Quantum Systems: Dynamics and Control[1]

Mazyar Mirrahimi[2], Pierre Rouchon[3], Alain Sarlette[4]

March 10, 2020

[1] See the web page: http://cas.ensmp.fr/~rouchon/MasterUPMC/index.html

[2] INRIA Paris, QUANTIC research team
[3] Mines ParisTech, QUANTIC research team
[4] INRIA Paris, QUANTIC research team

Outline

Discrete-time Stochastic Master Equations (SME)

Trace preserving Kraus map $\boldsymbol{K}_u$ depending on the classical control input u:

$$\boldsymbol{K}_u(\boldsymbol{\rho}) = \sum_\xi \boldsymbol{M}_{u,\xi} \boldsymbol{\rho} \boldsymbol{M}_{u,\xi}^\dagger \quad \text{with} \quad \sum_\xi \boldsymbol{M}_{u,\xi}^\dagger \boldsymbol{M}_{u,\xi} = \boldsymbol{I}.$$

Take a left stochastic matrix $\left[\eta_{y,\xi}\right]$ ($\eta_{y,\xi} \geq 0$ and $\sum_y \eta_{y,\xi} \equiv 1$, $\forall \xi$) and set $\boldsymbol{K}_{u,y}(\boldsymbol{\rho}) = \sum_\xi \eta_{y,\xi} \boldsymbol{M}_{u,\xi} \boldsymbol{\rho} \boldsymbol{M}_{u,\xi}^\dagger$. The associated Markov chain reads:

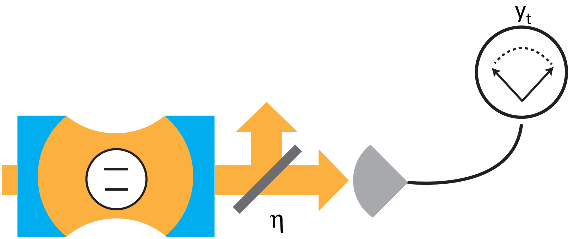
$$\boldsymbol{\rho}_{k+1} = \frac{\boldsymbol{K}_{u_k,y_k}(\boldsymbol{\rho}_k)}{\mathrm{Tr}\left(\boldsymbol{K}_{u_k,y_k}(\boldsymbol{\rho}_k)\right)} \quad \text{measurement } y_k \text{ with probability } \mathrm{Tr}\left(\boldsymbol{K}_{u_k,y_k}(\boldsymbol{\rho}_k)\right).$$

Classical input u, hidden state ρ, measured output y.
Ensemble average given by $\boldsymbol{K}_u$ since $\mathbb{E}\left(\boldsymbol{\rho}_{k+1} \mid \boldsymbol{\rho}_k, u_k\right) = \boldsymbol{K}_{u_k}(\boldsymbol{\rho}_k)$.
Markov model useful for:

1. Monte-Carlo simulations of quantum trajectories (decoherence, measurement back-action).
2. quantum filtering to get the quantum state ρ_k from ρ_0 and $(y_0, \ldots, y_{k-1})$ (Belavkin quantum filter developed for diffusive models).
3. feedback design and Monte-Carlo closed-loop simulations.

Outline

Markov process under continuous measurement



Inverse setup of photon-box: photons read out a qubit.

Two major differences

- measurement output taking values from a continuum of possible outcomes

$$dy_t = \sqrt{\eta}\,\mathrm{Tr}\left((\boldsymbol{L}+\boldsymbol{L}^\dagger)\rho_t\right) dt + dW_t.$$

- Time continuous dynamics.

Stochastic master equation: Markov process under continuous measurement

$$d\boldsymbol{\rho}_t = \left(-\frac{i}{\hbar}[\boldsymbol{H},\boldsymbol{\rho}_t] + \sum_\nu \boldsymbol{L}_\nu\boldsymbol{\rho}_t\boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger\boldsymbol{L}_\nu\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{L}_\nu^\dagger\boldsymbol{L}_\nu)\right)dt$$
$$+\sum_\nu \sqrt{\eta_\nu}\left(\boldsymbol{L}_\nu\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{L}_\nu^\dagger - \mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\boldsymbol{\rho}_t\right)\boldsymbol{\rho}_t\right)dW_{\nu,t},$$

where $W_{\nu,t}$ are independent Wiener processes, associated to measured signals

$$dy_{\nu,t} = dW_{\nu,t} + \sqrt{\eta_\nu}\,\mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\boldsymbol{\rho}_t\right)dt.$$

Wiener process W_t:

- $W_0 = 0$;
- $t \to W_t$ is almost surely everywhere continuous;
- For $0 \leq s_1 < t_1 \leq s_2 < t_2$, $W_{t_1} - W_{s_1}$ and $W_{t_2} - W_{s_2}$ are independent random variables satisfying $W_t - W_s \sim N(0, t-s)$.

