Quantum Systems: Dynamics and Control[1]

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Outline

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The Lindblad master differential equation (finite dimensional case)

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[\boldsymbol{H},\rho] + \sum_\nu \boldsymbol{L}_\nu \rho \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho + \rho \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu) \triangleq \mathcal{L}(\rho)$$

where

- $\boldsymbol{H}$ is the Hamiltonian that could depend on t (Hermitian operator on the underlying Hilbert space $\mathcal{H}$)
- the $\boldsymbol{L}_\nu$'s are operators on $\mathcal{H}$ that are not necessarily Hermitian.

• Structure invariance under a time-varying change of frame $\widetilde{\boldsymbol{\rho}} = \boldsymbol{U}_t^\dagger \boldsymbol{\rho} \boldsymbol{U}_t$ with $\boldsymbol{U}_t$ unitary: the new density operator $\widetilde{\boldsymbol{\rho}}$ obeys to a similar SME where $\widetilde{\boldsymbol{H}} = \boldsymbol{U}_t^\dagger \boldsymbol{H} \boldsymbol{U}_t + i\boldsymbol{U}_t^\dagger \left(\frac{d}{dt}\boldsymbol{U}_t\right)$ and $\widetilde{\boldsymbol{L}}_\nu = \boldsymbol{U}_t^\dagger \boldsymbol{L}_\nu \boldsymbol{U}_t$.

• Qualitative properties:

1. Positivity and trace conservation: if ρ_0 is a density operator, then $\rho(t)$ remains a density operator for all $t > 0$.
2. For any $t \geq 0$, the propagator $e^{t\mathcal{L}}$ is a Kraus map: exists a collection of operators $(M_{\mu,t})$ such that $\sum_\mu M_{\mu,t}^\dagger M_{\mu,t} = I$ with $e^{t\mathcal{L}}(\rho) = \sum_\mu M_{\mu,t}\rho M_{\mu,t}^\dagger$ (Kraus theorem characterizing completely positive linear maps).
3. Contraction for many distances such as the nuclear distance: take two trajectories ρ and ρ'; for any $0 \leq t_1 \leq t_2$,

$$\operatorname{Tr}\left(|\rho(t_2) - \rho'(t_2)|\right) \leq \operatorname{Tr}\left(|\rho(t_1) - \rho'(t_1)|\right)$$

where for any Hermitian operator A, $|A| = \sqrt{A^2}$ and $\operatorname{Tr}(|A|)$ corresponds to the sum of the absolute values of its eigenvalues.

Link between Lindblad differential equation and Kraus map

$$\rho_{k+1} = \sum_\mu \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger \quad \text{with} \quad \sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \boldsymbol{I}$$

$$\frac{d}{dt}\rho = -\tfrac{i}{\hbar}[\boldsymbol{H}, \rho] + \sum_\nu \boldsymbol{L}_\nu \rho \boldsymbol{L}_\nu^\dagger - \tfrac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho + \rho \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)$$

Take $dt > 0$ small. Set

$$\boldsymbol{M}_{dt,0} = \boldsymbol{I} - dt\left(\tfrac{i}{\hbar}\boldsymbol{H} + \tfrac{1}{2}\sum_\nu \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu\right), \quad \boldsymbol{M}_{dt,\nu} = \sqrt{dt}\,\boldsymbol{L}_\nu.$$

Since $\rho(t+dt) = \rho(t) + dt\left(\frac{d}{dt}\rho(t)\right) + O(dt^2)$, we have

$$\rho(t+dt) = \boldsymbol{M}_{dt,0}\rho(t)\boldsymbol{M}_{dt,0}^\dagger + \sum_\nu \boldsymbol{M}_{dt,\nu}\rho(t)\boldsymbol{M}_{dt,\nu}^\dagger + O(dt^2).$$

Since $\boldsymbol{M}_{dt,0}^\dagger \boldsymbol{M}_{dt,0} + \sum_\nu \boldsymbol{M}_{dt,\nu}^\dagger \boldsymbol{M}_{dt,\nu} = \boldsymbol{I} + 0(dt^2)$ the super-operator

$$\rho \mapsto \boldsymbol{M}_{dt,0}\rho \boldsymbol{M}_{dt,0}^\dagger + \sum_\nu \boldsymbol{M}_{dt,\nu}\rho \boldsymbol{M}_{dt,\nu}^\dagger$$

can be seen as an infinitesimal Kraus map.

