Quantum Systems: Dynamics and Control[1]

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Outline

Recall: the three basic features of quantum models[5]

1. Schrödinger: wave funct. $|\psi\rangle \in \mathcal{H}$ or density op. $\rho \sim |\psi\rangle\langle\psi|$

$$\frac{d}{dt}|\psi\rangle = -\frac{i}{\hbar}\boldsymbol{H}|\psi\rangle, \quad \frac{d}{dt}\rho = -\frac{i}{\hbar}[\boldsymbol{H}, \rho], \quad \boldsymbol{H} = \boldsymbol{H}_0 + u\boldsymbol{H}_1$$

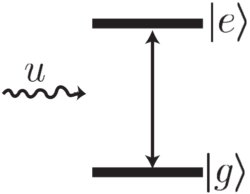
2. Entanglement and tensor product for composite systems (S, M):
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $\boldsymbol{H} = \boldsymbol{H}_S \otimes \boldsymbol{I}_M + \boldsymbol{H}_{int} + \boldsymbol{I}_S \otimes \boldsymbol{H}_M$
   - observable on sub-system M only: $\boldsymbol{O} = \boldsymbol{I}_S \otimes \boldsymbol{O}_M$.

3. Randomness and irreversibility induced by the measurement of observable $\boldsymbol{O}$ with spectral decomp. $\sum_\mu \lambda_\mu \boldsymbol{P}_\mu$:
   - measurement outcome μ with proba. $\mathbb{P}_\mu = \langle\psi|\boldsymbol{P}_\mu|\psi\rangle = \text{Tr}\left(\rho \boldsymbol{P}_\mu\right)$ depending on $|\psi\rangle$, ρ just before the measurement
   - measurement back-action if outcome $\mu = y$:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\boldsymbol{P}_y|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{P}_y|\psi\rangle}}, \quad \rho \mapsto \rho_+ = \frac{\boldsymbol{P}_y \rho \boldsymbol{P}_y}{\text{Tr}\left(\rho \boldsymbol{P}_y\right)}$$

[5] S. Haroche, J.M. Raimond: Exploring the Quantum: Atoms, Cavities and Photons. Oxford University Press, 2006.

2-level system (spin-1/2)



The simplest quantum system: a ground state $|g\rangle$ of energy ω_g; an excited state $|e\rangle$ of energy ω_e. The quantum state $|\psi\rangle \in \mathbb{C}^2$ is a linear superposition $|\psi\rangle = \psi_g|g\rangle + \psi_e|e\rangle$ and obeys to the Schrödinger equation (ψ_g and ψ_e depend on t).

Schrödinger equation for the uncontrolled 2-level system ($\hbar = 1$) :

$$\imath \frac{d}{dt}|\psi\rangle = \boldsymbol{H}_0|\psi\rangle = \left(\omega_e|e\rangle\langle e| + \omega_g|g\rangle\langle g|\right)|\psi\rangle$$

where $\boldsymbol{H}_0$ is the Hamiltonian, a Hermitian operator $\boldsymbol{H}_0^\dagger = \boldsymbol{H}_0$.
Energy is defined up to a constant: $\boldsymbol{H}_0$ and $\boldsymbol{H}_0 + \varpi(t)\boldsymbol{I}$, $\varpi(t) \in \mathbb{R}$ arbitrary, correspond to the same physical system. If $|\psi\rangle$ satisfies $i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}_0|\psi\rangle$ then $|\chi\rangle = e^{-i\vartheta(t)}|\psi\rangle$ with $\frac{d}{dt}\vartheta = \varpi$ obeys to $i\frac{d}{dt}|\chi\rangle = (\boldsymbol{H}_0 + \varpi\boldsymbol{I})|\chi\rangle$. Thus for any ϑ, $|\psi\rangle$ and $e^{-i\vartheta}|\psi\rangle$ represent the same physical system: The global phase of a quantum system $|\psi\rangle$ can be chosen arbitrarily at any time. Indeed, it is unobservable, it has no impact on measurement results nor dynamics.

