{5.26}$$

Consider then the above synchronization algorithm with

$$C < 1/2 \qquad \text{and} \qquad \delta \sim \epsilon^3. \tag{5.27}$$

We can fix then the time constant T in the algorithm large enough so that:

$$\limsup_{N\to\infty} \mathbb{E}\left(\Delta_N^2\right) \leq O(\epsilon^2). \tag{5.28}$$

We also have the following corollary

Corollary 5.2.2. *Under the assumptions of the Theorem 5.2.1, one has*

$$\limsup_{N\to\infty} \mathbb{P}(|\Delta_N| > \sqrt{\epsilon}) \leq O(\epsilon). \tag{5.29}$$

This corollary results from the Markov inequality:

$$\mathbb{P}(|\Delta_N| > \sqrt{\epsilon}) = \mathbb{P}(\Delta_N^2 > \epsilon) \leq \frac{\mathbb{E}\left(\Delta_N^2\right)}{\epsilon}.$$

Therefore applying (5.28), one deduces (5.29).

Remark 5.2.3. *Following the steps of the proof and changing the assumptions (5.26) and (5.27) to*

$$1/\omega \sim \epsilon \qquad \text{and} \qquad \delta \sim \epsilon^2, \tag{5.30}$$

one can show that, the detuning reaches an $O(\epsilon)$-neighborhood of 0 with a deviation of order $\sqrt{\epsilon}$.

We assume

$$1/\omega = \epsilon^2\kappa_1, \qquad \delta = \epsilon^3\kappa_2,$$

where $\kappa_1, \kappa_2 \sim 1$.

The proof of the Theorem 5.2.1 admits 2 main steps:

Step 1 We consider the evolution in the absence of the quantum jumps through the dynamics (5.22). We study the asymptotic regime of the dynamics. The constant time T will then be chosen to ensure the non-jumping system to reach an ϵ^3-neighborhood of the limit regime.

Step 2 In the second step, applying the result of the first step, we calculate the conditional expectation of Δ_{N+1} having fixed Δ_N. Finally, we sum up all these results in order to find the limit (5.28).

Through the two following subsections, we consider these two steps separately.

5.2.2 Convergence analysis: asymptotic regime of the non-jumping system

We are interested in the dynamics of the system (5.22). In this aim, we apply the Kapitsa shortcut method (see the Appendix C). Note that,

$$|b+\iota\epsilon\cos(\omega t)d\rangle\langle b+i\epsilon\cos(\omega t)d| = |b\rangle\langle b| + \frac{\epsilon^2}{2}|d\rangle\langle d| + i\epsilon\cos(\omega t)(|b\rangle\langle d| - |d\rangle\langle b|) + \frac{\epsilon^2}{2}\cos(2\omega t)|d\rangle\langle d|.$$

Applying the Kapitsa method, the variable ρ may be developed as

$$\rho = \widetilde{\rho} + O(\frac{\epsilon}{\omega}) = \widetilde{\rho} + O(\epsilon^3), \tag{5.31}$$

where $\widetilde{\rho}$ represents the unperturbed trajectory. Through the rest of this subsection we study the dynamics of $\widetilde{\rho}$.

The unperturbed part, $\widetilde{\rho}$, satisfies the dynamics:

$$\frac{d}{dt}\widetilde{\rho} = -i\frac{\Delta}{2}[\sigma_z,\widetilde{\rho}] - \frac{1}{2}\left\{|b\rangle\langle b| + \frac{\epsilon^2}{2}|d\rangle\langle d|, \widetilde{\rho}\right\} + \mathrm{Tr}\left(\left(|b\rangle\langle b| + \frac{\epsilon^2}{2}|d\rangle\langle d|\right)\widetilde{\rho}\right)\widetilde{\rho}. \tag{5.32}$$

In order to study the asymptotic behavior of (5.32), we begin with the case $\epsilon \equiv 0$ and we study first the system

$$\frac{d}{dt}\widehat{\rho} = -i\frac{\Delta}{2}[\sigma_z,\widehat{\rho}] - \frac{1}{2}\{|b\rangle\langle b|, \widehat{\rho}\} + \mathrm{Tr}\left(|b\rangle\langle b|\widehat{\rho}\right)\widehat{\rho}, \tag{5.33}$$

where $\{A,B\} = AB + BA$ is the anti-commutator.

The dynamics in the Bloch sphere coordinates, $X = \mathrm{Tr}(\sigma_x\widehat{\rho})$, $Y = \mathrm{Tr}(\sigma_y\widehat{\rho})$, $Z = \mathrm{Tr}(\sigma_z\widehat{\rho})$, are given as follows:

$$\begin{aligned}
\frac{d}{dt}X &= -\Delta Y - \frac{\sin(2\alpha)}{2} + \left(\frac{\sin(2\alpha)}{2}X + \frac{\cos(2\alpha)}{2}Z\right)X\\
\frac{d}{dt}Y &= \Delta X + \left(\frac{\sin(2\alpha)}{2}X + \frac{\cos(2\alpha)}{2}Z\right)Y\\
\frac{d}{dt}Z &= -\frac{\cos(2\alpha)}{2} + \left(\frac{\sin(2\alpha)}{2}X + \frac{\cos(2\alpha)}{2}Z\right)Z,
\end{aligned}$$

where we have applied $|b\rangle = \cos\alpha|g_1\rangle + \sin\alpha|g_2\rangle$. Taking

$$' = 2\frac{d}{dt}, \quad p = 2\Delta, \quad \beta = 2\alpha,$$

we have the following dynamical system

$$\begin{aligned} X' &= -pY - \sin\beta + (\sin\beta X + \cos\beta Z)X \\ Y' &= pX + (\sin\beta X + \cos\beta Z)Y \\ Z' &= -\cos\beta + (\sin\beta X + \cos\beta Z)Z. \end{aligned} \tag{5.34}$$

living on $\mathbb{R}^3$. Since the two transformations $(X, Y, Z, p, \beta) \mapsto (-X, -Y, -Z, \beta + \pi)$ and $(X, Y, Z, p, \beta) \mapsto (X, Y, -Z, \pi - \beta)$ leave the above equations unchanged, we can always consider, for the study of this dynamical system versus the parameter p and β, that the angle $\beta \in [0, \frac{\pi}{2}]$ and $p \in \mathbb{R}$. Since $X^2 + Y^2 + Z^2 = 1$ is invariant, these 3 differential equations define a dynamical system on the two dimensional sphere $\mathbb{S}^2$, the Bloch sphere.

Consider the element of Euclidean length $\delta s^2 = (\delta X)^2 + (\delta Y)^2 + (\delta Z)^2$ and its evolution along the dynamics defined by (5.34) on $\mathbb{S}^2$. We have

$$\left(\delta s^2\right)' = 2\left(\delta X \delta X' + \delta Y \delta Y' + \delta Z \delta Z'\right)$$

with $(\delta X', \delta Y', \delta Z')$ given by the first variation of (5.34):

$$\begin{aligned} \delta X' &= -p\delta Y + (\sin\beta X + \cos\beta Z)\delta X + X(\sin\beta\delta X + \cos\beta\delta Z) \\ \delta Y' &= p\delta X + (\sin\beta X + \cos\beta Z)\delta Y + Y(\sin\beta\delta X + \cos\beta\delta Z) \\ \delta Z' &= (\sin\beta X + \cos\beta Z)\delta Z + Z(\sin\beta\delta X + \cos\beta\delta Z). \end{aligned}$$

Since $X\delta X + Y\delta Y + Z\delta Z = 0$, we obtain the simple relation

$$\left(\delta s^2\right)' = 2(\sin\beta X + \cos\beta Z)\delta s^2. \tag{5.35}$$

Thus $\mathbb{S}^2$ splits into two hemispheres: the open hemisphere $\mathbb{S}^2_+$ corresponding to $\sin\beta X + \cos\beta Z > 0$ and where the dynamics is a strict dilation in any direction; the open hemisphere $\mathbb{S}^2_-$ corresponding to $\sin\beta X + \cos\beta Z < 0$ where the dynamics is a strict contraction (see [48]). The boundary between these two hemispheres is given by the intersection of the plane $\sin\beta X + \cos\beta Z = 0$ with $\mathbb{S}^2$. We have

$$(\sin\beta X + \cos\beta Z)' = -1 - p\sin\beta Y - (\sin\beta X + \cos\beta Z)^2.$$

Thus, when $|p\sin\beta| \leq 1$, $\mathbb{S}^2_+$ is negatively invariant and $\mathbb{S}^2_-$ positively invariant.

Assume first that $p \neq 0$ and $\beta \in]0, \frac{\pi}{2}[$ and consider the equilibrium on $\mathbb{S}^2$. Simple computations prove that we have only two equilibria associated to the point $M_+ \in \mathbb{S}^2_+$ and $M_- \in \mathbb{S}^2_-$ of coordinates (X_+, Y_+, Z_+) and (X_-, Y_-, Z_-) given by

$$\begin{aligned} X_\pm &= \pm\left(\frac{\cos\beta}{\sin\beta}\right)\frac{\sqrt{(p^2-1)^2+4p^2\cos^2\beta}-p^2-1}{\sqrt{2p^2}\sqrt{p^2-1+\sqrt{(p^2-1)^2+4p^2\cos^2\beta}}} \\ Y_\pm &= \frac{\sqrt{(p^2-1)^2+4p^2\cos^2\beta}-p^2-1}{2p\sin\beta} \\ Z_\pm &= \pm\sqrt{\frac{p^2-1+\sqrt{(p^2-1)^2+4p^2\cos^2\beta}}{2p^2}} \end{aligned} \tag{5.36}$$

When $p = 0$, the above formula can be extended by continuity to get the two equilibria:

$$X_\pm = \pm \sin\beta, \quad Y_\pm = 0, \quad Z_\pm = \pm\cos\beta.$$

When $\beta = 0$, similarly we obtain the two equilibria

$$X_\pm = 0, \quad Y_\pm = 0, \quad Z_\pm = \pm 1.$$

When $\beta = \frac{\pi}{2}$ the situation is slightly different:

- for $|p| < 1$ we have two equilibria
$$X_\pm = \pm\sqrt{1-p^2}, \quad Y_\pm = -p, \quad Z_\pm = 0.$$
- for $|p| = 1$ we have a unique equilibrium
$$X_\pm = 0, \quad Y_\pm = -p, \quad Z_\pm = 0.$$
- for $|p| > 1$ we have two equilibria
$$X_\pm = 0, \quad Y_\pm = -\frac{1}{p}, \quad Z_\pm = \pm\sqrt{1-\frac{1}{p^2}}.$$

With all the above properties we deduce the following lemma

Lemma 5.2.4. *Consider the differential equations (5.34) defining an autonomous dynamical system on the Bloch Sphere $\mathbb{S}^2$ with the parameters $p \in \mathbb{R}$ and $\beta \in [0, \frac{\pi}{2}]$. Then*

1. *for $(|p|, \beta) \neq (1, \pi/2)$, we have two distinct equilibrium points M_+ and M_- defined here above by (5.36). The two Lyapounov exponents at M_+ (resp. M_-) have strictly positive (resp. negative) real parts: M_+ is locally exponentially unstable (in all direction) and M_- is locally exponentially stable.*

2. *For $|p\sin\beta| < 1$, all the trajectories (except the unstable equilibrium M_+) converge asymptotically to the equilibrium point M_- that is exponentially stable: the attraction region of M_- is $\mathbb{S}^2/\{M_+\}$.*

Proof of Lemma 5.2.4. The first point result from (5.35) applied locally around M_+ and M_- and from $\sin\beta X_+ + \cos\beta Z_+ > 0$ whereas $\sin\beta X_- + \cos\beta Z_- < 0$.

The second point comes from the negative invariance of $\mathbb{S}^2_+$, positive invariance of $\mathbb{S}^2_-$ and the Poincare-Bendixon theory (see the Appendix F) for autonomous systems on the sphere: an hypothetic limit cycle C cannot intersect $\mathbb{S}^2_+$ and $\mathbb{S}^2_-$ simultaneously and thus must be included in $\mathbb{S}^2_+$ or $\mathbb{S}^2_-$; strict surface dilation (resp. contraction) in $\mathbb{S}^2_+$ (resp. $\mathbb{S}^2_-$) is incompatible with the existence of $C \subset \mathbb{S}^2_+$ (resp. $C \in \mathbb{S}^2_-$) because of the Gauss theorem (see Theorem F.0.4 of the Appendix F); since there is no limit cycle and since there exist only two equilibrium points, M_+ exponentially unstable in all direction and M_- exponentially stable, the attraction domain of M_- is the all sphere without the unstable point M_+.