Properties of the trace distance $D(\rho, \rho') = \mathrm{Tr}\left(|\rho - \rho'|\right)/2$.

1. Unitary invariance: for any unitary operator U ($U^\dagger U = I$), $D\big(U\rho U^\dagger, U\rho' U^\dagger\big) = D(\rho, \rho')$.
2. For any density operators ρ and ρ',

$$D(\rho, \rho') = \max_{\substack{P \text{ such that} \\ 0 \leq P = P^\dagger \leq I}} \mathrm{Tr}\left(P(\rho - \rho')\right).$$

3. Triangular inequality: for any density operators ρ, ρ' and ρ''

$$D(\rho, \rho'') \leq D(\rho, \rho') + D(\rho', \rho'').$$

Complement: Kraus maps are contractions for several "distances"[5]

For any Kraus map $\rho \mapsto \boldsymbol{K}(\rho) = \sum_\mu M_\mu \rho M_\mu^\dagger$ ($\sum_\mu M_\mu^\dagger M_\mu = I$)
$d(\boldsymbol{K}(\rho), \boldsymbol{K}(\sigma)) \leq d(\rho, \sigma)$ with

- trace distance: $d_{tr}(\rho, \sigma) = \frac{1}{2} \operatorname{Tr}\left(|\rho - \sigma|\right)$.
- Bures distance: $d_B(\rho, \sigma) = \sqrt{1 - F(\rho, \sigma)}$ with fidelity $F(\rho, \sigma) = \operatorname{Tr}\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)$.
- Chernoff distance: $d_C(\rho, \sigma) = \sqrt{1 - Q(\rho, \sigma)}$ where $Q(\rho, \sigma) = \min_{0 \leq s \leq 1} \operatorname{Tr}\left(\rho^s \sigma^{1-s}\right)$.
- Relative entropy: $d_S(\rho, \sigma) = \sqrt{\operatorname{Tr}\left(\rho(\log \rho - \log \sigma)\right)}$.
- χ^2-divergence: $d_{\chi^2}(\rho, \sigma) = \sqrt{\operatorname{Tr}\left((\rho - \sigma)\sigma^{-\frac{1}{2}}(\rho - \sigma)\sigma^{-\frac{1}{2}}\right)}$.
- Hilbert's projective metric: if $\operatorname{supp}(\rho) = \operatorname{supp}(\sigma)$
  $d_h(\rho, \sigma) = \log\left(\left\|\rho^{-\frac{1}{2}}\sigma\rho^{-\frac{1}{2}}\right\|_\infty \left\|\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\right\|_\infty\right)$
  otherwise $d_h(\rho, \sigma) = +\infty$.

[5] A good summary in M.J. Kastoryano PhD thesis: Quantum Markov Chain Mixing and Dissipative Engineering. University of Copenhagen, December 2011.