Take origin of energy such that ω_g (resp. ω_e) becomes $-\frac{\omega_e-\omega_g}{2}$ (resp. $\frac{\omega_e-\omega_g}{2}$) and set $\omega_{eg} = \omega_e - \omega_g$
The solution of $i\frac{d}{dt}|\psi\rangle = H_0|\psi\rangle = \frac{\omega_{eg}}{2}(|e\rangle\langle e| - |g\rangle\langle g|)|\psi\rangle$ is

$$|\psi\rangle_t = \psi_{g0} e^{\frac{i\omega_{eg} t}{2}} |g\rangle + \psi_{e0} e^{\frac{-i\omega_{eg} t}{2}} |e\rangle.$$

With a classical electromagnetic field described by $u(t) \in \mathbb{R}$, the systems follows the controlled Hamiltonian

$$\boldsymbol{H}(t) = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \frac{u(t)}{2}\boldsymbol{\sigma_x} = \frac{\omega_{eg}}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + \frac{u(t)}{2}(|e\rangle\langle g| + |g\rangle\langle e|)$$

The controlled Schrödinger equation $i\frac{d}{dt}|\psi\rangle = (\boldsymbol{H}_0 + u(t)\boldsymbol{H}_1)|\psi\rangle$ reads:

$$i\frac{d}{dt}\begin{pmatrix}\psi_e\\ \psi_g\end{pmatrix} = \frac{\omega_{eg}}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_e\\ \psi_g\end{pmatrix} + \frac{u(t)}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_e\\ \psi_g\end{pmatrix}.$$

with the 3 Pauli Matrices[6]

$$\boldsymbol{\sigma_x} = |e\rangle\langle g| + |g\rangle\langle e|, \ \boldsymbol{\sigma_y} = -i|e\rangle\langle g| + i|g\rangle\langle e|, \ \boldsymbol{\sigma_z} = |e\rangle\langle e| - |g\rangle\langle g|$$

[6] They correspond, up to multiplication by i, to the 3 imaginary quaternions.

Pauli matrices and some formulas

$$\boldsymbol{\sigma_x} = |e\rangle\langle g| + |g\rangle\langle e|,\ \boldsymbol{\sigma_y} = -i|e\rangle\langle g| + i|g\rangle\langle e|,\ \boldsymbol{\sigma_z} = |e\rangle\langle e| - |g\rangle\langle g|$$

$$\boldsymbol{\sigma_x}^2 = \boldsymbol{I}, \quad \boldsymbol{\sigma_x}\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_z}, \quad [\boldsymbol{\sigma_x}, \boldsymbol{\sigma_y}] = 2i\boldsymbol{\sigma_z}, \ \text{circular permutation} \ \ldots$$

- Since for any $\theta \in \mathbb{R}$, $e^{i\theta\boldsymbol{\sigma_x}} = \cos\theta + i\sin\theta\boldsymbol{\sigma_x}$ (idem for $\boldsymbol{\sigma_y}$ and $\boldsymbol{\sigma_z}$), the solution of $i\frac{d}{dt}|\psi\rangle = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z}|\psi\rangle$ is

$$|\psi\rangle_t = e^{\frac{-i\omega_{eg}t}{2}\boldsymbol{\sigma_z}}|\psi\rangle_0 = \left(\cos\left(\frac{\omega_{eg}t}{2}\right)\boldsymbol{I} - i\sin\left(\frac{\omega_{eg}t}{2}\right)\boldsymbol{\sigma_z}\right)|\psi\rangle_0$$

- For $\alpha, \beta = x, y, z$, $\alpha \neq \beta$ we have

$$\boldsymbol{\sigma_\alpha} e^{i\theta\boldsymbol{\sigma_\beta}} = e^{-i\theta\boldsymbol{\sigma_\beta}}\boldsymbol{\sigma_\alpha}, \qquad \left(e^{i\theta\boldsymbol{\sigma_\alpha}}\right)^{-1} = \left(e^{i\theta\boldsymbol{\sigma_\alpha}}\right)^{\dagger} = e^{-i\theta\boldsymbol{\sigma_\alpha}}.$$

  and also $e^{-\frac{i\theta}{2}\boldsymbol{\sigma_\alpha}}\boldsymbol{\sigma_\beta} e^{\frac{i\theta}{2}\boldsymbol{\sigma_\alpha}} = e^{-i\theta\boldsymbol{\sigma_\alpha}}\boldsymbol{\sigma_\beta} = \boldsymbol{\sigma_\beta} e^{i\theta\boldsymbol{\sigma_\alpha}}$

