Quantum Systems: Dynamics and Control[1]

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February 25, 2020

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Outline

1. Photon Box: a key example of indirect measurement
2. State evolution under measurement imperfections
3. Decoherence seen as unread measurements

Models of open quantum systems are based on three features[5]

1. Schrödinger: wave funct. $|\psi\rangle \in \mathcal{H}$ or density op. $\boldsymbol{\rho} \sim |\psi\rangle\langle\psi|$

$$\frac{d}{dt}|\psi\rangle = -\tfrac{i}{\hbar}\boldsymbol{H}|\psi\rangle, \quad \frac{d}{dt}\boldsymbol{\rho} = -\tfrac{i}{\hbar}[\boldsymbol{H}, \boldsymbol{\rho}], \quad \boldsymbol{H} = \boldsymbol{H}_0 + u\boldsymbol{H}_1$$

2. Entanglement and tensor product for composite systems (S, M):
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $\boldsymbol{H} = \boldsymbol{H}_S \otimes \boldsymbol{I}_M + \boldsymbol{H}_{int} + \boldsymbol{I}_S \otimes \boldsymbol{H}_M$
   - observable on sub-system M only: $\boldsymbol{O} = \boldsymbol{I}_S \otimes \boldsymbol{O}_M$.
3. Randomness and irreversibility induced by the measurement of observable $\boldsymbol{O}$ with spectral decomp. $\sum_\mu \lambda_\mu \boldsymbol{P}_\mu$:
   - measurement outcome μ with proba. $\mathbb{P}_\mu = \langle\psi|\boldsymbol{P}_\mu|\psi\rangle = \text{Tr}\left(\boldsymbol{\rho}\boldsymbol{P}_\mu\right)$ depending on $|\psi\rangle$, $\boldsymbol{\rho}$ just before the measurement
   - measurement back-action if outcome $\mu = y$:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\boldsymbol{P}_y|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{P}_y|\psi\rangle}}, \quad \boldsymbol{\rho} \mapsto \boldsymbol{\rho}_+ = \frac{\boldsymbol{P}_y\boldsymbol{\rho}\boldsymbol{P}_y}{\text{Tr}\left(\boldsymbol{\rho}\boldsymbol{P}_y\right)}$$

[5] S. Haroche, J.M. Raimond: Exploring the Quantum: Atoms, Cavities and Photons. Oxford University Press, 2006.

Outline

Composite system built with a harmonic oscillator and a qubit.

- System S corresponds to a quantized harmonic oscillator:

$$\mathcal{H}_S = \mathcal{H}_c = \left\{ \sum_{n=0}^{\infty} c_n |n\rangle \;\middle|\; (c_n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

  where $|n\rangle$ represents the Fock state associated to exactly n photons inside the cavity

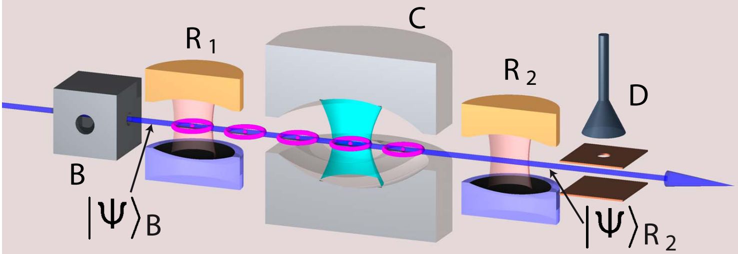
- Meter M is a qubit, a 2-level system: $\mathcal{H}_M = \mathcal{H}_a = \mathbb{C}^2$, each atom admits two energy levels and is described by a wave function $c_g|g\rangle + c_e|e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$;

- **State of the full system $|\Psi\rangle \in \mathcal{H}_S \otimes \mathcal{H}_M = \mathcal{H}_c \otimes \mathcal{H}_a$:**

$$|\Psi\rangle = \sum_{n=0}^{+\infty} c_{ng}|n\rangle \otimes |g\rangle + c_{ne}|n\rangle \otimes |e\rangle, \quad c_{ne}, c_{ng} \in \mathbb{C}.$$

  Orthonormal basis: $(|n\rangle \otimes |g\rangle, |n\rangle \otimes |e\rangle)_{n \in \mathbb{N}}$.

