Quantum Systems: Dynamics and Control[1]

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Outline

1. Quantum measurement and filtering
   - Projective measurement
   - Positive Operator Valued Measurement (POVM)
   - Stochastic processes attached to quantum measurement
   - Quantum Filtering
2. QND measurements and open-loop convergence
   - Martingales and convergence of Markov chains
   - Martingale behavior for QND measurement of photons
3. Feedback stabilization of photon number states

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Recall: measurements and backaction in LKB photon box

- projective measurement: meter qubit, Hilbert space $\mathbb{C}^2 = \text{span}(|g\rangle, |e\rangle)$, detected in $\mu \in \{g, e\}$ and projected in $|\mu\rangle$ with proba. $|\langle\psi|\mu\rangle|^2$
- non-projective measurement: cavity, measured indirectly through interaction with meter qubit, undergoes with proba. $\mathbb{P}_{\mu|\boldsymbol{\rho}} = \text{Tr}\left(\boldsymbol{M}_\mu \boldsymbol{\rho} \boldsymbol{M}_\mu^\dagger\right)$:

$$\boldsymbol{\rho}_+ = \boldsymbol{M}_\mu \boldsymbol{\rho} \boldsymbol{M}_\mu^\dagger \,/\, \mathbb{P}_{\mu|\boldsymbol{\rho}} \quad \text{associated to meas.result } \mu$$

- decoherence: interaction with environment which is not measured, e.g.

$$\rho_+ = \boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger \quad \text{or} \quad \rho_+ = \boldsymbol{M}_{-1} \rho \boldsymbol{M}_{-1}^\dagger + \boldsymbol{M}_{+1} \rho \boldsymbol{M}_{+1}^\dagger + \boldsymbol{M}_0 \rho \boldsymbol{M}_0^\dagger$$

- measurement errors: when “true” output $\mu \in \{g, e\}$ is read as $y \in \{g, e\}$ with probability $\eta_{y,\mu}$:

$$\boldsymbol{\rho}_+ = \mathbb{K}_y(\boldsymbol{\rho}) \,/\, \text{Tr}\left(\mathbb{K}_y(\boldsymbol{\rho})\right) \quad \text{with proba. } \text{Tr}\left(\mathbb{K}_y(\boldsymbol{\rho})\right) \;,$$

$$\text{where } \mathbb{K}_y(\boldsymbol{\rho}) = \textstyle\sum_\mu \eta_{y,\mu} \boldsymbol{M}_\mu \boldsymbol{\rho} \boldsymbol{M}_\mu^\dagger$$

These are the general forms of quantum measurement and associated evolution in discrete-time.

Projective measurement

For the system defined on Hilbert space $\mathcal{H}$, take

- an observable $\boldsymbol{O}$ (Hermitian operator) defined on $\mathcal{H}$:

$$\boldsymbol{O} = \sum_{\mu} \lambda_\mu \boldsymbol{P}_\mu,$$

  where λ_μ are the eigenvalues of $\boldsymbol{O}$ and $\boldsymbol{P}_\mu$ is the projection operator over the associated eigenspace.
  Often $\boldsymbol{P}_\mu = |\xi_\mu\rangle\langle\xi_\mu|$ rank-1 projection onto eigenstate $|\xi_\mu\rangle \in \mathcal{H}$.
- a quantum state given by the wave function $|\psi\rangle$ in $\mathcal{H}$.

Projective measurement of the physical observable $\boldsymbol{O} = \sum_\mu \lambda_\mu \boldsymbol{P}_\mu$ for the quantum state $|\psi\rangle$:

1. Probability of obtaining the value λ_μ is given by $\mathbb{P}_\mu = \langle\psi|\boldsymbol{P}_\mu|\psi\rangle$.
   (Note that $\sum_\mu \mathbb{P}_\mu = 1$ as $\sum_\mu \boldsymbol{P}_\mu = \boldsymbol{I}_{\mathcal{H}}$ identity operator on $\mathcal{H}$.)
2. After the measurement, the conditional (a posteriori) state $|\psi_+\rangle$ of the system, given the outcome λ_μ, is

$$|\psi_+\rangle = \frac{\boldsymbol{P}_\mu\,|\psi\rangle}{\sqrt{\mathbb{P}_\mu}} \quad \text{(collapse of the wave packet).}$$