- Similarly to the harmonic oscillator, energy annihilation and creation operators: $\boldsymbol{\sigma_-} = |g\rangle\langle e|$, $\boldsymbol{\sigma_+} = \boldsymbol{\sigma_-}^{\dagger} = |e\rangle\langle g|$

Density matrix and Bloch Sphere

Consider the density operator $\rho = |\psi\rangle\langle\psi|$. Thus ρ is an Hermitian operator, ≥ 0, that satisfies $\text{Tr}\,(\rho) = 1$, $\rho^2 = \rho$ and obeys to the Liouville equation:

$$\frac{d}{dt}\rho = -i[\boldsymbol{H}, \rho].$$

For a two level system $|\psi\rangle = \psi_g|g\rangle + \psi_e|e\rangle$ and

$$\rho = \frac{\boldsymbol{I} + x\boldsymbol{\sigma_x} + y\boldsymbol{\sigma_y} + z\boldsymbol{\sigma_z}}{2}$$

where
$$\begin{aligned}(x, y, z) &= (2\Re(\psi_g\psi_e^*), 2\Im(\psi_g\psi_e^*), |\psi_e|^2 - |\psi_g|^2) \in \mathbb{R}^3 \\ &= (\text{Tr}\,(\boldsymbol{\sigma_x}\rho)\,, \text{Tr}\,(\boldsymbol{\sigma_y}\rho)\,, \text{Tr}\,(\boldsymbol{\sigma_z}\rho))\end{aligned}$$

The Bloch vector $\vec{M} = (x, y, z)$ evolves on the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3 = \text{span}(\vec{e}_x, \vec{e}_y, \vec{e}_z)$, called the the Bloch Sphere, since $\text{Tr}\left(\rho^2\right) = x^2 + y^2 + z^2 = 1$. The Liouville equation with $\boldsymbol{H} = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \frac{u}{2}\boldsymbol{\sigma_x}$ corresponds to

$$\frac{d}{dt}\vec{M} = (u\vec{e}_x + \omega_{eg}\vec{e}_z) \times \vec{M}.$$

Exercise

Consider $\boldsymbol{H} = (u\boldsymbol{\sigma_x} + v\boldsymbol{\sigma_y} + w\boldsymbol{\sigma_z})/2$ with $(u, v, w) \in \mathbb{R}^3$.

1. For (u, v, w) constant and non zero, compute the solutions of

$$\frac{d}{dt}|\psi\rangle = -i\boldsymbol{H}|\psi\rangle, \quad \frac{d}{dt}\boldsymbol{U} = -i\boldsymbol{H}\boldsymbol{U} \text{ with } \boldsymbol{U}_0 = \boldsymbol{I}$$

   in term of $|\psi\rangle_0$, $\boldsymbol{\sigma} = (u\boldsymbol{\sigma_x} + v\boldsymbol{\sigma_y} + w\boldsymbol{\sigma_z})/\sqrt{u^2+v^2+w^2}$ and $\omega = \sqrt{u^2+v^2+w^2}$. Indication: use the fact that $\boldsymbol{\sigma}^2 = \boldsymbol{I}$.

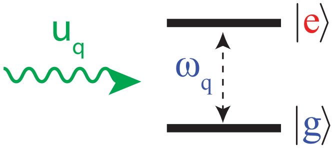
2. Assume that, (u, v, w) depends on t according to $(u, v, w)(t) = \omega(t)(\bar{u}, \bar{v}, \bar{w})$ with $(\bar{u}, \bar{v}, \bar{w}) \in \mathbb{R}^3/\{0\}$ constant of length 1. Compute the solutions of

$$\frac{d}{dt}|\psi\rangle = -i\boldsymbol{H}(t)|\psi\rangle, \quad \frac{d}{dt}\boldsymbol{U} = -i\boldsymbol{H}(t)\boldsymbol{U} \text{ with } \boldsymbol{U}_0 = \boldsymbol{I}$$

   in term of $|\psi\rangle_0$, $\overline{\boldsymbol{\sigma}} = \bar{u}\boldsymbol{\sigma_x} + \bar{v}\boldsymbol{\sigma_y} + \bar{w}\boldsymbol{\sigma_z}$ and $\theta(t) = \int_0^t \omega$.