Average dynamics: Lindblad master equation

$d\mathbb{E}(\boldsymbol{\rho}_t) =$
$\left(-\frac{i}{\hbar}[\boldsymbol{H},\mathbb{E}(\boldsymbol{\rho}_t)] + \sum_\nu \boldsymbol{L}_\nu\mathbb{E}(\boldsymbol{\rho}_t)\boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger\boldsymbol{L}_\nu\mathbb{E}(\boldsymbol{\rho}_t) + \mathbb{E}(\boldsymbol{\rho}_t)\boldsymbol{L}_\nu^\dagger\boldsymbol{L}_\nu)\right)dt.$

Ito stochastic calculus

Given a diffusive Stochastic Differential Equation (SDE)

$$dX_t = F(X_t, t)dt + \sum_\nu G_\nu(X_t, t)dW_{\nu,t},$$

we have the following chain rule:

Ito's rule

Defining $f_t = f(X_t)$ a C^2 function of X, we have

$$df_t = \left(\frac{\partial f}{\partial X}\bigg|_{X_t} F(X_t, t) + \frac{1}{2}\sum_\nu \frac{\partial^2 f}{\partial X^2}\bigg|_{X_t} (G_\nu(X_t, t), G_\nu(X_t, t)) \right) dt + \sum_\nu \frac{\partial f}{\partial X}\bigg|_{X_t} G_\nu(X_t, t)dW_{\nu,t}.$$

Furthermore

$$\frac{d}{dt}\mathbb{E}(f_t) = \mathbb{E}\left(\frac{\partial f}{\partial X}\bigg|_{X_t} F(X_t, t) + \frac{1}{2}\sum_\nu \frac{\partial^2 f}{\partial X^2}\bigg|_{X_t} (G_\nu(X_t, t), G_\nu(X_t, t)) \right).$$

Link to partial Kraus maps (1)

$$d\boldsymbol{\rho}_t = \left(-\frac{i}{\hbar}[\boldsymbol{H},\boldsymbol{\rho}_t] + \sum_\nu \boldsymbol{L}_\nu\boldsymbol{\rho}_t\boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger\boldsymbol{L}_\nu\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{L}_\nu^\dagger\boldsymbol{L}_\nu)\right)dt$$
$$+ \sum_\nu \sqrt{\eta_\nu}\left(\boldsymbol{L}_\nu\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{L}_\nu^\dagger - \mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\boldsymbol{\rho}_t\right)\boldsymbol{\rho}_t\right)dW_{\nu,t},$$

equivalent to

$$\boldsymbol{\rho}_{t+dt} = \frac{\boldsymbol{M}_{dy_t}\boldsymbol{\rho}_t\boldsymbol{M}_{dy_t}^\dagger + \sum_\nu(1-\eta_\nu)\boldsymbol{L}_\nu\boldsymbol{\rho}_t\boldsymbol{L}_\nu^\dagger dt}{\mathrm{Tr}\left(\boldsymbol{M}_{dy_t}\boldsymbol{\rho}_t\boldsymbol{M}_{dy_t}^\dagger + \sum_\nu(1-\eta_\nu)\boldsymbol{L}_\nu\boldsymbol{\rho}_t\boldsymbol{L}_\nu^\dagger dt\right)}$$

with

$$\boldsymbol{M}_{dy_t} = \boldsymbol{I} + (-\frac{i}{\hbar}\boldsymbol{H} - \frac{1}{2}\boldsymbol{L}_\nu^\dagger\boldsymbol{L}_\nu)dt + \sum_\nu \sqrt{\eta_\nu}dy_{\nu,t}\boldsymbol{L}_\nu.$$

Moreover, defining $dy_t = s_t\sqrt{dt} = (s_{\nu,t})\sqrt{dt}$:

$$\mathbb{P}(s_t \in [s, s+ds] \mid \boldsymbol{\rho}_t) = \mathrm{Tr}\left(\boldsymbol{M}_{s\sqrt{dt}}\boldsymbol{\rho}_t\boldsymbol{M}_{s\sqrt{dt}}^\dagger + \sum_\nu(1-\eta_\nu)\boldsymbol{L}_\nu\boldsymbol{\rho}_t\boldsymbol{L}_\nu^\dagger dt\right)\prod_\nu \frac{e^{-\frac{s_\nu^2}{2}}ds_\nu}{\sqrt{2\pi}}.$$

