Quantum Systems: Dynamics and Control[1]

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The Rotating Wave Approximation (RWA) recipes

Schrödinger dynamics $i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}(t)|\psi\rangle$, with

$$\boldsymbol{H}(t) = \boldsymbol{H}_0 + \sum_{k=1}^{m} u_k(t)\boldsymbol{H}_k, \qquad u_k(t) = \sum_{j=1}^{r} \boldsymbol{u}_{k,j} e^{i\omega_j t} + \boldsymbol{u}_{k,j}^{*} e^{-i\omega_j t}.$$

The Hamiltonian in interaction frame

$$\boldsymbol{H}_{\text{int}}(t) = \sum_{k,j} \left(\boldsymbol{u}_{k,j} e^{i\omega_j t} + \boldsymbol{u}_{k,j}^{*} e^{-i\omega_j t}\right) e^{i\boldsymbol{H}_0 t} \boldsymbol{H}_k e^{-i\boldsymbol{H}_0 t}$$

We define the first order Hamiltonian

$$\boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}} = \overline{\boldsymbol{H}_{\text{int}}} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \boldsymbol{H}_{\text{int}}(t)dt,$$

and the second order Hamiltonian

$$\boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}} = \boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}} - i\overline{\left(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}\right)\left(\int_t \left(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}\right)\right)}$$

Choose the amplitudes $\boldsymbol{u}_{k,j}$ and the frequencies ω_j such that the propagators of $\boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}}$ or $\boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}}$ admit simple explicit forms that are used to find $t \mapsto u(t)$ steering $|\psi\rangle$ from one location to another one.

Outline

1 Averaging and control of a qubit

2 Averaging and control of spin/spring systems
- The spin/spring model
- Resonant interaction
- Dispersive interaction

3 Resonant control: Law-Eberly method

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RWA and resonant control

In $i\frac{d}{dt}|\psi\rangle = \left(\frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \frac{u}{2}\boldsymbol{\sigma_x}\right)|\psi\rangle$, set $H_0 = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z}$ and $\epsilon H_1 = \frac{u}{2}\boldsymbol{\sigma_x}$ and consider $|\psi\rangle = e^{-\frac{i\omega_{eg}t}{2}\boldsymbol{\sigma_z}}|\phi\rangle$ to eliminate the drift H_0 and to get the Hamiltonian in the interaction frame:

$$i\frac{d}{dt}|\phi\rangle = \frac{u}{2}e^{\frac{i\omega_{eg}t}{2}\boldsymbol{\sigma_z}}\boldsymbol{\sigma_x}e^{-\frac{i\omega_{eg}t}{2}\boldsymbol{\sigma_z}}|\phi\rangle = \boldsymbol{H}_{\rm int}|\phi\rangle$$

with $\boldsymbol{H}_{\rm int} = \frac{u}{2}e^{i\omega_{eg}t}\overbrace{\frac{\boldsymbol{\sigma_x}+i\boldsymbol{\sigma_y}}{2}}^{\boldsymbol{\sigma_+}=|e\rangle\langle g|} + \frac{u}{2}e^{-i\omega_{eg}t}\overbrace{\frac{\boldsymbol{\sigma_x}-i\boldsymbol{\sigma_y}}{2}}^{\boldsymbol{\sigma_-}=|g\rangle\langle e|}$

Applying the resonant control $u = \boldsymbol{u}e^{i\omega_{eg}t} + \boldsymbol{u}^*e^{-i\omega_{eg}t}$ gives

$$H_{int} = \left(\frac{\boldsymbol{u}e^{2i\omega_{eg}t} + \boldsymbol{u}^*}{2}\right)\boldsymbol{\sigma_+} + \left(\frac{\boldsymbol{u} + \boldsymbol{u}^*e^{-2i\omega_{eg}t}}{2}\right)\boldsymbol{\sigma_-}.$$

When $|\boldsymbol{u}| \ll \omega_{eg}$ and $\left|\frac{d}{dt}\boldsymbol{u}\right| \ll |\boldsymbol{u}|\left|\frac{d}{dt}e^{2i\omega_{eg}t}\right|$, the variable $|\phi\rangle$ moves with a timescale of order $1/|\boldsymbol{u}|$ while H_{int} involves terms at a fast timescale $1/\omega_{eg}$.

Averaging tells us how we can average out this fast timescale and concentrate on the effect of slowly varying $\boldsymbol{u}$.

Second order approximation and Bloch-Siegert shift

The decomposition of $\boldsymbol{H}_{\text{int}}$,

$$\boldsymbol{H}_{\text{int}} = \underbrace{\frac{\boldsymbol{u}^*}{2}\boldsymbol{\sigma}_{\boldsymbol{+}} + \frac{\boldsymbol{u}}{2}\boldsymbol{\sigma}_{\boldsymbol{-}}}_{\overline{\boldsymbol{H}_{\text{int}}}} + \underbrace{\frac{\boldsymbol{u}e^{2i\omega_{\text{eg}}t}}{2}\boldsymbol{\sigma}_{\boldsymbol{+}} + \frac{\boldsymbol{u}^*e^{-2i\omega_{\text{eg}}t}}{2}\boldsymbol{\sigma}_{\boldsymbol{-}}}_{\boldsymbol{H}_{\text{int}}-\overline{\boldsymbol{H}_{\text{int}}}},$$

provides the first order approximation (RWA)

$\boldsymbol{H}^{1^{\text{st}}}_{\text{rwa}} = \overline{\boldsymbol{H}_{\text{int}}} = \lim_{T\to\infty}\frac{1}{T}\int_0^T \boldsymbol{H}_{\text{int}}(t)dt, = \frac{\boldsymbol{u}^*\boldsymbol{\sigma}_{\boldsymbol{+}}+\boldsymbol{u}\boldsymbol{\sigma}_{\boldsymbol{-}}}{2}$.