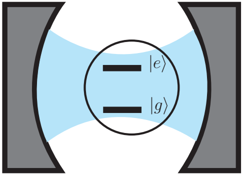
3. Explain why (u, v, w) colinear to the constant vector $(\bar{u}, \bar{v}, \bar{w})$ is crucial, for the computations in previous question.

- Hilbert space:
  $\mathcal{H}_M = \mathbb{C}^2 = \left\{ \psi_g |g\rangle + \psi_e |e\rangle,\ \psi_g, \psi_e \in \mathbb{C} \right\}.$
- Operators and commutations:
  $\boldsymbol{\sigma_-} = |g\rangle\langle e|,\ \boldsymbol{\sigma_+} = \boldsymbol{\sigma_-}^\dagger = |e\rangle\langle g|$
  $\boldsymbol{\sigma_x} = \boldsymbol{\sigma_-} + \boldsymbol{\sigma_+} = |g\rangle\langle e| + |e\rangle\langle g|;$
  $\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_-} - i\boldsymbol{\sigma_+} = i|g\rangle\langle e| - i|e\rangle\langle g|;$
  $\boldsymbol{\sigma_z} = \boldsymbol{\sigma_+}\boldsymbol{\sigma_-} - \boldsymbol{\sigma_-}\boldsymbol{\sigma_+} = |e\rangle\langle e| - |g\rangle\langle g|;$
  $\boldsymbol{\sigma_x}^2 = \boldsymbol{I},\ \boldsymbol{\sigma_x}\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_z},\ [\boldsymbol{\sigma_x}, \boldsymbol{\sigma_y}] = 2i\boldsymbol{\sigma_z},\ \ldots$
- Hamiltonian: $\boldsymbol{H}_M = \omega_q \boldsymbol{\sigma_z}/2 + \boldsymbol{u_q}\boldsymbol{\sigma_x}.$



Outline

Composite system: 2-level and harmonic oscillator



2-level system lives on $\mathbb{C}^2$ with $\boldsymbol{H_q} = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z}$
oscillator lives on $L^2(\mathbb{R},\mathbb{C}) \sim l^2(\mathbb{C})$ with

$$\boldsymbol{H_c} = -\frac{\omega_c}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega_c}{2}x^2 \sim \omega_c\left(\boldsymbol{N} + \frac{\boldsymbol{I}}{2}\right)$$

$$\boldsymbol{N} = \boldsymbol{a}^\dagger \boldsymbol{a} \text{ and } \boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P} \sim \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$$

The composite system lives on the tensor product
$\mathbb{C}^2 \otimes L^2(\mathbb{R},\mathbb{C}) \sim \mathbb{C}^2 \otimes l^2(\mathbb{C})$ with spin-spring Hamiltonian

$$\boldsymbol{H} = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} \otimes \boldsymbol{I_c} + \omega_c\,\boldsymbol{I_q} \otimes \left(\boldsymbol{N} + \frac{\boldsymbol{I}}{2}\right) + i\frac{\Omega}{2}\boldsymbol{\sigma_x} \otimes (\boldsymbol{a}^\dagger - \boldsymbol{a})$$

with the typical scales $\Omega \ll \omega_c, \omega_{eg}$ and $|\omega_c - \omega_{eg}| \ll \omega_c, \omega_{eg}$.
Shortcut notations:

$$\boldsymbol{H} = \underbrace{\frac{\omega_{eg}}{2}\boldsymbol{\sigma_z}}_{\boldsymbol{H_q}} + \underbrace{\omega_c\left(\boldsymbol{N} + \frac{\boldsymbol{I}}{2}\right)}_{\boldsymbol{H_c}} + \underbrace{i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a})}_{\boldsymbol{H_{int}}}$$

The spin-spring PDE

The Schrödinger system

$$i\frac{d}{dt}|\psi\rangle = \left(\tfrac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{N} + \frac{\mathbf{I}}{2}\right) + i\tfrac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a})\right)|\psi\rangle$$

corresponds to two coupled scalar PDE's:

$$i\frac{\partial\psi_e}{\partial t} = +\tfrac{\omega_{eg}}{2}\psi_e + \tfrac{\omega_c}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_e - i\frac{\Omega}{\sqrt{2}}\frac{\partial}{\partial x}\psi_g$$

$$i\frac{\partial\psi_g}{\partial t} = -\tfrac{\omega_{eg}}{2}\psi_g + \tfrac{\omega_c}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_g - i\frac{\Omega}{\sqrt{2}}\frac{\partial}{\partial x}\psi_e$$

since $\boldsymbol{N} = \boldsymbol{a}^\dagger\boldsymbol{a}$, $\boldsymbol{a} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$ and $|\psi\rangle = (\psi_e(x,t), \psi_g(x,t))$, $\psi_g(.,t), \psi_e(.,t) \in L^2(\mathbb{R},\mathbb{C})$ and $\|\psi_g\|^2 + \|\psi_e\|^2 = 1$.