Remark 5.2.5. *It is tempting to conjecture that, for all values of the parameters p and β ensuring two separate equilibria M_+ and M_- defined here above, we have a quasi-global convergence towards M_-, the locally exponentially stable equilibrium. This is not true since for $\beta = \pi/2$ and $|p| > 1$ we have the coexistence of the periodic orbit $X^2 + Y^2 = 1$ with $Z = 0$ with the two equilibria*

$$X_\pm = 0, \quad Y_\pm = -\frac{1}{p}, \quad Z_\pm = \pm\sqrt{1 - \frac{1}{p^2}}$$

and thus a trajectory starting with $Z > 0$ remains with $Z > 0$ for all the time and cannot converge to M_- since $Z_- < 0$.

We are now ready to study the unperturbed system (5.32) in the case where $\epsilon \neq 0$.

Under the assumption of the Theorem 5.2.1 on C, we know that the detuning Δ can not get larger than $1/2$ and therefore in the above notations $p < 1$. This trivially implies $|p \sin\beta| < 1$ and therefore we are in the settings of the second point of the Lemma 5.2.4. Hence, the system (5.33) admits two distinct equilibria $\overline{\rho}_-$ and $\overline{\rho}_+$ given by (5.36) in the Bloch sphere coordinates. Moreover the trajectories of the system, not starting at $\overline{\rho}_+$, necessarily converge towards the equilibria $\overline{\rho}_-$.

Applying this characterization of the dynamics, one easily gets

Lemma 5.2.6. *Under the assumption of the Theorem 5.2.1 for $|b\rangle$ and the assumption $|\Delta| < \frac{1}{2}$, and for small enough ϵ, the system (5.32) admits a locally asymptotically stable equilibrium $\overline{\rho}_\epsilon$ of the form*

$$\overline{\rho}_\epsilon = \overline{\rho}_- + \epsilon^2 \overline{\rho}_1 + O(\epsilon^4),$$

where $\overline{\rho}_-$ is given by (5.36) in the Bloch sphere coordinates. Moreover the trajectories starting at $|g_1\rangle\langle g_1|$ or $|g_2\rangle\langle g_2|$ converge towards this equilibrium.

For the proof of this lemma, note that, as $\alpha \neq 0$, $|g_1\rangle\langle g_1|$ and $|g_2\rangle\langle g_2|$ are not the equilibriums of the system (5.33). Thus, taking ϵ small enough, they will not be an equilibrium of (5.32) neither, and therefore the trajectories starting at $|g_1\rangle\langle g_1|$ and $|g_2\rangle\langle g_2|$ necessarily converge towards the perturbed asymptotically stable equilibrium $\overline{\rho}_\epsilon$.

The Lemma 5.2.6, together with (5.31), implies that the trajectories $\rho(t)$ of the system (5.22) starting at $|g_1\rangle\langle g_1|$ or $|g_2\rangle\langle g_2|$ converge to an $O(\epsilon^3)$-neighborhood of $\overline{\rho}_- + \epsilon^2\overline{\rho}_1$.

We may therefore choose the time constant T in the synchronization algorithm of the Subsection 5.2.1 such that

$$\rho(t) = \overline{\rho}_- + \epsilon^2 \overline{\rho}_1 + O(\epsilon^3), \qquad \forall t > T. \tag{5.37}$$

Through the nest subsection, we apply the result of this subsection in order to characterize the conditional evolution of the detuning.

5.2.3 Convergence analysis: conditional evolution of detuning

We are interested in the conditional expectations of Δ_{N+1} and Δ_{N+1}^2 knowing the value of Δ_N. Due to the synchronization algorithm $\Delta_{N+1} = \Delta_N - \delta \sin(2\alpha)\cos(\omega t)$, the value of Δ_{N+1} only depends on the phase $\phi = \omega t \mod (2\pi)$. We update Δ_{N+1} only if the time interval with respect to the previous jump is large enough to ensure that the solution of the no-jump dynamics (5.22) has reached its asymptotic regime (5.37). Thus $\mathrm{Tr}\left(|b + i\epsilon\cos(\omega t)d\rangle\langle b + i\epsilon\cos(\omega t)d|\,\rho\right)$ is given inserting the limit (5.37). The jump probability defined by (5.24) depends only on $\phi = \omega t \mod (2\pi)$. Since the probability of having a phase ϕ during the update Δ_N to Δ_{N+1} is proportional to

$$\mathrm{Tr}\left(|b + i\epsilon\cos(\phi)d\rangle\langle b + i\epsilon\cos(\phi)d|\,\rho\right),$$

this probability admits a density with respect to the Lebesgue measure on $[0, 2\pi]$, given by

$$p_{\phi,N} = \frac{1}{\mathcal{Z}_N(\epsilon)}\Big(\mathrm{Tr}\left(|b\rangle\langle b|\,\overline{\rho}_-\right) + \epsilon^2\cos^2(\varphi)\mathrm{Tr}\left(|d\rangle\langle d|\,\overline{\rho}_-\right) + \epsilon^2\mathrm{Tr}\left(|b\rangle\langle b|\,\overline{\rho}_1\right) \\ - \epsilon\cos(\varphi)\mathrm{Tr}\left(\sigma_y\overline{\rho}_-\right) + O(\epsilon^3)\Big), \qquad \phi \in [0, 2\pi), \quad (5.38)$$

where the index N in $p_{\phi,N}$ denotes, in particular, the dependence of $\overline{\rho}_-$ and $\overline{\rho}_1$ to the detuning Δ_N. Furthermore, the constant $\mathcal{Z}_N(\epsilon) > 0$ is a normalization constant given by the integral over $[0, 2\pi]$ of the term between parentheses. In particular, one can easily find the strictly positive constants $c_1, c_2 > 0$ such that

$$c_1\epsilon^2 < \mathcal{Z}_N(\epsilon) < c_2. \quad (5.39)$$

Removing the threshold C in the algorithm by allowing the detuning to get large, the value of Δ_{N+1}, having fixed Δ_N, is given as follows

$$\Delta_{N+1} = \Delta_N - \delta\sin(2\alpha)\cos(\varphi) \quad (5.40)$$

with a probability density $p_{\varphi,N}$.

Similarly for Δ_{N+1}^2 one has

$$\Delta_{N+1}^2 = \Delta_N^2 - 2\delta\sin(2\alpha)\cos(\varphi)\Delta_N + \delta^2\sin^2(2\alpha)\cos^2(\varphi), \quad (5.41)$$

with a probability density $p_{\varphi,N}$.

Inserting (5.38) into (5.41), we have

$$\mathbb{E}\left(\Delta_{N+1}^2 \mid \Delta_N\right) = \Delta_N^2 - \pi\epsilon\frac{\delta}{\mathcal{Z}_N(\epsilon)}\frac{\Theta_N}{2} + O\left(\frac{\delta^2}{\mathcal{Z}_N(\epsilon)}\right) + O\left(\frac{\delta\epsilon^3}{\mathcal{Z}_N(\epsilon)}\right), \quad (5.42)$$

where

$$\Theta_N = 4\Delta_N^2 + 1 - \sqrt{(4\Delta_N^2 - 1)^2 + 16\Delta_N^2\cos^2(2\alpha)}.$$

Note, in particular, that $\Theta_N > 0$ as $\alpha \neq 0$.

Now, taking into account the threshold C for the growth of the detuning Δ_{N+1}, we can easily see that

$$\Theta_N = \frac{16\Delta_N^2 \sin^2(2\alpha)}{4\Delta_N^2 + 1 + \sqrt{(4\Delta_N^2 - 1)^2 + 16\Delta_N^2 \cos^2(2\alpha)}} \geq \frac{8\sin^2(2\alpha)}{1 + 4C^2}\Delta_N^2.$$

Therefore, noting by

$$\varsigma = \pi\kappa_2 \frac{4\sin^2(2\alpha)}{1 + 4C^2} > 0, \tag{5.43}$$

where $\delta = \kappa_2\epsilon^3$, we have

$$\mathbb{E}\left(\Delta_{N+1}^2 \mid \Delta_N\right) \leq \Delta_N^2 - \frac{\epsilon^4}{\mathcal{Z}_N(\epsilon)}\varsigma\Delta_N^2 + c_3\frac{\epsilon^6}{\mathcal{Z}_N(\epsilon)}, \tag{5.44}$$

where $c_3 > 0$ is a fixed positive constant. Taking now the expectation of the both sides, we have

$$\mathbb{E}\left(\Delta_{N+1}^2\right) \leq \left(1 - \frac{\epsilon^4}{\mathcal{Z}_N(\epsilon)}\varsigma\right)\mathbb{E}\left(\Delta_N^2\right) + c_3\frac{\epsilon^6}{\mathcal{Z}_N(\epsilon)}, \tag{5.45}$$

where we have applied the relation $\mathbb{E}\left(\mathbb{E}\left(X|Y\right)\right) = \mathbb{E}\left(X\right)$. Noting that

$$0 < \frac{\epsilon^4}{c_2} \leq \frac{\epsilon^4}{\mathcal{Z}_N(\epsilon)} \leq \frac{\epsilon^2}{c_1},$$

where c_1 and c_2 are given by (5.39), the system (5.45) is a contracting one. Furthermore by noting that

$$\frac{\epsilon^6}{\mathcal{Z}_{N-1}(\epsilon)} + \left(1 - \frac{\epsilon^4}{\mathcal{Z}_{N-1}(\epsilon)}\varsigma\right) \times \frac{\epsilon^6}{\mathcal{Z}_{N-2}(\epsilon)} + \ldots + \prod_{k=1}^{N-1}\left(1 - \frac{\epsilon^4}{\mathcal{Z}_k(\epsilon)}\varsigma\right) \times \frac{\epsilon^6}{\mathcal{Z}_0(\epsilon)} =$$
$$\frac{\epsilon^2}{\varsigma}(1 - \prod_{k=0}^{N}\left(1 - \frac{\epsilon^4}{\mathcal{Z}_N(\epsilon)}\varsigma\right)) \leq \frac{\epsilon^2}{\varsigma},$$

we easily have the following lemma:

Lemma 5.2.7. *Considering the Monte-Carlo trajectories described by* (5.22)- (5.23) *and applying the synchronization algorithm of the Subsection* 5.2.1*, we have*

$$\mathbb{E}\left(\Delta_N^2\right) \leq \left(1 - \frac{\epsilon^4}{c_2}\varsigma\right)^N \Delta_0^2 + O(\epsilon^2),$$

where the positive constant ς *is given in* (5.43) *and* c_2 *in* (5.39).

This trivially finishes the proof of the Theorem 5.2.1 and we have

$$\limsup_{N\to\infty} \mathbb{E}\left(\Delta_N^2\right) \leq O(\epsilon^2).$$

Furthermore, note that as the detuning Δ_N gets near 0, the normalization constant $\mathcal{Z}_N(\epsilon)$ converges to an $O(\epsilon^2)$. This, in particular, leads to a higher convergence rate in the Lemma 5.2.7.

Appendix A

Basic Quantum notions

All the objects, notions and operators described in this section are mathematically well defined when the Hilbert spaces where the wave functions live are of finite dimensions. When the Hilbert spaces are of infinite dimensions, one has to be aware that these objects, notions and operators might also be defined in principle but one needs to explore the mathematical justifications depending strongly on the specific physical system under study (involving in particular its spectral decomposition). For clarity sake, we consider here only the finite dimensional case even if some constructions and objects (such as tensor product) admit a straightforward extension to infinite dimensional Hilbert spaces.

A.1 Bra, Ket and operators

We just recall here some basic notions of quantum mechanics. We refer to the excellent course [22] where these notions are explained in details. Bra $\langle \bullet |$ and Ket $| \bullet \rangle$ are co-vector and vector. The quantum state is described by the ket $|\psi\rangle$ an element of norm one and belonging to a Hilbert space $\mathcal{H}$. The quantum state is also called (probability amplitude) wave function. The Hermitian conjugate of a Ket is a Bra: $\langle \psi | = |\psi\rangle^\dagger$. The Hermitian product between two kets (vectors, i.e. elements of $\mathcal{H}$), $|\psi\rangle$ and $|\phi\rangle$ is denoted by

$$\langle \psi | \cdot | \phi \rangle = \langle \psi | \phi \rangle = \langle \phi | \psi \rangle^* \in \mathbb{C}$$

where * stands for complex conjugate. If we consider a Hilbert basis of $\mathcal{H}$, denoted by $|n\rangle$, $n = 1, \ldots, \dim(\mathcal{H})$, we have

$$\begin{aligned} |\psi\rangle &= \sum_n \psi_n |n\rangle , \qquad \forall n,\ \langle n|\psi\rangle = \psi_n \in \mathbb{C} \\ \langle\psi|\psi\rangle &= \sum_n |\psi_n|^2 = 1 \\ |\phi\rangle &= \sum_n \phi_n |n\rangle , \qquad \forall n,\ \langle n|\phi\rangle = \phi_n \in \mathbb{C} \\ \langle\phi|\phi\rangle &= \sum_n |\phi_n|^2 = 1 \\ \langle\psi|\phi\rangle &= \sum_n \psi_n^* \phi_n \end{aligned}$$

since for all m, n, $\langle m|n\rangle = \delta_{m,n}$.