Markov model (1)



- When atom exits B, $|\Psi\rangle_B$ of the full system is separable $|\Psi\rangle_B = |\psi\rangle \otimes |g\rangle$.
- Just before the measurement in D, the state is in general entangled (not separable):

$$|\Psi\rangle_{R_2} = \boldsymbol{U}_{SM}\big(|\psi\rangle \otimes |g\rangle\big) = \big(\boldsymbol{M}_g|\psi\rangle\big) \otimes |g\rangle + \big(\boldsymbol{M}_e|\psi\rangle\big) \otimes |e\rangle$$

where $\boldsymbol{U}_{SM}$ is a unitary transformation (Schrödinger propagator) defining the linear measurement operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ on $\mathcal{H}_S$. Since $\boldsymbol{U}_{SM}$ is unitary, $\boldsymbol{M}_g^\dagger \boldsymbol{M}_g + \boldsymbol{M}_e^\dagger \boldsymbol{M}_e = \boldsymbol{I}$.

Markov model (2)

Just before D, the field/atom state is **entangled**:

$$\boldsymbol{M}_g|\psi\rangle \otimes |g\rangle + \boldsymbol{M}_e|\psi\rangle \otimes |e\rangle$$

Denote by $\mu \in \{g, e\}$ the measurement outcome in detector D: with probability $\mathbb{P}_\mu = \left\langle \psi|\boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu|\psi \right\rangle$ we get μ. Just after the measurement outcome $\mu = y$, the state becomes separable:

$$|\Psi\rangle_D = \frac{1}{\sqrt{\mathbb{P}_y}} \left(\boldsymbol{M}_y|\psi\rangle\right) \otimes |y\rangle = \left(\frac{\boldsymbol{M}_y}{\sqrt{\left\langle \psi|\boldsymbol{M}_y^\dagger \boldsymbol{M}_y|\psi \right\rangle}} |\psi\rangle \right) \otimes |y\rangle.$$

Markov process: $|\psi_k\rangle \equiv |\psi\rangle_{t=k\Delta t}$, $k \in \mathbb{N}$, Δt sampling period,

$$|\psi_{k+1}\rangle = \begin{cases} \frac{\boldsymbol{M}_g|\psi_k\rangle}{\sqrt{\left\langle \psi_k|\boldsymbol{M}_g^\dagger \boldsymbol{M}_g|\psi_k \right\rangle}} & \text{with } y_k = g, \text{ probability } \mathbb{P}_g = \left\langle \psi_k|\boldsymbol{M}_g^\dagger \boldsymbol{M}_g|\psi_k \right\rangle; \\ \frac{\boldsymbol{M}_e|\psi_k\rangle}{\sqrt{\left\langle \psi_k|\boldsymbol{M}_e^\dagger \boldsymbol{M}_e|\psi_k \right\rangle}} & \text{with } y_k = e, \text{ probability } \mathbb{P}_e = \left\langle \psi_k|\boldsymbol{M}_e^\dagger \boldsymbol{M}_e|\psi_k \right\rangle. \end{cases}$$

Dispersive case

$$\boldsymbol{U}_{R_1} = \frac{1}{\sqrt{2}} \left(\boldsymbol{I} + |g\rangle\langle e| - |e\rangle\langle g| \right)$$
$$\boldsymbol{U}_{R_2} = \frac{1}{\sqrt{2}} \left(\boldsymbol{I} + e^{i\eta}|g\rangle\langle e| - e^{-i\eta}|e\rangle\langle g| \right)$$
$$\boldsymbol{U}_C = |g\rangle\langle g| e^{-i\phi(\boldsymbol{N})} + |e\rangle\langle e| e^{i\phi(\boldsymbol{N}+\boldsymbol{I})}$$

where $\phi(\boldsymbol{N}) = \vartheta_0 + \vartheta \boldsymbol{N}$.
With $\eta = 2(\varphi_0 - \vartheta_0) - \vartheta - \pi$, the measurement operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ are the following bounded operators:

$$\boldsymbol{M}_g = \cos(\varphi_0 + \boldsymbol{N}\vartheta), \quad \boldsymbol{M}_e = \sin(\varphi_0 + \boldsymbol{N}\vartheta)$$

up to irrelevant global phases.
Exercise: Show that $\boldsymbol{M}_g^\dagger \boldsymbol{M}_g + \boldsymbol{M}_e^\dagger \boldsymbol{M}_e = \boldsymbol{I}$.