Exercise: write the PDE for the controlled Hamiltonian

$$\tfrac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{N} + \tfrac{\boldsymbol{I}}{2}\right) + i\tfrac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a}) + u_c(\boldsymbol{a} + \boldsymbol{a}^\dagger) + u_q\boldsymbol{\sigma_x}$$

where $u_c, u_q \in \mathbb{R}$ are local control inputs associated to the oscillator and qubit, respectively.

The spin-spring ODE's

The Schrödinger system

$$i\frac{d}{dt}|\psi\rangle = \left(\frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{N} + \frac{\boldsymbol{I}}{2}\right) + i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a})\right)|\psi\rangle$$

corresponds also to an infinite set of ODE's

$$i\frac{d}{dt}\psi_{e,n} = ((n+1/2)\omega_c + \omega_{eg}/2)\psi_{e,n} + i\frac{\Omega}{2}\left(\sqrt{n}\,\psi_{g,n-1} - \sqrt{n+1}\,\psi_{g,n+1}\right)$$
$$i\frac{d}{dt}\psi_{g,n} = ((n+1/2)\omega_c - \omega_{eg}/2)\psi_{g,n} + i\frac{\Omega}{2}\left(\sqrt{n}\,\psi_{e,n-1} - \sqrt{n+1}\,\psi_{e,n+1}\right)$$

where $|\psi\rangle = \sum_{n=0}^{+\infty}\psi_{g,n}|g,n\rangle + \psi_{e,n}|e,n\rangle$, $\psi_{g,n}, \psi_{e,n} \in \mathbb{C}$.

Exercise: write the infinite set of ODE's for

$$\frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{N} + \frac{\boldsymbol{I}}{2}\right) + i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a}) + u_c(\boldsymbol{a} + \boldsymbol{a}^\dagger) + u_q\boldsymbol{\sigma_x}$$

where $u_c, u_q \in \mathbb{R}$ are local control inputs associated to the oscillator and qubit, respectively.

Dispersive case: approximate Hamiltonian for $\Omega \ll |\omega_c - \omega_{eg}|$.

$$\boldsymbol{H} \approx \boldsymbol{H}_{\text{disp}} = \tfrac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{N} + \tfrac{\boldsymbol{I}}{2}\right) - \frac{\chi}{2}\,\boldsymbol{\sigma_z}\left(\boldsymbol{N} + \tfrac{\boldsymbol{I}}{2}\right) \quad \text{with } \chi = \tfrac{\Omega^2}{2(\omega_c - \omega_{eg})}$$

The corresponding PDE is :

$$i\frac{\partial \psi_e}{\partial t} = +\frac{\omega_{eg}}{2}\psi_e + \frac{1}{2}(\omega_c - \frac{\chi}{2})(x^2 - \frac{\partial^2}{\partial x^2})\psi_e$$

$$i\frac{\partial \psi_g}{\partial t} = -\frac{\omega_{eg}}{2}\psi_g + \frac{1}{2}(\omega_c + \frac{\chi}{2})(x^2 - \frac{\partial^2}{\partial x^2})\psi_g$$

The propagator, the t-dependant unitary operator $\boldsymbol{U}$ solution of $i\frac{d}{dt}\boldsymbol{U} = \boldsymbol{H}\boldsymbol{U}$ with $\boldsymbol{U}(0) = \boldsymbol{I}$, reads:

$$\boldsymbol{U}(t) = e^{i\omega_{eg}t/2}\exp\left(-i(\omega_c + \chi/2)t(\boldsymbol{N} + \tfrac{\boldsymbol{I}}{2})\right) \otimes |g\rangle\langle g| + e^{-i\omega_{eg}t/2}\exp\left(-i(\omega_c - \chi/2)t(\boldsymbol{N} + \tfrac{\boldsymbol{I}}{2})\right) \otimes |e\rangle\langle e|$$

Exercise: write the infinite set of ODE's attached to the dispersive Hamiltonian $\boldsymbol{H}_{\text{disp}}$.