Any linear operator M from $\mathcal{H}$ into $\mathcal{H}$ reads, in the orthonormal frame $(|n\rangle)$,

$$M = \sum_{m,n} M_{m,n} |m\rangle \langle n| , \qquad M_{m,n} \in \mathbb{C}$$

where $M_{m,n} = \langle m|M|n\rangle$ is the Hermitian product between $|m\rangle$ and $M|n\rangle$. The operator M is Hermitian when $M = M^\dagger$ that reads $M_{m,n} = M_{n,m}^*$. The orthogonal projector P on a Hilbert subspace $\mathcal{H}_0$ of $\mathcal{H}$ is a Hermitian operator defined by the relation

$$P = \sum_k |\phi_k\rangle \langle\phi_k|$$

where $|\phi_k\rangle_{k\in\{1,\ldots,\dim(\mathcal{H}_0)\}}$ is any orthonormal basis of $\mathcal{H}_0$.

The operator U is unitary when $U^{-1} = U^\dagger$. Any operator $U = \exp(iH)$ is unitary as soon as H is Hermitian. We recall that

$$\exp(A) = \sum_{k=0}^{+\infty} \frac{A^k}{k!}$$

for any operator A.

Take a Hermitian operator M and consider its spectral decomposition

$$M = \sum_\nu \lambda_\nu P_\nu$$

where the λ_ν's are the eigenvalues of M ($\lambda_\nu \in \mathbb{R}$) and P_ν the orthogonal projector on the eigenspace associated to λ_ν. By construction we have $\mathbf{1} = \sum_\nu P_\nu$ where $\mathbf{1}$ is the identity operator. For any function $f : \mathbb{R} \mapsto \mathbb{R}$ we can define $f(M)$ by

$$f(M) = \sum_\nu f(\lambda_\nu) P_\nu .$$

Thus M and $f(M)$ commute and the image by f of the M-spectrum is the spectrum of $f(M)$. This definition of $f(M)$ is just a more intrinsic formulation of the usual construction based $M = U\Delta U^\dagger$ with U unitary and Δ diagonal: $f(M) = Uf(\Delta)U^\dagger$ with $f(\Delta)$ the diagonal matrix obtained by taking the image via f of the scalar elements forming the diagonal matrix Δ.

A.2 Schrödinger equation

In general the state $|\psi\rangle$ of the quantum system living in $\mathcal{H}$ depends on the time t. This dependance is described by the Schrödinger equation:

$$i\frac{d}{dt}|\psi\rangle = H(t)|\psi\rangle \tag{A.1}$$

where $H(t)$ is a time-varying Hermitian operator called the Hamiltonian (we set $\hbar = \frac{h}{2\pi}$ to one).

The evolution of $|\psi\rangle$ is unitary: if $|\psi\rangle$ and $|\phi\rangle$ are solutions of the same Schrödinger equation (A.1) then $\langle\psi|\phi\rangle_t$ is constant and equal to the initial value $\langle\psi|\phi\rangle_0$. This means that we can set $|\psi\rangle_t = U_t|\psi\rangle_0$, for any solution of (A.1) starting form $|\psi\rangle_0$ where the time dependant unitary operator U_t called also the *propagator* is solution of

$$i\frac{d}{dt}U_t = H(t)U_t, \qquad U_0 = \mathbf{1}. \tag{A.2}$$

Whenever the Hamiltonian H is time-invariant, and once we have the spectral decomposition of H, we have an explicit expression of U_t. Indeed, taking

$$H = \sum_\nu \omega_\nu P_\nu$$

where for each ν, ω_ν is a different eigenvalue and P_ν is the orthogonal projector onto the eigenspace associated to ω_ν, we have

$$U_t = e^{-itH} = \sum_\nu e^{-i\omega_\nu t}P_\nu$$

and thus

$$|\psi\rangle_t = \sum_\nu e^{-i\omega_\nu t}P_\nu|\psi\rangle_0 .$$

Since, for any angle θ, $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ represent the same quantum state, the Hamiltonian $H(t)$ is defined up-to an addition of $\lambda\mathbf{1}$, where λ is any real quantity (homogeneous to an energy). More precisely, take any time varying global phase θ_t. Then $|\psi\rangle_t$ and $|\phi\rangle_t = e^{i\theta_t}|\psi\rangle_t$ represent the same quantum system. This means that if the evolution of $|\psi\rangle$ is driven by the Hamiltonian $H(t)$, then the évolution of $|\phi\rangle$ is driven by $H(t) + \dot\theta_t\mathbf{1}$: Hamiltonians $H(t)$ and $H(t) + \dot\theta_t\mathbf{1}$ are equivalent since they are attached to the same system. Thus, in specific examples, we can always choose the origin of the energy in order to get the simplest computation and formulae.

Exercice A.2.1. *Show that if we replace* $H(t)$ *by* $H(t) - \frac{Tr(H(t))}{\dim(\mathcal{H})}\mathbf{1}$ *we ensure that* $\det(U_t) \equiv 1$. *(hint: use the Liouville formula* $i\frac{d}{dt}\det(U_t) = Tr(H(t))\det(U_t)$*)*

A.3 Composite systems and tensor product

A composite system is made of several sub-systems. It is very important to realize that the state space (Hilbert space) of a composite system is not the Cartesian product of the state space of its sub-systems, as it is the case for classical systems. It is their tensor product. This difference is essential.

Take a composite system of Hilbert space $\mathcal{H}$ made of two sub-systems with Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Then $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\dim(\mathcal{H}) = \dim(\mathcal{H}_1)\dim(\mathcal{H}_2)$. From Hilbert basis $(|n_1\rangle)_{n_1\in\{1,\ldots,\dim(\mathcal{H}_1)\}}$ of $\mathcal{H}_1$ and $(|n_2\rangle)_{n_2\in\{1,\ldots,\dim(\mathcal{H}_2)\}}$ of $\mathcal{H}_2$, we get a Hilbert basis of $\mathcal{H}$,

$$(|n_1 n_2\rangle)_{\substack{n_1\in\{1,\ldots,\dim(\mathcal{H}_1)\}\\ n_2\in\{1,\ldots,\dim(\mathcal{H}_2)\}}}$$

where $|n_1 n_2\rangle$ is used to denote $|n_1\rangle \otimes |n_2\rangle$. $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ contains all the tensor products $|\psi_1\rangle \otimes |\psi_2\rangle$ of elements $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$. But it contains much more elements that are not tensor products of elements of $\mathcal{H}_1$ and $\mathcal{H}_2$

Exercice A.3.1. *Prove that*

$$|\psi\rangle = |11\rangle + |22\rangle$$

cannot be expressed as a tensor product.

Take $|\psi\rangle, |\phi\rangle \in \mathcal{H}$. Then we have

$$\begin{aligned}
|\psi\rangle &= \sum_{n_1,n_2} \psi_{n_1,n_2} |n_1 n_2\rangle, \quad \langle n_1 n_2|\psi\rangle = \psi_{n_1,n_2} \in \mathbb{C}\\
|\phi\rangle &= \sum_{n_1,n_2} \phi_{n_1,n_2} |n_1 n_2\rangle, \quad \langle n_1 n_2|\phi\rangle = \phi_{n_1,n_2} \in \mathbb{C}\\
\langle\psi|\phi\rangle &= \sum_{n_1,n_2} \psi^*_{n_1,n_2}\phi_{n_1,n_2}.
\end{aligned}$$

Exercice A.3.2. *Prove from the above relationships that if* $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ *and* $|\phi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$ *with* $|\psi_1\rangle, |\phi_1\rangle \in \mathcal{H}_1$ *and* $|\psi_2\rangle, |\phi_2\rangle \in \mathcal{H}_2$*, then* $\langle\psi|\phi\rangle = \langle\psi_1|\phi_1\rangle\langle\psi_2|\phi_2\rangle$.

Consider M_1 a linear operator on $\mathcal{H}_1$ and M_2 a linear operator on $\mathcal{H}_2$. The tensor product $M_1 \otimes M_2$ defines a linear operator on $\mathcal{H}$ via the following relationships:

$$\begin{aligned}
|\psi\rangle &= \sum_{n_1,n_2} \psi_{n_1,n_2} |n_1 n_2\rangle, \quad \langle n_1 n_2|\psi\rangle = \psi_{n_1,n_2} \in \mathbb{C}\\
M_1 \otimes M_2 |\psi\rangle &= \sum_{n_1,n_2} \psi_{n_1,n_2} M_1 |n_1\rangle \otimes M_2 |n_2\rangle.
\end{aligned}$$

Thus when $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, then we have always

$$M_1 \otimes M_2 |\psi\rangle = M_1 |\psi_1\rangle \otimes M_2 |\psi_2\rangle .$$

There are many operators on $\mathcal{H}$ that are not tensor product of operators on $\mathcal{H}_1$ and $\mathcal{H}_2$.

Exercice A.3.3. *Show that the linear operator ($\mathbf{1}_1$ and $\mathbf{1}_2$ are the identity operator of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively)*

$$(|1\rangle\langle 2| + |2\rangle\langle 1|) \otimes \mathbf{1}_2 + \mathbf{1}_1 \otimes (|1\rangle\langle 2| + |2\rangle\langle 1|)$$

is not a tensor product $M_1 \otimes M_2$ (hint: consider the image of $|11\rangle$).

If U_1 and U_2 are unitary operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, then $U_1 \otimes U_2$ is also unitary and

$$(U_1 \otimes U_2)^{-1} = U_1^{-1} \otimes U_2^{-1} = U_1^\dagger \otimes U_2^\dagger = (U_1 \otimes U_2)^\dagger.$$

For any operators A_1 and A_2 on $\mathcal{H}_1$ and $\mathcal{H}_2$, we have[1]

$$\exp(A_1 \otimes \mathbf{1}_2 + \mathbf{1}_1 \otimes A_2) = \exp(A_1) \otimes \exp(A_2)$$

This results from the fact that $A_1 \otimes \mathbf{1}_2$ and $\mathbf{1}_1 \otimes A_2$ commute:

$$\exp(A_1 \otimes \mathbf{1}_2 + \mathbf{1}_1 \otimes A_2) = \exp(A_1 \otimes \mathbf{1}_2) \exp(\mathbf{1}_1 \otimes A_2).$$

Since $\exp(A_1 \otimes \mathbf{1}_2) = \exp(A_1) \otimes \mathbf{1}_2$ and $\exp(\mathbf{1}_1 \otimes A_2) = \mathbf{1}_1 \otimes \exp(A_2)$, we get

$$\exp(A_1 \otimes \mathbf{1}_2 + \mathbf{1}_1 \otimes A_2) = (\exp(A_1) \otimes \mathbf{1}_2)(\mathbf{1}_1 \otimes \exp(A_2)) = \exp(A_1) \otimes \exp(A_2).$$

This computation explains the shortcut notations of $A_1 + A_2$ instead of $A_1 \otimes \mathbf{1}_2 + \mathbf{1}_1 \otimes A_2$ and the rule

$$\exp(A_1 + A_2) = \exp(A_1) \exp(A_2) = \exp(A_2) \exp(A_1)$$

that is free from ambiguity since operators A_1 and A_2 act on different spaces and necessarily commute.

Take a composite system living on the tensor product $\mathcal{H} \otimes \mathcal{E}$ where $\mathcal{E}$ is another Hilbert space (typically the Hilbert space of the environment). The partial trace versus $\mathcal{E}$ is a superoperator that to any operator M on $\mathcal{H} \otimes \mathcal{E}$ associates an operator on $\mathcal{H}$, denoted by $\mathrm{Tr}_{\mathcal{E}}(M)$. It is defined as follows. Take any orthonormal basis of $\mathcal{H}$, $(|n\rangle)_n$, and of $\mathcal{E}$, $(|\nu\rangle)_\nu$. For the operator M defined by

$$M = \sum_{n_1,\nu_1,n_2,\nu_2} M_{n_1,\nu_1,n_2,\nu_2} |n_1\nu_1\rangle\langle n_2\nu_2|$$

its partial trace is given by

$$\mathrm{Tr}_{\mathcal{E}}(M) = \sum_{n_1,n_2,\nu} M_{n_1,\nu,n_2,\nu} |n_1\rangle\langle n_2| .$$

[1] Notice that in general $\exp(A_1 \otimes A_2) \neq \exp(A_1) \otimes \exp(A_2)$.

Exercice A.3.4. *Show that this definition is independent of the choice of the orthonormal frames* $(|n\rangle)_n$ *in* $\mathcal{H}$ *and* $(|\nu\rangle)_\nu$ *in* $\mathcal{E}$.