Resonant case: $\boldsymbol{U}_{SM} = \boldsymbol{U}_{R_2}\boldsymbol{U}_C\boldsymbol{U}_{R_1}$

$$\boldsymbol{U}_{R_1} = e^{-i\frac{\theta_1}{2}\boldsymbol{\sigma_y}} = \cos\left(\tfrac{\theta_1}{2}\right) + \sin\left(\tfrac{\theta_1}{2}\right)\left(|g\rangle\langle e| - |e\rangle\langle g|\right) \quad \text{and} \quad \boldsymbol{U}_{R_2} = \boldsymbol{I}$$

and

$$\boldsymbol{U}_C = |g\rangle\langle g|\cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}}\right) + |e\rangle\langle e|\cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}+\boldsymbol{I}}\right) + |g\rangle\langle e|\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)\boldsymbol{a}^\dagger - |e\rangle\langle g|\boldsymbol{a}\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)$$

The measurement operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ are the following bounded operators:

$$\boldsymbol{M}_g = \cos\left(\tfrac{\theta_1}{2}\right)\cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}}\right) - \sin\left(\tfrac{\theta_1}{2}\right)\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)\boldsymbol{a}^\dagger$$

$$\boldsymbol{M}_e = -\sin\left(\tfrac{\theta_1}{2}\right)\cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}+1}\right) - \cos\left(\tfrac{\theta_1}{2}\right)\boldsymbol{a}\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)$$

Exercise: Show that $\boldsymbol{M}_g^\dagger\boldsymbol{M}_g + \boldsymbol{M}_e^\dagger\boldsymbol{M}_e = \boldsymbol{I}$.

Outline

Updating ρ with detection inefficiency (1)

- Thus starting with $\boldsymbol{\rho} = |\psi\rangle\langle\psi|$, we have:

$$\boldsymbol{\rho}_{+,\mu} = |\psi_{+,\mu}\rangle\langle\psi_{+,\mu}| = \frac{1}{\mathrm{Tr}\left(\boldsymbol{M}_\mu \boldsymbol{\rho} \boldsymbol{M}_\mu^\dagger\right)} \boldsymbol{M}_\mu \boldsymbol{\rho} \boldsymbol{M}_\mu^\dagger$$

  when the atom collapses in $\mu \in \{g, e\}$ with proba. $\mathrm{Tr}\left(\boldsymbol{M}_\mu \boldsymbol{\rho} \boldsymbol{M}_\mu^\dagger\right)$.

- Two consecutive measurements with results μ_1 then μ_2:

$$\boldsymbol{\rho}_{+,\mu_1\mu_2} = \frac{\boldsymbol{M}_{\mu_2}\left(\frac{\boldsymbol{M}_{\mu_1}\boldsymbol{\rho}\boldsymbol{M}_{\mu_1}^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_{\mu_1}\boldsymbol{\rho}\boldsymbol{M}_{\mu_1}^\dagger\right)}\right)\boldsymbol{M}_{\mu_2}^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_{\mu_2}\left(\frac{\boldsymbol{M}_{\mu_1}\boldsymbol{\rho}\boldsymbol{M}_{\mu_1}^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_{\mu_1}\boldsymbol{\rho}\boldsymbol{M}_{\mu_1}^\dagger\right)}\right)\boldsymbol{M}_{\mu_2}^\dagger\right)} = \frac{\boldsymbol{M}_{\mu_2}\boldsymbol{M}_{\mu_1}\boldsymbol{\rho}\boldsymbol{M}_{\mu_1}^\dagger\boldsymbol{M}_{\mu_2}^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_{\mu_2}\boldsymbol{M}_{\mu_1}\boldsymbol{\rho}\boldsymbol{M}_{\mu_1}^\dagger\boldsymbol{M}_{\mu_2}^\dagger\right)}$$

  with proba.