Positive Operator Valued Measurement (POVM) (1)

System S of interest interacts with the meter M and the experimenter measures projectively the meter M.
Measurement process in three consecutive steps:

1. Initially the quantum state is separable

$$\mathcal{H}_S \otimes \mathcal{H}_M \ni |\Psi\rangle = |\psi_S\rangle \otimes |\psi_M\rangle$$

   with a well defined and known state $|\psi_M\rangle$ for M.

2. Then a Schrödinger evolution (unitary operator $\boldsymbol{U}_{S,M}$) of the composite system from $|\psi_S\rangle \otimes |\psi_M\rangle$ produces $\boldsymbol{U}_{S,M}\big(|\psi_S\rangle \otimes |\psi_M\rangle\big)$, entangled in general.[5]

3. Finally we make a projective measurement of the meter M: $\boldsymbol{O}_M = \boldsymbol{I}_S \otimes \left(\sum_\mu \lambda_\mu \boldsymbol{P}_\mu\right)$ the measured observable for the meter, usually $\boldsymbol{P}_\mu = |\xi_\mu\rangle\langle\xi_\mu|$ a rank-1 projection in $\mathcal{H}_M$ onto the eigenstate $|\xi_\mu\rangle \in \mathcal{H}_M$.

[5] A state is entangled if it cannot be written as $|\Psi\rangle = |\tilde{\psi}_S\rangle \otimes |\tilde{\psi}_M\rangle$ for some $|\tilde{\psi}_S\rangle, |\tilde{\psi}_M\rangle$. Entanglement leads to very peculiar quantum correlations.

We can always decompose in the basis of eigenstates $\{|\xi_\mu\rangle\}$:

$$\boldsymbol{U}_{S,M}\big(|\psi_S\rangle \otimes |\psi_M\rangle\big) = \sum_\mu \big(\boldsymbol{M}_\mu|\psi_S\rangle\big) \otimes |\xi_\mu\rangle$$

which define the measurement operators $\boldsymbol{M}_\mu$. Then $\sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \boldsymbol{I}_S$. The set $\{\boldsymbol{M}_\mu\}$ defines a Positive Operator Valued Measurement (POVM). Note $\boldsymbol{M}_\mu$ includes the known value of $|\psi_M\rangle$.

Projective meas. of $\boldsymbol{O}_M = \boldsymbol{I}_S \otimes \big(\sum_\mu \lambda_\mu |\xi_\mu\rangle\langle\xi_\mu|\big) = \sum_\mu \lambda_\mu \tilde{\boldsymbol{P}}_\mu$ on quantum state $\boldsymbol{U}_{S,M}\big(|\psi_S\rangle \otimes |\psi_M\rangle\big)$ in $\mathcal{H}_S \otimes \mathcal{H}_M$, summarized on $\mathcal{H}_S$:

1. The probability of obtaining the value λ_μ is given by $\mathbb{P}_\mu = \langle\psi_S|\boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu|\psi_S\rangle$
2. After the measurement, the conditional (a posteriori) state of the system on $\mathcal{H}_S$, given the outcome μ, is

$$|\psi_{S,+}\rangle = \frac{\boldsymbol{M}_\mu|\psi_S\rangle}{\sqrt{\mathbb{P}_\mu}}.$$

Stochastic processes attached to quantum measurement

- To the POVM $(\boldsymbol{M}_\mu)$ on $\mathcal{H}_S$ is attached a stochastic process of quantum state $|\psi\rangle$

$$|\psi_+\rangle = \frac{\boldsymbol{M}_\mu|\psi\rangle}{\sqrt{\mathbb{P}_\mu}} \text{ with probability } \mathbb{P}_\mu = \langle\psi|\boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu|\psi\rangle$$