Since $\int_t \boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}} = \frac{\boldsymbol{u}e^{2i\omega_{\text{eg}}t}}{4i\omega_{\text{eg}}}\boldsymbol{\sigma}_{\boldsymbol{+}} - \frac{\boldsymbol{u}^*e^{-2i\omega_{\text{eg}}t}}{4i\omega_{\text{eg}}}\boldsymbol{\sigma}_{\boldsymbol{-}}$, we have

$$\overline{\left(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}\right)\left(\int_t (\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}})\right)} = -\frac{|\boldsymbol{u}|^2}{8i\omega_{\text{eg}}}\boldsymbol{\sigma}_{\boldsymbol{z}}$$

(use $\boldsymbol{\sigma}_{\boldsymbol{+}}{}^2 = \boldsymbol{\sigma}_{\boldsymbol{-}}{}^2 = 0$ and $\boldsymbol{\sigma}_{\boldsymbol{z}} = \boldsymbol{\sigma}_{\boldsymbol{+}}\boldsymbol{\sigma}_{\boldsymbol{-}} - \boldsymbol{\sigma}_{\boldsymbol{-}}\boldsymbol{\sigma}_{\boldsymbol{+}}$).

The second order approximation reads:

$$\begin{aligned}
\boldsymbol{H}^{2^{\text{nd}}}_{\text{rwa}} &= \boldsymbol{H}^{1^{\text{st}}}_{\text{rwa}} - i\overline{\left(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}\right)\left(\int_t (\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}})\right)} \\
&= \boldsymbol{H}^{1^{\text{st}}}_{\text{rwa}} + \left(\frac{|\boldsymbol{u}|^2}{8\omega_{\text{eg}}}\right)\boldsymbol{\sigma}_{\boldsymbol{z}} = \frac{\boldsymbol{u}^*}{2}\boldsymbol{\sigma}_{\boldsymbol{+}} + \frac{\boldsymbol{u}}{2}\boldsymbol{\sigma}_{\boldsymbol{-}} + \left(\frac{|\boldsymbol{u}|^2}{8\omega_{\text{eg}}}\right)\boldsymbol{\sigma}_{\boldsymbol{z}}.
\end{aligned}$$

The 2nd order correction $\frac{|\boldsymbol{u}|^2}{4\omega_{\text{eg}}}(\boldsymbol{\sigma}_{\boldsymbol{z}}/2)$ is called the Bloch-Siegert shift.

Take the first order approximation

$$(\Sigma) \quad i\frac{d}{dt}|\phi\rangle = \frac{(\boldsymbol{u}^*\boldsymbol{\sigma_+} + \boldsymbol{u}\boldsymbol{\sigma_-})}{2}|\phi\rangle = \frac{(\boldsymbol{u}^*|e\rangle\langle g| + \boldsymbol{u}|g\rangle\langle e|)}{2}|\phi\rangle$$

with control $\boldsymbol{u} \in \mathbb{C}$.

1. Take constant control $\boldsymbol{u}(t) = \Omega_r e^{i\theta}$ for $t \in [0, T]$, $T > 0$. Show that $i\frac{d}{dt}|\phi\rangle = \frac{\Omega_r(\cos\theta\boldsymbol{\sigma_x}+\sin\theta\boldsymbol{\sigma_y})}{2}|\phi\rangle$.
2. Set $\Theta_r = \frac{\Omega_r}{2}T$. Show that the solution at T of the propagator $\boldsymbol{U}_t \in SU(2)$, $i\frac{d}{dt}\boldsymbol{U} = \frac{\Omega_r(\cos\theta\boldsymbol{\sigma_x}+\sin\theta\boldsymbol{\sigma_y})}{2}\boldsymbol{U}$, $\boldsymbol{U}_0 = \boldsymbol{I}$ is given by
$$\boldsymbol{U}_T = \cos\Theta_r\boldsymbol{I} - i\sin\Theta_r\left(\cos\theta\boldsymbol{\sigma_x} + \sin\theta\boldsymbol{\sigma_y}\right),$$
3. Take any constant $|\bar{\phi}\rangle$. Show that there exist Ω_r and θ such that $\boldsymbol{U}_T|g\rangle = e^{i\alpha}|\bar{\phi}\rangle$, where α is some global phase.
4. Prove that for any given two wave functions $|\phi_a\rangle$ and $|\phi_b\rangle$ there exists a piecewise constant control $[0, 2T] \ni t \mapsto \boldsymbol{u}(t) \in \mathbb{C}$ such that the solution of (Σ) with $|\phi\rangle_0 = |\phi_a\rangle$ satisfies $|\phi\rangle_T = e^{i\beta}|\phi_b\rangle$ for some global phase β.

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The spin/spring model

The Schrödinger system

$$i\frac{d}{dt}|\psi\rangle = \left(\tfrac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\mathbf{I}}{2}\right) + i\tfrac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a})\right)|\psi\rangle$$

corresponds to two coupled scalar PDE's:

$$i\frac{\partial\psi_e}{\partial t} = +\frac{\omega_{eg}}{2}\psi_e + \frac{\omega_c}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_e - i\frac{\Omega}{\sqrt{2}}\frac{\partial}{\partial x}\psi_g$$
$$i\frac{\partial\psi_g}{\partial t} = -\frac{\omega_{eg}}{2}\psi_g + \frac{\omega_c}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_g - i\frac{\Omega}{\sqrt{2}}\frac{\partial}{\partial x}\psi_e$$

since $\boldsymbol{a} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$ and $|\psi\rangle$ corresponds to $(\psi_e(x,t), \psi_g(x,t))$ where $\psi_e(.,t), \psi_g(.,t) \in L^2(\mathbb{R},\mathbb{C})$ and $\|\psi_e\|^2 + \|\psi_g\|^2 = 1$.