$$\mathbb{P}_{(\mu_1,\mu_2|\boldsymbol{\rho})} = \mathbb{P}_{(\mu_1|\boldsymbol{\rho})}\,\mathbb{P}_{(\mu_2|\mu_1,\boldsymbol{\rho})} = \mathrm{Tr}\left(\boldsymbol{M}_{\mu_2}\boldsymbol{M}_{\mu_1}\boldsymbol{\rho}\boldsymbol{M}_{\mu_1}^\dagger\boldsymbol{M}_{\mu_2}^\dagger\right)$$

What can we say for μ_2 when μ_1 is unknown?

Updating ρ with detection inefficiency (2)

- Distribution of the second measurement output:

$$\mathbb{P}_{(\mu_2|\boldsymbol{\rho})} = \sum_{\mu_1} \mathrm{Tr}\left(\boldsymbol{M}_{\mu_2}\boldsymbol{M}_{\mu_1}\boldsymbol{\rho}\boldsymbol{M}_{\mu_1}^\dagger\boldsymbol{M}_{\mu_2}^\dagger\right) = \mathrm{Tr}\left(\boldsymbol{M}_{\mu_2}\boldsymbol{\rho}_1\boldsymbol{M}_{\mu_2}^\dagger\right)$$

  with the linear Kraus map

$$\boldsymbol{\rho}_1 = \sum_{\mu_1} \boldsymbol{M}_{\mu_1}\boldsymbol{\rho}\boldsymbol{M}_{\mu_1}^\dagger = \boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + \boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger = \mathbb{K}(\boldsymbol{\rho}) = \sum_{\mu_1} \boldsymbol{\rho}_{+,\mu_1}\,\mathbb{P}_{\mu_1|\boldsymbol{\rho}}$$

- Iterating this argument, the distribution of further measurement outputs, knowing μ_2 but not μ_1, is given by just replacing

$$\boldsymbol{\rho}_{+,\mu_1\mu_2} = \frac{\boldsymbol{M}_{\mu_2}\boldsymbol{\rho}_{+,\mu_1}\boldsymbol{M}_{\mu_2}^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_{\mu_2}\boldsymbol{\rho}_{+,\mu_1}\boldsymbol{M}_{\mu_2}^\dagger\right)} \quad \text{by} \quad \boldsymbol{\rho}_{+,\mu_2} = \frac{\boldsymbol{M}_{\mu_2}\mathbb{K}(\boldsymbol{\rho})\boldsymbol{M}_{\mu_2}^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_{\mu_2}\mathbb{K}(\boldsymbol{\rho})\boldsymbol{M}_{\mu_2}^\dagger\right)}.$$

  i.e. in fact just replacing $\boldsymbol{\rho}_{+,\mu_1}$ by $\rho_1 = \mathbb{K}(\boldsymbol{\rho})$.

"True" value of μ_1 is inaccessible through any future measurement.

Updating ρ with detection errors (1)

- Two consecutive measurements with results μ_1 then μ_2:

$$\boldsymbol{\rho}_{+,\mu_1\mu_2} = \frac{\boldsymbol{M}_{\mu_2}\left(\frac{\boldsymbol{M}_{\mu_1}\boldsymbol{\rho}\boldsymbol{M}_{\mu_1}^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_{\mu_1}\boldsymbol{\rho}\boldsymbol{M}_{\mu_1}^\dagger\right)}\right)\boldsymbol{M}_{\mu_2}^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_{\mu_2}\left(\frac{\boldsymbol{M}_{\mu_1}\boldsymbol{\rho}\boldsymbol{M}_{\mu_1}^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_{\mu_1}\boldsymbol{\rho}\boldsymbol{M}_{\mu_1}^\dagger\right)}\right)\boldsymbol{M}_{\mu_2}^\dagger\right)} = \frac{\boldsymbol{M}_{\mu_2}\boldsymbol{M}_{\mu_1}\boldsymbol{\rho}\boldsymbol{M}_{\mu_1}^\dagger\boldsymbol{M}_{\mu_2}^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_{\mu_2}\boldsymbol{M}_{\mu_1}\boldsymbol{\rho}\boldsymbol{M}_{\mu_1}^\dagger\boldsymbol{M}_{\mu_2}^\dagger\right)}$$

  with proba.