- Knowing the state $|\psi\rangle$, the conditional expectation value for any observable $\boldsymbol{A}$ on $\mathcal{H}_S$ after applying the POVM is

$$\mathbb{E}\left(\langle\psi_+|\boldsymbol{A}|\psi_+\rangle \;\middle|\; |\psi\rangle\right) = \langle\psi|(\sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{A}\boldsymbol{M}_\mu)|\psi\rangle = \text{Tr}\left(\boldsymbol{A}\,\mathbb{K}(|\psi\rangle\langle\psi|)\right)$$

with Kraus map $\mathbb{K}(\rho) = \sum_\mu \boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger$ with $\rho = |\psi\rangle\langle\psi|$ density operator corresponding to $|\psi\rangle$.

- Imperfection and errors described by left stochastic matrix $(\eta_{y,\mu})$, $\sum_y \eta_{y,\nu} \equiv 1$, where $\eta_{y,\mu}$ is the probability of detector outcome y knowing that the ideal detection should be μ. Then Bayes law yields

$$\mathbb{E}\left(\langle\psi_+|\boldsymbol{A}|\psi_+\rangle \;\middle|\; |\psi\rangle, y\right) = \frac{\text{Tr}\left(\boldsymbol{A}\mathbb{K}_y(\rho)\right)}{\text{Tr}\left(\mathbb{K}_y(\rho)\right)}$$

with completely positive linear maps $\mathbb{K}_y(\rho) = \sum_\mu \eta_{y,\mu} \boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger$ depending on y. Probability to detect y knowing ρ is $\text{Tr}\left(\mathbb{K}_y(\rho)\right)$.

Stochastic Master Equation (SME) and quantum filtering

Discrete-time open quantum models are Markov processes

$$\rho_{k+1} = \frac{\mathbb{K}_{y_k}(\rho_k)}{\mathrm{Tr}\left(\mathbb{K}_{y_k}(\rho_k)\right)}, \text{ with proba. } \mathbb{P}_{y_k}(\rho_k) = \mathrm{Tr}\left(\mathbb{K}_{y_k}(\rho_k)\right).$$

Each $\mathbb{K}_y$ is a linear completely positive map depending on meas. outcomes, $\mathbb{K}_y(\rho) = \sum_\mu \boldsymbol{K}_{y,\mu} \boldsymbol{\rho} \boldsymbol{K}_{y,\mu}^\dagger$, with $\sum_{y,\mu} \boldsymbol{K}_{y,\mu}^\dagger \boldsymbol{K}_{y,\mu} = \boldsymbol{I}$.

When discarding meas. outcomes, state update follows Kraus map (quantum channel, completely positive trace-preserving map (CPTP), ensemble average)

$$\rho_{k+1} = \mathbb{K}(\rho_k) = \sum_y \mathbb{K}_y(\rho_k) = \sum_{y,\mu} \boldsymbol{K}_{y,\mu} \rho_k \boldsymbol{K}_{y,\mu}^\dagger.$$

Quantum filtering (Belavkin quantum filters)

data: initial quantum state ρ_0, past measurement outcomes $\boldsymbol{y}_\ell$ for $\ell \in \{0, \ldots, k-1\}$;

goal: estimation of ρ_k via the recurrence (quantum filter)

$$\rho_{\ell+1} = \frac{\mathbb{K}_{\boldsymbol{y}_\ell}(\rho_\ell)}{\mathrm{Tr}\left(\mathbb{K}_{\boldsymbol{y}_\ell}(\rho_\ell)\right)}, \quad \ell = 0, \ldots, k-1.$$

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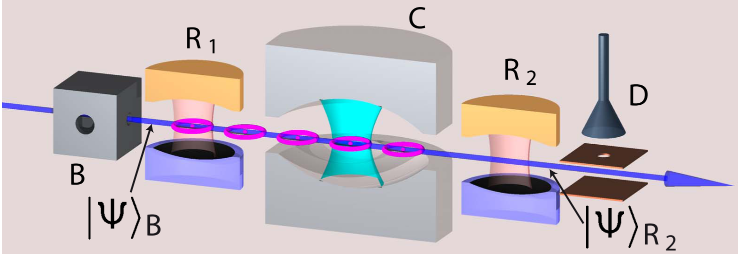