Resonant case: passage to the interaction frame

In $\frac{\boldsymbol{H}}{\hbar} = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right) + i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a})$, take $\omega_{eg} = \omega_c + \Delta = \omega + \Delta$ with $|\Omega|, |\Delta| \ll \omega$. Then $\boldsymbol{H} = \boldsymbol{H}_0 + \epsilon\boldsymbol{H}_1$ where ϵ is a small parameter and

$$\frac{\boldsymbol{H}_0}{\hbar} = \frac{\omega}{2}\boldsymbol{\sigma_z} + \omega\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right)$$

$$\epsilon\frac{\boldsymbol{H}_1}{\hbar} = \frac{\Delta}{2}\boldsymbol{\sigma_z} + i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a}).$$

$\boldsymbol{H}_{\text{int}}$ is obtained by setting $|\psi\rangle = e^{-i\omega t\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right)} e^{\frac{-i\omega t}{2}\boldsymbol{\sigma_z}}|\phi\rangle$ in $i\hbar\frac{d}{dt}|\psi\rangle = \boldsymbol{H}|\psi\rangle$ to get $i\hbar\frac{d}{dt}|\phi\rangle = \boldsymbol{H}_{\text{int}}|\phi\rangle$ with

$$\frac{\boldsymbol{H}_{\text{int}}}{\hbar} = \frac{\Delta}{2}\boldsymbol{\sigma_z} + i\frac{\Omega}{2}\left(e^{-i\omega t}\boldsymbol{\sigma_-} + e^{i\omega t}\boldsymbol{\sigma_+}\right)\left(e^{i\omega t}\boldsymbol{a}^\dagger - e^{-i\omega t}\boldsymbol{a}\right)$$

where we used

$$e^{\frac{i\theta}{2}\boldsymbol{\sigma_z}}\,\boldsymbol{\sigma_x} e^{-\frac{i\theta}{2}\boldsymbol{\sigma_z}} = e^{-i\theta}\boldsymbol{\sigma_-} + e^{i\theta}\boldsymbol{\sigma_+}, \quad e^{i\theta\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right)}\,\boldsymbol{a}\,e^{-i\theta\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right)} = e^{-i\theta}\boldsymbol{a}$$

Resonant case: first order (Jaynes-Cummings Hamiltonian)

The secular terms in $\boldsymbol{H}_{\text{int}}$ are given by (RWA, first order approximation) $\boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}} = \frac{\Delta}{2}\boldsymbol{\sigma_z} + i\frac{\Omega}{2}\left(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}\right)$. Since quantum state $|\phi\rangle = e^{+i\omega t\left(\boldsymbol{a}^\dagger\boldsymbol{a}+\frac{\mathbf{I}}{2}\right)} e^{\frac{\pm i\omega t}{2}\boldsymbol{\sigma_z}}|\psi\rangle$ obeys approximatively to $i\hbar\frac{d}{dt}|\phi\rangle = \boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}}|\phi\rangle$, the original quantum state $|\psi\rangle$ is governed by

$$i\frac{d}{dt}|\psi\rangle = \left(\frac{\omega_{\text{eg}}}{2}\boldsymbol{\sigma_z} + \omega\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\mathbf{I}}{2}\right) + i\frac{\Omega}{2}\left(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}\right)\right)|\psi\rangle$$

The Jaynes-Cummings Hamiltonian ($\omega_{\text{eg}} = \omega_c + \Delta$ with $|\Delta| \ll \omega_c$) reads:

$$\boldsymbol{H}_{JC}/\hbar = \frac{\omega_{\text{eg}}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\mathbf{I}}{2}\right) + i\frac{\Omega}{2}\left(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}\right)$$

The corresponding PDE is (case $\Delta = 0$) :

$$i\frac{\partial\psi_e}{\partial t} = +\frac{\omega}{2}\psi_e + \frac{\omega}{2}(x^2 - \frac{\partial^2}{\partial x^2})\psi_e - i\frac{\Omega}{2\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)\psi_g$$

$$i\frac{\partial\psi_g}{\partial t} = -\frac{\omega}{2}\psi_g + \frac{\omega}{2}(x^2 - \frac{\partial^2}{\partial x^2})\psi_g + i\frac{\Omega}{2\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)\psi_e$$

For $\omega \gg |\Delta| \gg |\Omega|$, the dominant term in $\boldsymbol{H}^{1^{\text{st}}}_{\text{rwa}}$ is an isolated qubit. To make the interaction dominant, we go to the interaction frame with

$$\frac{\boldsymbol{H}_0}{\hbar} = \frac{\omega_{\text{eg}}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\mathbf{I}}{2}\right), \quad \epsilon\frac{\boldsymbol{H}_1}{\hbar} = i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a}).$$

By setting $|\psi\rangle = e^{-i\omega_c t\left(\boldsymbol{a}^\dagger\boldsymbol{a}+\frac{\mathbf{I}}{2}\right)} e^{\frac{-i\omega_{\text{eg}}t}{2}\boldsymbol{\sigma_z}}|\phi\rangle$ we get $i\hbar\frac{d}{dt}|\phi\rangle = \boldsymbol{H}_{\text{int}}|\phi\rangle$ with

$$\begin{aligned}\frac{\boldsymbol{H}_{\text{int}}}{\hbar} &= i\frac{\Omega}{2}\left(e^{-i\omega_{\text{eg}}t}\boldsymbol{\sigma_-} + e^{i\omega_{\text{eg}}t}\boldsymbol{\sigma_+}\right)\left(e^{i\omega_c t}\boldsymbol{a}^\dagger - e^{-i\omega_c t}\boldsymbol{a}\right) \\ &= i\frac{\Omega}{2}\left(e^{-i\Delta t}\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - e^{i\Delta t}\boldsymbol{\sigma_+}\boldsymbol{a} + e^{i(2\omega_c+\Delta)t}\boldsymbol{\sigma_+}\boldsymbol{a}^\dagger - e^{-i(2\omega_c+\Delta)t}\boldsymbol{\sigma_-}\boldsymbol{a}\right)\end{aligned}$$