Complement: non-commutative consensus and Hilbert's metric[6] [7]

The Schrödinger approach $d_h(\rho,\sigma) = \log\left(\left\|\rho^{-\frac{1}{2}}\sigma\rho^{-\frac{1}{2}}\right\|_\infty \left\|\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\right\|_\infty\right)$

$$\boldsymbol{K}(\rho) = \sum M_\mu \rho M_\mu^\dagger, \quad \sum M_\mu^\dagger M_\mu = I$$
$$\frac{d}{dt}\rho = -i[H,\rho] + \sum L_\mu \rho L_\mu^\dagger - \frac{1}{2}L_\mu^\dagger L_\mu \rho - \frac{1}{2}\rho L_\mu^\dagger L_\mu$$

Contraction ratio: $\tanh\left(\frac{\Delta(\boldsymbol{K})}{4}\right)$ with $\Delta(\boldsymbol{K}) = \max_{\rho,\sigma>0} d_h(\boldsymbol{K}(\rho),\boldsymbol{K}(\sigma))$

The Heisenberg approach (dual of Schrödinger approach):

$$\boldsymbol{K}^*(A) = \sum M_\mu^\dagger A M_\mu, \quad \boldsymbol{K}^*(I) = I$$
$$\frac{d}{dt}A = i[H,A] + \sum L_\mu^\dagger A L_\mu - \frac{1}{2}L_\mu^\dagger L_\mu A - \frac{1}{2}A L_\mu^\dagger L_\mu, \quad A = I \text{ steady-state.}$$

"Contraction of the spectrum":

$$\lambda_{min}(A) \leq \lambda_{min}(\boldsymbol{K}^*(A)) \leq \lambda_{max}(\boldsymbol{K}^*(A)) \leq \lambda_{max}(A).$$

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[6] R. Sepulchre et al.: Consensus in non-commutative spaces. CDC 2010.

[7] D. Reeb et al.: Hilbert's projective metric in quantum information theory. J. Math. Phys. 52, 082201 (2011).

Outline

The driven and damped classical oscillator

Dynamics in the (x', p') phase plane with $\omega \gg \kappa, \sqrt{u_1^2 + u_2^2}$:

$$\frac{d}{dt}x' = \omega p', \quad \frac{d}{dt}p' = -\omega x' - \kappa p' - 2u_1 \sin(\omega t) + 2u_2 \cos(\omega t)$$

Define the frame rotating at ω by $(x', p') \mapsto (x, p)$ with

$$x' = \cos(\omega t)x + \sin(\omega t)p, \quad p' = -\sin(\omega t)x + \cos(\omega t)p.$$

Removing highly oscillating terms (rotating wave approximation), from

$$\frac{d}{dt}x = -\kappa \sin^2(\omega t)x + 2u_1 \sin^2(\omega t) + (\kappa p - 2u_2) \sin(\omega t) \cos(\omega t)$$
$$\frac{d}{dt}p = -\kappa \cos^2(\omega t)p + 2u_2 \cos^2(\omega t) + (\kappa x - 2u_1) \sin(\omega t) \cos(\omega t)$$

we get, with $\alpha = x + ip$ and $u = u_1 + iu_2$:

$$\frac{d}{dt}\alpha = -\frac{\kappa}{2}\alpha + u.$$

With $x' + ip' = \alpha' = e^{-i\omega t}\alpha$, we have $\frac{d}{dt}\alpha' = -(\frac{\kappa}{2} + i\omega)\alpha' + ue^{-i\omega t}$

Driven and damped quantum oscillator ($n_{th} = 0$)

- The Lindblad master equation:

$$\frac{d}{dt}\rho = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + \kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a}\right).$$

- Consider $\rho = \boldsymbol{D}_{\overline{\alpha}}\xi\boldsymbol{D}_{-\overline{\alpha}}$ with $\overline{\alpha} = 2u/\kappa$ and $\boldsymbol{D}_{\overline{\alpha}} = e^{\overline{\alpha}\boldsymbol{a}^\dagger - \overline{\alpha}^*\boldsymbol{a}}$. We get

$$\frac{d}{dt}\xi = \kappa\left(\boldsymbol{a}\xi\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\xi - \tfrac{1}{2}\xi\boldsymbol{a}^\dagger\boldsymbol{a}\right)$$

since $\boldsymbol{D}_{-\overline{\alpha}}\boldsymbol{a}\boldsymbol{D}_{\overline{\alpha}} = \boldsymbol{a} + \overline{\alpha}$.