Partial traces are related to usual traces:

$$\mathrm{Tr}\,(M) = \mathrm{Tr}_{\mathcal{H}}\,(\mathrm{Tr}_{\mathcal{E}}\,(M)) = \mathrm{Tr}_{\mathcal{E}}\,(\mathrm{Tr}_{\mathcal{H}}\,(M)).$$

We also have $\mathrm{Tr}_{\mathcal{E}}\left(M^\dagger\right) = (\mathrm{Tr}_{\mathcal{E}}\,(M))^\dagger$ and if $M = A \otimes B$ then $\mathrm{Tr}_{\mathcal{E}}\,(A \otimes B) = \mathrm{Tr}\,(B)A$. Finally, for any operators M on $\mathcal{H} \times \mathcal{E}$ and A on $\mathcal{H}$, we have:

$$\mathrm{Tr}\,(\mathrm{Tr}_{\mathcal{E}}\,(M)A) = \mathrm{Tr}\,(M(A \otimes I)).$$

A.4 Density operator

There are mainly two situations where the quantum state $|\psi\rangle$ cannot be used directly:

- a statistical mixture of identical quantum systems, living in $\mathcal{H}$. Each $|\psi\rangle$ belongs to the unit sphere of $\mathcal{H}$; the only way to represent statistical mixture consists in taking average of the projectors onto each $|\psi\rangle$: set k the index associated to this statistical mixture, one defines then the density operator ρ as

  $$\rho = \sum_k p_k \,|\psi_k\rangle\,\langle\psi_k|$$

  where $p_k \in (0,1)$ is the probability of $|\psi_k\rangle$ and thus $\sum_k p_k = 1$.

- the true quantum state belongs to a tensor product $\mathcal{H} \otimes \mathcal{E}$ and we do not have access to $\mathcal{E}$; denoting by $|\Psi\rangle \in \mathcal{H} \otimes \mathcal{E}$ this quantum state, the density operator is then defined by a partial trace versus $\mathcal{E}$ of the projector $|\Psi\rangle\,\langle\Psi|$:

  $$\rho = \mathrm{Tr}_{\mathcal{E}}\,(|\Psi\rangle\,\langle\Psi|).$$

The density operator is then always Hermitian, semi-definite positive and with $\mathrm{Tr}\,(\rho) = 1$. When additionally $\mathrm{Tr}\,(\rho^2) = 1$, ρ is a projector onto a pure state $\rho = |\psi\rangle\,\langle\psi|$, one says briefly that ρ is a pure state.

Exercice A.4.1. *Take* $|\Psi\rangle \in \mathcal{H} \otimes \mathcal{E}$ *and assume that* $\rho = Tr_{\mathcal{E}}\,(|\Psi\rangle\,\langle\Psi|)$ *is a pure state,* $\rho = |\psi\rangle\,\langle\psi|$ *with* $|\psi\rangle \in \mathcal{H}$. *Then prove that* $|\Psi\rangle = |\psi\rangle \otimes |\xi\rangle$ *with* $|\xi\rangle \in \mathcal{E}$.

If we assume that

- either for the mixture case, each $|\psi_k\rangle$ admits a time evolution (A.1) relying on the same Hamiltonian H.

- or that $|\Psi\rangle$ admits a time evolution (A.1) with Hamiltonian $H \otimes I$,

then the time evolution of the density operator ρ is given by the Liouville equations

$$i\frac{d}{dt}\rho = [H, \rho] = H\rho - \rho H$$

where H may depend on t. Thus, the spectrum of ρ is invariant since ρ_t and ρ_0 are related by $\rho_t U_t = U_t \rho_0$ where U_t is the propagator defined in (A.2). In particular, for any integer exponent m, $\mathrm{Tr}\left((\rho_t)^m\right) = \mathrm{Tr}\left((\rho_0)^m\right)$.

A.5 Observables and measurement

To each measurement process is attached a Hermitian operator M on $\mathcal{H}$, called also a physical observable. Take its spectral decomposition

$$M = \sum_\nu \lambda_\nu P_\nu$$

where the λ_ν's are the eigen-values of M ($\lambda_\nu \in \mathbb{R}$) and P_ν the orthogonal projector on the eigenspace associated to λ_ν. In this spectral decomposition $\lambda_{\nu_1} \neq \lambda_{\nu_2}$ as soon as $\nu_1 \neq \nu_2$: each ν corresponds to a different value of the measurement process.

Take now $|\psi\rangle \in \mathcal{H}$. Then the measurement process attached to M yields to λ_ν with probability $\langle\psi|P_\nu|\psi\rangle$. Indeed, assume that we have, at our disposal, a large number n of identical systems with the same quantum state $|\psi\rangle$. For each system, we measure M and obtain the value $\lambda_{\nu_1}, \ldots, \lambda_{\nu_n}$. Set

$$n_\nu = \#\{\lambda_\mu \mid \lambda_\mu = \lambda_\nu\}.$$

Then for n large and each ν, we have $\frac{n_\nu}{n} \approx \langle\psi|P_\nu|\psi\rangle$. This is consistent with the fact that, independently of $|\psi\rangle$, we have $\sum_\nu n_\nu = n$ and $\sum_\nu P_\nu = I$. Notice also that the arithmetic mean value of the n measures is approximatively $\langle\psi|M|\psi\rangle$ since we have, for n large,

$$\frac{\sum_{k=1}^n \lambda_{\nu_k}}{n} = \frac{\sum_\nu n_\nu \lambda_\nu}{n} \approx \sum_\nu \langle\psi|P_\nu|\psi\rangle\lambda_\nu = \langle\psi|M|\psi\rangle.$$

Moreover just after the measure number k that yields λ_{ν_k}, the state $|\psi\rangle$ is drastically changed to $\frac{1}{\langle\psi|P_\nu|\psi\rangle} P_{\nu_k}|\psi\rangle$. This is the famous "collapse of the wave packet" associated to any measurement process and on which is based the Copenhagen interpretation of the wave function $|\psi\rangle$.

Example A.5.1. *The measurement of $\sigma_z = -|g\rangle\langle g| + |e\rangle\langle e|$ for the first qubit of a 2-qubit system (see Section 2.4 for the definition of an n-qubit system) corresponds to the operator (observable) $M = \sigma_z \otimes I_d$. On the 2-qubit system*

$$|\psi\rangle = \psi_{gg}|gg\rangle + \psi_{ge}|ge\rangle + \psi_{eg}|eg\rangle + \psi_{ee}|ee\rangle$$

the measurement of σ_z for the first qubit, gives, in average,

$$\langle\psi|M|\psi\rangle = -(|\psi_{gg}|^2 + |\psi_{ge}|^2) + (|\psi_{eg}|^2 + |\psi_{ee}|^2)$$

i.e., gives either -1 with a probability $|\psi_{gg}|^2+|\psi_{ge}|^2$, or $+1$ with a probability $|\psi_{eg}|^2+|\psi_{ee}|^2$. If, just before the measurement of σ_z on the first qubit, the quantum state is

$$|\psi\rangle = \psi_{gg}\,|gg\rangle + \psi_{ge}\,|ge\rangle + \psi_{eg}\,|eg\rangle + \psi_{ee}\,|ee\rangle\,,$$

then, just after the measurement, the quantum state changes to

- *either* $\frac{\psi_{gg}|gg\rangle+\psi_{ge}|ge\rangle}{\sqrt{|\psi_{gg}|^2+|\psi_{ge}|^2}} = |g\rangle \otimes \left(\frac{\psi_{gg}|g\rangle+\psi_{ge}|e\rangle}{\sqrt{|\psi_{gg}|^2+|\psi_{ge}|^2}}\right)$ *if the measurement outcome is -1,*
- *or* $\frac{\psi_{eg}|eg\rangle+\psi_{ee}|ee\rangle}{\sqrt{|\psi_{eg}|^2+|\psi_{ee}|^2}} = |e\rangle \otimes \left(\frac{\psi_{eg}|g\rangle+\psi_{ee}|e\rangle}{\sqrt{|\psi_{eg}|^2+|\psi_{ee}|^2}}\right)$ *if the measurement outcome is $+1$*

For systems with quantum states described by a density operator ρ, the measurement process attached to the Hermitian operator M with spectral decomposition $M = \sum_\nu \lambda_\nu P_\nu$ becomes:

- the probability to get λ_ν, as the measurement outcome, is $\mathrm{Tr}\,(\rho P_\nu)$ and just after this measurement ρ collapses to $\frac{1}{\mathrm{Tr}(\rho P_\nu)} P_\nu \rho P_\nu$ (notice that $\mathrm{Tr}\,(\rho P_\nu) = \mathrm{Tr}\,(P_\nu \rho P_\nu)$ since $P_\nu^2 = P_\nu$).
- the average value of a large number of measurements of M on the same quantum state ρ is given by $\mathrm{Tr}\,(\rho M)$.

A.6 Pauli Matrices

The Pauli matrices are 2×2 Hermitian matrices defined here below:

$$\sigma_x = |e\rangle\langle g| + |g\rangle\langle e|\,,\ \sigma_y = -i\,|e\rangle\langle g| + i\,|g\rangle\langle e|\,,\ \sigma_z = |e\rangle\langle e| - |g\rangle\langle g|\,.$$

They satisfy the following relations ($\mathbf{1}$ denotes the 2×2 identity matrix here):

$$\sigma_x^2 = \mathbf{1},\quad \sigma_y^2 = \mathbf{1},\quad \sigma_z^2 = \mathbf{1},\quad \sigma_x\sigma_y = i\sigma_z,\quad \sigma_y\sigma_z = i\sigma_x,\quad \sigma_z\sigma_x = i\sigma_y.$$

For any angle $\theta \in \mathbb{R}$ we have

$$e^{i\theta\sigma_\alpha} = \cos\theta\mathbf{1} + i\sin\theta\sigma_\alpha,\quad \text{for}\quad \alpha = x, y, z.$$

Thus the solution of the Schrödinger equation ($\Omega \in \mathbb{R}$)

$$i\frac{d}{dt}\,|\psi\rangle = \frac{\Omega}{2}\sigma_z\,|\psi\rangle$$

reads

$$|\psi\rangle_t = e^{\frac{-i\Omega t}{2}\sigma_z}\,|\psi\rangle_0 = \left(\cos\left(\frac{\Omega t}{2}\right) - i\sin\left(\frac{\Omega t}{2}\right)\sigma_z\right)\,|\psi\rangle_0\,.$$

For $\alpha, \beta = x, y, z$, $\alpha \neq \beta$ we have the useful formulas:

$$\sigma_\alpha e^{i\theta\sigma_\beta} = e^{-i\theta\sigma_\beta}\sigma_\alpha, \qquad \left(e^{i\theta\sigma_\alpha}\right)^{-1} = \left(e^{i\theta\sigma_\alpha}\right)^\dagger = e^{-i\theta\sigma_\alpha}$$

and also

$$e^{-\frac{i\theta}{2}\sigma_\alpha}\sigma_\beta e^{\frac{i\theta}{2}\sigma_\alpha} = e^{-i\theta\sigma_\alpha}\sigma_\beta = \sigma_\beta e^{i\theta\sigma_\alpha}.$$

Take $\sigma = a\sigma_x + b\sigma_y + c\sigma_z$ with $a, b, c \in \mathbb{R}$ such that $a^2 + b^2 + c^2 = 1$. Then $\sigma^2 = 1$. Thus for any angle $\theta \in \mathbb{R}$, we have

$$e^{i\theta\sigma} = \cos\theta\mathbf{1} + i\sin\theta\ \sigma.$$

Appendix B

Linear quantum operations

A linear quantum operation $\mathbb{T}$ is a linear superoperator acting on the space of the density matrices in the system's Hilbert space $\mathcal{S}$, and satisfying the following properties:

- $\mathbb{T}$ is trace-preserving or decreasing. This is, $0 \leq \mathrm{Tr}\,(\mathbb{T}\rho) \leq 1$ for any density matrix ρ.
- $\mathbb{T}$ is completely positive. That is, not only does $\mathbb{T}$ map positive operators to positive operators in the system's Hilbert space $\mathcal{S}$, but so does $\mathbf{1}_{\mathcal{H}} \otimes \mathbb{T}$ for positive operators in $\mathcal{H} \otimes \mathcal{S}$. Here $\mathcal{H}$ is the Hilbert space of a second arbitrary system and $\mathbf{1}_{\mathcal{H}}$ is its identity operator.

Concerning the last property, it may seem that positivity of a superoperator would be sufficient to represent a physical process. However, in practice, the considered system can be entangled to another system before the physical process acts on it. It must still be the case that the total state of both systems remains a physical state with a positive density operator. This justifies the last property.