$$\mathbb{P}_{(\mu_1,\mu_2|\boldsymbol{\rho})} = \mathbb{P}_{(\mu_1|\boldsymbol{\rho})}\,\mathbb{P}_{(\mu_2|\mu_1,\boldsymbol{\rho})} = \mathrm{Tr}\left(\boldsymbol{M}_{\mu_2}\boldsymbol{M}_{\mu_1}\boldsymbol{\rho}\boldsymbol{M}_{\mu_1}^\dagger\boldsymbol{M}_{\mu_2}^\dagger\right)$$

- Detection errors on first measurement:
  $\mathbb{P}(y_1 = e/\mu_1 = g) = \eta_{e,g} \in [0,1]$ the probability of erroneous assignation to e when the atom collapses in g, and similarly η_{y_1,μ_1} for other values of y_1 and μ_1 (given by the contrast of the Ramsey fringes).

What can we say for μ_2 when y_1 is known but μ_1 is unknown?

Updating ρ with detection inefficiency (2)

- Distribution of the second measurement output:[6]

$$\begin{aligned}\mathbb{P}_{(\mu_2|\boldsymbol{\rho},y_1)} &= \sum_{\mu_1} \mathbb{P}_{(\mu_1,\mu_2|\boldsymbol{\rho},y_1)} = \sum_{\mu_1} \frac{\mathbb{P}_{(y_1|\mu_1,\mu_2,\boldsymbol{\rho})}\mathbb{P}_{(\mu_1,\mu_2|\boldsymbol{\rho})}}{\mathbb{P}_{y_1|\boldsymbol{\rho}}} \\ &= \frac{\sum_{\mu_1} \eta_{y_1,\mu_1} \operatorname{Tr}\left(\boldsymbol{M}_{\mu_2}\boldsymbol{M}_{\mu_1}\boldsymbol{\rho}\boldsymbol{M}_{\mu_1}^\dagger\boldsymbol{M}_{\mu_2}^\dagger\right)}{\mathbb{P}_{y_1|\boldsymbol{\rho}}} = \operatorname{Tr}\left(\boldsymbol{M}_{\mu_2}\,\rho_{+,y_1}\,\boldsymbol{M}_{\mu_2}^\dagger\right)\end{aligned}$$

where $\mathbb{P}_{y_1|\boldsymbol{\rho}} = \operatorname{Tr}\left(\sum_{\mu_1} \eta_{y_1,\mu_1}\boldsymbol{M}_{\mu_1}\boldsymbol{\rho}\boldsymbol{M}_{\mu_1}^\dagger\right)$ and we define

$$\boldsymbol{\rho}_{+,y_1} = \frac{\sum_{\mu_1} \eta_{y_1,\mu_1}\boldsymbol{M}_{\mu_1}\boldsymbol{\rho}\boldsymbol{M}_{\mu_1}^\dagger}{\mathbb{P}_{y_1|\boldsymbol{\rho}}} .$$

- Repeating such arguments, the distribution of all future measurement outputs is obtained by just

replacing $\boldsymbol{\rho}_{+,\mu_1}$ by $\boldsymbol{\rho}_{+,y_1}$

[6] Use the Bayes law $\mathbb{P}(A|B,C) = P(B|A,C)P(A|C)\,/\,P(B|C)$ with $A = (\mu_1,\mu_2)$, $B = y_1$ and $C = \rho$. In the next line, use the Markov model $\mathbb{P}_{(y_1|\mu_1,\mu_2,\rho)} = \mathbb{P}_{y_1|\mu_1} = \eta_{y_1,\mu_1}$.

Updating ρ with detection inefficiency (3)

The "true" value of μ_1 is again inaccessible through any future measurement.