- Informal convergence proof with the strict Lyapunov function $V(\xi) = \mathrm{Tr}\left(\xi\boldsymbol{N}\right)$:

$$\frac{d}{dt}V(\xi) = -\kappa V(\xi) \Rightarrow V(\xi(t)) = V(\xi_0)e^{-\kappa t}.$$

Since $\xi(t)$ is Hermitian and non-negative, $\xi(t)$ tends to $|0\rangle\langle 0|$ when $t \mapsto +\infty$.

The rigorous underlying convergence result

Theorem

Consider with $u \in \mathbb{C}$, $\kappa > 0$, *the following Cauchy problem*

$$\frac{d}{dt}\rho = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + \kappa \left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a}\right), \quad \rho(0) = \rho_0.$$

Assume that the initial state ρ_0 *is a density operator with finite energy* $\mathrm{Tr}\left(\rho_0 \boldsymbol{N}\right) < +\infty$. *Then exists a unique solution to the Cauchy problem in the Banach space* $\mathcal{K}^1(\mathcal{H})$*, the set of trace class operators on* $\mathcal{H}$*. It is defined for all* $t > 0$ *with* $\rho(t)$ *a density operator (Hermitian, non-negative and trace-class) that remains in the domain of the Lindblad super-operator*

$$\rho \mapsto [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + \kappa \left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a}\right).$$

This means that $t \mapsto \rho(t)$ *is differentiable in the Banach space* $\mathcal{K}^1(\mathcal{H})$*. Moreover* $\rho(t)$ *converges for the trace-norm towards* $|\overline{\alpha}\rangle\langle\overline{\alpha}|$ *when* t *tends to* $+\infty$*, where* $|\overline{\alpha}\rangle$ *is the coherent state of complex amplitude* $\overline{\alpha} = \frac{2u}{\kappa}$.

Link with the classical oscillator

Lemma

Consider with $u \in \mathbb{C}$, $\kappa > 0$, the following Cauchy problem

$$\frac{d}{dt}\rho = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + \kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a}\right), \quad \rho(0) = \rho_0.$$

1. *for any initial density operator ρ_0 with $\mathrm{Tr}\left(\rho_0 \boldsymbol{N}\right) < +\infty$, we have $\frac{d}{dt}\alpha = -\frac{\kappa}{2}(\alpha - \overline{\alpha})$ where $\alpha = \mathrm{Tr}\left(\rho \boldsymbol{a}\right)$ and $\overline{\alpha} = \frac{2u}{\kappa}$.*
2. *Assume that $\rho_0 = |\beta_0\rangle\langle\beta_0|$ where β_0 is some complex amplitude. Then for all $t \geq 0$, $\rho(t) = |\beta(t)\rangle\langle\beta(t)|$ remains a coherent state of amplitude $\beta(t)$ solution of the following equation: $\frac{d}{dt}\beta = -\frac{\kappa}{2}(\beta - \overline{\alpha})$ with $\beta(0) = \beta_0$.*

Statement 2 relies on:

$$\boldsymbol{a}|\beta\rangle = \beta|\beta\rangle, \quad |\beta\rangle = e^{-\frac{\beta\beta^*}{2}} e^{\beta\boldsymbol{a}^\dagger}|0\rangle \quad \frac{d}{dt}|\beta\rangle = \left(-\tfrac{1}{2}(\beta^*\dot{\beta} + \beta\dot{\beta}^*) + \dot{\beta}\boldsymbol{a}^\dagger\right)|\beta\rangle.$$