Markov process: $|\psi_k\rangle \equiv |\psi\rangle_{t=k\Delta t}$, $k \in \mathbb{N}$, Δt sampling period,

$$|\psi_{k+1}\rangle = \begin{cases} \frac{\boldsymbol{M}_g |\psi_k\rangle}{\sqrt{\langle\psi_k|\boldsymbol{M}_g^\dagger \boldsymbol{M}_g|\psi_k\rangle}} & \text{with } y_k = g \text{, probability } \mathbb{P}_g = \left\langle \psi_k|\boldsymbol{M}_g^\dagger \boldsymbol{M}_g|\psi_k \right\rangle; \\ \frac{\boldsymbol{M}_e |\psi_k\rangle}{\sqrt{\langle\psi_k|\boldsymbol{M}_e^\dagger \boldsymbol{M}_e|\psi_k\rangle}} & \text{with } y_k = e \text{, probability } \mathbb{P}_e = \left\langle \psi_k|\boldsymbol{M}_e^\dagger \boldsymbol{M}_e|\psi_k \right\rangle, \end{cases}$$

with

$$\boldsymbol{M}_g = \cos(\varphi_0 + \boldsymbol{N}\vartheta), \quad \boldsymbol{M}_e = \sin(\varphi_0 + \boldsymbol{N}\vartheta).$$

QND measurement of photons

Markov process: density operator $\rho_k = |\psi_k\rangle\langle\psi_k|$ as state.

$$\rho_{k+1} = \begin{cases} \frac{\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger\right)} & \text{with } y_k = g \text{, probability } \mathbb{P}_g = \mathrm{Tr}\left(\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger\right); \\ \frac{\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger\right)} & \text{with } y_k = e \text{, probability } \mathbb{P}_e = \mathrm{Tr}\left(\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger\right), \end{cases}$$

with

$$\boldsymbol{M}_g = \cos(\varphi_0 + \boldsymbol{N}\vartheta), \quad \boldsymbol{M}_e = \sin(\varphi_0 + \boldsymbol{N}\vartheta).$$

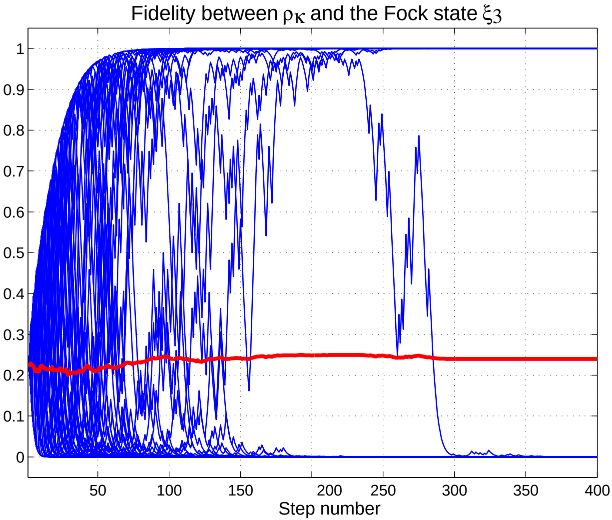
Experimental data

Quantum Non-Demolition (QND) measurement

The measurement operators $\boldsymbol{M}_{g,e}$ commute with the photon-number observable $\boldsymbol{N}$: photon-number states $|n\rangle\langle n|$ are fixed points of the measurement process. We say that the measurement is QND for the observable $\boldsymbol{N}$.

Asymptotic behavior: numerical simulations

100 Monte-Carlo simulations of $\mathrm{Tr}\left(\rho_k |3\rangle\langle 3|\right)$ versus k



Some definitions [6]

Convergence of a random process

Consider (X_k) a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in a metric space $\mathcal{X}$. The random process X_k is said to,

1. converge in probability towards the random variable X if for all $\epsilon > 0$,

$$\lim_{k\to\infty} \mathbb{P}\left(|X_k - X| > \epsilon\right) = \lim_{k\to\infty} \mathbb{P}\left(\omega \in \Omega \mid |X_k(\omega) - X(\omega)| > \epsilon\right) = 0;$$

2. converge almost surely towards the random variable X if

$$\mathbb{P}\left(\lim_{k\to\infty} X_k = X\right) = \mathbb{P}\left(\omega \in \Omega \mid \lim_{k\to\infty} X_k(\omega) = X(\omega)\right) = 1;$$

3. converge in mean towards the random variable X if $\lim_{k\to\infty} \mathbb{E}\left(|X_k - X|\right) = 0$.