Thus $\boldsymbol{H}^{1^{\text{st}}}_{\text{rwa}} = \overline{\boldsymbol{H}_{\text{int}}} = 0$: no secular term. We have to compute $\boldsymbol{H}^{2^{\text{nd}}}_{\text{rwa}} = \overline{\boldsymbol{H}_{\text{int}}} - i\overline{\left(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}\right)\left(\int_t(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}})\right)}$ where $\int_t(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}/\hbar$ corresponds to

$$\frac{-\Omega}{2}\left(\frac{e^{-i\Delta t}}{\Delta}\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger + \frac{e^{i\Delta t}}{\Delta}\boldsymbol{\sigma_+}\boldsymbol{a} - \frac{e^{i(2\omega_c+\Delta)t}}{2\omega_c+\Delta}\boldsymbol{\sigma_+}\boldsymbol{a}^\dagger - \frac{e^{-i(2\omega_c+\Delta)t}}{2\omega_c+\Delta}\boldsymbol{\sigma_-}\boldsymbol{a}\right)$$

The secular terms in $\boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}}$ are

$$\frac{-\Omega^2}{4\Delta}\left(\boldsymbol{\sigma_-\sigma_+ a^\dagger a} - \boldsymbol{\sigma_+\sigma_- a a^\dagger}\right) + \frac{-\Omega^2}{4(\omega_c+\omega_{\text{eg}})}\left(\boldsymbol{\sigma_-\sigma_+ a a^\dagger} - \boldsymbol{\sigma_+\sigma_- a^\dagger a}\right)$$

Since $|\Omega| \ll |\Delta| \ll \omega_{\text{eg}}, \omega_c$, we have $\frac{\Omega^2}{4(\omega_c+\omega_{\text{eg}})} \ll \frac{\Omega^2}{4\Delta}$

$$\boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}}/\hbar \approx \frac{\Omega^2}{4\Delta}\left(\boldsymbol{\sigma_z}\left(\boldsymbol{N} + \frac{\boldsymbol{I}}{2}\right) + \frac{\boldsymbol{I}}{2}\right).$$

Since quantum state $|\phi\rangle = e^{+i\omega_c t\left(\boldsymbol{N}+\frac{\boldsymbol{I}}{2}\right)} e^{\frac{+i\omega_{\text{eg}} t}{2}\boldsymbol{\sigma_z}}|\psi\rangle$ obeys approximatively to $i\hbar\frac{d}{dt}|\phi\rangle = \boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}}|\phi\rangle$, the original quantum state $|\psi\rangle$ is governed by $i\frac{d}{dt}|\psi\rangle = \left(\frac{\boldsymbol{H}_{disp}}{\hbar} + \frac{\Omega^2}{8\Delta}\right)|\psi\rangle$ with

$$\boldsymbol{H}_{disp}/\hbar = \frac{\omega_{\text{eg}}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{N} + \frac{\boldsymbol{I}}{2}\right) - \frac{\chi}{2}\boldsymbol{\sigma_z}\left(\boldsymbol{N} + \frac{\boldsymbol{I}}{2}\right) \quad \text{and } \chi = \frac{-\Omega^2}{2\Delta}$$

The corresponding PDE is :

$$i\frac{\partial\psi_e}{\partial t} = +\frac{\omega_{\text{eg}}}{2}\psi_e + \frac{1}{2}(\omega_c - \frac{\chi}{2})(x^2 - \frac{\partial^2}{\partial x^2})\psi_e$$

$$i\frac{\partial\psi_g}{\partial t} = -\frac{\omega_{\text{eg}}}{2}\psi_g + \frac{1}{2}(\omega_c + \frac{\chi}{2})(x^2 - \frac{\partial^2}{\partial x^2})\psi_g$$

Exercise: resonant spin-spring system with controls

Consider the resonant spin-spring model with $\Omega \ll |\omega|$:

$$\frac{\boldsymbol{H}}{\hbar} = \frac{\omega}{2}\boldsymbol{\sigma_z} + \omega\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right) + i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a}) + u(\boldsymbol{a} + \boldsymbol{a}^\dagger)$$

with a real control input $u(t) \in \mathbb{R}$:

1. Show that with the resonant control $u(t) = \boldsymbol{u}e^{-i\omega t} + \boldsymbol{u}^* e^{i\omega t}$ with complex amplitude $\boldsymbol{u}$ such that $|\boldsymbol{u}| \ll \omega$, the first order RWA approximation yields the following dynamics in the interaction frame:
$$i\frac{d}{dt}|\psi\rangle = \left(i\frac{\Omega}{2}(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}) + \boldsymbol{u}\boldsymbol{a}^\dagger + \boldsymbol{u}^*\boldsymbol{a}\right)|\psi\rangle$$
2. Set $\mathbf{v} \in \mathbb{C}$ solution of $\frac{d}{dt}\mathbf{v} = -i\boldsymbol{u}$ and consider the following change of frame $|\phi\rangle = D_{-\mathbf{v}}|\psi\rangle$ with the displacement operator $D_{-\mathbf{v}} = e^{-\mathbf{v}\boldsymbol{a}^\dagger + \mathbf{v}^*\boldsymbol{a}}$. Show that, up to a global phase change, we have, with $\tilde{\boldsymbol{u}} = i\frac{\Omega}{2}\mathbf{v}$,
$$i\frac{d}{dt}|\phi\rangle = \left(\frac{i\Omega}{2}(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}) + (\tilde{\boldsymbol{u}}\boldsymbol{\sigma_+} + \tilde{\boldsymbol{u}}^*\boldsymbol{\sigma_-})\right)|\phi\rangle$$
3. Take the orthonormal basis $\{|g,n\rangle, |e,n\rangle\}$ with $n \in \mathbb{N}$ being the photon number and where for instance $|g,n\rangle$ stands for the tensor product $|g\rangle \otimes |n\rangle$. Set $|\phi\rangle = \sum_n \phi_{g,n}|g,n\rangle + \phi_{e,n}|e,n\rangle$ with $\phi_{g,n}, \phi_{e,n} \in \mathbb{C}$ depending on t and $\sum_n |\phi_{g,n}|^2 + |\phi_{e,n}|^2 = 1$. Show that, for $n \geq 0$
$$i\frac{d}{dt}\phi_{g,n+1} = i\frac{\Omega}{2}\sqrt{n+1}\phi_{e,n} + \tilde{\boldsymbol{u}}^*\phi_{e,n+1}, \quad i\frac{d}{dt}\phi_{e,n} = -i\frac{\Omega}{2}\sqrt{n+1}\phi_{g,n+1} + \tilde{\boldsymbol{u}}\phi_{g,n}$$
and $i\frac{d}{dt}\phi_{g,0} = \tilde{\boldsymbol{u}}^*\phi_{e,0}$.
4. Assume that $|\phi\rangle_0 = |g,0\rangle$. Construct an open-loop control $[0,T] \ni t \mapsto \tilde{\boldsymbol{u}}(t)$ such that $|\phi\rangle_T \approx |g,1\rangle$ (hint: use an impulse for $t \in [0,\epsilon]$ followed by 0 on $[\epsilon, T]$ with $\epsilon \ll T$ and well chosen T).
5. Generalize the above open-loop control when the goal state $|\phi\rangle_T$ is $|g,n\rangle$ with any arbitrary photon number n.

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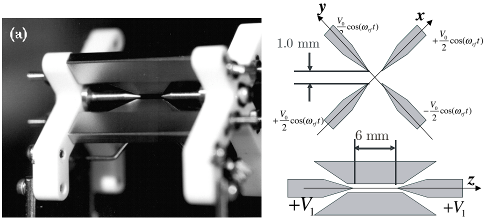
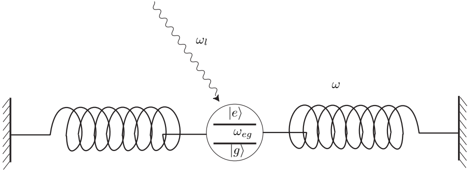
A single trapped ion



1D ion trap, picture borrowed from S. Haroche course at CDF.



A classical cartoon of spin-spring system.

A single trapped ion

A composite system:

internal degree of freedom+vibration inside the 1D trap

Hilbert space:

$$\mathbb{C}^2 \otimes L^2(\mathbb{R}, \mathbb{C})$$

Hamiltonian:

$$\frac{\boldsymbol{H}}{\hbar} = \omega_m \left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\mathbf{I}}{2} \right) + \frac{\omega_{\text{eg}}}{2} \boldsymbol{\sigma_z} + \left(u_l e^{i(\omega_l t - \eta_l(\boldsymbol{a}+\boldsymbol{a}^\dagger))} + u_l^* e^{-i(\omega_l t - \eta_l(\boldsymbol{a}+\boldsymbol{a}^\dagger))} \right) \boldsymbol{\sigma_x}$$

Parameters:
ω_m: harmonic oscillator of the trap,
ω_{eg}: optical transition of the internal state,
ω_l: lasers frequency,
$\eta_l = \omega_l / c$: Lamb-Dicke parameter.
Scales:

$$|\omega_l - \omega_{\text{eg}}| \ll \omega_{\text{eg}}, \quad \omega_m \ll \omega_{\text{eg}}, \quad |u_l| \ll \omega_{\text{eg}}, \quad \left| \frac{d}{dt} u_l \right| \ll \omega_{\text{eg}} |u_l|.$$

PDE formulation

The Schrödinger equation $i\hbar\frac{d}{dt}|\psi\rangle = \boldsymbol{H}|\psi\rangle$, with

$$\frac{\boldsymbol{H}}{\hbar} = \omega_m\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\mathbf{I}}{2}\right) + \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \left(u_l e^{i(\omega_l t - \eta_l(\boldsymbol{a}+\boldsymbol{a}^\dagger))} + u_l^* e^{-i(\omega_l t - \eta_l(\boldsymbol{a}+\boldsymbol{a}^\dagger))}\right)\boldsymbol{\sigma_x}$$

can be written in the form

$$i\frac{\partial\psi_g}{\partial t} = \frac{\omega_m}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_g - \frac{\omega_{eg}}{2}\psi_g + \left(u_l e^{i(\omega_l t - \sqrt{2}\eta_l x)} + u_l^* e^{-i(\omega_l t - \sqrt{2}\eta_l x)}\right)\psi_e,$$

$$i\frac{\partial\psi_e}{\partial t} = \frac{\omega_m}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_e + \frac{\omega_{eg}}{2}\psi_e + \left(u_l e^{i(\omega_l t - \sqrt{2}\eta_l x)} + u_l^* e^{-i(\omega_l t - \sqrt{2}\eta_l x)}\right)\psi_g.$$

- This system is approximately controllable in $(L^2(\mathbb{R},\mathbb{C}))^2$:
  S. Ervedoza and J.-P. Puel, Annales de l'IHP (c), 26(6): 2111-2136, 2009.