Link to partial Kraus maps (2)

- $\mathbb{P}$ defines a probability density up to a correction of order dt^2:

$$\int \mathbb{P}(s_t \in [s, s+ds] \mid \rho_t) = 1 + O(dt^2).$$

- Mean value of measured signal

$$\int s_\nu \mathbb{P}(s_t \in [s, s+ds] \mid \rho_t) = \sqrt{\eta_\nu}\,\mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\rho_t\right)\sqrt{dt} + O(dt^{3/2}).$$

- Variance of measured signal

$$\int s_\nu^2\, \mathbb{P}(s_t \in [s, s+ds] \mid \rho_t) = 1 + O(dt).$$

Compatible with $dy_{\nu,t} = dW_{\nu,t} + \sqrt{\eta_\nu}\,\mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\rho_t\right) dt$.

Link to partial Kraus maps (3)

$$d\boldsymbol{\rho}_t = \left(-\frac{i}{\hbar}[\boldsymbol{H}, \boldsymbol{\rho}_t] + \sum_\nu \boldsymbol{L}_\nu \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \boldsymbol{\rho}_t + \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)\right) dt$$
$$+ \sum_\nu \sqrt{\eta_\nu} \left(\boldsymbol{L}_\nu \boldsymbol{\rho}_t + \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger - \mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\boldsymbol{\rho}_t\right) \boldsymbol{\rho}_t\right) dW_{\nu,t},$$

equivalent to

$$\boldsymbol{\rho}_{t+dt} = \frac{\boldsymbol{M}_{dy_t} \boldsymbol{\rho}_t \boldsymbol{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \boldsymbol{L}_\nu \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger dt}{\mathrm{Tr}\left(\boldsymbol{M}_{dy_t} \boldsymbol{\rho}_t \boldsymbol{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \boldsymbol{L}_\nu \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger dt\right)}$$

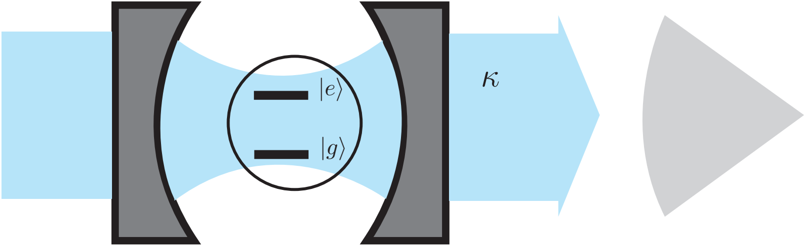
- Indicates that the solution remains in the space of semi-definite positive Hermitian matrices;
- Provides a time-discretized numerical scheme preserving non-negativity of $\boldsymbol{\rho}$.

Theorem

The above master equation admits a unique solution remaining for all $t \geq 0$ in $\{\boldsymbol{\rho} \in \mathbb{C}^{N\times N} \ : \ \boldsymbol{\rho} = \boldsymbol{\rho}^\dagger, \ \boldsymbol{\rho} \geq 0, \ \mathrm{Tr}(\boldsymbol{\rho}) = 1\}$.

Outline

Dispersive measurement of a qubit



Inverse setup of photon-box: photons read out a qubit.

Approximate model

Cavity's dynamics are removed (singular perturbation techniques) to achieve a qubit SME:

$$d\rho_t = -\frac{i}{\hbar}[\boldsymbol{H}, \rho_t]dt + \frac{\Gamma_m}{4}(\boldsymbol{\sigma_z}\rho_t\boldsymbol{\sigma_z} - \rho_t)dt + \frac{\sqrt{\eta\Gamma_m}}{2}(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\,(\boldsymbol{\sigma_z}\rho_t)\,\rho_t)dW_t,$$

$$dy_t = dW_t + \sqrt{\eta\Gamma_m}\,\mathrm{Tr}\,(\boldsymbol{\sigma_z}\rho_t)\,dt.$$

Quantum Non-Demolition measurement

$$d\boldsymbol{\rho}_t = -\frac{i}{\hbar}[\boldsymbol{H}, \boldsymbol{\rho}_t]dt + \frac{\Gamma_m}{4}(\boldsymbol{\sigma_z}\boldsymbol{\rho}_t\boldsymbol{\sigma_z} - \boldsymbol{\rho}_t)dt + \frac{\sqrt{\eta\Gamma_m}}{2}(\boldsymbol{\sigma_z}\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\left(\boldsymbol{\sigma_z}\boldsymbol{\rho}_t\right)\boldsymbol{\rho}_t)dW_t,$$

$$dy_t = dW_t + \sqrt{\eta\Gamma_m}\,\mathrm{Tr}\left(\boldsymbol{\sigma_z}\boldsymbol{\rho}_t\right)dt.$$

Uncontrolled case: $\boldsymbol{H}/\hbar = \omega_{\mathrm{eg}}\boldsymbol{\sigma_z}/2$.