[6] see e.g. C.W. Gardiner: Handbook of stochastic methods ... [3rd ed], Springer, 2004

Some definitions

Markov process

The sequence $(X_k)_{k=1}^{\infty}$ is called a Markov process, if for all k and ℓ satisfying $k > \ell$ and any measurable function $f(x)$ with $\sup_x |f(x)| < \infty$,

$$\mathbb{E}\left(f(X_k) \mid X_1, \ldots, X_\ell\right) = \mathbb{E}\left(f(X_k) \mid X_\ell\right).$$

Martingales

The sequence $(X_k)_{k=1}^{\infty}$ is called respectively a *supermartingale*, a *submartingale* or a martingale, if $\mathbb{E}\left(|X_k|\right) < \infty$ for $k = 1, 2, \cdots$, and

$$\mathbb{E}\left(X_k \mid X_1, \ldots, X_\ell\right) \le X_\ell \qquad (\mathbb{P} \text{ almost surely}), \qquad k \ge \ell$$

or respectively

$$\mathbb{E}\left(X_k \mid X_1, \ldots, X_\ell\right) \ge X_\ell \qquad (\mathbb{P} \text{ almost surely}), \qquad k \ge \ell,$$

or finally,

$$\mathbb{E}\left(X_k \mid X_1, \ldots, X_\ell\right) = X_\ell \qquad (\mathbb{P} \text{ almost surely}), \qquad k \ge \ell.$$

Martingales asymptotic behavior

Stochastic version of Lasalle invariance principle for Lyapunov function of deterministic dynamics.

H.J. Kushner invariance Theorem

Let $\{X_k\}$ be a Markov chain on the compact state space S. Suppose that there exists a non-negative function $V(x)$ satisfying $\mathbb{E}\left(V(X_{k+1}) \mid X_k = x\right) - V(x) = -\sigma(x)$, where $\sigma(x) \geq 0$ is a positive continuous function of x. Then the ω-limit set (in the sense of almost sure convergence) of X_k is included in the following set

$$I = \{X \mid \sigma(X) = 0\}.$$

Trivially, the same result holds true for $V(x)$ bounded from above and $\mathbb{E}\left(V(X_{k+1}) \mid X_k = x\right) - V(x) = \sigma(x)$ with $\sigma(x) \geq 0$.

Asymptotic behavior: QND photon measurement

Theorem

Consider $\boldsymbol{M}_g = \cos(\varphi_0 + \boldsymbol{N}\vartheta)$ and $\boldsymbol{M}_e = \sin(\varphi_0 + \boldsymbol{N}\vartheta)$

$$\rho_{k+1} = \begin{cases} \frac{\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger\right)} & \text{with } y_k = g \text{, probability } \mathbb{P}_g = \mathrm{Tr}\left(\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger\right); \\ \frac{\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger\right)} & \text{with } y_k = e \text{, probability } \mathbb{P}_e = \mathrm{Tr}\left(\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger\right), \end{cases}$$

with an initial density matrix ρ_0 defined on the subspace $\mathrm{span}\{|n\rangle \mid n = 0, 1, \cdots, n^{\max}\}$. Also, assume the non-degeneracy $\cos^2(\varphi_m) \neq \cos^2(\varphi_n)$ $\forall n \neq m \in \{0, 1, \cdots, n^{\max}\}$, where $\varphi_n = \varphi_0 + n\vartheta$. Then

- for any $n \in \{0, \ldots, n^{\max}\}$, $\mathrm{Tr}\left(\rho_k |n\rangle\langle n|\right) = \langle n|\rho_k|n\rangle$ is a martingale
- ρ_k converges with proba. 1 to one of the $n^{\max} + 1$ Fock states $|n\rangle\langle n|$ with $n \in \{0, \ldots, n^{\max}\}$.
- the probability to converge towards the Fock state $|n\rangle\langle n|$ is given by $\mathrm{Tr}\left(\rho_0 |n\rangle\langle n|\right) = \langle n|\rho_0|n\rangle$.