Law-Eberly method

Main idea

Control is superposition of 3 mono-chromatic plane waves with:

1. frequency ω_{eg} (ion transition frequency) and amplitude u;
2. frequency $\omega_{eg} - \omega_m$ (red shift by a vibration quantum) and amplitude u_r;
3. frequency $\omega_{eg} + \omega_m$ (blue shift by a vibration quantum) and amplitude u_b;

Control Hamiltonian:

$$\frac{\boldsymbol{H}}{\hbar} = \omega_m \left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\mathbf{I}}{2} \right) + \frac{\omega_{eg}}{2} \boldsymbol{\sigma_z} + \left(u e^{i(\omega_{eg} t - \eta(\boldsymbol{a}+\boldsymbol{a}^\dagger))} + u^* e^{-i(\omega_{eg} t - \eta(\boldsymbol{a}+\boldsymbol{a}^\dagger))} \right) \boldsymbol{\sigma_x}$$
$$+ \left(u_b e^{i((\omega_{eg}+\omega_m)t - \eta_b(\boldsymbol{a}+\boldsymbol{a}^\dagger))} + u_b^* e^{-i((\omega_{eg}+\omega_m)t - \eta_b(\boldsymbol{a}+\boldsymbol{a}^\dagger))} \right) \boldsymbol{\sigma_x}$$
$$+ \left(u_r e^{i((\omega_{eg}-\omega_m)t - \eta_r(\boldsymbol{a}+\boldsymbol{a}^\dagger))} + u_r^* e^{-i((\omega_{eg}-\omega_m)t - \eta_r(\boldsymbol{a}+\boldsymbol{a}^\dagger))} \right) \boldsymbol{\sigma_x}.$$

Lamb-Dicke parameters:

$$\eta = \eta_{eg} \approx \eta_r \approx \eta_b \ll 1.$$

Law-Eberly method: rotating frame

Rotating frame: $|\psi\rangle = e^{-i\omega_m t\left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\mathbf{I}}{2}\right)} e^{\frac{-i\omega_{\text{eg}} t}{2}\boldsymbol{\sigma_z}} |\phi\rangle$

$$\frac{\boldsymbol{H}_{\text{int}}}{\hbar} = e^{i\omega_m t\left(\boldsymbol{a}^\dagger \boldsymbol{a}\right)} \left(u e^{i\omega_{\text{eg}} t} e^{-i\eta(\boldsymbol{a}+\boldsymbol{a}^\dagger)} + u^* e^{-i\omega_{\text{eg}} t} e^{i\eta(\boldsymbol{a}+\boldsymbol{a}^\dagger)}\right)$$
$$e^{-i\omega_m t\left(\boldsymbol{a}^\dagger \boldsymbol{a}\right)} \left(e^{i\omega_{\text{eg}} t}|e\rangle\langle g| + e^{-i\omega_{\text{eg}} t}|g\rangle\langle e|\right)$$
$$+ e^{i\omega_m t\left(\boldsymbol{a}^\dagger \boldsymbol{a}\right)} \left(u_b e^{i(\omega_{\text{eg}}+\omega_m)t} e^{-i\eta_b(\boldsymbol{a}+\boldsymbol{a}^\dagger)} + u_b^* e^{-i(\omega_{\text{eg}}+\omega_m)t} e^{i\eta_b(\boldsymbol{a}+\boldsymbol{a}^\dagger)}\right)$$
$$e^{-i\omega_m t\left(\boldsymbol{a}^\dagger \boldsymbol{a}\right)} \left(e^{i\omega_{\text{eg}} t}|e\rangle\langle g| + e^{-i\omega_{\text{eg}} t}|g\rangle\langle e|\right)$$
$$+ e^{i\omega_m t\left(\boldsymbol{a}^\dagger \boldsymbol{a}\right)} \left(u_r e^{i(\omega_{\text{eg}}-\omega_m)t} e^{-i\eta_r(\boldsymbol{a}+\boldsymbol{a}^\dagger)} + u_r^* e^{-i(\omega_{\text{eg}}-\omega_m)t} e^{i\eta_r(\boldsymbol{a}+\boldsymbol{a}^\dagger)}\right)$$
$$e^{-i\omega_m t\left(\boldsymbol{a}^\dagger \boldsymbol{a}\right)} \left(e^{i\omega_{\text{eg}} t}|e\rangle\langle g| + e^{-i\omega_{\text{eg}} t}|g\rangle\langle e|\right)$$

Law-Eberly method: RWA

Commutation of exponentials in $(\boldsymbol{a} + \boldsymbol{a}^\dagger)$ and $(\boldsymbol{a}^\dagger \boldsymbol{a})$ is non-trivial.

- Approximation $e^{i\epsilon(\boldsymbol{a}+\boldsymbol{a}^\dagger)} \approx 1 + i\epsilon(\boldsymbol{a} + \boldsymbol{a}^\dagger)$ for $\epsilon = \pm\eta, \eta_b, \eta_r$

Then averaging: neglecting highly oscillating terms of frequencies $2\omega_{\mathrm{eg}}$, $2\omega_{\mathrm{eg}} \pm \omega_m$, $2(\omega_{\mathrm{eg}} \pm \omega_m)$ and $\pm\omega_m$, as

$$|u|, |u_b|, |u_r| \ll \omega_m, \quad \left|\frac{d}{dt}u\right| \ll \omega_m |u|, \quad \left|\frac{d}{dt}u_b\right| \ll \omega_m |u_b|, \quad \left|\frac{d}{dt}u_r\right| \ll \omega_m |u_r|.$$