Outline

Parameters $\omega \gg \kappa, |u|$ and $n_{\text{th}} > 0$:

$$\frac{d}{dt}\boldsymbol{\rho} = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \boldsymbol{\rho}] + (1+n_{\text{th}})\kappa\left(\boldsymbol{a}\boldsymbol{\rho}\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}^\dagger\boldsymbol{a}\right) + n_{\text{th}}\kappa\left(\boldsymbol{a}^\dagger\boldsymbol{\rho}\boldsymbol{a} - \tfrac{1}{2}\boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}\boldsymbol{a}^\dagger\right).$$

Key issue: $\lim_{t\mapsto+\infty}\boldsymbol{\rho}(t) = ?$.
With $\bar{\alpha} = 2u/k$, we have

$$\frac{d}{dt}\boldsymbol{\rho} = (1+n_{\text{th}})\kappa\left((\boldsymbol{a}-\bar{\alpha})\boldsymbol{\rho}(\boldsymbol{a}-\bar{\alpha})^\dagger - \tfrac{1}{2}(\boldsymbol{a}-\bar{\alpha})^\dagger(\boldsymbol{a}-\bar{\alpha})\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}(\boldsymbol{a}-\bar{\alpha})^\dagger(\boldsymbol{a}-\bar{\alpha})\right) + n_{\text{th}}\kappa\left((\boldsymbol{a}-\bar{\alpha})^\dagger\boldsymbol{\rho}(\boldsymbol{a}-\bar{\alpha}) - \tfrac{1}{2}(\boldsymbol{a}-\bar{\alpha})(\boldsymbol{a}-\bar{\alpha})^\dagger\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}(\boldsymbol{a}-\bar{\alpha})(\boldsymbol{a}-\bar{\alpha})^\dagger\right).$$

Using the unitary change of frame $\boldsymbol{\xi} = \boldsymbol{D}_{-\bar{\alpha}}\boldsymbol{\rho}\boldsymbol{D}_{\bar{\alpha}}$ based on the displacement $\boldsymbol{D}_{\bar{\alpha}} = e^{\bar{\alpha}\boldsymbol{a}^\dagger - \bar{\alpha}^\dagger\boldsymbol{a}}$, we get the following dynamics on $\boldsymbol{\xi}$

$$\frac{d}{dt}\boldsymbol{\xi} = (1+n_{\text{th}})\kappa\left(\boldsymbol{a}\boldsymbol{\xi}\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\xi} - \tfrac{1}{2}\boldsymbol{\xi}\boldsymbol{a}^\dagger\boldsymbol{a}\right) + n_{\text{th}}\kappa\left(\boldsymbol{a}^\dagger\boldsymbol{\xi}\boldsymbol{a} - \tfrac{1}{2}\boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{\xi} - \tfrac{1}{2}\boldsymbol{\xi}\boldsymbol{a}\boldsymbol{a}^\dagger\right)$$

since $\boldsymbol{a} + \bar{\alpha} = \boldsymbol{D}_{-\bar{\alpha}}\boldsymbol{a}\boldsymbol{D}_{\bar{\alpha}}$.

Asymptotic convergence towards the thermal equilibrium

The thermal mixed state $\xi_{\text{th}} = \frac{1}{1+n_{\text{th}}}\left(\frac{n_{\text{th}}}{1+n_{\text{th}}}\right)^{\boldsymbol{N}}$ is an equilibrium of

$$\frac{d}{dt}\xi = \kappa(1+n_{\text{th}})\left(\boldsymbol{a}\xi\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\xi - \tfrac{1}{2}\xi\boldsymbol{a}^\dagger\boldsymbol{a}\right) + \kappa n_{\text{th}}\left(\boldsymbol{a}^\dagger\xi\boldsymbol{a} - \tfrac{1}{2}\boldsymbol{a}\boldsymbol{a}^\dagger\xi - \tfrac{1}{2}\xi\boldsymbol{a}\boldsymbol{a}^\dagger\right)$$

with $\text{Tr}\left(\boldsymbol{N}\xi_{\text{th}}\right) = n_{\text{th}}$. Following [8], set ζ the solution of the Sylvester equation: $\xi_{\text{th}}\zeta + \zeta\xi_{\text{th}} = \xi - \xi_{\text{th}}$. Then $V(\xi) = \text{Tr}\left(\xi_{\text{th}}\zeta^2\right)$ is a strict Lyapunov function. It is based on the following computations that can be made rigorous with an adapted Banach space for ξ:

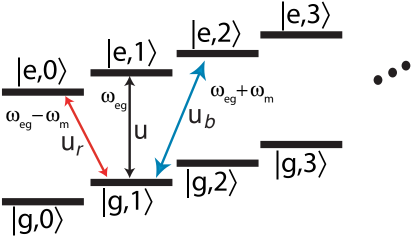
$$\frac{d}{dt}V(\xi) = -\kappa(1+n_{\text{th}})\,\text{Tr}\left([\zeta,\boldsymbol{a}]\xi_{\text{th}}[\zeta,\boldsymbol{a}]^\dagger\right) - \kappa n_{\text{th}}\,\text{Tr}\left([\zeta,\boldsymbol{a}^\dagger]\xi_{\text{th}}[\zeta,\boldsymbol{a}^\dagger]^\dagger\right) \leq 0.$$

When $\frac{d}{dt}V = 0$, ζ commutes with $\boldsymbol{a}$, $\boldsymbol{a}^\dagger$ and $\boldsymbol{N}$. It is thus a constant function of $\boldsymbol{N}$. Since $\xi_{\text{th}}\zeta + \zeta\xi_{\text{th}} = \xi - \xi_{\text{th}}$, we get $\xi = \xi_{\text{th}}$.

[8] PR and A. Sarlette: Contraction and stability analysis of steady-states for open quantum systems described by Lindblad differential equations. Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on, 10-13 Dec. 2013, 6568-6573.

Outline

The Law-Eberly Hamiltionian (cf. lect no 5)



$$\frac{\boldsymbol{H}}{\hbar} = u|g\rangle\langle e| + u^*|e\rangle\langle g| + \bar{u}_b|g\rangle\langle e|\boldsymbol{a} + \bar{u}_b^*|e\rangle\langle g|\boldsymbol{a}^\dagger + \bar{u}_r|g\rangle\langle e|\boldsymbol{a}^\dagger + \bar{u}_r^*|e\rangle\langle g|\boldsymbol{a}$$

Take $\bar{u} = \bar{u}_b = 0$ and $\mathbb{C} \ni \bar{u}_r \neq 0$ constant with the dissipation channel $\boldsymbol{\sigma_-} = |g\rangle\langle e|$ with inverse qubit life time κ. The Lindblad equation reads:

$$\frac{d}{dt}\boldsymbol{\rho} = -i\left[\bar{u}_r|g\rangle\langle e|\boldsymbol{a}^\dagger + \bar{u}_r^*|e\rangle\langle g|\boldsymbol{a}\ ,\ \boldsymbol{\rho}\right] + \kappa\left(|g\rangle\langle e|\ \boldsymbol{\rho}\ |e\rangle\langle g| - \tfrac{1}{2}\Big(|e\rangle\langle e|\ \boldsymbol{\rho} + \boldsymbol{\rho}\ |e\rangle\langle e|\Big)\right).$$

Then $\lim_{t\mapsto+\infty}\boldsymbol{\rho}(t) = |g\rangle\langle g| \otimes |0\rangle\langle 0| = |g0\rangle\langle g0|$.
Proof based on the fact that for any integer $\bar{n}$.

$$\frac{d}{dt}\operatorname{Tr}\left(\left(|g\bar{n}\rangle\langle g\bar{n}| + \sum_{0\leq n\leq \bar{n}-1}|n\rangle\langle n|\right)\boldsymbol{\rho}\right) = \kappa\,\langle e\bar{n}|\boldsymbol{\rho}|e\bar{n}\rangle \geq 0$$