Proof based on QND super-martingales

- For any function f, $V_f(\rho) = \text{Tr}\left(f(\boldsymbol{N})\rho\right)$ is a martingale: $\mathbb{E}\left(V_f(\rho_{k+1}) \mid \rho_k\right) = V_f(\rho_k)$ (basic computation).
- $V(\rho) = \sum_{n \neq m} \sqrt{\langle n|\rho|n\rangle \langle m|\rho|m\rangle} \geq 0$ is a strict super-martingale:

$$\mathbb{E}\left(V(\rho_{k+1}) \mid \rho_k\right) = \sum_{n \neq m} \left(|\cos\phi_n \cos\phi_m| + |\sin\phi_n \sin\phi_m|\right) \sqrt{\langle n|\rho|n\rangle \langle m|\rho|m\rangle} \leq rV(\rho_k)$$

  with $r = \max_{n \neq m} \left(|\cos\phi_n \cos\phi_m| + |\sin\phi_n \sin\phi_m|\right) < 1.$
- $V(\rho) = 0$ implies that there exists n such that $\rho = |n\rangle\langle n|$.

Interpretation: For large k, $V(\rho_k)$ is very close to 0, thus ρ_k very close to $|n\rangle\langle n|$ for an a priori random n. Information extracted by measurement makes state "less uncertain" *a posteriori* but not more predictable *a priori*.

Asymptotic behavior: discarding measurement results

Theorem

Consider $\rho_{k+1} = \boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger$ with the same definitions and assumptions as in the previous theorem.
Then $\boldsymbol{\rho}_k$ converges exponentially towards $\boldsymbol{\rho} = \text{diag}(\rho_0)$.

Proof: Deterministic system, one easily checks that

- $\langle n|\rho_{k+1}|n\rangle = \langle n|\rho_k|n\rangle$
- $\langle n|\rho_{k+1}|m\rangle = \left(|\cos\phi_n \cos\phi_m| + |\sin\phi_n \sin\phi_m|\right) \langle n|\rho_k|m\rangle \leq$
  $r\ \langle n|\rho_k|m\rangle$ with
  $r = \max_{n\neq m}\left(|\cos\phi_n \cos\phi_m| + |\sin\phi_n \sin\phi_m|\right)\ <\ 1.$

Interpretation: Diagonal ρ is equivalent to a classical probability distribution over the values of n. This distribution is not modified in absence of measurement results.
However, the QND measurement process for $\boldsymbol{N}$, even without recording the output, perturbs future measurements of other observables (off-diagonal terms in the $\boldsymbol{N}$ eigenbasis).

Exercice

Consider the Markov chain $\rho_{k+1} = \mathbb{K}_{y_k}(\rho_k) \,/\, \mathbb{P}_{y,k}$ where $y_k = g$ (resp. $y_k = e$) with probability $\mathbb{P}_{g,k} = \text{Tr}\left(\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger\right)$ (resp. $p_{e,k} = \text{Tr}\left(\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger\right)$). The Kraus operators are now given by (resonant interaction)

$$\boldsymbol{M}_g = \cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right) - \sin\left(\frac{\theta_1}{2}\right)\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)\boldsymbol{a}^\dagger$$

$$\boldsymbol{M}_e = -\sin\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}+1}\right) - \cos\left(\frac{\theta_1}{2}\right)\boldsymbol{a}\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)$$

with $\theta_1 = 0$. Assume the initial state to be defined on the subspace $\{|n\rangle\}_{n=0}^{n^{\max}}$ and that the cavity state at step k is described by the density operator ρ_k.