We have the following theorem called the Kraus representation theorem (see [57, page 368] for a proof):

Theorem B.0.1. *Any linear quantum operation satisfying the above conditions, can be expressed in the form*

$$\mathbb{T}\rho = \sum_j \mathcal{K}_j \rho \mathcal{K}_j^\dagger$$

with

$$\mathbf{1}_{\mathcal{S}} - \sum_j \mathcal{K}_j^\dagger \mathcal{K}_j \geq 0.$$

The above formula is known as the Kraus representation or the operator-sum representation of the linear quantum operation and the operators $\mathcal{K}_j$ are known as the measurement operators. Moreover, $\mathbb{T}$ is trace-preserving ($Tr(\mathbb{T}\rho) = Tr(\rho)$ for any density operator ρ) if, and only if, $\sum_j \mathcal{K}_j^\dagger \mathcal{K}_j = \mathbf{1}_{\mathcal{S}}$.

As soon as we make the additional assumption of a trace-preserving quantum operation, we can also prove some contraction properties. In this aim, we first define the quantum Fidelity and quantum trace distance between two density matrices:

Definition B.0.2. *Consider two well-defined density matrices ρ and σ. The quantum trace distance $D(\rho,\sigma)$ and the quantum fidelity $F(\rho,\sigma)$ are then defined as follows:*

$$D(\rho,\sigma)=\frac{1}{2}Tr\left(|\rho-\sigma|\right),\qquad F(\rho,\sigma)=Tr\left(\sqrt{\rho^{1/2}\sigma\rho^{1/2}}\right),$$

where $|A|\equiv\sqrt{A^\dagger A}$ is the positive square root of $A^\dagger A$.

Remark B.0.3. *One can prove that (see [57, Chapter 9]) as soon as one of the density matrices is a projector state $\sigma=|\psi\rangle\langle\psi|$, the fidelity between ρ and σ is given by the standard form*

$$F(\rho,\sigma)=\sqrt{Tr\left(\rho\sigma\right)}=\sqrt{\langle\psi|\,\rho\,|\psi\rangle}.$$

We have the following contraction properties for trace-preserving quantum operations:

Theorem B.0.4. *Suppose that $\mathbb{T}$ is a trace-preserving quantum operation. Let ρ and σ be two well-defined density operators. Then*

$$D(\mathbb{T}\rho,\mathbb{T}\sigma)\leq D(\rho,\sigma)\quad\text{and}\quad F(\mathbb{T}\rho,\mathbb{T}\sigma)\geq F(\rho,\sigma).$$

The proof of this theorem is beyond the scope of these notes and we refer to [57, Chapter 9] for a rigorous proof.

Appendix C

Single-frequency Averaging

We summarize here the basic result and approximations used in these notes for single-frequency systems. One can consult [61, 31, 7] for much more elaborated results. We emphasize a particular computational trick that simplifies notably second order calculations. This trick is a direct extension of a computation explained in [42] and done by the soviet physicist Kapitsa for deriving the average motion of a particle in a highly oscillating force field.

Consider the oscillating system of dimension n;

$$\frac{dx}{dt} = \varepsilon f(x,t,\varepsilon), \quad x \in \mathbb{R}^n$$

with f smooth and of period T versus t, where ε is a small parameter. For x bounded and $|\varepsilon|$ small enough, there exists a time-periodic change of variables, close to identity, of the form

$$x = z + \varepsilon w(z,t,\varepsilon)$$

with w smooth function and T-periodic versus t, such that, the differential equation in the z frame reads:

$$\frac{dz}{dt} = \varepsilon \overline{f}(z,\varepsilon) + \varepsilon^2 f_1(z,t,\varepsilon)$$

with

$$\overline{f}(z,\varepsilon) = \frac{1}{T}\int_0^T f(z,t,\varepsilon)\, dt$$

and f_1 smooth and T-periodic versus t.

Thus we can approximate on interval $[0, \frac{T}{\epsilon}]$ the trajectories of the oscillating system $\frac{dx}{dt} = \varepsilon f(x,t,\varepsilon)$ by those of the average one $\frac{dz}{dt} = \varepsilon \overline{f}(z,\varepsilon)$. More precisely, if $x(0) = z(0)$ then $x(t) = z(t) + O(|\varepsilon|)$ for all $t \in [0, \frac{T}{\epsilon}]$. Since this approximation is valid on intervals of length T/ε, we say that this approximation is of order one. One also speaks of secular approximation.

The function $w(z,t,\varepsilon)$ appearing in this change of variables is given by a t-primitive of $f-\bar{f}$. If we replace x by $z+\varepsilon w$ in $\frac{d}{dt}x=\varepsilon f$ we get

$$\left(I_d+\varepsilon\frac{\partial w}{\partial z}\right)\frac{d}{dt}z=\varepsilon f-\varepsilon\frac{\partial w}{\partial t}=\varepsilon\bar{f}+\varepsilon\left(f-\bar{f}-\frac{\partial w}{\partial t}\right).$$

Since for each z, the function $\int_0^t\left(f(z,\tau,\varepsilon)-\bar{f}(z,\varepsilon)\right)d\tau$ is T-periodic, we set

$$w(z,t,\varepsilon)=\int_0^t\left(f(z,\tau,\varepsilon)-\bar{f}(z,\varepsilon)\right)\ d\tau+c(z,\varepsilon)$$

where the integration "constant" $c(z,\varepsilon)$ can be set arbitrarily. We will see that a clever choice for c corresponds to w with a null time-average. We have

$$\left(I_d+\varepsilon\frac{\partial w}{\partial z}(z,t,\varepsilon)\right)\frac{d}{dt}z=\varepsilon\bar{f}(z,\varepsilon)+\varepsilon\left(f(z+\epsilon w(z,t,\varepsilon),t,\varepsilon)-f(z,t,\varepsilon)\right)$$

and thus

$$\frac{d}{dt}z=\varepsilon\left(I_d+\varepsilon\frac{\partial w}{\partial z}(z,t,\varepsilon)\right)^{-1}\left(\ \bar{f}(z,\varepsilon)+f(z+\epsilon w(z,t,\varepsilon),t,\varepsilon)-f(z,t,\varepsilon)\ \right).$$

We obtain the form we were looking for, $\frac{d}{dt}z=\varepsilon\bar{f}+\varepsilon^2 f_1$, with

$$f_1(z,t,\varepsilon)=\frac{1}{\epsilon}\left(\left(\left(I_d+\varepsilon\frac{\partial w}{\partial z}(z,t,\varepsilon)\right)^{-1}-I_d\right)\bar{f}(z,\varepsilon)\right.$$
$$\left.+\left(I_d+\varepsilon\frac{\partial w}{\partial z}(z,t,\varepsilon)\right)^{-1}\frac{f(z+\epsilon w(z,t,\varepsilon),t,\varepsilon)-f(z,t,\varepsilon)}{\varepsilon}\right).$$

Notice that

$$f_1(z,t,\varepsilon)=\frac{\partial f}{\partial z}(z,t,\varepsilon)w(z,t,\varepsilon)-\frac{\partial w}{\partial z}(z,t,\varepsilon)\bar{f}(z,\varepsilon)+O(\varepsilon).$$

The second order approximation is then obtained by taking the time-average of f_1. Its justification is still based on a time-periodic change of variables of type $z=\zeta+\varepsilon^2\varpi(\zeta,t,\varepsilon)$, i.e., close to identity but up-to second order in ε.

If we adjust $c(z,\epsilon)$ in order to have w of null time-average, then the time-average of $\frac{\partial w}{\partial z}$ is also zero. Thus, up to order one terms in ε, the time-average of f_1 is identical to the time average of $\frac{\partial f}{\partial z}w$. For this particular choice of w, the second order approximation reads

$$\frac{d}{dt}x=\varepsilon\bar{f}+\varepsilon^2\overline{\frac{\partial f}{\partial x}w}$$

where the symbol "$\bar{}$" stands for time-average. The solutions of the oscillating system $\frac{d}{dt}x=\varepsilon f$ and those of the second order approximation here above remain close on time intervals of length $\frac{T}{\varepsilon^2}$.

A suggestive manner to compute this second order approximation and very efficient on physical examples is due to Kapitsa [42, page 147]. One decomposes $x = \bar{x} + \delta x$ in a non-oscillating part $\bar{x}$ of order 0 in ε and an oscillating part δx of order 1 in ε and of null time-average. One has

$$\frac{d}{dt}\bar{x} + \frac{d}{dt}\delta x = \varepsilon f(\bar{x} + \delta x, t, \varepsilon).$$

Since $\delta x = O(\varepsilon)$, we have

$$f(\bar{x} + \delta x, t, \varepsilon) = f(\bar{x}, t, \varepsilon) + \frac{\partial f}{\partial x}(\bar{x}, t, \varepsilon)\delta x + O(\varepsilon^2).$$

Thus

$$\frac{d}{dt}\bar{x} + \frac{d}{dt}\delta x = \varepsilon f(\bar{x}, t, \varepsilon) + \varepsilon\frac{\partial f}{\partial x}(\bar{x}, t, \varepsilon)\delta x + O(\varepsilon^3).$$

Since $\frac{d}{dt}\bar{x} = \varepsilon\bar{f}(\bar{x}, \varepsilon) + O(\varepsilon^2)$, identification of oscillating terms of null time-average and of first order in ϵ provides

$$\frac{d}{dt}(\delta x) = \varepsilon(f(\bar{x}, t, \varepsilon) - \bar{f}(\bar{x}, \varepsilon)).$$

This equation can be integrated in time since $\bar{x}$ is almost constant. The integration constant is fixed by the constraint on the time-average of δx. Finally,

$$\delta x = \varepsilon \int_0^t \left(f(\bar{x}, \tau, \varepsilon) - \bar{f}(\bar{x}, \varepsilon)\right)\ d\tau + \varepsilon c(\bar{x}, \varepsilon)$$

is a function of $(\bar{x}, t, \varepsilon)$, $\delta x = \delta x(\bar{x}, t, \varepsilon)$, T-periodic versus t and of null time-average (good choice of $c(\bar{x}, \varepsilon)$). Let us plug this function $\delta x(\bar{x}, t, \varepsilon)$ into the differential equation for $\bar{x}$,

$$\frac{d}{dt}\bar{x} = \varepsilon\bar{f}(\bar{x}, \varepsilon) + \varepsilon\frac{\partial f}{\partial x}(\bar{x}, t, \varepsilon)\delta x(\bar{x}, t, \varepsilon) + O(\varepsilon^3),$$

And let us take its time-average. We get

$$\frac{d}{dt}\bar{x} = \varepsilon\bar{f}(\bar{x}, \varepsilon) + \varepsilon^2\bar{f}_1(\bar{x}, \varepsilon)$$

with

$$\varepsilon\bar{f}_1(\bar{x}, \varepsilon) = \frac{1}{T}\int_0^T \frac{\partial f}{\partial x}(\bar{x}, t, \varepsilon)\delta x(\bar{x}, t, \varepsilon)\ dt$$

We recover then exactly the previous second order approximation.

Appendix D

Singular perturbation theory

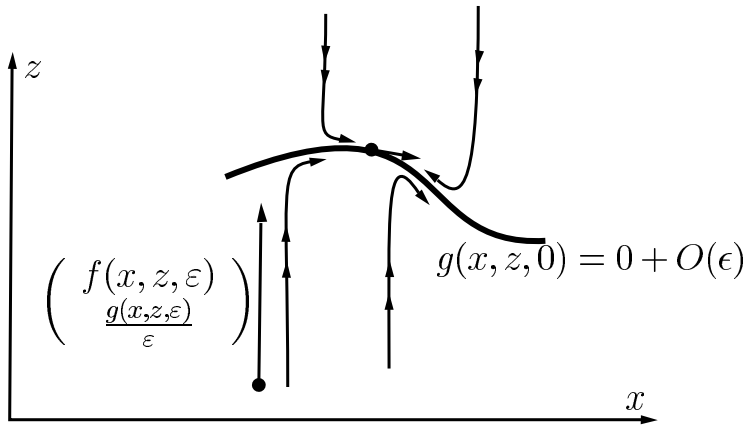

Figure D.1: Slow/fast system in Tikhonov normal; under assumptions stated in theorem D.0.5, the slow approximation (also called quasi-static or adiabatic elimination), consists in setting directly ε to 0 in the equation defining (Σ_ε); this yields to a differential-algebraic system $\frac{d}{dt}x = f(x,z,0)$ where z is an implicit function of x defined by $0 = g(x,z,0)$.

Let us start with the standard case illustrated on figure D.1 and the approximation result known as Tikhonov theorem [39].

Theorem D.0.5. *Consider the singularly perturbed system :*

$$(\Sigma_\varepsilon): \quad \frac{d}{dt}x = f(x,z,\varepsilon), \qquad \varepsilon\frac{d}{dt}z = g(x,z,\varepsilon)$$

where (x,y) belongs to an open subset of $\mathbb{R}^n \times \mathbb{R}^p$, f and g are smooth functions, ε is a small positive parameter. Assume that

- *$g(x,z,0) = 0$ admits a solution $z = \rho(x)$, with ρ smooth function of x and such that $\frac{\partial g}{\partial z}(x,\rho(x),0)$ is a stable matrix (eigenvalues with strictly negative real parts).*

- *the reduced slow sub-system* $\frac{d}{dt}x = f(x, \rho(x), 0), x(0) = x_0$ *admits a unique solution* $x^0(t)$ *defined for* $t \in [0, T]$, $0 < T < +\infty$ *for some* $T > 0$.