Reformulation with linear quantum maps : set

$$\mathbb{K}_g(\boldsymbol{\rho}) = \eta_{g,g}\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + \eta_{g,e}\boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger, \quad \mathbb{K}_e(\boldsymbol{\rho}) = \eta_{e,g}\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + \eta_{e,e}\boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger.$$

Then $\quad \boldsymbol{\rho}_{+,y} = \dfrac{\mathbb{K}_y(\boldsymbol{\rho})}{\mathrm{Tr}\left(\mathbb{K}_y(\boldsymbol{\rho})\right)} \quad$ when we detect $y \in e, g$.

The probability to detect y knowing $\boldsymbol{\rho}$ is $\mathbb{P}_{y|\rho} = \mathrm{Tr}\left(\mathbb{K}_y(\boldsymbol{\rho})\right)$.

When we neglect the measurement result, we logically get back

$$\rho_+ = \sum_y \boldsymbol{\rho}_{+,y}\, \mathbb{P}_{y|\rho} = \mathbb{K}_g(\boldsymbol{\rho}) + \mathbb{K}_e(\boldsymbol{\rho}) = \mathbb{K}(\boldsymbol{\rho}) = \boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + \boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger.$$

ρ plays the role of a probability measure for all future measurement outcomes, given all past observations and initial measure ρ_0.

- The pure state $\boldsymbol{\rho} = |\psi\rangle\langle\psi|$ of rank($\boldsymbol{\rho}$) =1 is a special case, implying the minimal possible uncertainty on measurements of a quantum system.

  In general, $\boldsymbol{\rho}$ becomes a mixed state (rank($\boldsymbol{\rho}$) >1), through classical uncertainties.
- the update $\boldsymbol{\rho}_+ = \mathbb{K}(\boldsymbol{\rho})$ when μ_1 is lost, represents the law of total probabilities
- the update $\boldsymbol{\rho}_{+,y}$ with detection errors represents the Bayes law on probability measures

This underlies the general models for open quantum systems.

Outline

Decoherence: the environment is like an unread meter (1)

Limit of Markovian environment: over sampling period $\Delta T \to 0$, the photon box interacts with some external system (ancilla) which was initialized in a possibly imprecise state; the ancilla state is never read, and reset / replaced after the interaction.

Example: resonant interaction with weakly excited spin.

- Before interaction: the spin ancilla has been reset to

$$\cos\tfrac{\theta_1}{2}|g\rangle \pm \sin\tfrac{\theta_1}{2}|e\rangle$$

  with $\pm$ unknown (unread meas. on ancilla before interaction).

- Resonant interaction with $\Theta \ll 1$:

$$\begin{aligned}
\boldsymbol{M}_{g,\pm} &= \cos\tfrac{\theta_1}{2}\cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}}\right) \mp \sin\tfrac{\theta_1}{2}\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)\boldsymbol{a}^\dagger \\
&\approx \cos\tfrac{\theta_1}{2}\left(1-\tfrac{\Theta^2}{8}\boldsymbol{N}\right) \mp \tfrac{\Theta}{2}\sin\tfrac{\theta_1}{2}\boldsymbol{a}^\dagger \\
\boldsymbol{M}_{e,\pm} &= \mp\sin\tfrac{\theta_1}{2}\cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}+1}\right) - \cos\tfrac{\theta_1}{2}\boldsymbol{a}\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right) \\
&\approx \mp\sin\tfrac{\theta_1}{2}\left(1-\tfrac{\Theta^2}{8}(\boldsymbol{N}+1)\right) - \tfrac{\Theta}{2}\cos\tfrac{\theta_1}{2}\boldsymbol{a}
\end{aligned}$$

Decoherence: the environment is like an unread meter (2)

- Cavity update without ever measuring the environment ancilla:

$$\begin{aligned}\rho_+ &= \tfrac{1}{2}(\boldsymbol{M}_{g,+}\rho\boldsymbol{M}_{g,+}^\dagger + \boldsymbol{M}_{e,+}\rho\boldsymbol{M}_{e,+}^\dagger) + \tfrac{1}{2}(\boldsymbol{M}_{g,-}\rho\boldsymbol{M}_{g,-}^\dagger + \boldsymbol{M}_{e,-}\rho\boldsymbol{M}_{e,-}^\dagger)\\ &\approx \boldsymbol{M}_{-1}\rho\boldsymbol{M}_{-1}^\dagger + \boldsymbol{M}_{+1}\rho\boldsymbol{M}_{+1}^\dagger + \boldsymbol{M}_0\rho\boldsymbol{M}_0^\dagger + O(\Theta^3)\end{aligned}$$