1. Show that

$$\mathbb{E}\left(\text{Tr}\left(\boldsymbol{N}\rho_{k+1}\right) \mid \rho_k\right) = \text{Tr}\left(\boldsymbol{N}\rho_k\right) - \text{Tr}\left(\sin^2\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)\rho_k\right).$$

2. Assume that for any integer n, $\Theta\sqrt{n}/\pi$ is irrational. Then prove that almost surely ρ_k tends to the vacuum state $|0\rangle\langle 0|$ whatever its initial condition.
3. When $\Theta\sqrt{n}/\pi$ is rational for some integer n, describe the possible ω-limit sets for ρ_k.

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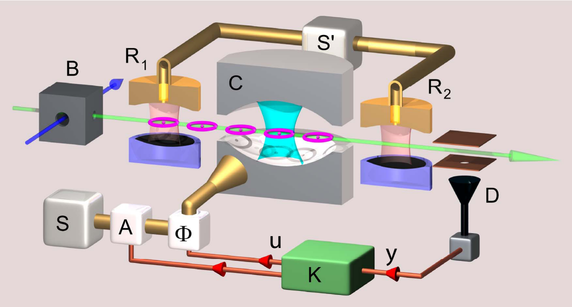
Quantum feedback

Question: how to stabilize deterministically a given photon-number state $|\bar{n}\rangle\langle\bar{n}|$?



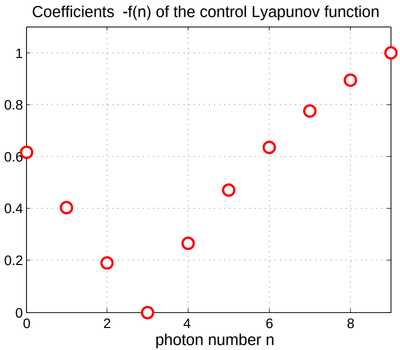
Controlled Markov chain:

$$\rho_{k+\frac{1}{2}} = \mathbb{M}_{y_k}(\rho_k), \qquad \rho_{k+1} = \mathbb{D}_{u_k}(\rho_{k+\frac{1}{2}}),$$

where $\mathbb{M}_y(\rho) = \boldsymbol{M}_y \rho \boldsymbol{M}_y^\dagger / \operatorname{Tr}\left(\boldsymbol{M}_y \rho \boldsymbol{M}_y^\dagger\right)$ and $\mathbb{D}_u(\rho) = \boldsymbol{D}_u \rho \boldsymbol{D}_u^\dagger$ with $\boldsymbol{D}_u = e^{u\boldsymbol{a}^\dagger - u^* \boldsymbol{a}}$, the displacement unitary operator of complex amplitude u.

Control Lyapunov function

Idea: $$\overline{V}(\rho) = V(\rho) + \sum_{n\geq 0} f(n)\,\mathrm{Tr}\left(\rho|n\rangle\langle n|\right),$$



Bounded quantum-state stabilizing feedback: take

$$u_k := \underset{|u|\leq u_{\max}}{\mathrm{argmin}}\left\{\mathbb{E}\left(\overline{V}(\rho_{k+1})|\rho_k, u_k = u\right)\right\}$$
$$= \underset{|u|\leq u_{\max}}{\mathrm{argmin}}\left\{\mathrm{Tr}\left(\boldsymbol{M}_g\boldsymbol{\rho}_k\boldsymbol{M}_g\right)\overline{V}\left(\mathbb{D}_u\left(\mathbb{M}_g(\boldsymbol{\rho}_k)\right)\right) + \mathrm{Tr}\left(\boldsymbol{M}_e\boldsymbol{\rho}_k\boldsymbol{M}_e\right)\overline{V}\left(\mathbb{D}_u\left(\mathbb{M}_e(\boldsymbol{\rho}_k)\right)\right)\right\}.$$

Quantum-state feedback (stabilization around 3-photon state)

Experiment: C. Sayrin et. al., Nature 477, 73-77, 2011.
Theory: I. Dotsenko et al. Physical Review A, 80: 013805-013813, 2009.
H. Amini et. al., Automatica, 49 (9): 2683-2692, 2013.