Resonant case: approximate Hamiltonian for $\omega_c = \omega_{eg} = \omega$.

The Hamiltonian becomes (Jaynes-Cummings Hamiltonian):

$$\boldsymbol{H} \approx \boldsymbol{H}_{JC} = \tfrac{\omega}{2}\boldsymbol{\sigma_z} + \omega\left(\boldsymbol{N} + \frac{\mathbf{I}}{2}\right) + i\tfrac{\Omega}{2}(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}).$$

The corresponding PDE is :

$$i\frac{\partial\psi_e}{\partial t} = +\frac{\omega}{2}\psi_e + \frac{\omega}{2}(x^2 - \frac{\partial^2}{\partial x^2})\psi_e - i\tfrac{\Omega}{2\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)\psi_g$$

$$i\frac{\partial\psi_g}{\partial t} = -\frac{\omega}{2}\psi_g + \frac{\omega}{2}(x^2 - \frac{\partial^2}{\partial x^2})\psi_g + i\tfrac{\Omega}{2\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)\psi_e$$

Exercise: Write the infinite set of ODE's attached to the Jaynes-Cummings Hamiltonian $\boldsymbol{H}$.

Exercise: For $\boldsymbol{H}_{JC} = \frac{\omega}{2}\boldsymbol{\sigma_z} + \omega\left(\boldsymbol{N} + \frac{\boldsymbol{I}}{2}\right) + i\frac{\Omega}{2}(\boldsymbol{\sigma_{\text{-}}}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_{+}}\boldsymbol{a})$ show that the propagator, the t-dependant unitary operator $\boldsymbol{U}$ solution of $i\frac{d}{dt}\boldsymbol{U} = \boldsymbol{H}_{JC}\boldsymbol{U}$ with $\boldsymbol{U}(0) = \boldsymbol{I}$, reads
$\boldsymbol{U}(t) = e^{-i\omega t\left(\frac{\boldsymbol{\sigma_z}}{2} + \boldsymbol{N} + \frac{1}{2}\right)} e^{\frac{\Omega t}{2}(\boldsymbol{\sigma_{\text{-}}}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_{+}}\boldsymbol{a})}$ where for any angle θ,

$$e^{\theta(\boldsymbol{\sigma_{\text{-}}}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_{+}}\boldsymbol{a})} = |g\rangle\langle g| \otimes \cos(\theta\sqrt{\boldsymbol{N}}) + |e\rangle\langle e| \otimes \cos(\theta\sqrt{\boldsymbol{N}+\boldsymbol{I}}) - \boldsymbol{\sigma_{+}} \otimes \boldsymbol{a}\frac{\sin(\theta\sqrt{\boldsymbol{N}})}{\sqrt{\boldsymbol{N}}} + \boldsymbol{\sigma_{\text{-}}} \otimes \frac{\sin(\theta\sqrt{\boldsymbol{N}})}{\sqrt{\boldsymbol{N}}}\boldsymbol{a}^\dagger$$

Hint: show that

$$\left[\frac{\boldsymbol{\sigma_z}}{2} + \boldsymbol{N}\,,\ \boldsymbol{\sigma_{\text{-}}}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_{+}}\boldsymbol{a}\right] = 0$$

$$\left(\boldsymbol{\sigma_{\text{-}}}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_{+}}\boldsymbol{a}\right)^{2k} = (-1)^k\left(|g\rangle\langle g| \otimes \boldsymbol{N}^k + |e\rangle\langle e| \otimes (\boldsymbol{N}+\boldsymbol{I})^k\right)$$

$$\left(\boldsymbol{\sigma_{\text{-}}}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_{+}}\boldsymbol{a}\right)^{2k+1} = (-1)^k\left(\boldsymbol{\sigma_{\text{-}}} \otimes \boldsymbol{N}^k\boldsymbol{a}^\dagger - \boldsymbol{\sigma_{+}} \otimes \boldsymbol{a}\boldsymbol{N}^k\right)$$

and compute the series defining the exponential of an operator.