First order approximation:

$$\frac{\boldsymbol{H}_{\mathrm{rwa}}}{\hbar} = u|g\rangle\langle e| + u^*|e\rangle\langle g| + \bar{u}_b \boldsymbol{a}|g\rangle\langle e| + \bar{u}_b^* \boldsymbol{a}^\dagger|e\rangle\langle g| + \bar{u}_r \boldsymbol{a}^\dagger|g\rangle\langle e| + \bar{u}_r^* \boldsymbol{a}|e\rangle\langle g|$$

where

$$\bar{u}_b = -i\eta_b u_b \quad \text{and} \quad \bar{u}_r = -i\eta_r u_r$$

PDE form

$$i\frac{\partial \phi_g}{\partial t} = \left(u + \frac{\bar{u}_b}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right) + \frac{\bar{u}_r}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)\right)\phi_e$$

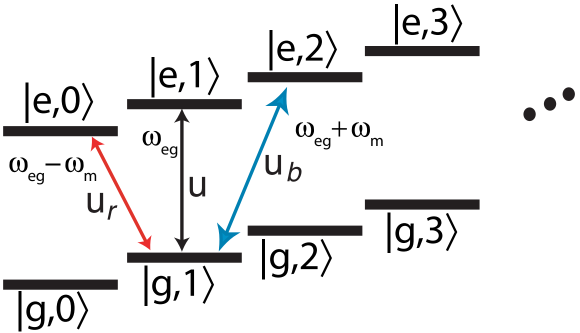
$$i\frac{\partial \phi_e}{\partial t} = \left(u^* + \frac{\bar{u}_b^*}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right) + \frac{\bar{u}_r^*}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)\right)\phi_g$$

Hilbert basis: $\{|g,n\rangle, |e,n\rangle\}_{n=0}^{\infty}$

Dynamics:

$$i\frac{d}{dt}\phi_{g,n} = u\phi_{e,n} + \bar{u}_r\sqrt{n}\phi_{e,n-1} + \bar{u}_b\sqrt{n+1}\phi_{e,n+1}$$
$$i\frac{d}{dt}\phi_{e,n} = u^*\phi_{g,n} + \bar{u}_r^*\sqrt{n+1}\phi_{g,n+1} + \bar{u}_b^*\sqrt{n}\phi_{g,n-1}$$

Physical interpretation:



Law-Eberly method: spectral controllability

Truncation to n-phonon space:

$$\mathcal{H}_n = \text{span}\{|g,0\rangle, |e,0\rangle, \ldots, |g,n\rangle, |e,n\rangle\}$$

We consider $|\phi\rangle_0, |\phi\rangle_T \in \mathcal{H}_n$ and we look for u, $\bar{u}_b$ and $\bar{u}_r$, s.t.

$$\text{for } |\phi\rangle(t=0) = |\phi\rangle_0 \text{ we have } |\phi\rangle(t=T) = |\phi\rangle_T.$$

- If u^1, $\bar{u}_b^1$ and $\bar{u}_r^1$ bring $|\phi\rangle_0$ to $|g,0\rangle$ at time $T/2$,
- and u^2, $\bar{u}_b^2$ and $\bar{u}_r^2$ bring $|\phi\rangle_T$ to $|g,0\rangle$ at time $T/2$,

then

$$\begin{aligned} u &= u^1, \quad & u_b &= u_b^1, \quad & u_r &= u_r^1 \quad \text{for } t \in [0, T/2], \\ u &= -u^2, \quad & u_b &= -u_b^2, \quad & u_r &= -u_r^2 \quad \text{for } t \in [T/2, T], \end{aligned}$$

bring $|\phi\rangle_0$ to $|\phi\rangle_T$ at time T.

Take $|\phi_0\rangle \in \mathcal{H}_n$ and $\overline{T} > 0$:

- For $t \in [0, \frac{\overline{T}}{2}]$, $\bar{u}_r(t) = \bar{u}_b(t) = 0$, and

$$\bar{u}(t) = \frac{2i}{\overline{T}} \arctan \left| \frac{\phi_{e,n}(0)}{\phi_{g,n}(0)} \right| e^{i \arg(\phi_{g,n}(0)\phi^*_{e,n}(0))}$$

  implies $\phi_{e,n}(\overline{T}/2) = 0$;

- For $t \in [\frac{\overline{T}}{2}, \overline{T}]$, $\bar{u}_b(t) = \bar{u}(t) = 0$, and

$$\bar{u}_r(t) = \frac{2i}{\overline{T}\sqrt{n}} \arctan \left| \frac{\phi_{g,n}(\frac{\overline{T}}{2})}{\phi_{e,n-1}(\frac{\overline{T}}{2})} \right| e^{i \arg\left(\phi_{g,n}(\frac{\overline{T}}{2})\phi^*_{e,n-1}(\frac{\overline{T}}{2})\right)}$$

  implies that $\phi_{e,n}(\overline{T}) = 0$ and that $\phi_{g,n}(\overline{T}) = 0$.

The two pulses $\bar{u}$ and $\bar{u}_r$ lead to some $|\phi\rangle(\overline{T}) \in \mathcal{H}_{n-1}$.

Law-Eberly method

Repeating n times, we have

$$|\phi\rangle(n\overline{T}) \in \mathcal{H}_0 = \text{span}\{|g,0\rangle, \langle e,0|\}.$$

- for $t \in [n\overline{T}, (n+\frac{1}{2})\overline{T}]$, the control

$$\bar{u}_r(t) = \bar{u}_b(t) = 0,$$
$$\bar{u}(t) = \frac{2i}{\overline{T}} \arctan \left| \frac{\phi_{e,0}(n\overline{T})}{\phi_{g,0}(n\overline{T})} \right| e^{i \arg\left(\phi_{g,0}(n\overline{T})\phi^*_{e,0}(n\overline{T})\right)}$$

implies $|\phi\rangle_{(n+\frac{1}{2})\overline{T}} = e^{i\theta}|g,0\rangle$.