Interpretation as a Markov process with Kraus operators

$$\boldsymbol{M}_{dy_t} = \boldsymbol{I} - \left(i\frac{\omega_{\mathrm{eg}}}{2}\boldsymbol{\sigma_z} + \frac{\Gamma_m}{8}\boldsymbol{I}\right)dt + \frac{\sqrt{\eta\Gamma_m}}{2}\boldsymbol{\sigma_z}dy_t,$$

$$\sqrt{(1-\eta)dt}\,\boldsymbol{L} = \frac{\sqrt{(1-\eta)\Gamma_m dt}}{2}\boldsymbol{\sigma_z}.$$

QND measurement

Kraus operators $\boldsymbol{M}_{dy_t}$ and $\sqrt{(1-\eta)dt}\,\boldsymbol{L}$ commute with observable $\boldsymbol{\sigma_z}$: qubit states $|g\rangle\langle g|$ and $|e\rangle\langle e|$ are fixed points of the measurement process. The measurement is QND for the observable $\boldsymbol{\sigma_z}$.

QND measurement: asymptotic behavior

Theorem

Consider the SME

$$d\rho_t = -\frac{i}{\hbar}[\boldsymbol{H}, \rho_t]dt + \frac{\Gamma_m}{4}(\boldsymbol{\sigma_z}\rho_t\boldsymbol{\sigma_z} - \rho_t)dt + \frac{\sqrt{\eta\Gamma_m}}{2}(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\operatorname{Tr}(\boldsymbol{\sigma_z}\rho_t)\rho_t)dW_t,$$

with $\boldsymbol{H} = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z}$ and $\eta > 0$.

- For any initial state ρ_0, the solution ρ_t converges almost surely as $t \to \infty$ to one of the states $|g\rangle\langle g|$ or $|e\rangle\langle e|$.
- The probability of convergence to $|g\rangle\langle g|$ (respectively $|e\rangle\langle e|$) is given by $p_g = \operatorname{Tr}(|g\rangle\langle g|\rho_0)$ (respectively $\operatorname{Tr}(|e\rangle\langle e|\rho_0)$).
- The convergence rate is given by $\eta\Gamma_M/2$.

Proof based on the Lyapunov function $V(\rho) = \sqrt{\operatorname{Tr}(\boldsymbol{\sigma_z}^2\rho) - \operatorname{Tr}^2(\boldsymbol{\sigma_z}\rho)}$ with

$$\frac{d}{dt}\mathbb{E}(V(\rho)) = -\frac{\eta\Gamma_M}{2}\mathbb{E}(V(\rho))$$

Matlab open-loop simulations: `ModelQubit.m`

Quantum feedback

Question: how to stabilize deterministically a single qubit state $|g\rangle\langle g|$ or $|e\rangle\langle e|$?
Controlled SME:

$$d\rho_t = -\frac{i}{\hbar}[\boldsymbol{H}, \rho_t]dt + \frac{\Gamma_m}{4}(\boldsymbol{\sigma_z}\rho_t\boldsymbol{\sigma_z} - \rho_t)dt \\ + \frac{\sqrt{\eta\Gamma_m}}{2}(\boldsymbol{\sigma_z}\rho_t + \rho_r\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\,(\boldsymbol{\sigma_z}\rho_t)\,\rho_t)dW_t,$$

with

$$\boldsymbol{H} = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \frac{u(\rho_t)}{2}\boldsymbol{\sigma_x},$$
$$u(\rho) = -\alpha\,\mathrm{Tr}\left(i[\boldsymbol{\sigma_x}, \rho]\rho_{tag}\right) + \beta(1 - \mathrm{Tr}\left(\rho\rho_{tag}\right)), \quad \alpha, \beta > 0 \;\text{ and }\; \beta^2 < 8\alpha\eta,$$

globally stabilizes the target state $\rho_{tag} = |g\rangle\langle g|$ or $|e\rangle\langle e|$.

Matlab closed-loop simulations: `FeedbackQubit.m`