Then, for $\varepsilon > 0$ *small enough,* (Σ_ε) *admits a unique solution* $(x^\varepsilon(t), z^\varepsilon(t))$ *defined on* $[0, T]$ *with initial condition* $(x^\varepsilon(0), z^\varepsilon(0)) = (x_0, z_0)$ *as soon as* z_0 *belongs to the attraction domain of the equilibrium* $\rho(x_0)$ *for the fast sub-system,* $\varepsilon\frac{d}{dt}\zeta = g(x_0, \zeta, 0)$. *Moreover we have, for any* $\eta > 0$,

$$\lim_{\varepsilon \to 0^+} \left(\max_{t \in [\eta, T]} \left(\|x^\varepsilon(t) - x^0(t)\| + \|z^\varepsilon(t) - z^0(t)\| \right) \right) = 0.$$

This theorem just means that the approximation of (Σ_ε) via

$$(\Sigma_0): \quad \frac{d}{dt}x = f(x, z, 0), \qquad 0 = g(x, z, 0)$$

is valid for time intervals of length of order 0 versus ε. To get approximation of higher order and thus valid on longer time-intervals one has to construct higher order approximations. They are based on center manifold tools [20].

Let us consider such higher order approximations when the fast dynamics is almost linear, i.e., when $g(x, z, \varepsilon) = -Az + \varepsilon h(x, z)$ and f is independent of ε. The system admits thus the special form

$$\frac{d}{dt}x = f(x, z), \qquad \frac{d}{dt}z = -\frac{1}{\varepsilon}Az + h(x, z)$$

where x and z are respectively the slow and fast states (Tikhonov coordinates), all the eigenvalues of the matrix A have strictly positive real parts, and ε is small strictly positive parameter. Therefore the invariant attractive manifold admits for equation

$$z = \varepsilon A^{-1} h(x, 0) + O(\varepsilon^2) \tag{D.1}$$

and the restriction of the dynamics on this slow invariant manifold reads

$$\frac{d}{dt}x = f(x, \varepsilon A^{-1}h(x, 0)) + O(\varepsilon^2) = f(x, 0) + \varepsilon \left.\frac{\partial f}{\partial z}\right|_{(x,0)} A^{-1}h(x, 0) + O(\varepsilon^2).$$

A Taylor expansion of the attractive invariant manifold, which would satisfy an equation of the form $z = I(x, \epsilon)$, can be conducted through center manifold techniques as explained in [20]). The second order term is then given by (see, e.g., [29]):

$$z = \varepsilon A^{-1}h(x, 0) + \varepsilon^2 A^{-1} \left(\left.\frac{\partial h}{\partial z}\right|_{(x,0)} A^{-1}h(x, 0) - A^{-1} \left.\frac{\partial h}{\partial x}\right|_{(x,0)} f(x, 0) \right) + O(\varepsilon^3), \tag{D.2}$$

and so on.

Roughly speaking, an approximation of order ν in ε of the slow invariant manifold provides an approximation on time intervals of length of order $\frac{1}{\varepsilon^\nu}$ as sketched below:

- $z = 0$ is an approximation of order 0; the slow reduced model $\frac{d}{dt}x = f(x, 0)$ is valid on time intervals of length 1.
- $z = \varepsilon A^{-1}h(x, 0)$ is an approximation of order 1: the slow reduced model $\frac{d}{dt}x = f(x, \varepsilon A^{-1}h(x, 0))$ is valid on time intervals of length $\frac{1}{\varepsilon}$.
- $z = \varepsilon A^{-1}h(x, 0) + \varepsilon^2 A^{-1}\left(\frac{\partial h}{\partial z}|_{(x,0)}A^{-1}h(x, 0) - A^{-1}\frac{\partial h}{\partial x}|_{(x,0)}f(x, 0)\right)$ is an approximation of order 2: the slow reduced model

$$\frac{d}{dt}x = f\left(x,\ \varepsilon A^{-1}h(x, 0) + \varepsilon^2 A^{-1}\left(\frac{\partial h}{\partial z}|_{(x,0)}A^{-1}h(x, 0) - A^{-1}\frac{\partial h}{\partial x}|_{(x,0)}f(x, 0)\right)\right)$$

  is valid on time intervals of length $\frac{1}{\varepsilon^2}$.

Theorem D.0.5 provides a mathematically precise setting of approximation of order 0. However, we have not been able to find in the literature similar precise settings for approximation of order 1 or 2, as sketched here above.

Appendix E

Stability of deterministic systems of finite dimension

Stability formalizes the following intuitive notion: An equilibrium $\bar{x}$ of a dynamical system $\frac{d}{dt}x = v(x)$ $(v(\bar{x}) = 0)$ is said to be stable if a small initial off-set between x and $\bar{x}$ will remain small for all time. To summarize: small causes produce small effects. The proof of the results recalled in this appendix can be found in [39].

E.1 Stability of equilibrium

Definition E.1.1 (Stability (in the Lyapunov sense) and instability)**.** *Take $\mathbb{R}^n$ $(n > 0)$, a time-depend C^1 vector field $\mathbb{R}^n \ni x \mapsto v(x,t) \in \mathbb{R}^n$ and the differential equation $\frac{d}{dt}x = v(x,t)$. Assume that $\bar{x} \in \mathbb{R}^n$ is an equilibrium (steady-state): $v(\bar{x},t) = 0$ for all $t \in \mathbb{R}$.*

The steady-state $\bar{x} \in \mathbb{R}^n$ is said to be stable (in the Lyapunov sense) if, and only if, for all $\epsilon > 0$, there exists $\eta > 0$ such that for any initial condition x^0 satisfying $\|x^0 - \bar{x}\| \leq \eta$, the solution of $\frac{d}{dt}x = v(x,t)$ starting from x^0 at $t = 0$, is defined for $t > 0$ and verifies $\|x(t) - \bar{x}\| \leq \epsilon$ for all $t \geq 0$. If $\bar{x}$ is not stable, it is said to be unstable.

Definition E.1.2 (Asymptotic stability)**.** *With assumptions of Definition E.1.1, the steady-state $\bar{x}$ is said to be locally asymptotically stable if, and only if, it is stable and there exists $\epsilon > 0$ such that any solution $x(t)$ of $\frac{d}{dt}x = v(x,t)$, starting at $t = 0$ from x^0 such that $\|x^0 - \bar{x}\| \leq \epsilon$, converges towards $\bar{x}$ as t tends to $+\infty$.*

This notion of stability is illustrated on Figure E.1. When $\bar{x}$ est asymptotically stable, one also says that the system forgets its initial condition. Whenever in the Definition E.1.2, the parameter ϵ can be chosen arbitrary large, $\bar{x}$ is said to be globally asymptotically stable: the attraction region of $\bar{x}$ is $\mathbb{R}^n$.

For autonomous systems (v independent of t), it is possible to deduce local asymptotic stability of an equilibrium from the first order approximation of the differential system around $\bar{x}$ also called the linear tangent system at $\bar{x}$.

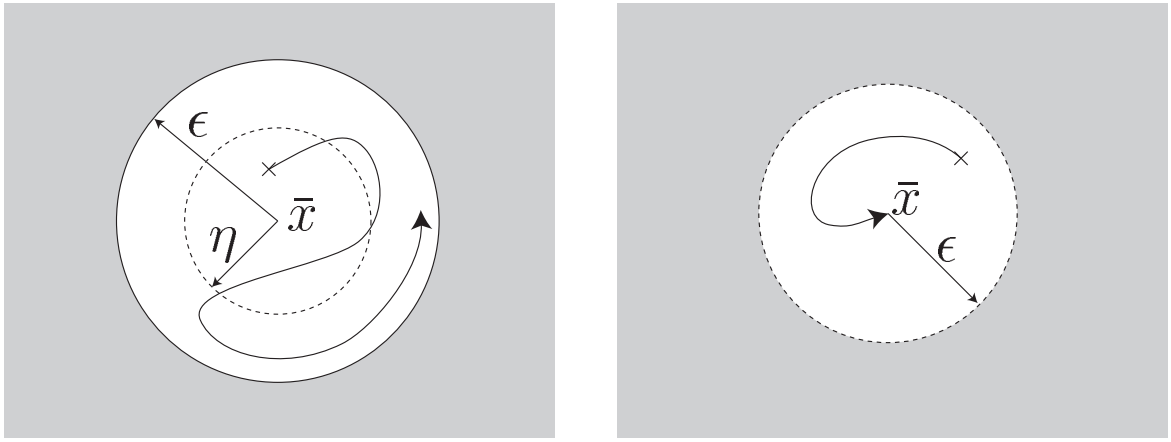

Figure E.1: Stability (left) and asymptotic stability (right).

Definition E.1.3 (Hyperbolic steady-state)**.** *The steady state $\bar{x}$ of the C^1 autonomous system $\frac{d}{dt}x = v(x)$ is said hyperbolic if, and only if, all the eigenvalues of the Jacobian matrix*

$$\frac{\partial v}{\partial x}(\bar{x}) = \left(\frac{\partial v_i}{\partial x_j}(\bar{x})\right)_{1\leq i,j\leq n}$$

admit non-zero real parts.

Theorem E.1.4 (Local asymptotic stability of a hyperbolic steady-state (first Lyapunov method))**.** *Take $\mathbb{R}^n \ni x \mapsto v(x) \in \mathbb{R}^n$ continuously differentiable versus x. The steady-state $\bar{x} \in \mathbb{R}^n$ ($v(\bar{x}) = 0$) is locally asymptotically stable if all the eigenvalues of*

$$\frac{\partial v}{\partial x}(\bar{x}) = \left(\frac{\partial v_i}{\partial x_j}\right)_{1\leq i,j\leq n}$$

admit strictly negative real parts. If one of the eigenvalues of the Jacobian matrix $\frac{\partial v}{\partial x}(\bar{x})$ admit a strictly positive part, then the equilibrium $\bar{x}$ is unstable (in the Lyapunov sense).

E.2 Lyapunov function and Lasalle's invariance principle

Let us consider the damped harmonic oscillator in the phase plane (x_1, x_2):

$$\frac{d}{dt}x_1 = x_2, \quad \frac{d}{dt}x_2 = -\gamma x_2 - \omega^2 x_1$$

with parameters $\gamma, \omega > 0$. Consider its energy

$$V(x_1, x_2) = \frac{\omega^2}{2}(x_1)^2 + \frac{1}{2}(x_2)^2$$

For all solution $t \mapsto (x_1(t), x_2(t))$ of the above differential system, we have

$$\frac{d}{dt}\left(V(x_1(t), x_2(t))\right) = \frac{\partial V}{\partial x_1}\frac{d}{dt}x_1 + \frac{\partial V}{\partial x_2}\frac{d}{dt}x_2 = -2\gamma(x_2)^2 \leq 0.$$

Thus $t \mapsto V(x_1(t), x_2(t))$ is decreasing. Since $V \geq 0$, it converges as $t \mapsto +\infty$. It is intuitive to guess that $\frac{d}{dt}V$ will converge then to 0 and thus that x_2 tends to 0 ($\gamma > 0$). It is also natural to guess that the time derivative of x_2 also converges to 0: $\frac{d}{dt}x_2 = -\gamma x_2 - \omega^2 x_1$ should converge to 0 and thus, since $\omega > 0$, x_1 tends to 0. The above heuristic arguments could be made rigorous using compactness, uniform continuity, and Barbalat's lemma[1]. It is important to notice that, during these computations, we never use the explicit formulae for $x_1(t)$ and $x_2(t)$ based on time exponentials. This is the main interest of such function V called Lyapunov function: it provides precious information on solutions of differential equations without the analytic knowledge of their time-dependencies.