Exercise: resonant spin-spring system with controls

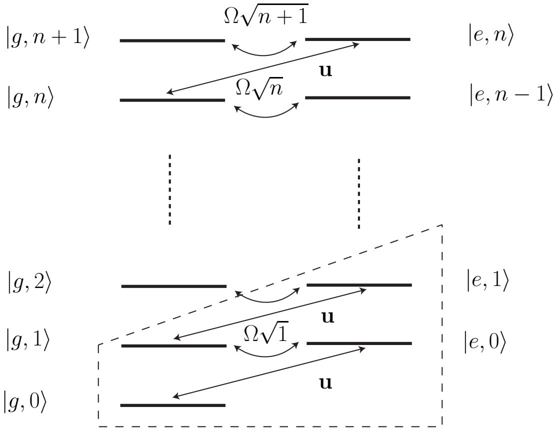
Consider the resonant spin-spring model with $\Omega \ll |\omega|$:

$$\frac{\boldsymbol{H}}{\hbar} = \frac{\omega}{2}\boldsymbol{\sigma_z} + \omega\left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right) + i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a}) + u(\boldsymbol{a} + \boldsymbol{a}^\dagger)$$

with a real control input $u(t) \in \mathbb{R}$:

1. Show that with the resonant control $u(t) = \boldsymbol{u}e^{i\omega t} + \boldsymbol{u}^* e^{-i\omega t}$ with complex amplitude $\boldsymbol{u}$ such that $|\boldsymbol{u}| \ll \omega$, the first order RWA approximation yields the following dynamics in the interaction frame:
$$i\frac{d}{dt}|\psi\rangle = \left(i\frac{\Omega}{2}(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}) + \boldsymbol{u}\boldsymbol{a}^\dagger + \boldsymbol{u}^*\boldsymbol{a}\right)|\psi\rangle$$
2. Set $\mathbf{v} \in \mathbb{C}$ solution of $\frac{d}{dt}\mathbf{v} = -i\boldsymbol{u}$ and consider the following change of frame $|\phi\rangle = D_{-\mathbf{v}}|\psi\rangle$ with the displacement operator $D_{-\mathbf{v}} = e^{-\mathbf{v}\boldsymbol{a}^\dagger + \mathbf{v}^*\boldsymbol{a}}$. Show that, up to a global phase change, we have, with $\tilde{\boldsymbol{u}} = i\frac{\Omega}{2}\mathbf{v}$,
$$i\frac{d}{dt}|\phi\rangle = \left(\frac{i\Omega}{2}(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}) + (\tilde{\boldsymbol{u}}\boldsymbol{\sigma_+} + \tilde{\boldsymbol{u}}^*\boldsymbol{\sigma_-})\right)|\phi\rangle$$
3. Take the orthonormal basis $\{|g,n\rangle, |e,n\rangle\}$ with $n \in \mathbb{N}$ being the photon number and where for instance $|g,n\rangle$ stands for the tensor product $|g\rangle \otimes |n\rangle$. Set $|\phi\rangle = \sum_n \phi_{g,n}|g,n\rangle + \phi_{e,n}|e,n\rangle$ with $\phi_{g,n}, \phi_{e,n} \in \mathbb{C}$ depending on t and $\sum_n |\phi_{g,n}|^2 + |\phi_{e,n}|^2 = 1$. Show that, for $n \geq 0$
$$i\frac{d}{dt}\phi_{g,n+1} = i\frac{\Omega}{2}\sqrt{n+1}\phi_{e,n} + \tilde{\boldsymbol{u}}^*\phi_{e,n+1}, \quad i\frac{d}{dt}\phi_{e,n} = -i\frac{\Omega}{2}\sqrt{n+1}\phi_{g,n+1} + \tilde{\boldsymbol{u}}\phi_{g,n}$$
and $i\frac{d}{dt}\phi_{g,0} = \tilde{\boldsymbol{u}}^*\phi_{e,0}$.
4. Assume that $|\phi\rangle_0 = |g,0\rangle$. Construct an open-loop control $[0,T] \ni t \mapsto \tilde{\boldsymbol{u}}(t)$ such that $|\phi\rangle_T \approx |g,1\rangle$ (hint: use an impulse for $t \in [0,\epsilon]$ followed by 0 on $[\epsilon, T]$ with $\epsilon \ll T$ and well chosen T).
5. Generalize the above open-loop control when the goal state $|\phi\rangle_T$ is $|g,n\rangle$ with any arbitrary photon number n.

Control for resonant spin-spring: schematic



Schematic of Jaynes-Cummings model

Control for resonant spin-spring: real case

We consider $|\phi\rangle_0$ and $|\phi\rangle_T$ in $\mathcal{H}_n$ such that:

$$\langle g,k \mid \phi_0\rangle\,,\langle e,k \mid \phi_0\rangle \in \mathbb{R} \qquad \text{and} \qquad \langle g,k \mid \phi_T\rangle\,,\langle e,k \mid \phi_T\rangle \in \mathbb{R},$$

and we consider pure imaginary controls: $\tilde{u} = i\tilde{v}$, $\tilde{v} \in \mathbb{R}$.

Model in the real case:

$$\begin{aligned}
\frac{d}{dt}\phi_{g,0} &= -\tilde{v}\phi_{e,0} \\
\frac{d}{dt}\phi_{g,n+1} &= -\frac{\Omega}{2}\sqrt{n+1}\phi_{e,n} - \tilde{v}\phi_{e,n+1}, \\
\frac{d}{dt}\phi_{e,n} &= \tfrac{\Omega}{2}\sqrt{n+1}\phi_{g,n+1} + \tilde{v}\phi_{g,n}.
\end{aligned}$$