one photon annihilation during ΔT with probability $\approx \mathrm{Tr}\left(\boldsymbol{M}_{-1}\rho\boldsymbol{M}_{-1}^\dagger\right)$ and corresponding state update (backaction),

$$\boldsymbol{M}_{-1} = \tfrac{\Theta}{2}\cos\tfrac{\theta_1}{2}\,\boldsymbol{a}$$

one photon creation during ΔT with probability $\approx \mathrm{Tr}\left(\boldsymbol{M}_{+1}\rho\boldsymbol{M}_{+1}^\dagger\right)$ and backaction,

$$\boldsymbol{M}_{+1} = \tfrac{\Theta}{2}\sin\tfrac{\theta_1}{2}\,\boldsymbol{a}^\dagger$$

zero photon annihilation during ΔT with probability $\approx \mathrm{Tr}\left(\boldsymbol{M}_0\rho\boldsymbol{M}_0^\dagger\right)$ and backaction,

$$\boldsymbol{M}_0 = \boldsymbol{I} - \tfrac{1}{2}(\boldsymbol{M}_{-1}^\dagger\boldsymbol{M}_{-1} + \boldsymbol{M}_{+1}^\dagger\boldsymbol{M}_{+1})$$

A standard decoherence channel: cavity decay

This result is a general model for cavity decoherence, exact in the limit $\Delta T \to 0$:

$$\rho_+ \approx \; \boldsymbol{M}_{-1}\rho\boldsymbol{M}_{-1}^\dagger + \boldsymbol{M}_{+1}\rho\boldsymbol{M}_{+1}^\dagger + \boldsymbol{M}_0\rho\boldsymbol{M}_0^\dagger$$

$$\text{with } \boldsymbol{M}_{-1} = \sqrt{\tfrac{\Delta T(1+n_{th})}{T_{cav}}}\,\boldsymbol{a}\,,$$

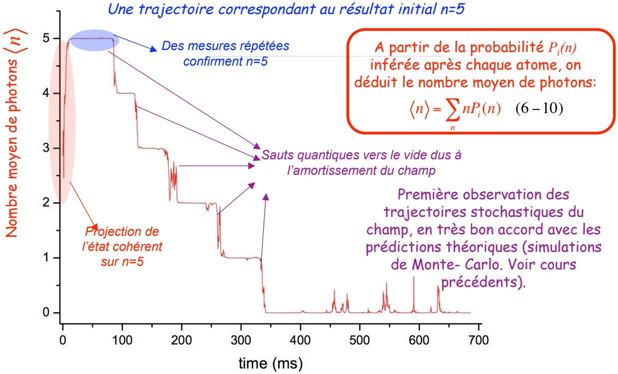
$$\boldsymbol{M}_{+1} = \sqrt{\tfrac{\Delta T\, n_{th}}{T_{cav}}}\,\boldsymbol{a}^\dagger\,,$$

$$\boldsymbol{M}_0 = \boldsymbol{I} - \tfrac{1}{2}(\boldsymbol{M}_{-1}^\dagger \boldsymbol{M}_{-1} + \boldsymbol{M}_{+1}^\dagger \boldsymbol{M}_{+1})$$

- n_{th} the average photons in the cavity in steady state (thermal photons, vanishes with the environment temperature);
- T_{cav} the expected lifetime of a single photon when $n_{th} = 0$;
- $\Delta T \ll T_{cav}$ sampling period e.g. between consecutive atoms

($n_{th} \approx 0.05$, $T_{cav} = 100$ ms and $\Delta T \approx 100$ μs for the LKB photon Box)

Valeur moyenne du nombre de photons le long d'une longue séquence de mesure: observation d'une trajectoire stochastique



See the quantum Monte Carlo simulations of the Matlab script: `RealisticModelPhotonBox.m`.

[7] From Serge Haroche, Collège de France, notes de cours 2007/2008.