Theorem E.2.1 (Lyapunov function and Lasalle invariance principle)**.** *Take $\Omega \subset \mathbb{R}^n$ an open and non-empty subset of $\mathbb{R}^n$ and $\Omega \ni x \mapsto v(x) \in \mathbb{R}^n$ continuously differentiable function of x. Consider $\Omega \ni x \mapsto V(x) \in \mathbb{R}$ a continuously differentiable function of x and assume that*

1. *there exits $c \in \mathbb{R}$ such that the subset $V_c = \{x \in \Omega \mid V(x) \leq c\}$ of $\mathbb{R}^n$ is compact (bounded and closed) and non-empty.*
2. *V is a decreasing time function for solutions of $\frac{d}{dt}x = v(x)$ inside V_c:*

$$\forall x \in V_c, \quad \frac{d}{dt}V(x) = \nabla V(x) \cdot v(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x)\ v_i(x) \leq 0$$

Then for any initial condition $x^0 \in V_c$, the solution of $\frac{d}{dt}x = v(x)$ remains in V_c, is defined for all $t > 0$ (no explosion in finite time) and converges towards the largest invariant set included in

$$\left\{ x \in V_c \mid \frac{d}{dt}V(x) = 0 \right\}.$$

Such functions V decreasing along trajectories, $\frac{d}{dt}V \leq 0$, are called Lyapunov functions. A subset $\mathcal{E} \subset \mathbb{R}^n$ is said to be invariant if, and only if, any solution of $\frac{d}{dt}x = v(x)$ starting in $\mathcal{E}$ remains in $\mathcal{E}$ for all time t where it is defined. Thus the largest invariant set considered in Theorem E.2.1 is characterized by the over-determined system of $n+1$ scalar equations

$$\frac{d}{dt}\xi_1 = v_1(\xi), \quad \ldots, \quad \frac{d}{dt}\xi_n = v_n(\xi), \qquad \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(\xi)\ v_i(\xi) = 0$$

[1] Take h a uniformly continuous function from $[0, +\infty[$ to $\mathbb{R}$: if $\int_0^t h(s)ds$ admits a limit for t tending to ∞, then necessarily, h tends to 0 as t tends to ∞.

and n scalar unknowns $(\xi_1(t), \ldots, \xi_n(t)) \in V_c$. To get this Lasalle invariant set, the method consists in successive time-differentiations of the last static equation to get new static equations. The collection of these static equations characterizes then the limit set that captures asymptotically the trajectories starting in V_c.

When $\bar{x} \in V_c$ is an equilibrium, $\xi(t) = \bar{x}$ is a solution of this over-determined system. If $\bar{x}$ is its unique solution, then the limit set is reduced to $\{\bar{x}\}$ and $\bar{x}$ is locally asymptotically stable.

If, in the Theorem E.2.1, $\Omega = \mathbb{R}^n$ and V tends to $+\infty$ as $\|x\|$ tends to $+\infty$ (V is radially unbounded) then the trajectory $x(t)$ are bounded for all $t > 0$ since they are contained in V_c with $c = V(x(0))$. If additionally, the equilibrium $\bar{x}$ is the unique solution of the above over-determined system, then all trajectories converge to $\bar{x}$: the equilibrium is then globally asymptotically stable.

The Lasalle's invariance principle can be extended to almost-periodic systems.

Theorem E.2.2 (Lasalle's invariance principle for almost periodic systems)**.** *Take $\Omega \subset \mathbb{R}^n$ an open and non-empty subset of $\mathbb{R}^n$ and $\Omega \times \mathbb{R} \ni (x,t) \mapsto v(x,t) \in \mathbb{R}^n$ a continuously differentiable function of (x,t) that is periodic versus t. Consider $\Omega \ni x \mapsto V(x) \in \mathbb{R}$ a continuously differentiable function of x. Assume that*

1. *there exits $c \in \mathbb{R}$ such that the subset $V_c = \{x \in \Omega \mid V(x) \leq c\}$ of $\mathbb{R}^n$ is compact (bounded and closed) and non-empty.*
2. *V is a decreasing time function for solutions of $\frac{d}{dt}x = v(x,t)$ inside V_c:*

$$\forall t \in \mathbb{R}, \quad \frac{d}{dt}V(t) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x)\ v_i(x,t) \leq 0$$

Then for any initial condition $x^0 \in V_c$, the solution of $\frac{d}{dt}x = v(x,t)$ remains in V_c, is defined for all $t > 0$ (non explosion in finite time) and converges towards the largest invariant set included in

$$\left\{ x \in V_c \mid \frac{d}{dt}V(x) = 0 \right\}.$$

Here a set $S \subset V_c$ is said to be invariant for the time-periodic system $\frac{d}{dt}x = v(x,t)$ if, for all $x_0 \in S$ there exists a time $t_0 > 0$ such that the solution starting from x_0 at time t_0 remains in the set S for all $t \geq t_0$.

This invariant set is also characterized by the over-determined system:

$$\frac{d}{dt}\xi_1 = v_1(\xi,t), \quad \ldots, \quad \frac{d}{dt}\xi_n = v_n(\xi,t), \qquad \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(\xi)\ v_i(\xi,t) = 0.$$

Finally note that, the above Theorem can be extended to almost periodic time-dependent systems.[2]

[2] An almost periodic time function $f(x,t)$ is equal by definition to $F(x, \varpi_1 t, \ldots, \varpi_p t)$ where the function F is a 2π-periodic function of each of its last p arguments and the ϖ_j's form a set of p different pulsations.

Appendix F

Poincaré-Bendixon theory

In this Appendix, we consider a generic 2-dimensional autonomous differential equation and we propose a few theoretical results characterizing all the possible asymptotic behaviors for such a dynamical system. We note that, all the results that we propose in this appendix for the case of a dynamical system on a plane can also be applied in the case of a dynamical system on a sphere. In fact the sphere can be seen as the plane $\mathbb{R}^2$ where the point at infinity has been added. Then, in Figure F.0.3, cases 1 and (2a) coincide.

Theorem F.0.3 (Poincaré)**.** *The autonomous dynamical system* $\frac{d}{dt}x = v(x)$ *with* $x \in \mathbb{R}^2$*, can only admit a few types of possible asymptotic behaviors. Given an initial condition* $x^0 \in \mathbb{R}^2$*, we consider* $t \mapsto x(t)$ *solution of* $\frac{d}{dt}x = v(x)$ *starting at* x^0 *for* $t = 0$*. Then, for any time* $t \geq 0$*, we can only have one of the possibilities below (see also Figure F.0.3):*

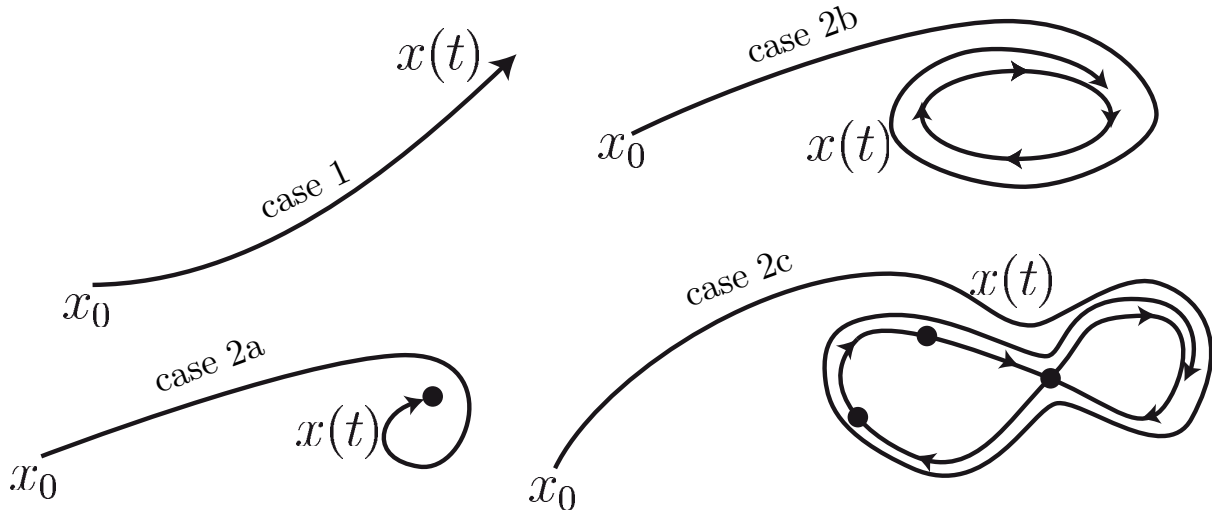

Figure F.1: The four possible asymptotic behaviors for the trajectories of an autonomous dynamical system defined on a plane.

1. *If $x(t)$ is not bounded, then either $x(t)$ explodes in finite time, or if $x(t)$ does not explode in finite time and therefore it is defined for $t > 0$ and $\lim_{t\to\infty} \|x(t)\| = \infty$.*

2. *If $x(t)$ remains bounded for positive times then it is defined for all $t > 0$ and we can distinguish three cases:*

   (a) *either $x(t)$ converges to an equilibrium point (in infinite time of x_0 is not an equilibrium point itself);*

   (b) *or, $x(t)$ converges towards a periodic trajectory (limit cycle);*

   (c) *or finally, $x(t)$ winds up around a closed curve of the plane formed by the trajectories that leave from an equilbirium at $t = -\infty$ and catches at $t = +\infty$, another equilibrium point (heteroclinic orbit if the two equilibria are different and homoclinic orbit if they are identical).*

In summary, whenever the trajectories remain bounded, they converge either toward a point or towards a closed curve of the plane, the curve being tangent to the vector field $v(x)$. Under an additional assumption we can specify the nature of the limit:

Theorem F.0.4 (Bendixon's criteria)**.** *Let $\mathbb{R}^2 \ni x \mapsto v(x) \in \mathbb{R}^2$ be a continuous and differentiable function. We assume that $div(v)(x) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial x} < 0$ for almost all $x \in \mathbb{R}^2$. Let $t \mapsto x(t)$ be a solution of $\frac{d}{dt}x = v(x)$ that remains bounded for positive times t. Then, its limit when t tends to infinity is necessarily an equilibrium point, i.e., a solution $\bar{x} \in \mathbb{R}^2$ of $v(\bar{x}) = 0$.*

In fact, by applying the Gauss theorem, one can easily prove the non-existence of periodic, heteroclinic or homoclinic orbits. The only possible asymptotic regime is therefore the case 2a.

We also have the following theorem:

Theorem F.0.5 (Poincaré-Bendixon)**.** *Let $\mathbb{R}^2 \ni x \mapsto v(x) \in \mathbb{R}^2$ be a function of class C^1. We consider the dynamical system $\frac{d}{dt}x = v(x)$. We suppose that there exists a compact set Ω of the plane such that*

- *all the trajectories admitting an initial condition inside Ω remain inside Ω for all $t > 0$ (Ω is positively invariant);*

- *either Ω does not contain any equilibrium point, or Ω contains a unique equilibrium point $\bar{x}$ such that all the eigenvalues of the linearized system around $\bar{x}$ have strictly positive real parts;*

then Ω admits necessarily a periodic orbit (limit cycle).

The idea behind this theorem is that the bounded trajectories can not converge towards a heteroclinic orbit (as there are at most one equilibrium point). Moreover as the eventual equilibrium point $\bar{x}$ is unstable in all directions, we can not have a homoclinic orbit nor a stable equilibrium. Therefore the only remaining possibility would be a limit cycle.

Appendix G

Pontryaguin Maximum Principe

This appendix is a summary of the necessary optimality conditions called Pontryaguin Maximum Principle (PMP) for finite dimensional systems (for tutorial exposures see [18] or [3]).

Take a control system of the form $\frac{d}{dt}x = f(x,u)$, $x \in \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$ with a cost to maximize of the form $J = \int_0^T c(x,u)dt$ $(T > 0)$, initial condition $x(0) = x^a$ and final condition $x(T) = x^b$. The functions $f \in \mathbb{R}^n$ and $c \in \mathbb{R}$ are assumed to be C^1 functions of their arguments. If the couple $[0,T] \ni t \mapsto (x(t),u(t)) \in \mathbb{R}^n \times U$ is optimal, then there exists a never vanishing and absolutely continuous function[1] $[0,T] \ni t \mapsto p \in \mathbb{R}^n$ and a constant $p_0 \in]-\infty, 0]$ such that:

(i) with $\mathbb{H}(x,p,u) = p_0 c(x,u) + \sum_{i=1}^n p_i f_i(x,u)$, x and p are solutions of

$$\frac{d}{dt}x = \frac{\partial \mathbb{H}}{\partial p}(x,p,u), \quad \frac{d}{dt}p = -\frac{\partial \mathbb{H}}{\partial x}(x,p,u),$$

(ii) for almost all $t \in [0,T]$

$$\mathbb{H}(x(t),p(t),u(t)) = \overline{\mathbb{H}}(x(t),p(t)) \quad \text{where} \quad \overline{\mathbb{H}}(x,p) = \max_{v \in U} \mathbb{H}(x,p,v).$$

(iii) $\overline{\mathbb{H}}(x(t),p(t))$ is independent of t and its value $\bar{h}$, depends on T if the final time is fixed to T or $\bar{h} = 0$ if T is free (as for minimum time problem with U bounded and $c = -1$).

Conditions (i), (ii) and (iii) form the Pontryaguin Maximum Principle (PMP). Couples $[0,T] \ni t \mapsto (x(t),u(t))$ satisfying these conditions are called extremals: if $p_0 = 0$ the extremal is called abnormal; if $p_0 < 0$ the extremal is called normal. Strictly abnormal

[1] An absolutely continuous function $[0,T] \ni t \mapsto z \in \mathbb{R}^m$ satisfies, by definition, the following condition: for all $\epsilon > 0$, there exits $\eta > 0$ such that, for any ordered sequence $0 \leq t_1 \leq \ldots \leq t_k \leq T$ of arbitrary length k fulfilling $\sum_{i=1}^{k-1} |t_{i+1} - t_i| \leq \eta$, we have $\sum_{i=1}^{k-1} |z(t_{i+1}) - z(t_i)| \leq \epsilon$. Such functions are differentiable versus t, for almost all $t \in [0,T]$ and, moreover we have $z(t) = z(0) + \int_0^t z(s)ds$.

extremals are abnormal ((x,p) satisfies (i), (ii) and (iii) with $p_0 = 0$) and not normal ((x,p) never satisfies (i), (ii) and (iii) for $p_0 < 0$). Abnormal extremals do not depend on the cost $c(x,u)$ but only on the system itself $\frac{d}{dt}x = f(x,u)$: they are strongly related to system controllability (for driftless systems where $f(x,u)$ is linear versus x, see [14]).

Assume that we have a normal extremal (x,u), i.e. satisfying conditions (i), (ii) and (iii) with $p_0 < 0$. Assume also that $u \mapsto \mathbb{H}(x,p,u)$ is differentiable, α concave, bounded from above, infinite at infinity and that $U = \mathbb{R}^m$. Then condition (ii) is then equivalent to $\frac{\partial \mathbb{H}}{\partial u} = 0$. Replacing p by p/p_0, PMP conditions (i), (ii) and (iii) coincide with the usual *first order stationary conditions* († means transpose here):

$$\frac{d}{dt}x = f, \quad \frac{d}{dt}p = -\left(\frac{\partial f}{\partial x}\right)^\dagger p - \left(\frac{\partial c}{\partial x}\right)^\dagger, \quad \left(\frac{\partial f}{\partial u}\right)^\dagger p + \left(\frac{\partial c}{\partial u}\right)^\dagger = 0 \tag{G.1}$$

with the boundary condtions $x(0) = x^a$, $x(T) = x^b$. From static equations in (G.1) we can express generally u as a function of (x,p), denoted here by $u = k(x,p)$. Then $\overline{\mathbb{H}}(x,p) = \mathbb{H}(x,p,k(x,p))$ and the first order stationary conditions form an Hamiltonian system

$$\frac{d}{dt}x = \frac{\partial \overline{\mathbb{H}}}{\partial p}(x,p), \quad \frac{d}{dt}p = -\frac{\partial \overline{\mathbb{H}}}{\partial x}(x,p)$$

since $\frac{\partial \overline{\mathbb{H}}}{\partial p} = \frac{\partial \mathbb{H}}{\partial p} + \frac{\partial \mathbb{H}}{\partial u}\frac{\partial k}{\partial p} = \frac{\partial \mathbb{H}}{\partial p}$ because $\frac{\partial \mathbb{H}}{\partial u} \equiv 0$ (idem for $\frac{\partial \overline{\mathbb{H}}}{\partial x}$). In general, this Hamiltonian system is not integrable in the Arnol'd-Liouville sense and numerical methods are then used.

These first order stationary conditions can be obtained directly using standard variation calculus based on the Lagrange method. The adjoint state p is the Lagrange multipliers associated to the constraint $\frac{d}{dt}x = f(x,u)$. Assume T given and consider the Lagrangian $L(x,\dot{x},p,u) = c(x,u) + \sum_{i=1}^n p_i(f_i(x,u) - \dot{x}_i)$ associated to

$$\begin{array}{c} \max \\ u,\, x \end{array} \quad \int_0^T c(x,u)dt.$$
$$f(x,u) - \frac{d}{dt}x = 0$$
$$x(0) = x^a,\ x(T) = x^b$$

The first variation $\delta\mathcal{L}$ of $\mathcal{L} = \int_0^T L(x,\dot{x},p,u)dt$ should vanish for any variation δx, δp and δu such that $\delta x(0) = \delta x(T) = 0$:

- $\delta\mathcal{L} = 0$ for any δp yields to $\frac{d}{dt}x = f(x,u)$;
- $\delta\mathcal{L} = 0$ for any δx with $\delta x(0) = \delta x(T) = 0$ yields to $\frac{d}{dt}p = -\left(\frac{\partial f}{\partial x}\right)^\dagger p - \left(\frac{\partial c}{\partial x}\right)^\dagger$
- $\delta\mathcal{L} = 0$ for any δu yields to $\frac{\partial c}{\partial u} + \sum_i p_i \frac{\partial f_i}{\partial u} = 0$

We recover the stationary conditions (G.1).

It is then simple to show that the stationary conditions for

$$\begin{array}{c} \max \\ u, x \\ f(x,u) - \frac{d}{dt}x = 0 \\ x(0) = x^a \end{array} \qquad \int_0^T c(x,u)dt + l(x(T)),$$

where the final condition $x(T) = x^b$ is replaced by a final cost $l(x(T)$ (l a C^1 function), remain unchanged except for the boundary conditions that become

$$x(0) = x^a, \quad p(T) = \left(\frac{\partial l}{\partial x}\right)^{\dagger}(x(T)).$$

Appendix H

Markov chains, martingales and convergence theorems

This Appendix has for aim to give a very brief overview of some definitions and some theorems in the theory of random processes. The stability Theorems H.0.10, H.0.11 and H.0.13 can be seen as stochastic analogues of deterministic Lyapunov function techniques.

We start the appendix by defining three types of convergence for random processes:

Definition H.0.6. *Consider (X_n) a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in a metric space $\mathcal{X}$. The random process X_n is said to,*

- *converge in probability towards the random variable X if for all $\epsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\left(|X_n - X| > \epsilon\right) = \lim_{n\to\infty} \mathbb{P}\left(\omega \in \Omega \mid |X_n(\omega) - X(\omega)| > \epsilon\right) = 0;$$

- *converge almost surely towards the random variable X if*

$$\mathbb{P}\left(\lim_{n\to\infty} X_n = X\right) = \mathbb{P}\left(\omega \in \Omega \mid \lim_{n\to\infty} X_n(\omega) = X(\omega)\right) = 1;$$

- *converge in mean towards the random variable X if*

$$\lim_{n\to\infty} \mathbb{E}\left(|X_n - X|\right) = 0.$$

We can prove that the almost sure convergence and the convergence in mean imply the convergence in probability. However no such relation can be proved between the convergence in mean and the almost sure convergence in general.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}$ be a nondecreasing family of sub-σ-algebras. We have the following definitions

Definition H.0.7. *The sequence $(X_n, \mathcal{F}_n)_{n=1}^{\infty}$ is called a Markov process with respect to $F = (\mathcal{F}_n)_{n=1}^{\infty}$, if for $n' > n$ and any measurable function $f(x)$ with $\sup_x |f(x)| < \infty$,*

$$\mathbb{E}\left(f(X_{n'}) \mid \mathcal{F}_n\right) = \mathbb{E}\left(f(X_{n'}) \mid X_n\right).$$

Definition H.0.8. *The sequence $(X_n, \mathcal{F}_n)_{n=1}^{\infty}$ is called respectively a supermartingale, a submartingale or a martingale, if $\mathbb{E}(|X_n|) < \infty$ for $n = 1, 2, \cdots$, and*

$$\mathbb{E}(X_n \mid \mathcal{F}_m) \leq X_m \qquad (\mathbb{P} \text{ almost surely}), \qquad n \geq m,$$

or

$$\mathbb{E}(X_n \mid \mathcal{F}_m) \geq X_m \qquad (\mathbb{P} \text{ almost surely}), \qquad n \geq m,$$

or finally,

$$\mathbb{E}(X_n \mid \mathcal{F}_m) = X_m \qquad (\mathbb{P} \text{ almost surely}), \qquad n \geq m.$$

Remark H.0.9. *A time-continuous version of the above definitions can also be considered for $(X_t, \mathcal{F}_t)_{t\geq 0}$, where $F = (\mathcal{F}_t)_{t\geq 0}$, is non decreasing family of sub-σ-alegbras of $\mathcal{F}$.*

The following theorem characterizes the convergence of bounded martingales:

Theorem H.0.10 (Doob's first martingale convergence theorem)**.** *Let $(X_n, \mathcal{F}_n)_{n<\infty}$ be a submartingale such that (x^+ is the positive part of x)*

$$\sup_n \mathbb{E}\left(X_n^+\right) < \infty.$$

Then $\lim_n X_n$ $(= X_\infty)$ exists with probability 1, and $\mathbb{E}(X_\infty^+) < \infty$.

For a proof we refer to [47, Chapter 2, Page 43].

Here, we recall two results that are often referred as the stochastic versions of the Lyapunov stability theory and the LaSalle's invariance principle. For detailed discussions and proofs we refer to [41, Sections 8.4 and 8.5]. The first theorem is the following:

Theorem H.0.11 (Doob's Inequality)**.** *Let $\{X_n\}$ be a Markov chain on state space $\mathcal{X}$. Suppose that there is a non-negative function $V(x)$ satisfying $\mathbb{E}(V(X_1) \mid X_0 = x) - V(x) = -k(x)$, where $k(x) \geq 0$ on the set $\{x : V(x) < \lambda\} \equiv Q_\lambda$. Then*

$$\mathbb{P}\left(\sup_{\infty > n \geq 0} V(X_n) \geq \lambda \mid X_0 = x\right) \leq \frac{V(x)}{\lambda}.$$

Corollary H.0.12. *Consider the same assumptions as in Theorem H.0.11. Assume moreover that there exists $\bar{x} \in \mathcal{X}$ such that $V(\bar{x}) = 0$ and that $V(x) \neq 0$ for all x different from $\bar{x}$. Then the Theorem H.0.11 implies that the Markov process X_n is **stable in probability** around $\bar{x}$, i.e.*

$$\lim_{x \to \bar{x}} \mathbb{P}\left(\sup_n \|X_n - \bar{x}\| \geq \epsilon \mid X_0 = x\right) = 0, \qquad \forall \epsilon > 0.$$

For the statement of the second Theorem (which can be seen as a stochastic version of the LaSalle's invariance principle), we need to use the language of probability measures rather than the random processes. Therefore, we deal with the space M of probability measures on the state space $\mathcal{X}$. Let $\mu_0 = \sigma$ be the initial probability distribution (everywhere through these notes we have dealt with the case where μ_0 is a Dirac on a state ρ_0 of the state space of density matrices). Then, the probability distribution of X_n, given initial distribution σ, is to be denoted by $\mu_n(\sigma)$. Note that for $m \geq 0$, the Markov property implies: $\mu_{n+m}(\sigma) = \mu_n(\mu_m(\sigma))$.

Theorem H.0.13 (Kushner's invariance Theorem)**.** *Consider the same assumptions as that of the Theorem H.0.11. Let $\mu_0 = \sigma$ be concentrated on a state $x_0 \in Q_\lambda$ (Q_λ being defined as in Theorem H.0.11), i.e. $\sigma(x_0) = 1$. Assume that $0 \leq k(X_n) \to 0$ in Q_λ implies that $X_n \to \{x \mid k(x) = 0\} \cap Q_\lambda \equiv K_\lambda$. Under the conditions of Theorem H.0.11, for trajectories never leaving Q_λ, X_n converges to K_λ almost surely. Also, the associated conditioned probability measures $\tilde{\mu}_n$ tend to the largest invariant set of measures $M_\infty \subset M$ whose support set is in K_λ. Finally, for the trajectories never leaving Q_λ, X_n converges, in probability, to the support set of M_∞.*

In the simple case where the set K_λ is reduced to the equilibrium point $\bar{x}$ such that $V(\bar{x}) = 0$, we have the following corollary:

Corollary H.0.14. *Consider the same assumptions as in Theorem H.0.13 and assume moreover that $\bar{x} \in \mathcal{X}$ is the only point in Q_λ such that $V(\bar{x}) = 0$ and furthermore that the set K_λ defined in Theorem H.0.13 is reduced to $\{\bar{x}\}$. Then the equilibrium $\bar{x}$ is* ***globally stable in probability*** *in the set Q_λ, i.e.*

$$\bar{x} \text{ is stable in probability and moreover} \quad \mathbb{P}\left(\lim_{n\to\infty} X_n = \bar{x} \mid X_n \text{ never leaves } Q_\lambda\right) = 1.$$

Note that the Theorem H.0.13 is much stronger that this corollary as by the invariance property we can reduce the limit set to a much smaller set than K_λ. Therefore, even in the case where K_λ is not reduced to $\{\bar{x}\}$, we may hope by the invariance property, to be able to show the convergence (at least in probability) towards the equilibrium $\bar{x}$. This is what happens in the Section 5.1 of these notes.

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